Generalised Solution Concepts in Games without Expected Utility

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Abstract

We propose a general framework for strategic interaction that relaxes the expected utility assumption and instead works directly with players' conditional preference relations (Gilboa & Schmeidler, 2003; Perea, 2025). By leveraging their epistemic foundations, we show that three classical solution concepts admit natural generalisations to this broader setting without altering their underlying reasoning principles. First, we introduce the iterated elimination of never-optimal choices as an analogue of the iterated elimination of strictly dominated choices, characterised by common belief in rationality. We also prove that it always yields a non-empty solution. Second, we generalise Nash equilibrium, preserving its characterisation by common belief in rationality and simple beliefs. Third, we extend correlated equilibrium, characterised by common belief in rationality and a common prior. While the latter two solution concepts are not guaranteed to exist in all games, we identify a sufficient condition for their existence, which we call strong continuity. This property requires the set of beliefs where a choice is weakly preferred to another to be closed, for any player and any choice pair. We also show that this condition is equivalent to imposing a continuous utility representation on the game, but weaker than imposing an expected utility representation. Our results demonstrate that the foundations of game theory are robust to the relaxation of expected utility, opening the door to richer and more flexible models of strategic interaction.

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1 Introduction

Imagine Alex standing in his hallway on a Friday evening, shoes in hand, agonising over whether to go to a party. Meanwhile, Barbara is already at the venue, facing a dilemma of her own. Should she make a move on Chris – who is already there – or wait in the hope that Alex will show up? Barbara likes Alex more, but she does not want to spend the evening alone. She will only wait for Alex if she believes that there is an at least 50% chance that he will show up, otherwise she will just make a move on Chris. Alex's preferences are less conventional: if he is more than 75% sure that Barbara will wait for him, he wants to go so they can get together; and if he is less than 25% sure, he also wants to go so he can drink himself into forgetting her. Yet in the murky middle – when he believes the probability of Barbara waiting for him lies between 25% and 75% – the stress of uncertainty pushes him to stay home. This simple, intuitive scenario reveals an interesting puzzle: Alex's preferences over his choices depend on his beliefs in a way that no expected-utility function can capture, and thus cannot be represented within a standard normal-form game.

The expected utility assumption has long been a cornerstone of game theory (von Neumann & Morgenstern, 1944/1947; Savage, 1954; Anscombe & Aumann, 1963). Nevertheless, systematic deviations from this principle are well-documented in the empirical literature (Allais, 1953; Ellsberg, 1961), and recent theoretical findings also suggest that its axiomatic basis imposes very strong conditions on the preferences (Perea, 2025). This paper proposes an alternative game-theoretic framework with more general preference structures, conditional preference relations (Gilboa & Schmeidler, 2003; Perea, 2025). These structures assign a ranking over the player's choices to any probabilistic belief she may hold about her opponents' choices, allowing for much more flexibility in specifying the preferences than the expected utility assumption. We explore how three standard solution concepts from game theory can be meaningfully extended to this new framework.

A key methodological principle of our approach is that the generalisation of a solution concept should preserve the essence of its epistemic foundations. That is, we view solution concepts as being fully defined by the reasoning principles that underlie them. In this sense, our generalised concepts are not just analogous to their expected-utility counterparts – they are epistemically equivalent. The justification for calling them generalisations lies precisely in this shared reasoning.

The first solution concept we consider is the iterated elimination of strictly dominated choices, also known as correlated rationalisability (Bernheim, 1984; Pearce, 1984; Brandenburger & Dekel, 1987). We rely on its characterisation by common belief in rationality (Brandenburger & Dekel, 1987; Tan & da Costa Werlang, 1988), and provide a generalisation called the iterated elimination of never-optimal choices. Next, we generalise Nash equilibrium (Nash, 1950, 1951), which we view as being characterised by common belief in rationality and simple beliefs (Perea, 2012). We similarly generalise correlated equilibrium (Aumann, 1974), using its characterisation by common belief in rationality and a common prior (Dekel & Siniscalchi, 2015; Bach & Perea, 2020b; Aumann, 1987).

Besides generalising these solution concepts, we also provide some existence results. We first prove that the iterated elimination of never-optimal choices yields a solution in any game. Next, we show that the continuity axiom of Gilboa and Schmeidler (2003) – which we refer to as *strong* continuity to avoid confusion with the weaker continuity notion of Perea (2025) – provides a sufficient condition for the existence of Nash and correlated equilibria. Strong continuity simply means that the set of beliefs under which a choice is weakly preferred to another is closed, which seems rather plausible from a behavioural perspective. It is much weaker than the expected utility assumption, which also requires other axioms (preservation of indifference and strict preference, three-choice and four-choice linear preference intensity) in addition to continuity (Perea, 2025). These additional axioms thus turn out to be unnecessary for equilibrium existence. We also show that strongly continuous preferences are exactly the ones that admit a continuous (but not necessarily expected) utility representation.

The paper is organised as follows. Section 2 introduces our basic framework of games with conditional preference relations. Next, we present the proposed solution concepts: the iterated elimination of never-optimal choices in Section 3, the Nash equilibrium in Section 4, and the correlated equilibrium in Section 5. In Section 6, we provide an example of a specific game in our framework and demonstrate how the proposed solution concepts can be applied to it. Finally, in Section 7 we discuss our results. Appendix A includes the proofs of our propositions and some additional lemmata.

2 Basic setting

This section introduces the framework we use in this paper. The key concept of our game-theoretic analysis is of course the concept of a game. We define a game as a tuple $(I, \boldsymbol{C}, \boldsymbol{\succsim})$, where:

- I is the non-empty and finite set of players[1].
- $\boldsymbol{C} = (C_i)_{i \in I}$, where C_i is the non-empty and finite set of choices available for player $i \in I$.
- $\boldsymbol{\succsim} = \big((\succsim_i^p)_{p \in \Delta(\boldsymbol{C}_{-i})}\big)_{i \in I}$, where $\succsim_i^p$ is player i's complete and transitive preference relation over C_i, when her probabilistic belief about her opponents' choice combinations is[2] $p \in \Delta(\boldsymbol{C}_{-i})$. We also refer to $\succsim_i = (\succsim_i^p)_{p \in \Delta(\boldsymbol{C}_{-i})}$ as the conditional preference relation of player i.

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[1] Note that whenever we use a bold symbol without subscript (like $\boldsymbol{x}$), it always denotes a *profile* or *vector* with length $|I|$, with a unique element (x_i) corresponding to each player $i \in I$. We also use the common game-theoretic notation for truncated profiles, $\boldsymbol{x}_{-i} = (x_j)_{j \in I \setminus \{i\}}$. When each x_i is a set, $\boldsymbol{x}$ may either refer to the vector of sets $(x_i)_{i \in I}$ or the set of vectors $\times_{i \in I} x_i$, depending on the context.

[2] For any finite set X, $\Delta(X)$ denotes the set of probability distributions on X, which can be viewed as a simplex. However, in contexts where one set is a subset of another (e.g. $Y \subseteq X$), $\Delta(Y)$ is sometimes understood as the subset of $\Delta(X)$ consisting of distributions that assign probability zero to every element of $X \setminus Y$. This notational convention keeps several definitions more compact and improves readability.

The only point where this formulation differs from the textbook definition of a normal-form game is how we treat the preferences. Instead of specifying a payoff function, we use the concept of conditional preference relations (Gilboa & Schmeidler, 2003; Perea, 2025). This is a key element of our approach. Unlike in the classic game-theoretic framework, where players have preferences over choice *profiles*, here they have preferences over only their *own* choices, and these preferences depend on their beliefs about the choices of others. This allows for more flexibility, enabling us to relax the expected utility assumption.

For the remainder of this paper, we fix the game $(I, \boldsymbol{C}, \succsim)$ as our analytic basis, and every concept we define henceforth is to be interpreted within the context of this given game. We do this to avoid having to write '*In game* $(I, \boldsymbol{C}, \succsim)$' at the beginning of each definition that follows.

2.1 Continuity

A central concept in our analysis is the notion of continuous preferences. In the literature, there are two non-equivalent definitions of what continuity means for a conditional preference relation. To distinguish between them, we refer to the notion of Perea (2025) as *weak continuity*, and to the notion of Gilboa and Schmeidler (2003) as *strong continuity.* The two concepts are defined as follows.

Definition 1 (Weakly continuous preferences)**.** *Player i's conditional preference relation $\succsim_i$ is weakly continuous if for any pair of choices $c_i, c_i' \in C_i$ and pair of beliefs $p, p' \in \Delta(\boldsymbol{C}_{-i})$ we have:*

$$c_i \succ_i^p c_i' \text{ and } c_i' \succ_i^{p'} c_i \implies \exists \lambda \in]0,1[\colon c_i \sim_i^{\lambda \cdot p + (1-\lambda) \cdot p'} c_i'$$

Definition 2 (Strongly continuous preferences)**.** *Player i's conditional preference relation $\succsim_i$ is strongly continuous if $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \succsim_i^p c_i'\}$ is a closed set for any pair of choices $c_i, c_i' \in C_i$.*

As our terminology suggests, the continuity notion of Gilboa and Schmeidler (2003) is a stronger property than that of Perea (2025). To illustrate this, consider a two-player game where player 1's preference between her choices c_1 and c_1' depends only on the probability $p(c_2)$ that player 2 chooses choice c_2. Player 1 strictly prefers c_1 to c_1' if $p(c_2)$ is a rational number, and she is indifferent between them whenever $p(c_2)$ is irrational. These preferences are trivially weakly continuous, because $c_1' \succ_i^p c_1$ never occurs, so the requirement in Definition 1 is satisfied. However, the set $\{\, p \in \Delta(C_2) : c_1' \succsim_i^p c_1 \,\}$ is not closed, so the preferences are not strongly continuous. Therefore, a weakly continuous conditional preference relation is not necessarily strongly continuous[3]. On the other hand, strong continuity does imply weak continuity. We prove this claim by characterising the extra requirement that distinguishes the two.

