Epistemic Game Theory: Incomplete Information

Part III: Symmetric Beliefs

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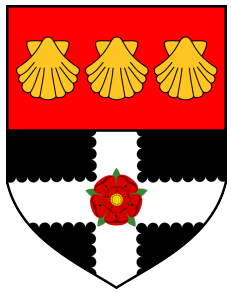


The Beliefs of All Agents stem from a Joint Source

- A classical idea in economics is the common prior assumption (CPA): accordingly, all beliefs stem from a joint source.
- In other words, differences in beliefs among agents can be entirely explained by differences in information.
- In games with **complete information**, it has been shown that common belief in rationality together with the CPA epistemically characterizes Correlated Equilibirum.
- Also, the CPA has been furnished with a conceptually illuminating characterization by means of symmetric beliefs.
- In this part things are extended to **incomplete information**.
- It turns out that the CPA together with common belief in rationality is equivalent to Bayesian Equilibrium.

Example: The Moonlight Serenade

- *You* had a fight with *Barbara* and contemplate about three ways of apologizing to her:
  - perform a moonlight serenade outside her house,
  - bring her a box of her favourite chocolate,
  - send your common friend Chris to apologize for you.

- When the doorbell rings *Barbara* can open up or ignore the bell.

- *Your* preferences are captured by the following decision problem:

$\Gamma_y(u_y)$

	open	*ignore*
serenade	4	0
chocolate	0	4
Chris	3	3

Example: The Moonlight Serenade

- *You* are uncertain about *Barbara*'s preferences and whether she will be in an angry or a forgiving mood.
- *Her* preferences are captured by the following two decision problems:

$\Gamma_B(u_B^{an})$

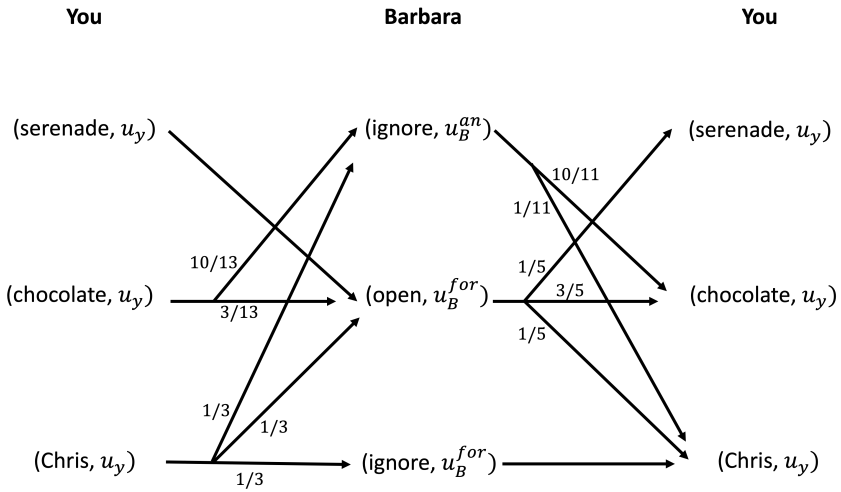
	serenade	*chocolate*	*Chris*
open	0	0	0
ignore	1	1	1