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[3] This example also demonstrates that weak continuity is an extremely permissive assumption: it even allows for pathological and behaviourally implausible preferences such as in this Dirichlet example. In this sense, strong continuity is only a mild additional requirement.

Proposition 1. *Player i's conditional preference relation* $\succsim_i$ *is strongly continuous if and only if it is weakly continuous and* $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \sim_i^p c_i'\}$ *is a closed set for any* $c_i, c_i' \in C_i$.

As is common in decision theory, we can also use a utility function to represent the preferences of players. The finiteness of the choice sets and the transitivity of the preferences guarantee that such a representation exists for any belief. Although we mostly use the preference ranking notation to emphasise our epistemic approach, it will sometimes be convenient to think in terms of utilities. A utility representation is defined as follows.

Definition 3 (Utility representation). *A utility function* $u_i : C_i \times \Delta(\boldsymbol{C}_{-i}) \to \mathbb{R}$ *represents conditional preference relation* $\succsim_i$ *if:*

$$c_i \succsim_i^p c_i' \iff u_i(c_i, p) \geq u_i(c_i', p) \quad \forall c_i, c_i' \in C_i, p \in \Delta(\boldsymbol{C}_{-i})$$

While the finiteness and the transitivity trivially ensure that *some* utility representation exists, the existence of a representation that is *continuous* in beliefs is not guaranteed. This continuity plays a crucial role in our analysis, so it is important to understand which conditional preferences have a continuous representation. Interestingly, these are exactly the ones that satisfy the above-defined condition of *strong* continuity.

Proposition 2. *Player i's conditional preference relation* $\succsim_i$ *is strongly continuous if and only if there exists a utility function that is continuous in* $\Delta(\boldsymbol{C}_{-i})$ *and represents* $\succsim_i$.

2.2 Epistemic models

To describe how players reason about the game, we use essentially the same framework as Tan and da Costa Werlang (1988). We encode players' infinite belief hierarchies by *types*, where each type carries a probabilistic belief about opponents' choices and types. The notion of types traces back to Harsányi (1967, 1968a, 1968b), though his formulation is slightly different. The object that contains types and their associated beliefs is called an epistemic model of the game.

Definition 4 (Epistemic model). *An epistemic model of the game is a tuple* $(\boldsymbol{T}, \boldsymbol{b})$*, where:*

- T_i *is the non-empty and finite set of types for player* $i \in I$
- $b_i : T_i \to \Delta(\boldsymbol{C}_{-i} \times \boldsymbol{T}_{-i})$ *is the belief function of player* $i \in I$

We next introduce a relation between epistemic models that will be used in subsequent definitions. We say one epistemic model is a *submodel* of another when it can be obtained by restricting each player's type set without altering the underlying belief functions. Loosely speaking, this means that one is contained in the other.

Definition 5 (Submodel)**.** *An epistemic model* $(\boldsymbol{T}', \boldsymbol{b}')$ *is a submodel of epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$ *if for any* $i \in I$ *we have* $T_i' \subseteq T_i$ *and* $b_i'(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) = b_i(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) \quad \forall (t_i, \boldsymbol{t}_{-i}) \in \boldsymbol{T}', \boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}$*.*

Next, we define the notion of optimality. A choice is called optimal under a given belief if it is (one of) the player's most preferred choice(s) under that belief. We can also define optimality for a given type in an epistemic model. In that case, we need to take the marginal of that type's belief on the opponents' choice combinations.

Definition 6 (Optimal choice under belief)**.** *A choice* $c_i \in C_i$ *is optimal for player* $i \in I$ *under belief* $p \in \Delta(\boldsymbol{C}_{-i})$ *if* $c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i$*.*

Definition 7 (Optimal choice for type)**.** *In epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$*, choice* $c_i \in C_i$ *is optimal for type* $t_i \in T_i$ *if it is optimal under belief*[4] $\operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t_i)$*.*

In the following sections, we define several properties (e.g. common belief in rationality, simple beliefs, common prior) that types in an epistemic model may or may not have. We will call a choice *rational* under such a property if it is optimal for some type that has the property.

Definition 8 (Rational choice under property)**.** *For player* $i \in I$*, choice* $c_i \in C_i$ *is rational under a given property of types if there exists an epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$ *and a type* $t_i \in T_i$ *such that:*

- t_i *has that property*
- c_i *is optimal for* t_i

3 Iterated elimination of never-optimal choices

3.1 Epistemic background

In this section, we provide a generalised version of the iterated elimination of strictly dominated choices (Bernheim, 1984; Pearce, 1984; Brandenburger & Dekel, 1987). This solution concept has been epistemically characterised by common belief in rationality (Brandenburger & Dekel, 1987; Tan & da Costa Werlang, 1988), which is defined in three steps. First, we define (onefold) belief in rationality. Intuitively, a player believes in rationality if she is certain that her opponents will select choices that are optimal under their beliefs.

Definition 9 (Type with belief in rationality)**.** *In epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$*, type* $t_i \in T_i$ *has (onefold) belief in rationality if for any* $(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) \in \boldsymbol{C}_{-i} \times \boldsymbol{T}_{-i}$ *we have:*

$$b_i(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) > 0 \implies c_j \text{ is optimal for } t_j \quad \forall j \in I \setminus \{i\}$$

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[4] For a distribution $p \in \Delta(X \times Y)$, $\operatorname{marg}_X p$ is defined as $(\operatorname{marg}_X p)(x) := \sum_{y \in Y} p(x, y) \quad \forall x \in X$.

Next, we define $k$fold belief in rationality. A player has twofold belief in rationality if she believes that her opponents believe in rationality, threefold if she believes that her opponents believe that their opponents believe in rationality, and so on. Formally, we have:

Definition 10 (Type with $k$fold belief in rationality)**.** *In epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$*, type* $t_i \in T_i$ *has* k*fold belief in rationality (where*[5] $k \in \mathbb{N}_+ \setminus \{1\}$*) if for any* $(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) \in \boldsymbol{C}_{-i} \times \boldsymbol{T}_{-i}$ *we have:*

$$b_i(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) > 0 \implies t_j \textit{ has } k-1\textit{fold belief in rationality} \quad \forall j \in I \setminus \{i\}$$

Finally, common belief in rationality simply requires $k$fold belief for any k.

Definition 11 (Type with common belief in rationality)**.** *In epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$*, type* $t_i \in T_i$ *has common belief in rationality if it has* k*fold belief in rationality for any* $k \in \mathbb{N}_+$*.*

3.2 Proposed generalisation

Having defined these epistemic notions, we can now turn to the generalised solution concept we propose. It relies on the same logic as the iterated elimination of strictly dominated choices: in each round, we eliminate the choices which are not optimal under any belief about the opponents' remaining choices from the previous round. However, since the basis of elimination in this case is not strategic dominance, we needed to give our procedure a different name. We call it the iterated elimination of never-optimal choices, as this name captures its essence. After providing a formal definition of this solution concept, we also show that its existence is guaranteed in any game.

Definition 12 (Iterated elimination of never-optimal choices)**.** *Consider the following sets, defined recursively for each* $k \in \mathbb{N}_0$ *and* $i \in I$*:*

$$\begin{aligned} C_i^0 &= C_i \\ C_i^k &= \left\{c_i \in C_i^{k-1} : \exists p \in \Delta(\boldsymbol{C}_{-i}^{k-1}) \textit{ such that } c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i\right\} \end{aligned}$$

The iterated elimination of never-optimal choices (IENC) is a procedure that returns:

$$IENC_i = \bigcap_{k \in \mathbb{N}_0} C_i^k \quad \forall i \in I$$

Proposition 3. *The* $IENC_i$ *set is non-empty for any player* $i \in I$*.*

To show that this concept is indeed a generalisation of the iterated elimination of strictly dominated choices, we prove that it can be characterised by common belief in rationality.

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[5] To avoid confusion, we use $\mathbb{N}_+$ when we refer to the set of positive integers, and $\mathbb{N}_0$ when we refer to the set of non-negative integers.

Proposition 4. *For a player $i \in I$, choice $c_i \in C_i$ is rational under common belief in rationality if and only if $c_i \in IENC_i$.*

4 Nash equilibrium

4.1 Epistemic background

The next solution concept we generalise is Nash equilibrium (Nash, 1950, 1951). We view this solution concept as being epistemically characterised by common belief in rationality and simple beliefs (Perea, 2012). We have already defined the former criterion, so here we focus on the latter. An epistemic model is called simple if each player has only one type, and beliefs can be generated by a profile of probability distributions on the choices of each individual player. A type has simple beliefs if it is contained in a simple epistemic (sub)model. For a more intuitive interpretation of what simple beliefs mean, see Perea (2007, 2012).