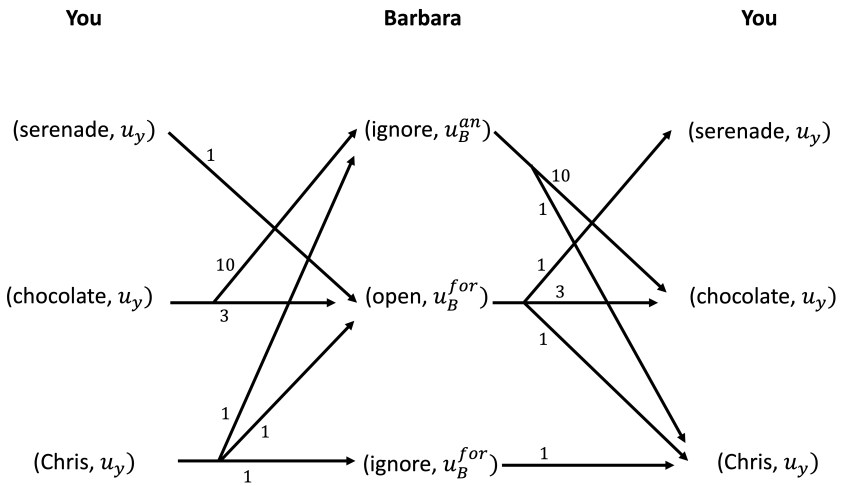
$\Gamma_B(u_B^{for})$

	serenade	*chocolate*	*Chris*
open	0	1	0
ignore	1	0	1

Example: The Moonlight Serenade

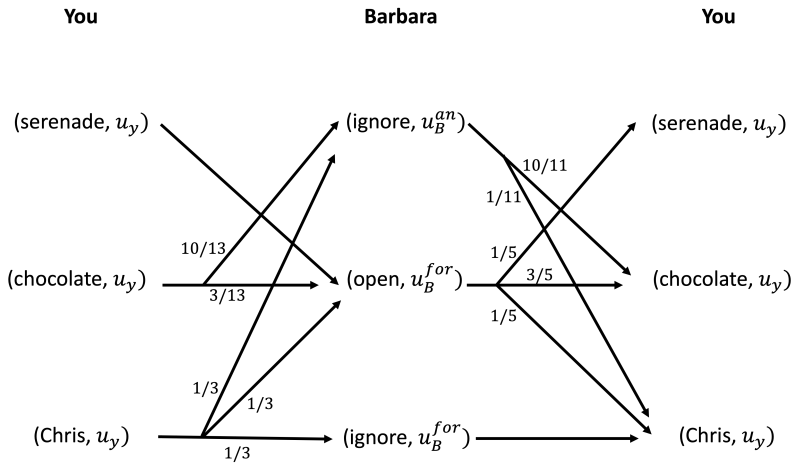
A beliefs diagram with an associated weighted beliefs diagram:

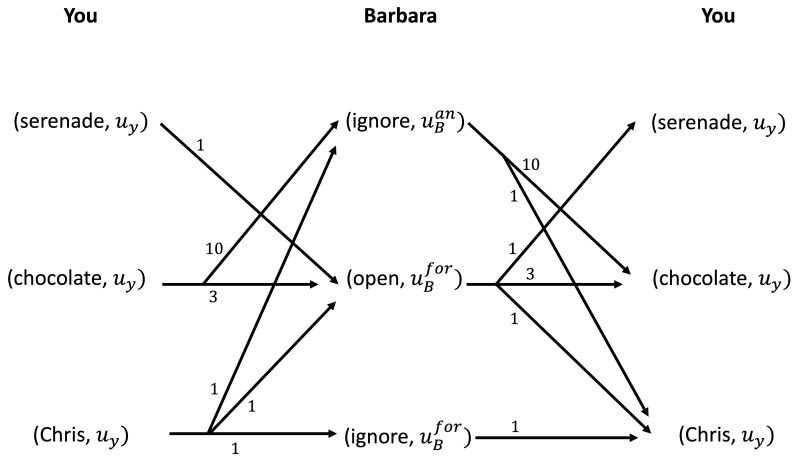




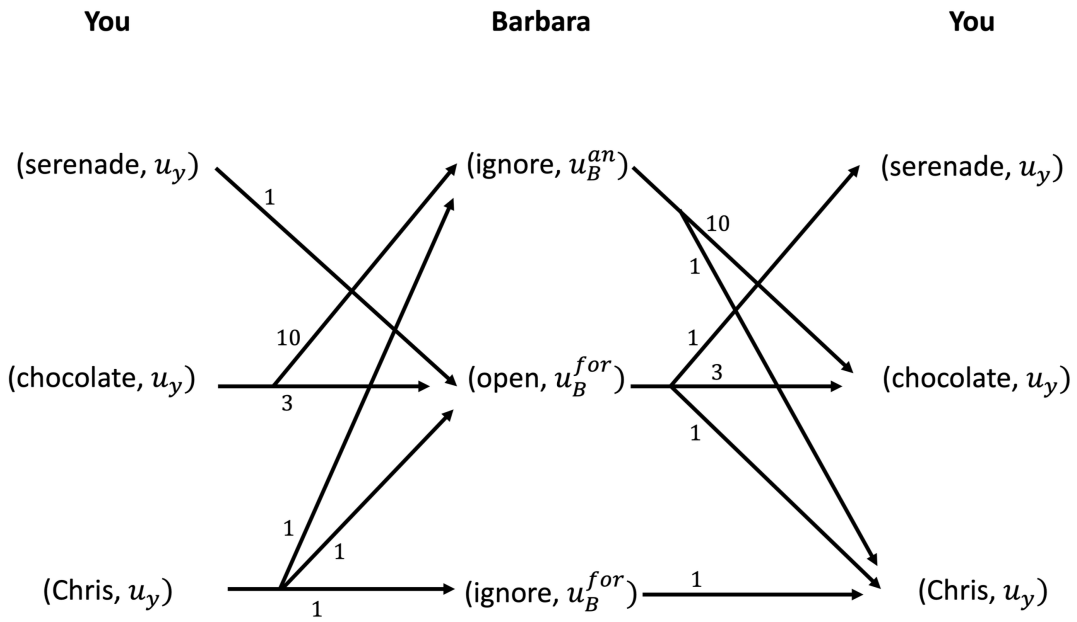
Example: The Moonlight Serenade

In fact all the relative weights of the outgoing arrows in the weighted beliefs diagram correspond precisely to the probabilities of these outgoing arrows in the induced beliefs diagram on top.





Example: The Moonlight Serenade

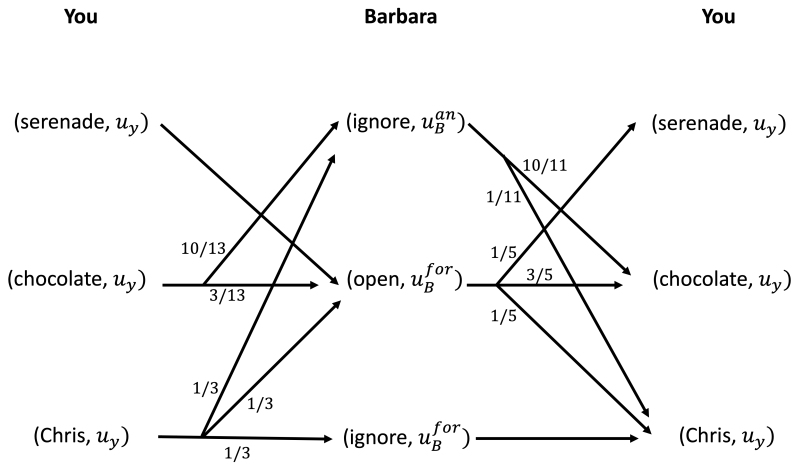


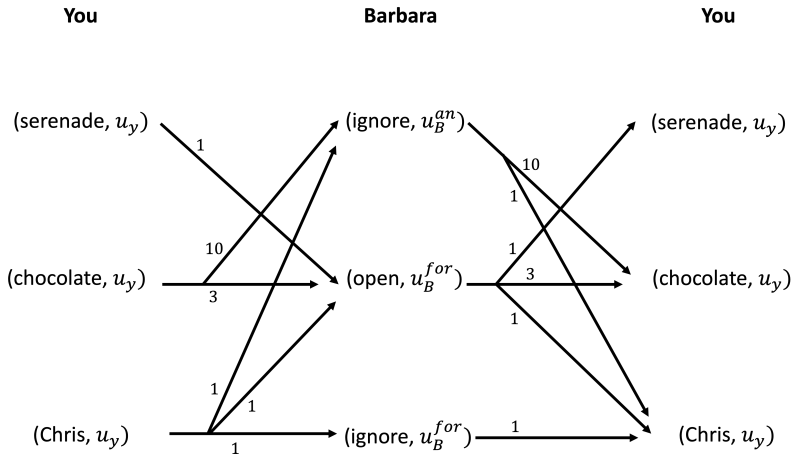
Observe that the weighted beliefs diagram is symmetric:

- For every arrow from a choice utility function pair (c_y, u_y) of yours to a choice utility function pair (c_B, u_B) of Barbara, the symmetric counterpart, which is an arrow from (c_B, u_B) to (c_y, u_y), also exists.
- For every arrow from a choice utility function pair (c_B, u_B) of Barbara to a choice utility function pair (c_y, u_y) of yours, the symmetric counterpart, which is an arrow from (c_y, u_y) to (c_B, u_B), also exists.
- Every arrow carries the same weight as its symmetric counterpart.

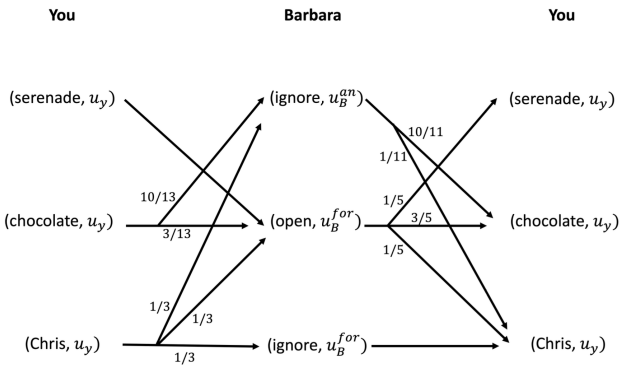
Example: The Moonlight Serenade

As the beliefs diagram on top is induced by the weighted beliefs diagram below that is symmetric, all belief hierarchies in the beliefs diagram are said to be symmetric.