Definition 13 (Simple epistemic model)**.** *An epistemic model $(\boldsymbol{T}, \boldsymbol{b})$ is simple if $\boldsymbol{T} = \{\boldsymbol{t}\}$ is a singleton and there exists a profile of distributions $\boldsymbol{q} \in \times_{j \in I} \Delta(C_j)$ such that:*

$$b_i(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) = \prod_{j \in I \setminus \{i\}} q_j(c_j) \quad \forall i \in I, \boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}$$

Definition 14 (Type with simple beliefs)**.** *In epistemic model $(\boldsymbol{T}, \boldsymbol{b})$, type $t_i \in T_i$ has simple beliefs if there exists a submodel $(\boldsymbol{T}', \boldsymbol{b}')$ of $(\boldsymbol{T}, \boldsymbol{b})$ such that $t_i \in T'_i$ and $(\boldsymbol{T}', \boldsymbol{b}')$ is simple.*

4.2 Proposed generalisation

Next, we present our generalised version of the Nash equilibrium. Unlike most game theory textbooks, we do not formulate it in terms of mixed strategies. In our framework, a Nash equilibrium is a profile of probabilistic beliefs $\boldsymbol{q} \in \times_{j \in I} \Delta(C_j)$ where each player $i \in I$ is believed never to choose a choice that is not optimal under q_{-i}.

Definition 15 (Nash equilibrium)**.** *A profile of distributions $\boldsymbol{q} \in \times_{j \in I} \Delta(C_j)$ is a Nash equilibrium if for any player $i \in I$ and any choice $c_i \in C_i$ we have*[6]*:*

$$q_i(c_i) > 0 \implies c_i \succsim_i^{q_{-i}} c'_i \quad \forall c'_i \in C_i$$

Before jumping to our existence result, we also define the closely related concept of the game's best response correspondence. This correspondence takes a profile of beliefs $\boldsymbol{q} \in \times_{j \in I} \Delta(C_j)$ on the

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[6] With a slight abuse of notation, $q_{-i} \in \Delta(\boldsymbol{C}_{-i})$ is defined as $q_{-i}(\boldsymbol{c}_{-i}) = \prod_{j \in I \setminus \{i\}} q_j(c_j) \quad \forall \boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}$.

players' choice sets. For each player $i \in I$, it returns the set of beliefs which assign zero probability to the choices not optimal under belief q_{-i}.

Definition 16 (Best response correspondence)**.** *A correspondence* $\boldsymbol{B} : \times_{j \in I} \Delta(C_j) \rightrightarrows \times_{j \in I} \Delta(C_j)$ *is the game's best response correspondence if for any player* $i \in I$ *and profile of distributions* $\boldsymbol{q} \in \times_{j \in I} \Delta(C_j)$ *we have:*

$$B_i(\boldsymbol{q}) = \Delta\left(\left\{c_i \in C_i : c_i \succsim_i^{q_{-i}} c_i' \quad \forall c_i' \in C_i\right\}\right)$$

When the expected utility assumption holds, Nash (1950, 1951) has shown that a Nash equilibrium exists in any finite game. Unfortunately, this famous result does not apply to our generalised framework. However, we can provide a sufficient condition for the existence of a Nash equilibrium which is weaker than the expected utility assumption: when the preferences of all players are strongly continuous, a Nash equilibrium is guaranteed to exist.

Proposition 5. *If the conditional preference relation* $\succsim_i$ *of each player* $i \in I$ *is strongly continuous, then a Nash equilibrium exists.*

To show that our version of Nash equilibrium is indeed a generalisation of the original one, we prove that it can also be characterised by common belief in rationality and simple beliefs. We follow a choice-based approach, showing that a choice is rational under common belief in rationality and simple beliefs iff it is rational under Nash equilibrium. But before proving this claim, we need to define what we mean by 'rational under Nash equilibrium'.

Definition 17 (Rational choice under Nash equilibrium)**.** *For player* $i \in I$*, choice* $c_i \in C_i$ *is rational under Nash equilibrium if there exists a Nash equilibrium* $\boldsymbol{q} \in \times_{i \in I} \Delta(C_i)$ *such that* $c_i \succsim_i^{q_{-i}} c_i' \quad \forall c_i' \in C_i$*.*

Proposition 6. *For a player* $i \in I$*, choice* $c_i \in C_i$ *is rational under simple beliefs and common belief in rationality if and only if it is rational under Nash equilibrium.*

5 Correlated equilibrium

5.1 Epistemic background

Finally, we also present a generalisation of the concept of correlated equilibrium (Aumann, 1974). The epistemic characterisation of this solution concept relies on common belief in rationality and the common prior assumption (Dekel & Siniscalchi, 2015; Bach & Perea, 2020b; Aumann, 1987). The latter posits that each player's beliefs are Bayesian updates of a single shared prior distribution. For the formal definition of this condition, we use a two-step approach, analogous to the case of simple beliefs. Note that we use a slight abuse of notation in the definitions that follow. Whenever we omit some variables of the π function, we mean a marginal distribution. In particular, we replace $(\text{marg}_{T_i}\, \pi)(t_i)$ by $\pi(t_i)$, $(\text{marg}_{C_i \times T_i}\, \pi)(c_i, t_i)$ by $\pi(c_i, t_i)$, and so on.

Definition 18 (Common prior)**.** *In epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$*, a distribution* $\pi \in \Delta(\boldsymbol{C} \times \boldsymbol{T})$ *is a common prior if for any* $i \in I$ *we have:*

- $\pi(t_i) > 0 \quad \forall t_i \in T_i$
- $\pi(c_i, t_i) > 0 \implies b_i(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) = \frac{\pi(\boldsymbol{c}, \boldsymbol{t})}{\pi(c_i, t_i)} \quad \forall(\boldsymbol{c}, \boldsymbol{t}) \in \boldsymbol{C} \times \boldsymbol{T}$

Definition 19 (Type with common prior)**.** *In epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$*, type* $t_i \in T_i$ *has a common prior if there exists a submodel* $(\boldsymbol{T}', \boldsymbol{b}')$ *of* $(\boldsymbol{T}, \boldsymbol{b})$ *such that* $t_i \in T_i'$ *and there exists a common prior in* $(\boldsymbol{T}', \boldsymbol{b}')$*.*

5.2 Proposed generalisation

Now we can give our generalised definition of a correlated equilibrium. Following the original formulation of Aumann (1974), we define this concept as a common prior on *choice-type combinations*, and not simply on *choice profiles*, as it appears in some textbooks. This important distinction has been emphasised by Bach and Perea (2020b), who refer to the textbook version as the *canonical* correlated equilibrium. They show that the two definitions are not equivalent because the choices that are rational under correlated equilibrium are not necessarily rational under canonical correlated equilibrium.

Definition 20 (Correlated equilibrium)**.** *A common prior* $\pi \in \Delta(\boldsymbol{C} \times \boldsymbol{T})$ *in epistemic model* $(\boldsymbol{T}, \boldsymbol{b})$ *is a correlated equilibrium of the game if for any player* $i \in I$*, type* $t_i \in T_i$*, belief* $p \in \Delta(\boldsymbol{C}_{-i})$*, and choice* $c_i \in C_i$ *we have:*

$$\left(\pi(c_i, t_i) > 0 \quad \textit{and} \quad p(\boldsymbol{c}_{-i}) = \frac{\pi(\boldsymbol{c}, t_i)}{\pi(c_i, t_i)} \quad \forall \boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}\right) \implies c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i$$

While the existence of a correlated equilibrium is not guaranteed in our generalised framework, the strong continuity of the preferences once again provides a sufficient condition.

Proposition 7. *If the conditional preference relation* $\succsim_i$ *of each player* $i \in I$ *is strongly continuous, then a correlated equilibrium exists.*

Finally, we prove that this version of correlated equilibrium is indeed a generalisation of the original. We show that a choice is rational under a common prior and common belief in rationality iff it is rational under correlated equilibrium.

Definition 21 (Rational choice under correlated equilibrium)**.** *For player $i \in I$, choice $c_i \in C_i$ is rational under correlated equilibrium if there exists a correlated equilibrium $\pi \in \Delta(\boldsymbol{C} \times \boldsymbol{T})$ and a choice-type pair $(c_i', t_i) \in C_i \times T_i$ such that $\pi(c_i', t_i) > 0$ and $c_i \succsim_i^p c_i'' \quad \forall c_i'' \in C_i$, where $p \in \Delta(\boldsymbol{C}_{-i})$ is defined as $p(\boldsymbol{c}_{-i}) = \frac{\pi(c_i', \boldsymbol{c}_{-i}, t_i)}{\pi(c_i', t_i)} \quad \forall \boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}$.*

Proposition 8. *For a player $i \in I$, choice $c_i \in C_i$ is rational under a common prior and common belief in rationality if and only if it is rational under correlated equilibrium.*

6 Example

To demonstrate how our framework operates in practice, we now present an example of a game with conditional preference relations that do not have an expected utility representation, and apply the generalised solution concepts to it. This example was carefully constructed with a twofold purpose. First, it illustrates how our general framework accommodates all kinds of preferences, even seemingly strange or unrealistic ones. To highlight this flexibility, we have deliberately chosen somewhat intricate functions in the game's definition. Second, we constructed the game so that once the iterated elimination of never-optimal choices is applied, the resulting reduced game is an exact description of the motivating Alex-Barbara story from Section 1. We make this connection explicit in Subsection 6.4 below. But for now, let us just focus on the game in its pure mathematical form. It is a two-player game with three actions for each player, i.e. $I = \{1, 2\}$, $C_1 = \{a, b, c\}$, and $C_2 = \{d, e, f\}$. The conditional preference relation of player 1 is described as:

$$\begin{aligned}
&a \succ_1^p b \succ_1^p c && \text{if} \quad p(d) > 4 \cdot p(e)^2 - 5 \cdot p(e) + \frac{7}{4} \\
&a \sim_1^p b \succ_1^p c && \text{if} \quad p(d) = 4 \cdot p(e)^2 - 5 \cdot p(e) + \frac{7}{4} \\
&b \succ_1^p a \succ_1^p c && \text{if} \quad \frac{1}{2} - \frac{\sin(2000 \cdot p(e))}{20} - \frac{2 \cdot p(e)}{3} < p(d) < 4 \cdot p(e)^2 - 5 \cdot p(e) + \frac{7}{4} \\
&b \succ_1^p c \succ_1^p a && \text{if} \quad p(d) < \frac{1}{2} - \frac{\sin(2000 \cdot p(e))}{20} - \frac{2 \cdot p(e)}{3} \\
& && \text{and } 2 \cdot p(d)^2 + 2 \cdot p(e)^2 + 2 \cdot p(d) \cdot p(e) - p(d) - p(e) + \frac{55}{384} \geq 0 \\
&b \succ_1^p a \sim_1^p c && \text{otherwise}
\end{aligned} \tag{1}$$

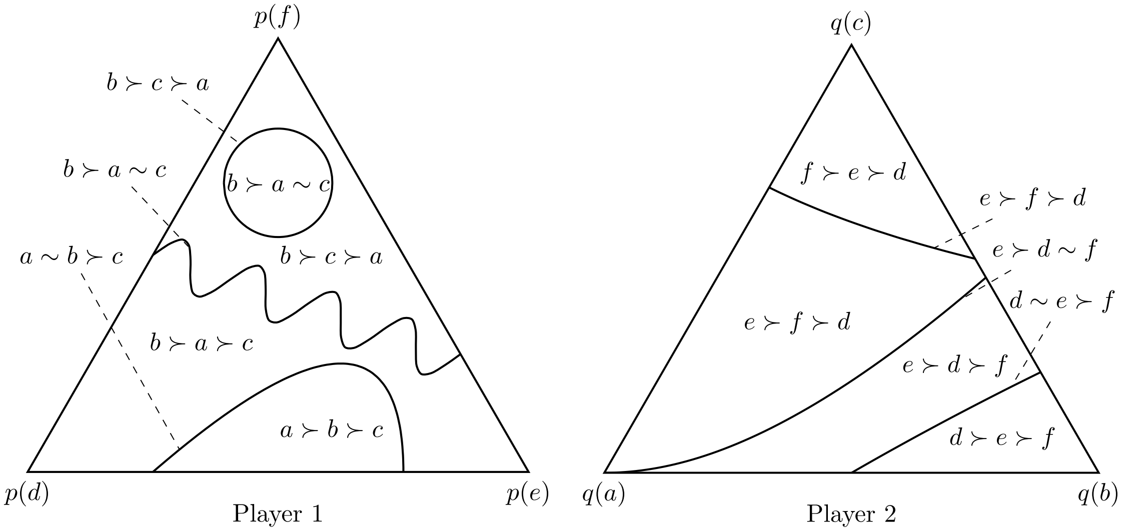


Figure 1: The two players' conditional preference relations

The conditional preference relation of player 2 is:

$$
\begin{aligned}
&d \succ_2^q e \succ_2^q f \quad \text{if} \quad q(a) > 1 - \ln\left(q(b) + \frac{1}{2}\right) - q(b) \\
&d \sim_2^q e \succ_2^q f \quad \text{if} \quad q(a) = 1 - \ln\left(q(b) + \frac{1}{2}\right) - q(b) \\
&e \succ_2^q d \succ_2^q f \quad \text{if} \quad 1 - q(b)^3 - q(b)^2 - q(b) < q(a) < 1 - \ln\left(q(b) + \frac{1}{2}\right) - q(b) \\
&e \succ_2^q d \sim_2^q f \quad \text{if} \quad q(a) = 1 - q(b)^3 - q(b)^2 - q(b) \\
&e \succ_2^q f \succ_2^q d \quad \text{if} \quad \frac{1 - 2 \cdot q(b)^2 - q(b)}{2 \cdot q(b) + 3} \leq q(a) < 1 - q(b)^3 - q(b)^2 - q(b) \\
&f \succ_2^q e \succ_2^q d \quad \text{if} \quad q(a) < \frac{1 - 2 \cdot q(b)^2 - q(b)}{2 \cdot q(b) + 3}
\end{aligned}
\tag{2}
$$

Since both players have three actions, their beliefs about their opponent's choices are elements of a two-simplex. This allows us to provide a graphical representation of the players' conditional preference relations, as shown in Figure 1. Looking at the figure, it is easy to see that these preferences are indeed very unconventional. Under the expected utility assumption, areas with different preferences would have to be bordered by straight lines that intersect in the same point (Perea, 2025). This is clearly not the case here, demonstrating that even such 'strange' preferences can be handled in our general framework.

At the left side of Figure 1, we observe that the conditional preference relation of player 1 is weakly continuous: for any pair of beliefs under which two choices are ranked oppositely, there exists an intermediate belief under which the player is indifferent between them. However, the preference is not strongly continuous since the set $\{p \in \Delta(C_2) : a \succsim_1^p c\}$ is not closed (it contains the interior of the disc but not its boundary). On the right side of the figure, player 2's conditional preference relation is not even weakly continuous. This is evident from the fact that no belief yields indifference between e and f, yet there are beliefs under which e is strictly preferred to f, and others where the reverse holds.

6.1 Iterated elimination of never-optimal choices

Now let us apply the iterated elimination of never-optimal choices to this example. We initialise this procedure by setting $C_1^0 = C_1 = \{a, b, c\}$ and $C_2^0 = C_2 = \{d, e, f\}$. Next, we get from C_1^0 and C_2^0 to C_1^1 and C_2^1 by eliminating the choices that are not optimal for the given player under any belief. For player 1, we can see from Figure 1 that this is the case for choice c: it is never ranked on the top of the player's ranking. However, we cannot eliminate anything for player 2, since her choices d, e and f are all optimal under some belief. Therefore, we have $C_1^1 = \{a, b\}$ and $C_2^1 = \{d, e, f\}$. In Figure 2, we also show this first round of elimination graphically: we omit choice c from player 1's conditional preference relation, and also omit the beliefs that assign positive probability to c from player 2's conditional preference relation. Thus the two-simplex on the right collapses to a line segment.

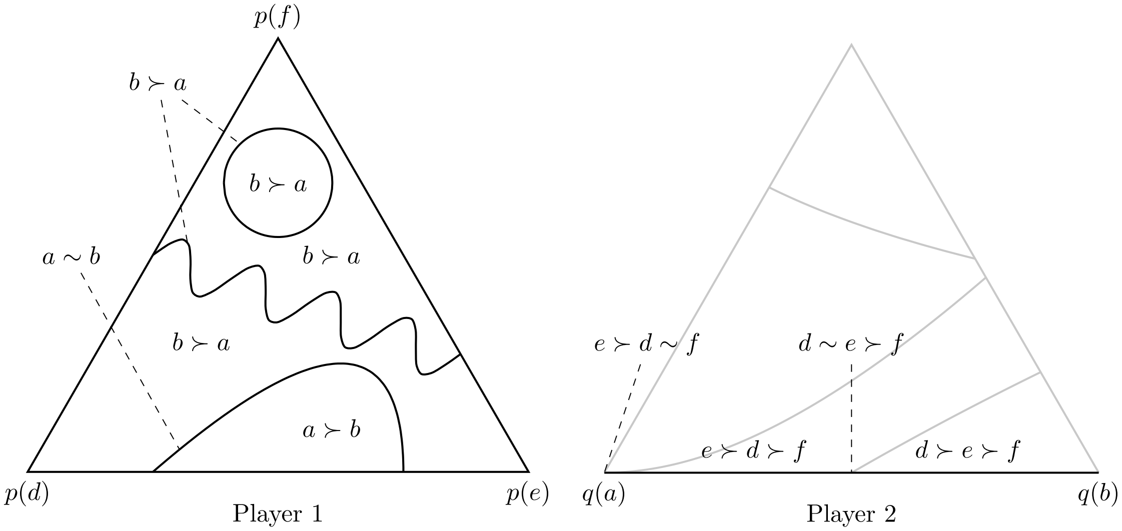
In the next step of the procedure, we obtain C_1^2 and C_2^2 from C_1^1 and C_2^1 by eliminating the choices that are never-optimal in Figure 2. For player 1, the remaining choices a and b are both optimal under some beliefs, so we cannot eliminate them from her choice set: $C_1^2 = \{a, b\}$. On the other hand, choice f has become never-optimal for player 2 since the beliefs under which it was optimal were omitted in the previous round. Therefore, we can eliminate f and conclude that $C_2^2 = \{d, e\}$. This step is shown in Figure 3.