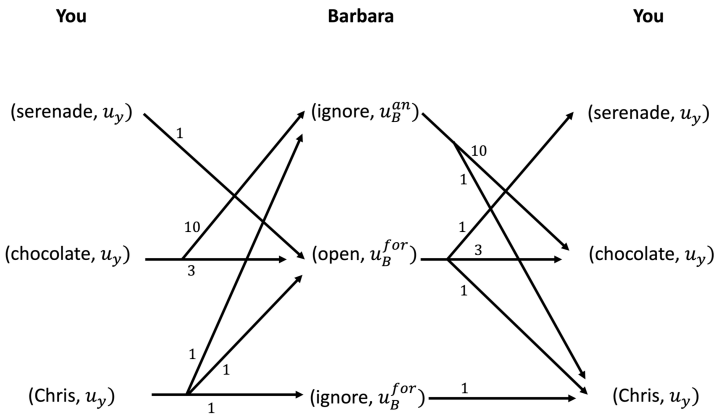
Example: The Moonlight Serenade



This beliefs diagram can be transformed into the following epistemic model:

- $T_{you} = \{t_y^1, t_y^2, t_y^3\}$ and $T_{Barbara} = \{t_B^1, t_B^2, t_B^3\}$,
- $v_{you}[t_y^1] = v_{you}[t_y^2] = v_{you}[t_y^3] = u_y$
- $v_{Barbara}[t_B^1] = u_B^{an}$ and $v_{Barbara}[t_B^2] = v_{Barbara}[t_B^3] = u_B^{for}$,
- $b_{you}[t_y^1] = (open, t_B^2)$,
- $b_{you}[t_y^2] = \frac{10}{13} \cdot (ignore, t_B^1) + \frac{3}{13} \cdot (open, t_B^2)$,
- $b_{you}[t_y^3] = \frac{1}{3} \cdot (ignore, t_B^1) + \frac{1}{3} \cdot (open, t_B^2) + \frac{1}{3} \cdot (ignore, t_B^3)$,
- $b_{Barbara}[t_B^1] = \frac{10}{11} \cdot (chocolate, t_y^2) + \frac{1}{11} \cdot (Chris, t_y^3)$,
- $b_{Barbara}[t_B^2] = \frac{1}{5} \cdot (serenade, t_y^1) + \frac{3}{5} \cdot (chocolate, t_y^2) + \frac{1}{5} \cdot (Chris, t_y^3)$,
- $b_{Barbara}[t_B^3] = (Chris, t_y^3)$.

Example: The Moonlight Serenade



Constructing a common prior $\pi \in \Delta(C \times T)$ for the epistemic model:

	weights	probabilities $\pi(\cdot)$
$\left((serenade, t_y^1), (open, t_B^2)\right)$	1	$\frac{1}{17}$
$\left((chocolate, t_y^2), (ignore, t_B^1)\right)$	10	$\frac{10}{17}$
$\left((chocolate, t_y^2), (open, t_B^2, u_B^{for})\right)$	3	$\frac{3}{17}$
$\left((Chris, t_y^3), (ignore, t_B^1)\right)$	1	$\frac{1}{17}$
$\left((Chris, t_y^3), (open, t_B^2)\right)$	1	$\frac{1}{17}$
$\left((Chris, t_y^3), (ignore, t_B^3)\right)$	1	$\frac{1}{17}$
$\sum$	17	1

Example: The Moonlight Serenade

	weights	probabilities $\pi(\cdot)$
$\left((serenade, t_y^1), (open, t_B^2)\right)$	1	$\frac{1}{17}$
$\left((chocolate, t_y^2), (ignore, t_B^1)\right)$	10	$\frac{10}{17}$
$\left((chocolate, t_y^2), (open, t_B^2)\right)$	3	$\frac{3}{17}$
$\left((Chris, t_y^3), (ignore, t_B^1)\right)$	1	$\frac{1}{17}$
$\left((Chris, t_y^3), (open, t_B^2)\right)$	1	$\frac{1}{17}$
$\left((Chris, t_y^3), (ignore, t_B^3)\right)$	1	$\frac{1}{17}$
$\sum$	17	1

It can be verified that all beliefs in the epistemic model indeed stem from π as a common prior, e.g.:

$$b_{you}[t_y^1](open, t_B^2) = \frac{\pi\left((serenade, t_y^1), (open, t_B^2)\right)}{\pi(serenade, t_y^1)} = \frac{\frac{1}{17}}{\frac{1}{17}} = 1$$

$$b_{you}[t_y^2](ignore, t_B^1) = \frac{\pi\left((chocolate, t_y^2), (ignore, t_B^1)\right)}{\pi(chocolate, t_y^2)} = \frac{\frac{10}{17}}{\frac{10}{17} + \frac{3}{17}} = \frac{10}{13}$$

$$b_{you}[t_y^2](open, t_B^2) = \frac{\pi\left((chocolate, t_y^2), (open, t_B^2)\right)}{\pi(chocolate, t_y^2)} = \frac{\frac{3}{17}}{\frac{10}{17} + \frac{3}{17}} = \frac{3}{13}$$

Outline

- Symmetric Beliefs
- Common Prior Assumption
- Bayesian Equilibrium
- One Theory per Choice Utility Pair

SYMMETRIC BELIEFS

Doxastic Reflection

- According to symmetric beliefs there is a symmetry between what a player believes about an opponent and what he believes the opponent to believe about him.

- It is as if there exists a doxastic mirror in the mind of the agent.

When do Belief Hierarchies satisfy Symmetry?

Let Γ be a game with incomplete information.

- A *weighted beliefs diagram* starts from a beliefs diagram, removes the probabilities at the forked arrows (if there are any) and assigns to every arrow a from a pair (c_i, u_i) to a tuple $(c_j, u_j)_{j \neq i}$ a positive weight $w(a)$.

- For every opponent $j \neq i$ the *symmetric counterpart* to an incoming arrow a is the outgoing arrow to a tuple $(c_k, u_k)_{k \neq j}$ that uses the same choice utility function pairs as a.

- A weighted beliefs diagram diagram is *symmetric*, if for every arrow a each of its symmetric counterparts (one for every opponent) is also part of the weighted beliefs diagram and carries the same weight as a.

- The weighted beliefs diagram induces a beliefs diagram in which the probability of an arrow a leaving a choice utility function pair (c_i, u_i) is qual to

$$p(a) = \frac{w(a)}{\sum_{\text{arrows } a' \text{ leaving } (c_i, u_i)} w(a')}.$$

- A belief hierarchy is *symmetric*, if it is induced by a beliefs diagram that itself is induced by a symmetric weighted beliefs diagram.

Decision Rule

Definition 1

Let Γ be a game with incomplete information, $i \in I$ some player, $c_i \in C_i$ some choice of player i, and $u_i \in U_i$ some utility function of player i. The choice c_i is *rational under common belief in rationality and symmetric beliefs* given u_i, if there exists an epistemic model $\mathcal{M}^\Gamma$ of Γ with some type $t_i \in T_i$ of player i such that

- t_i expresses common belief in rationality,
- the induced belief hierarchy of t_i is symmetric,
- $v_i[t_i] = u_i$,
- c_i is optimal for t_i.

Common Prior Assumption

Differences in Beliefs due to Differences in Information

- According to the classical CPA agents start out with identical prior beliefs and then condition on the information they receive.
- If the agents' beliefs differ, then this is due to them having received distinct information.
- Alternatively phrased, all beliefs root back to the same doxastic source.

The Common Prior Assumption

Let Γ be a game with incomplete information.

- Consider a beliefs diagram and identify every choice utility function pair (c_i, u_i) with a unique choice type pair (c_i, t_i) such that $v_i[t_i] = u_i$ and construct a corresponding type-based beliefs diagram.

- A *common prior* $\pi \in \times_{j \in I}(C_j \times T_j)$ is a probability distribution on all players' choices type combinations.

- The type-based beliefs diagram is induced by π, if for all $\big((c_i, t_i), (c_{-i}, t_{-i})\big) \in \times_{j \in I}(C_j \times T_j)$ and for all $i \in I$, the corresponding arrow $a\big((c_i, t_i) \to (c_{-i}, t_{-i})\big)$ exists if and only if $\pi\big((c_i, t_i), (c_{-i}, t_{-i})\big) > 0$ and carries the probability

$$p(a) = \frac{\pi\big((c_i, t_i), (c_{-i}, t_{-i})\big)}{\pi(c_i, t_i)}$$

- A belief hiearchy is induced by a *common prior*, if there exists a common prior $\pi \in \times_{j \in I}(C_j \times T_j)$ such that the type-based beliefs diagram is induced by π.

Illustration

- Consider the following game in one-person perspective form:

$\Gamma_A(u_A)$

	d	e	f
a	3	2	1
b	2	1	3
c	0	0	0

$\Gamma_A(u'_A)$

	d	e	f
a	1	3	1
b	2	1	1
c	0	0	0

$\Gamma_B(u_B)$

	a	b	c
d	3	2	1
e	2	1	3
f	0	0	0

$\Gamma_B(u'_B)$

	a	b	c
d	1	3	1
e	2	1	1
f	0	0	0

- Suppose the following epistemic model of this game:
  - $T_{Alice} = \{t_A, t'_A\}$ and $T_{Bob} = \{t_B, t'_B\}$,
  - $v_{Alice}[t_A] = u_A$ and $v_{Alice}[t_B] = u'_A$,
  - $v_{Bob}[t_B] = u_B$ and $v_{Bob}[t'_B] = u'_B$,
  - $b_{Alice}[t_A] = \frac{1}{2} \cdot (d, t_B) + \frac{1}{2} \cdot (e, t'_B)$,
  - $b_{Alice}[t'_A] = (e, t'_B)$,
  - $b_{Bob}[t_B] = (a, t_A,)$,
  - $b_{Bob}[t'_B] = \frac{1}{2} \cdot (a, t_A) + \frac{1}{2} \cdot (a, t'_A)$.