Now looking at Figure 3, it is clear that we cannot eliminate further choices from neither player's choice set, as both a and b are optimal for some belief of player 1, and the same goes for d and e and player 2. This means that $C_1^3 = \{a, b\}$ and $C_2^3 = \{d, e\}$. By an inductive reasoning, we can conclude that $C_1^k = \{a, b\}$ and $C_2^k = \{d, e\}$ for any $k \in \mathbb{N}_+ \setminus \{1, 2\}$. By definition, this implies that:

$$IENC_1 = \{a, b\}, \quad IENC_2 = \{d, e\}$$

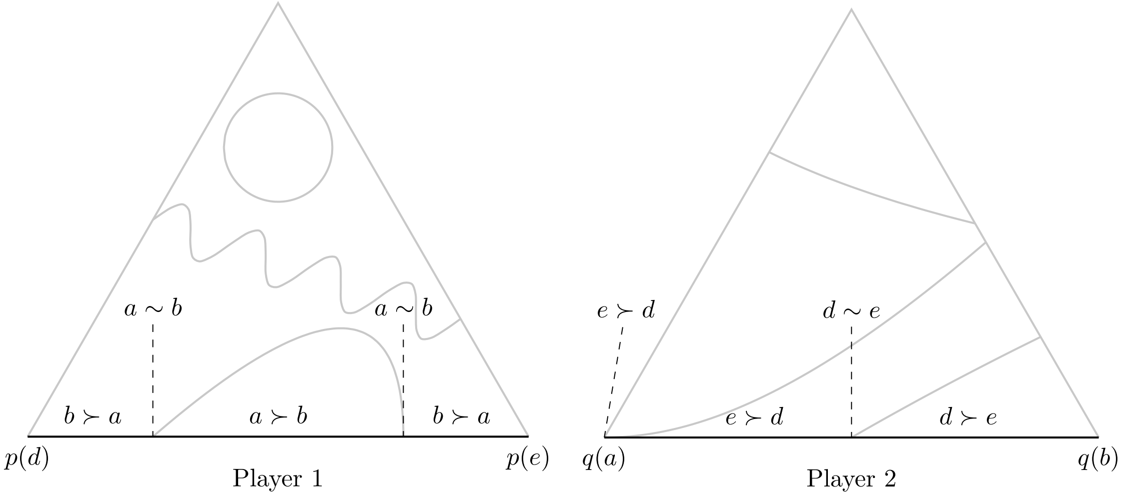
6.2 Nash equilibrium

The next solution concept we apply to this example game is Nash equilibrium. If we tried to find this in the game's original form from Figure 1, our analysis would be rather complicated. Furthermore, since we have demonstrated that neither player's preferences are strongly continuous,



Player 1 Player 2

Figure 2: The two players' conditional preference relations after the first iteration of the IENC



Player 1 Player 2

Figure 3: Two players' conditional preference relations after the second iteration of the IENC

even the existence of a Nash equilibrium cannot be guaranteed. However, we can rely on our epistemic results to provide a simpler way of obtaining this game's Nash equilibria. In Proposition 6 we have shown that the rational choices under Nash equilibrium are precisely the choices that are rational under common belief in rationality and simple beliefs. Similarly, in Proposition 4 we have characterised the $IENC$ sets as the choices that are rational under common belief in rationality. Together, these results imply that any choice that is rational under Nash equilibrium must be an element of the corresponding $IENC$ set. And since the Nash equilibrium never assigns a positive probability to a choice that is not rational under Nash equilibrium[7], we can ignore the choices we have eliminated in the IENC procedure. This practically means that, instead of trying to find the Nash equilibria of the original game from Figure 1, we can find the Nash equilibria of the reduced game from Figure 3. The results we get will be the same.

Having said this, let us now use Figure 3 to obtain the best response correspondences of the two players. For player 1, choice a is optimal whenever she believes that the probability of player 2 playing e is between $\frac{1}{4}$ and $\frac{3}{4}$, and choice b is optimal in the remaining cases. For $p(e) = \frac{1}{4}$ and $p(e) = \frac{3}{4}$, both a and b are optimal[8]. Therefore, the best response correspondence of player 1 can be described as:

$$B_1(p)(b) = 1 - B_1(p)(a) = \begin{cases} 0 & \text{if } \frac{1}{4} < p(e) < \frac{3}{4} \\ 1 & \text{if } p(e) < \frac{1}{4} \text{ or } p(e) > \frac{3}{4} \\ [0,1] & \text{if } p(e) = \frac{1}{4} \text{ or } p(e) = \frac{3}{4} \end{cases}$$

Similarly, we can characterise player 2's best response correspondence as:

$$B_2(q)(e) = 1 - B_2(q)(d) = \begin{cases} 0 & \text{if } q(b) > \frac{1}{2} \\ 1 & \text{if } q(b) < \frac{1}{2} \\ [0,1] & \text{if } q(b) = \frac{1}{2} \end{cases}$$

It is easy to see that the game's Nash equilibria are the fixed points of its best response correspondence $\boldsymbol{B}$ (in fact, we use this in the proof of Proposition 5). Therefore, like in standard game theory textbooks, we can find these equilibria by plotting B_1 and B_2 in the same coordinate system and checking where they intersect. This is what we do in Figure 4. Note that, unlike in games with expected utility, where such graphs admit only one 'step', here the best response correspondence of player 1 jumps from 1 to 0 and then back up to 1 again. This is yet another demonstration that our framework is richer. The figure shows that the game has three Nash equilibria:

- (p^*, q^*), where $p^*(e) = 1 - p^*(d) = 0$ and $q^*(b) = 1 - q^*(a) = 1$

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[7] This follows directly from Definitions 15 and 17.

[8] Although we did not explicitly show that the two boundaries are $\frac{1}{4}$ and $\frac{3}{4}$, the reader can easily check.

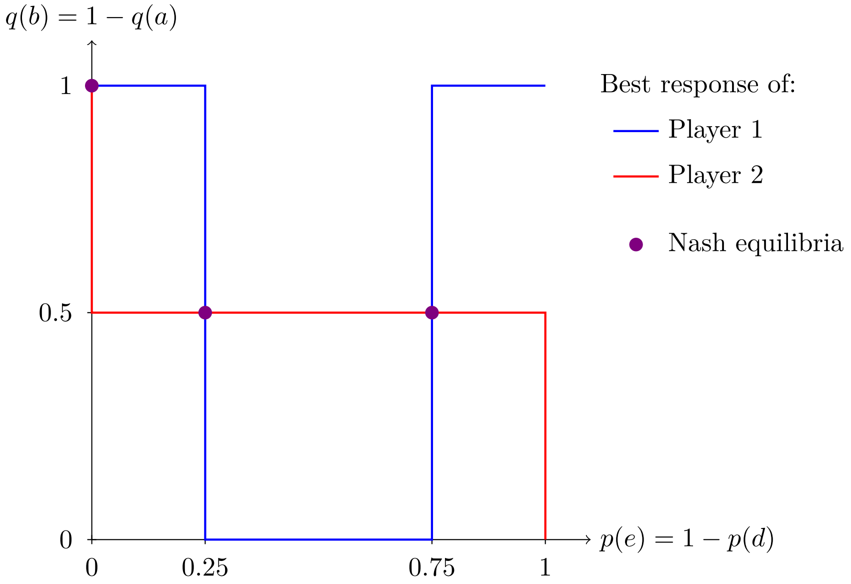

Figure 4: The players' best response correspondences and the game's Nash equilibria

- (p^{**}, q^{**}), where $p^{**}(e) = 1 - p^{**}(d) = \frac{1}{4}$ and $q^{**}(b) = 1 - q^{**}(a) = \frac{1}{2}$
- (p^{***}, q^{***}), where $p^{***}(e) = 1 - p^{***}(d) = \frac{3}{4}$ and $q^{***}(b) = 1 - q^{***}(a) = \frac{1}{2}$

Note that neither player had strongly continuous preferences in the original game, yet we find some Nash equilibria. Therefore, this example demonstrates that the reverse of Proposition 5 is not true: strong continuity is a sufficient but not necessary condition of a Nash equilibrium's existence.

6.3 Correlated equilibrium

Finally, we turn to the correlated equilibria of the game. Using a similar argument as above, Propositions 4 and 8 imply that a correlated equilibrium can never assign a positive probability to a choice that is not in the corresponding *IENC* set. Therefore, we can once again ignore the original game and focus on its reduced form from Figure 3 instead.

Since we have defined correlated equilibrium analogously to Aumann (1974), each correlated equilibrium has an associated epistemic model. But due to the infinite number of possible epistemic models with common priors, characterising the full set of correlated equilibria would be very complicated. The game's *canonical* correlated equilibria would be somewhat easier to characterise, but we have deliberately decided to stick to the original definition. Therefore, instead of identifying *all* correlated equilibria of the game, here we simply show *one* example of a correlated equilibrium. Consider an epistemic model $(\boldsymbol{T}, \boldsymbol{b})$ with type sets $T_1 = \{t_1^a, t_1^b\}$ and $T_2 = \{t_2^c, t_2^d\}$. Suppose that the belief function $\boldsymbol{b}$ is defined in such a way that the following distribution[9] π is a common prior in $(\boldsymbol{T}, \boldsymbol{b})$:

$$\pi(a, t_1^a, d, t_2^d) = \frac{1}{5}, \quad \pi(a, t_1^a, e, t_2^e) = \frac{3}{10},$$
$$\pi(b, t_1^b, d, t_2^d) = \frac{2}{5}, \quad \pi(b, t_1^b, e, t_2^e) = \frac{1}{10}$$

We will show that this π is a correlated equilibrium of the game. We can achieve this by checking whether, for each choice-type combination with a positive probability in π, the choice is optimal for the corresponding conditional belief. There are only four choice-type combinations with positive probabilities in π: (a, t_1^a), (b, t_1^b), (d, t_2^d), and (e, t_2^e). We will consider these one by one.

According to π, the (marginal) probability of choice-type combination (a, t_1^a) is $\frac{1}{5} + \frac{3}{10} = \frac{1}{2}$. Conditional on this, the probabilities of the opponent's choices are $\pi(d|a, t_1^a) = \frac{1/5}{1/2} = \frac{2}{5}$ and $\pi(e|a, t_1^a) = \frac{3/10}{1/2} = \frac{3}{5}$. Since $\frac{3}{5} > \frac{1}{4}$ and $\frac{3}{5} < \frac{3}{4}$, we can see from Figure 3 that choice a is indeed optimal for player 1 under this belief.

Similarly, the marginal probability of (b, t_1^b) is $\frac{2}{5} + \frac{1}{10} = \frac{1}{2}$, so the corresponding conditional probabilities are $\pi(d|b, t_1^b) = \frac{2/5}{1/2} = \frac{4}{5}$ and $\pi(e|b, t_1^b) = \frac{1/10}{1/2} = \frac{1}{5}$. Since $\frac{1}{5} < \frac{1}{4}$, Figure 3 shows that b is indeed optimal for player 1.

Next, let us consider the two choice-type combinations of player 2. In the case of (d, t_2^d), the conditional beliefs are $\pi(a|d, t_2^d) = \frac{1/5}{1/5+2/5} = \frac{1}{3}$ and $\pi(b|d, t_2^d) = \frac{2/5}{1/5+2/5} = \frac{2}{3}$. From here, $\frac{2}{3} > \frac{1}{2}$ and Figure 3 implies that d is indeed optimal for player 2. Similarly, for (e, t_2^e) we have $\pi(a|e, t_2^e) = \frac{3/10}{3/10+1/10} = \frac{3}{4}$ and $\pi(b|e, t_2^e) = \frac{1/10}{3/10+1/10} = \frac{1}{4}$ as the conditional beliefs, so $\frac{1}{4} < \frac{1}{2}$ confirms e's optimality.

Therefore, we have shown that π is indeed a correlated equilibrium of the game. We could of course find many other examples as well. The reason we chose this specific example was that the distribution on choice profiles generated by π is not a convex combination of the distributions generated by the game's three Nash equilibria from Subsection 6.2. This demonstrates that, just like in games with expected utility, the set of correlated equilibria is richer than just the convex hull of the Nash equilibria.

6.4 Alex and Barbara

We now return to our motivating example from the first paragraph of Section 1. We can represent this situation by a two-player game between Alex and Barbara. Alex's choice set is $C_{\text{Alex}} = \{\text{stay}, \text{go}\}$, while Barbara's choice set is $C_{\text{Barbara}} = \{\text{wait}, \text{approach Chris}\}$. Alex's preferences over

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[9] For the choice-type combinations that do not appear below, π assigns zero probability.

his choices depend on his belief $p \in \Delta(C_{\text{Barbara}})$ about Barbara's choice in the following way:

$$\begin{aligned}
&\text{stay} \succ^p_{\text{Alex}} \text{go} \quad \text{if } \frac{1}{4} < p(\text{wait}) < \frac{3}{4} \\
&\text{stay} \sim^p_{\text{Alex}} \text{go} \quad \text{if } p(\text{wait}) = \frac{1}{4} \text{ or } p(\text{wait}) = \frac{3}{4} \\
&\text{stay} \prec^p_{\text{Alex}} \text{go} \quad \text{otherwise}
\end{aligned} \tag{3}$$

Similarly, Barbara's preferences depend on her belief $q \in \Delta(C_{\text{Alex}})$ about whether Alex stays:

$$\begin{aligned}
&\text{wait} \succ^q_{\text{Barbara}} \text{approach Chris} \quad \text{if } q(\text{stay}) < \frac{1}{2} \\
&\text{wait} \sim^q_{\text{Barbara}} \text{approach Chris} \quad \text{if } q(\text{stay}) = \frac{1}{2} \\
&\text{wait} \prec^q_{\text{Barbara}} \text{approach Chris} \quad \text{otherwise}
\end{aligned} \tag{4}$$

If we now relabel Alex and Barbara as players 1 and 2, 'stay' and 'go' as a and b, and 'wait' and 'approach Chris' as d and e, then what we get is exactly the reduced game from Figure 3. Thus, while the original game from Figure 1 does not correspond to any realistic scenario, the game that emerges after iterated elimination has a clear and intuitive real-life interpretation. It is a precise description of the Alex-Barbara situation introduced in Section 1. This was in fact one of our goals when calibrating the example.

This equivalence also allows us to translate our previous findings into the context of the motivating story. Subsection 6.1 showed that IENC returns precisely the reduced game, so both of Alex's and Barbara's choices are rational under common belief in rationality. Subsection 6.2 then established that the game has three Nash equilibria: a pure one where Alex is believed to go to the party and Barbara is believed to wait for him, and two others where Alex is believed to go with probability $\frac{1}{2}$, and Barbara is believed to wait either with probability $\frac{1}{4}$ or $\frac{3}{4}$. This also means that both of Alex's and Barbara's choices are rational under simple beliefs and common belief in rationality. Finally, our findings from Subsection 6.3 reveal a possible correlated equilibrium of the game, and demonstrate that both of Alex's and Barbara's choices are rational under a common prior and common belief in rationality. However, for the sake of true love, the authors hope that Alex chooses to go and Barbara chooses to wait for him.

7 Discussion

This paper demonstrates that the fundamental pillars of game theory do not rely on the expected utility assumption. We show that classic solution concepts like iterated elimination, Nash equilibrium, and correlated equilibrium can be generalised to a much broader setting with conditional preference relations. Our epistemic approach preserves the conceptual core of each solution concept.

The generalised versions rest on the same reasoning as their classical counterparts, making them epistemically equivalent.

Of course, one might naturally ask what we gain by abandoning the convenient simplicity of expected utility. The standard model of normal-form games with payoff matrices is indeed simple and tractable. However, this simplicity also forces the analyst to assume that players' preferences obey strong mathematical constraints. As Perea (2025) has shown, expected utility is not an inherent form of rationality. Rather, it is a highly specific condition, characterised by a set of demanding axioms (preservation of indifference and strict preference, three-choice and four-choice linear preference intensity) in addition to continuity. By imposing expected utility, game theorists risk excluding plausible preferences simply because they violate these restrictive axioms.

The primary motivation behind our new framework is to avoid this risk. We wish to represent realistic preferences that the expected utility model is structurally incapable of representing. In addition to the well-known empirical evidence (Allais, 1953; Ellsberg, 1961), our story about Alex and Barbara also serves as an example of such preferences. Alex wishes to go to the party if he is highly confident that Barbara will wait for him or if he is highly confident that she will not, but he prefers to stay at home when he is uncertain. An expected utility maximiser cannot exhibit such preferences, even though they appear completely natural and intuitive from a behavioural perspective. Our framework accommodates these preferences, as well as other violations of the expected utility hypothesis.

Moreover, our existence results demonstrate that this gain in behavioural flexibility does not come at the cost of solvability. The iterated elimination of never-optimal choices yields a non-empty solution in any game, and Nash and correlated equilibria also exist if the preferences are strongly continuous. Therefore, strong continuity is the only property of expected utility we need for equilibrium existence.

There are several promising directions for further research. We could conduct a similar analysis with other well-known solution concepts such as trembling-hand perfect equilibrium (Selten, 1975) or proper equilibrium (Myerson, 1978). Both refinements have epistemic characterisations (Blume, Brandenburger & Dekel, 1991) in the expected-utility setting, albeit only for two-player games. We could also extend our analysis to games with incomplete information (Harsányi, 1967, 1968a, 1968b). In such games, analogous solution concepts have been characterised epistemically (Battigalli, 2003; Battigalli & Siniscalchi, 2003; Bach & Perea, 2021, 2020a, 2017), but not without the assumption of expected utility. Finally, one could design laboratory or field experiments that elicit conditional preference relations directly, and test whether observed behaviour converges to our generalised equilibria. This would provide critical empirical validation and may help identify which preference axioms best capture real-world strategic decision-making.

A Proofs

Proposition 1. *Player i's conditional preference relation $\succsim_i$ is strongly continuous if and only if it is weakly continuous and $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \sim_i^p c_i'\}$ is a closed set for any $c_i, c_i' \in C_i$.*

Proof. We first prove the 'if' part, and then the 'only if' part.

Proof of the 'if' part: We use proof by contradiction. Suppose that $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \sim_i^p c_i'\}$ is a closed set for any $c_i, c_i' \in C_i$, and that $\succsim_i$ is weakly but not strongly continuous. The latter means that for some $c_i, c_i' \in C_i$ there exists a convergent sequence of beliefs $(p^k)_{k\in\mathbb{N}_0}$ such that $c_i \succsim_i^{p^k} c_i' \quad \forall k \in \mathbb{N}_0$, but for $p^\infty := \lim_{k\to\infty} p^k$ we have $c_i' \succ_i^{p^\infty} c_i$.

By definition, the preference $c_i \succsim_i^{p^k} c_i'$ can mean one of two things. Either the player is indifferent between the two choices, or she strictly prefers c_i. Suppose that the former is the case for infinitely many elements of the (p^k) sequence. This implies that there exists an infinite subsequence $(\widetilde{p}^\ell)_{\ell\in\mathbb{N}_0}$ of (p^k) such that $c_i \sim_i^{\tilde{p}^\ell} c_i' \quad \forall \ell \in \mathbb{N}_0$. This subsequence must converge to the same value as (p^k), so we have $\lim_{\ell\to\infty} \widetilde{p}^\ell = p^\infty$. But then the closedness of $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \sim_i^p c_i'\}$ would imply that $c_i \sim_i^{p^\infty} c_i'$, which contradicts our initial assumption.