- The beliefs in this epistemic model are induced by common prior π, where

$$\pi\big((a, t_A), (d, t_B)\big) = \pi\big((a, t_A), (e, t'_B)\big) = \pi\big((a, t'_A), (e, t'_B)\big) = \frac{1}{3}$$

Symmetric Beliefs characterize the CPA

Theorem 2

Let Γ be a game with incomplete information. A belief hierarchy is symmetric, if and only if, it is induced by a common prior.

Bayesian Equilibrium

The Classical Notion of Bayesian Equilibrium

- Let Γ be a game with incomplete information and $\mathcal{M}^{\Gamma}$ an epistemic model of Γ.
- A common prior $\pi \in \times_{j \in I}(C_j \times T_j)$ constitutes a *Bayesian Equilibrium (BE)*, if for every player $i \in I$ it is the case that for every pair $(c_i, t_i) \in C_i \times T_i$ such that $\pi(c_i, t_i) > 0$, the choice c_i is optimal for t_i and t_i's belief is given by $\pi\big(\cdot \mid (c_i, t_i)\big)$.
- A choice is c_i is *rational under Bayesian Equilibrium* given a utility function u_i, if there exists an epistemic model $\mathcal{M}^{\Gamma}$ of Γ with a common prior π and a choice type pair (c_i', t_i) such that $\pi(c_i', t_i) > 0$, $v_i[t_i] = u_i$, and c_i is optimal for t_i.

Illustration

$\Gamma_A(u_A)$

	d	e	f
a	3	2	1
b	2	1	3
c	0	0	0

$\Gamma_A(u'_A)$

	d	e	f
a	1	3	1
b	2	1	1
c	0	0	0

$\Gamma_B(u_B)$

	a	b	c
d	3	2	1
e	2	1	3
f	0	0	0

$\Gamma_B(u'_B)$

	a	b	c
d	1	3	1
e	2	1	1
f	0	0	0

- Epistemic model:
  - $T_{Alice} = \{t_A, t'_A\}$ and $T_{Bob} = \{t_B, t'_B\}$,
  - $v_{Alice}[t_A] = u_A$ and $v_{Alice}[t_B] = u'_A$,
  - $v_{Bob}[t_B] = u_B$ and $v_{Bob}[t'_B] = u'_B$,
  - $\pi\big((a, t_A), (d, t_B)\big) = \pi\big((a, t_A), (e, t'_B)\big) = \pi\big((a, t'_A), (e, t'_B)\big) = \frac{1}{3}$
- Thus,
  - $b_{Alice}[t_A] = \frac{1}{2} \cdot (d, t_B) + \frac{1}{2} \cdot (e, t'_B)$,
  - $b_{Alice}[t'_A] = (e, t'_B)$,
  - $b_{Bob}[t_B] = (a, t_A,)$,
  - $b_{Bob}[t'_B] = \frac{1}{2} \cdot (a, t_A) + \frac{1}{2} \cdot (a, t'_A)$.
- π constitutes a Bayesian Equilibrium.
- a is rational under Bayesian Equilibrium given u_A (and u'_A),
- d is rational under Bayesian Equilibrium given u_B,
- e is rational under Bayesian Equilibrium given u'_B,

Characterization of Bayesian Equilibrium

Theorem 3

Let Γ be a game with incomplete information, $i \in I$ some player, $c_i \in C_i$ some choice of player i, and $u_i \in U_i$ some utility function of player i. The choice c_i is rational under common belief in rationality and symmetric beliefs given u_i, if and only if, c_i is rational under Bayesian Equilibrium given u_i.

One Theory per Choice Utility Pair

Idea

- The epistemic condition of one theory per choice utility pair restricts reasoning in the following way:

  For every rational choice utiltiy pair (c_j, u_j) deemed possible there is a unique belief that supports c_j given u_j.