The other possibility is that $c_i \sim_i^{p^k} c_i'$ only holds for finitely many values of k, which necessarily implies that we have a strict preference for infinitely many values of k. Therefore, there exists an infinite subsequence $(\widehat{p}^\ell)_{\ell\in\mathbb{N}_0}$ of (p^k) such that $c_i \succ_i^{\hat{p}^\ell} c_i' \quad \forall \ell \in \mathbb{N}_0$. We of course have $\lim_{\ell\to\infty} \widehat{p}^\ell = p^\infty$. We have assumed that $c_i' \succ_i^{p^\infty} c_i$, so the player's preferences are reversed between p^∞ and each $\widehat{p}^\ell$. Then the weak continuity of the preferences implies that for each $\ell \in \mathbb{N}_0$ there exists a $\lambda^\ell \in]0,1[$ such that the belief $\lambda^\ell \cdot \widehat{p}^\ell + (1-\lambda^\ell) \cdot p^\infty$ leads to an indifference. Let us define the $(\widetilde{p}^\ell)_{\ell\in\mathbb{N}_0}$ sequence as $\widetilde{p}^\ell = \lambda^\ell \cdot \widehat{p}^\ell + (1-\lambda^\ell) \cdot p^\infty$ for each $\ell \in \mathbb{N}_0$. Each element of this $(\widetilde{p}^\ell)$ sequence is a convex combination of p^∞ and the corresponding element of a sequence that converges to p^∞, indicating that $\lim_{\ell\to\infty} \widetilde{p}^\ell = p^\infty$. But note that $c_i \sim_i^{\tilde{p}^\ell} c_i' \quad \forall \ell \in \mathbb{N}_0$, so the closedness of $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \sim_i^p c_i'\}$ implies that $c_i \sim_i^{p^\infty} c_i'$. We have thus reached the final contradiction, proving the 'if' part of our proposition.

Proof of the 'only if' part: The strong continuity of $\succsim_i$ trivially implies that $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \sim_i^p c_i'\}$ is closed because this set is equal to $\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i \succsim_i^p c_i'\} \cap \{p \in \Delta(\boldsymbol{C}_{-i}) : c_i' \succsim_i^p c_i\}$, and the intersection of two closed sets must also be closed. It remains to show that the strong continuity of $\succsim_i$ also implies its weak continuity. Suppose for contradiction that $\succsim_i$ is strongly but not weakly continuous. The latter means that there exists a pair of choices $c_i, c_i' \in C_i$ and a pair of beliefs $p, p' \in \Delta(\boldsymbol{C}_{-i})$ such that $c_i \succ_i^p c_i'$, $c_i' \succ_i^{p'} c_i$, and $c_i \not\sim_i^{\lambda\cdot p+(1-\lambda)\cdot p'} c_i' \quad \forall \lambda \in]0,1[$. Now let us define the set L as follows:

$$L = \{\lambda \cdot p + (1-\lambda) \cdot p' : \lambda \in [0,1]\}$$

This set L is a closed line segment, which means that it is a closed set. We also know from the strong continuity of $\succsim_i$ that the set $\{p'' \in \Delta(\boldsymbol{C}_{-i}) : c_i \succsim_i^{p''} c_i'\}$ is closed. Now let us consider the intersection of these two closed sets, $S = L \cap \{p'' \in \Delta(\boldsymbol{C}_{-i}) : c_i \succsim_i^{p''} c_i'\}$. This must be closed as well, and it is also non-empty because p is an element of both intersected sets. Moreover, we know that $c_i \succ_i^{p''} c_i' \quad \forall p'' \in S$ because the indifference is ruled out by the assumption above.

We can similarly show that the set $S' = L \cap \{p'' \in \Delta(\boldsymbol{C}_{-i}) : c_i' \succsim_i^{p''} c_i\}$ is closed and non-empty, and that $c_i' \succ_i^{p''} c_i \quad \forall p'' \in S'$. The latter also implies that $S \cap S' = \emptyset$, while it is easy to see that $S \cup S' = L$. Therefore, we have found that the closed line segment L is the union of two disjoint, closed and non-empty sets (S and S'). This contradicts the fact that any line segment is a connected set, and thus concludes our proof. $\square$

Proposition 2. *Player i's conditional preference relation $\succsim_i$ is strongly continuous if and only if there exists a utility function that is continuous in $\Delta(\boldsymbol{C}_{-i})$ and represents $\succsim_i$.*

Proof. We first prove the 'if' part, and then the 'only if' part.

Proof of the 'if' part: Let u_i be an arbitrary continuous utility function that represents $\succsim_i$. Suppose for contradiction that $\succsim_i$ is not strongly continuous, i.e. there exists a pair of choices $c_i, c_i' \in C_i$ and a convergent sequence of beliefs $(p^k)_{k \in \mathbb{N}_0}$ such that $c_i \succsim_i^{p^k} c_i' \quad \forall k \in \mathbb{N}_0$, but for $p^\infty = \lim_{k\to\infty} p^k$ we have $c_i' \succ_i^{p^\infty} c_i$. If we use the fact that $\succsim_i$ is represented by u_i, we can rewrite these statements as $u_i(c_i, p^k) \geq u_i(c_i', p^k) \quad \forall k \in \mathbb{N}_0$ and $u_i(c_i', p^\infty) > u_i(c_i, p^\infty)$. However, this would mean that u_i is not continuous, so we have reached a contradiction. This proves that $\succsim_i$ is strongly continuous when a continuous representation exists.

Proof of the 'only if' part: Suppose that $\succsim_i$ is strongly continuous. We will show by construction that a continuous utility function u_i representing $\succsim_i$ exists. Let us consider the elements of i's choice set C_i in an arbitrary order. We denote the choice at the kth position in this order by c_i^k, where of course we have $k \in \{1, ..., |C_i|\}$. We also define the $S_{k\ell}$ sets for each $k, \ell \in \{1, ..., |C_i|\}$ in the following way:

$$S_{k\ell} = \left\{p \in \Delta(\boldsymbol{C}_{-i}) : c_i^k \sim_i^p c_i^\ell\right\}$$

Finally, let us define the function $d_{k\ell} : \Delta(\boldsymbol{C}_{-i}) \to \mathbb{R}$ in the following way[10]:

$$d_{k\ell}(p) = \begin{cases} \min\limits_{q \in S_{k\ell}} \|p - q\| & \text{if } S_{k\ell} \neq \emptyset \\ \\ 1 & \text{if } S_{k\ell} = \emptyset \end{cases}$$

Together with Proposition 1, the strong continuity of $\succsim_i$ implies that $S_{k\ell}$ is a closed set, so

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[10] In this definition, $\|p - q\|$ denotes the Euclidean distance between vectors p and q.

the minimum at the top branch of $d_{k\ell}$ indeed exists. These definitions enable us to construct the continuous utility function u_i that represents $\succsim_i$. We will follow a recursive approach. In the first step, we define $u_i(c_i^1, \cdot)$ as:

$$u_i(c_i^1, p) = 0 \quad \forall p \in \Delta(\boldsymbol{C}_{-i})$$

Next, we consider each $k \in \{2, ..., |C_i|\}$ one by one, in ascending order. We define each $u_i(c_i^k, \cdot)$ in the following way[11]:

$$u_i(c_i^k, p) = \begin{cases} u_i(c_i^\ell, p) & \text{if } \exists \ell \in \{1, ..., k-1\} : c_i^k \sim_i^p c_i^\ell \\[1em] u_i(c_i^\ell, p) + d_{k\ell}(p) & \text{if } c_i^k \succ_i^p c_i^\ell \succsim_i^p ... \\ \dfrac{d_{kj}(p)}{d_{kj}(p) + d_{k\ell}(p)} \cdot u_i(c_i^\ell, p) + \dfrac{d_{k\ell}(p)}{d_{kj}(p) + d_{k\ell}(p)} \cdot u_i(c_i^j, p) & \text{if } ... \succsim_i^p c_i^j \succ_i^p c_i^k \succ_i^p c_i^\ell \succsim_i^p ... \\ u_i(c_i^\ell, p) - d_{k\ell}(p) & \text{if } ... \succsim_i^p c_i^\ell \succ_i^p c_i^k \end{cases}$$

This utility function clearly represents $\succsim_i$ since the order of utilities is consistent with the ranking of choices. What remains to show is that each $u_i(c_i^k, \cdot)$ is continuous. We prove this by induction. The function $u_i(c_i^1, \cdot)$ is constant, so it is trivially continuous. Next, we need to prove that $u_i(c_i^k, \cdot)$ is continuous, using the inductive hypothesis that $u_i(c_i^1, \cdot)$, $u_i(c_i^2, \cdot)$, ..., and $u_i(c_i^{k-1}, \cdot)$ are all continuous. This proof is a bit trickier since we have a piecewise function with four branches. For values of p that fall on the same branch, the function is indeed continuous because each $d_{k\ell}$ is a continuous function, and the inductive hypothesis also implies that each $u_i(c_i^\ell, \cdot)$ is continuous. The remaining question is what happens when we 'jump' from one branch to the other.