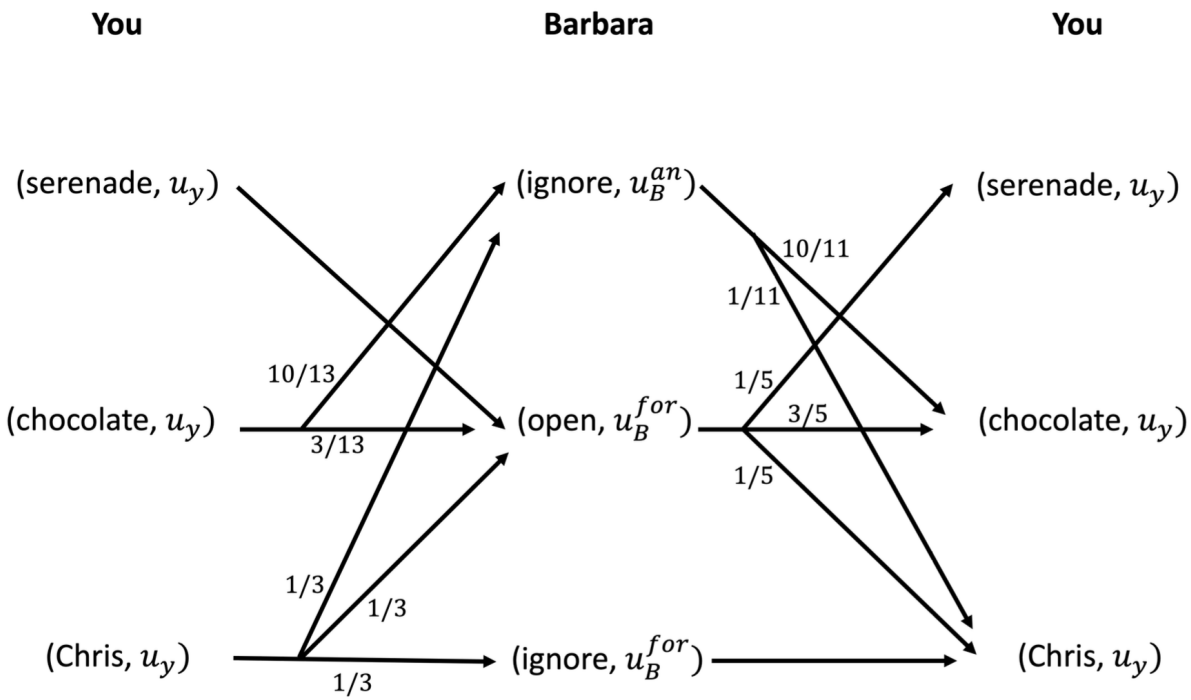
- Intuitively, a reasoner satisfying one theory per choice utility pair never uses in his entire belief hierarchy distinct first-order beliefs to rationally explain the same choice utiltiy pair for any player.

- Nonetheless, it is possible to support the same choice c_j with two different beliefs given two different utility functions u_j and u_j'.

- A belief hierarchy satisfies one theory per choice utility pair, if it can be generated by a beliefs diagram where every choice utility pair only appears once.

Example: The Moonlight Serenade

- Consider the following beliefs diagram for *The Moonlight Serenade*:



- As every choice utility pair only appears once in this beliefs diagram, all belief hierarchies induced by it satisfy one theory per choice utility pair.

Example: The Moonlight Serenade

The corresponding epistemic model thus only contains belief hierarchies satisfying the one theory per choice utility pair:

- $T_{you} = \{t_y^1, t_y^2, t_y^3\}$ and $T_{Barbara} = \{t_B^1, t_B^2, t_B^3\}$,
- $v_{you}[t_y^1] = v_{you}[t_y^2] = v_{you}[t_y^3] = u_y$
- $v_{Barbara}[t_B^1] = u_B^{an}$ and $v_{Barbara}[t_B^2] = v_{Barbara}[t_B^3] = u_B^{for}$,
- $b_{you}[t_y^1] = (open, t_B^2)$,
- $b_{you}[t_y^2] = \frac{10}{13} \cdot (ignore, t_B^1) + \frac{3}{13} \cdot (open, t_B^2)$,
- $b_{you}[t_y^3] = \frac{1}{3} \cdot (ignore, t_B^1) + \frac{1}{3} \cdot (open, t_B^2) + \frac{1}{3} \cdot (ignore, t_B^3)$,
- $b_{Barbara}[t_B^1] = \frac{10}{11} \cdot (chocolate, t_y^2) + \frac{1}{11} \cdot (Chris, t_y^3)$,
- $b_{Barbara}[t_B^2] = \frac{1}{5} \cdot (serenade, t_y^1) + \frac{3}{5} \cdot (chocolate, t_y^2) + \frac{1}{5} \cdot (Chris, t_y^3)$,
- $b_{Barbara}[t_B^3] = (Chris, t_y^3)$.

Behavioural Implications of Adding One Theory Per Choice Utility Pair to CBR and Symmetry

- Consider a symmetric belief hierarchy β that expresses CBR and satisfies one theory per choice utility pair.
- By Theorem 3, β is induced by a Bayesian Equilibrium, that is, by a common prior $\pi \in \times_{j \in I}(C_j \times T_j)$.
- By one theory per choice utility pair, for all (c_j, u_j) deemed possible in β there exists a unique type $t_j^{(c_j,u_j)}$ supporting c_j given $u_j = v_j[t_j^{(c_j,u_j)}]$.
- Replace every such choice type pair $(c_j, t_j^{(c_j,u_j)})$ simply by the associated choice utility pair (c_j, u_j).
- Moreover, the common prior π then induces a common prior $\hat{\pi} \in \times_{j \in I}(C_j \times U_j)$ given by

$$\hat{\pi}\big((c_j, u_j)_{j \in I}\big) := \pi\big((c_j, t_j^{(c_j,u_j)})_{j \in I}\big)$$

for all $(c_j, u_j)_{j \in I}$ deemed possible in β.

Behavioural Implications of Adding One Theory Per Choice Utility Pair to CBR and Symmetry

- Since π forms a Bayesian Equilibrium, for all $(c_j, t_j^{c_j,u_j})$ such that $\pi(c_j, t_j^{c_j,u_j}) > 0$, the associated choice c_j is optimal given utility function u_j and conditional belief $\pi\big(\cdot \mid (c_j, t_j^{c_j,u_j})\big)$.
- Then, by construction, $\hat{\pi}$ induced by π satisfies an optimality condition.
- Accordigly, for all (c_j, u_j) such that $\hat{\pi}(c_j, u_j) > 0$, the associated choice c_j is optimal given utility function u_j and conditional belief $\hat{\pi}\big(\cdot \mid (c_j, u_j)\big)$.
- A common prior $\hat{\in} \times_{j \in I} (C_j \times U_j)$ satisfying this optimality property is called *Canonical Bayesian Equilibrium*.
- A choice c_i is *rational under Canonical Bayesian Equilibrium* given a utility function u_i, if there exists a common prior $\hat{\pi}$ and a choice utility pair (c_i', u_i') such that $\hat{\pi}(c_i', u_i') > 0$ and c_i is optimal given u_i and conditional belief $\hat{\pi}\big(\cdot \mid (c_i', u_i')\big)$.

Illustration

$\Gamma_A(u_A)$

	d	e	f
a	3	2	1
b	2	1	3
c	0	0	0

$\Gamma_A(u'_A)$

	d	e	f
a	1	3	1
b	2	1	1
c	0	0	0

$\Gamma_B(u_B)$

	a	b	c
d	3	2	1
e	2	1	3
f	0	0	0

$\Gamma_B(u'_B)$

	a	b	c
d	1	3	1
e	2	1	1
f	0	0	0

- Consider the comon prior $\hat{\pi} \in \times_{j \in I}(C_j \times U_j)$ given by

$$\hat{\pi}\big((a, u_A), (d, u_B)\big) = \hat{\pi}\big((a, u_A), (e, u'_B)\big) = \hat{\pi}\big((a, u'_A), (e, u'_B)\big) = \frac{1}{3}$$

- Thus,
  - $\hat{\pi}\big(\cdot \mid (a, u_A)\big) = \frac{1}{2} \cdot (d, u_B) + \frac{1}{2} \cdot (e, u'_B)$,
  - $\hat{\pi}\big(\cdot \mid (a, u'_A)\big) = (e, u'_B)$,
  - $\hat{\pi}\big(\cdot \mid (d, u_B)\big) = (a, u_A,)$,
  - $\hat{\pi}\big(\cdot \mid (e, u'_B)\big) = \frac{1}{2} \cdot (a, u_A) + \frac{1}{2} \cdot (a, u'_A)$.
- $\hat{\pi}$ constitutes a Canonical Bayesian Equilibrium.
- a is rational under Bayesian Equilibrium given u_A (and u'_A),
- d is rational under Bayesian Equilibrium given u_B,
- e is rational under Bayesian Equilibrium given u'_B,

Characterization of Canonical Bayesian Equilibrium

Theorem 4

Let Γ be a game with incomplete information, $i \in I$ some player, $c_i \in C_i$ some choice of player i, and $u_i \in U_i$ some utility function of player i. The choice c_i is rational under common belief in rationality, symmetric beliefs, and one theory per choice utility pair given u_i, if and only if, c_i is rational under Canonical Bayesian Equilibrium given u_i.

The Bigger Picture

- Solution concepts for static games in relation to **informational assumptions** and *epistemic conditions*:

Epistemic Conditions	**Complete Information** Solution Concepts	**Incomplete Information** Solution Concepts
CBR	ISD	GISD
⇓ / ⇑	⇓ / ⇑	⇓ / ⇑
CBR & *Common Prior Assumption*	Correlated Equilibrium	Bayesian Equilibrium
⇓ / ⇑	⇓ / ⇑	⇓ / ⇑
CBR & *Simple Belief Hierarchy*	Nash Equilibrium	Generalized Nash Equilibrium

- **Remark:** the characterization of Nash equilibrium & Generalized Nash Equilibrium can be weakened to ones without CBR.