We have assumed that $\succsim_i$ is strongly continuous, so Proposition 1 tells us that it is also weakly continuous. Therefore, the set of beliefs where one choice is strictly preferred to the other cannot directly border the set of beliefs where this preference is reversed: these two sets must be separated by beliefs where the player is indifferent between the two. This means that in the piecewise function defined above, there can be no direct 'jumps' between the three bottom branches; the top branch must always be touched when we switch between two orders. However, note that when $c_i^k \sim_i^p c_i^\ell$, we must necessarily have $p \in S_{k\ell}$ and hence $d_{k\ell}(p) = 0$. And for $d_{k\ell}(p) = 0$, all three of the bottom branches yield exactly[12] $u_i(c_i^\ell, p)$, the value of the top branch. This means that there are no discontinuities when jumping between different branches either, so $u_i(c_i^k, \cdot)$ is continuous for any $k \in \{1, ..., |C_i|\}$. This concludes our proof. $\square$

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[11] By $c_i^k \succ_i^p c_i^\ell \succsim_i^p$ ..., we mean that among the choices in $\{c_i^1, ..., c_i^k\}$, the most preferred one is c_i^k, followed directly by c_i^ℓ. Similarly, ... $\succsim_i^p c_i^\ell \succ_i^p c_i^k$ means that c_i^k is the least preferred choice, preceded directly by c_i^ℓ. Finally, ... $\succsim_i^p c_i^j \succ_i^p c_i^k \succ_i^p c_i^\ell \succsim_i^p$ ... denotes the case where c_i^k is ranked directly between c_i^j and c_i^ℓ.

[12] In the case of the third branch, the value of $u_i(c_i^j, p)$ is also possible if we approach the other boundary, but this is only a matter of indexing.

Lemma 1. *In an epistemic model where each type has onefold belief in rationality, each type has common belief in rationality.*

Proof. Suppose that $(\boldsymbol{T}, \boldsymbol{b})$ is an epistemic model where each type has onefold belief in rationality. We will use induction to show that these types also have $k$fold belief in rationality for any $k \in \mathbb{N}_+$. For $k = 1$ this is true by definition. It remains to show that it is also true for an arbitrary k, using the inductive hypothesis that it is true for $k-1$.

Consider an arbitrary type $t_i \in T_i$, and a $(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) \in \boldsymbol{C}_{-i} \times \boldsymbol{T}_{-i}$ pair that satisfies $b_i(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) > 0$. Our inductive hypothesis suggests that each type (including the ones in the $\boldsymbol{t}_{-i}$ profile) has $k-1$fold belief in rationality. This means by definition that t_i must have $k$fold belief in rationality. And since we defined t_i as an arbitrary type, this concludes our proof. □

Proposition 3. *The $IENC_i$ set is non-empty for any player $i \in I$.*

Proof. The definition of C_i^k indicates that for any $k \in \mathbb{N}_+$ we have $C_i^k \subseteq C_i^{k-1}$. Since $IENC_i$ is the intersection of these C_i^k sets and these sets are finite, our proposition is equivalent to the claim that each C_i^k is non-empty. We show this by induction on k. By assumption, $C_i^0 = C_i$ is non-empty for any $i \in I$. Next, we need to show that C_i^k is non-empty, using the inductive hypothesis that C_j^{k-1} is non-empty for any $j \in I$. This hypothesis indicates that $\Delta(\boldsymbol{C}_{-i}^{k-1})$ must be non-empty as well. Now consider an arbitrary element of this set, $p \in \Delta(\boldsymbol{C}_{-i}^{k-1})$. Due to the transitivity of $\succsim_i^p$, there must be at least one choice $c_i \in C_i$ for which we have $c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i$.

Trivially, we have $c_i \in C_i^0$. Let us now use another induction to prove that $c_i \in C_i^\ell$ for any $\ell \in \{1, ..., k\}$, using the inductive hypothesis that $c_i \in C_i^{\ell-1}$. We know that $C_j^{k-1} \subseteq C_j^{\ell-1} \quad \forall j \in I$, which also means that $\Delta(\boldsymbol{C}_{-i}^{k-1}) \subseteq \Delta(\boldsymbol{C}_{-i}^{\ell-1})$, and thus $p \in \Delta(\boldsymbol{C}_{-i}^{\ell-1})$. Together with our inductive hypothesis and the fact that c_i is optimal for i under belief p, this implies that c_i satisfies all the requirements to be in the C_i^ℓ set. Therefore, we have shown that $c_i \in C_i^k$, so the C_i^k set must be non-empty for any $k \in \mathbb{N}_0$ and $i \in I$. Due to our arguments above, this also proves that $IENC_i$ is non-empty for any $i \in I$. □

Lemma 2. *For player $i \in I$, choice $c_i \in C_i$ is rational under up to $k$fold*[13] *belief in rationality if and only if $c_i \in C_i^{k+1}$*

Proof. Let $\varrho_i^k(\boldsymbol{T}, \boldsymbol{b})$ denote the types of i from the given epistemic model that have $k$fold belief in rationality, and R_i^k denote the choices that are rational for i under up to $k$fold belief in rationality.

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[13] That is, under ℓfold belief in rationality for each $\ell \in \{1, 2, ..., k\}$, where $k \in \mathbb{N}_+$.

Formally, we have:

$$\begin{aligned}\varrho_i^k(\boldsymbol{T},\boldsymbol{b}) &= \{t_i \in T_i : t_i \text{ has } k\text{fold belief in rationality}\}\\ R_i^k &= \left\{c_i \in C_i : \exists(\boldsymbol{T},\boldsymbol{b}), t_i \in \bigcap_{\ell=1}^{k} \varrho_i^\ell(\boldsymbol{T},\boldsymbol{b}) \text{ such that } c_i \text{ is optimal for } t_i\right\}\end{aligned}$$

What we need to prove is that $R_i^k = C_i^{k+1} \quad \forall i \in I, k \in \mathbb{N}_+$. We show this in two steps: first that $R_i^k \subseteq C_i^{k+1}$, and then that $C_i^{k+1} \subseteq R_i^k$.

Proof that $R_i^k \subseteq C_i^{k+1}$: We show this by induction. For an arbitrary $c_i \in R_i^1$, we know by definition that there exists a type t_i with onefold belief in rationality such that $c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i, p = \operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t_i)$. Since $\operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t_i)$ is a distribution on $\boldsymbol{C}_{-i} = \boldsymbol{C}_{-i}^0$, this already indicates by definition that $c_i \in C_i^1$. t_i's onefold belief in rationality means that $b_i(t_i)$ only assigns positive probabilities to (c_j, t_j) pairs where each c_j is optimal for t_j. For such pairs, $c_j \in C_j^1$ must necessarily hold since there exists a belief (in particular, $b_j(t_j)$) under which c_j is optimal. In other words, the support of $\operatorname{marg}_{C_j} b_i(t_i)$ must be a subset of C_j^1. Since this is true for any $j \in I \setminus \{i\}$, we have $\operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t_i) \in \Delta\left(\boldsymbol{C}_{-i}^1\right)$. Together with $c_i \in C_i^1$, the definition of C_i^2 implies that we also have $c_i \in C_i^2$. This concludes our proof for $k = 1$.

Next, we need to prove that $R_i^k \subseteq C_i^{k+1}$, using the inductive hypothesis that $R_j^{k-1} \subseteq C_j^k \quad \forall j \in I$. For an arbitrary $c_i \in R_i^k$, we know that there exists a type t_i with up to $k$fold belief in rationality such that $c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i, p = \operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t_i)$. Since t_i has up to $k$fold belief in rationality, we know that that $b_i(t_i)$ only assigns positive probabilities to (c_j, t_j) pairs where each c_j is optimal for t_j, and t_j has up to $k-1$fold belief in rationality. By definition, we then have $c_j \in R_j^{k-1}$. From the inductive hypothesis it follows that $c_j \in C_j^k$, so the support of $\operatorname{marg}_{C_j} b_i(t_i)$ is a subset of C_j^k. Since this is true for any $j \in I \setminus \{i\}$, we can conclude that $\operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t_i) \in \Delta\left(\boldsymbol{C}_{-i}^k\right)$. It can easily be seen that $R_i^k \subseteq R_i^{k-1}$, so together with the inductive hypothesis and $c_i \in R_i^k$ we get $c_i \in C_i^k$. From here, the definition of C_i^{k+1} implies $c_i \in C_i^{k+1}$. This is precisely what we wanted to show.

Proof that $C_i^{k+1} \subseteq R_i^k$: Our proof starts by constructing an epistemic model $(\boldsymbol{T},\boldsymbol{b})$. First, we define each type set as $T_i = \{t^{c_i} : c_i \in C_i^1\} \quad \forall i \in I$. This means that we have a separate type for each choice in the given player's C_i^1 set. For any $i \in I$, let $p_i : C_i^1 \to \Delta(\boldsymbol{C}_{-i})$ be an arbitrary function that satisfies the following criteria:

$$\begin{aligned}c_i \in C_i^k \setminus C_i^{k+1} &\implies p_i(c_i) \in \left\{p \in \Delta(\boldsymbol{C}_{-i}^{k-1}) : c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i\right\} \quad \forall k \in \mathbb{N}_+\\ c_i \in IENC_i &\implies p_i(c_i) \in \{p \in \Delta(\boldsymbol{IENC}_{-i}) : c_i \succsim_i^p c_i' \quad \forall c_i' \in C_i\}\end{aligned} \tag{5}$$

We know from Proposition 3 and C_i^k's definition that the sets above must always be non-empty, so such a p_i function exists. This enables us to define the b_i function of any player $i \in I$ as:

$$b_i(t^{c_i})(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) = \begin{cases} \frac{p_i(c_i)(\boldsymbol{c}_{-i})}{|\boldsymbol{T}_{-i}|} & \text{if } c_i \notin C_i^2 \\ p_i(c_i)(\boldsymbol{c}_{-i}) & \text{if } c_i \in C_i^2 \text{ and } t_j = t^{c_j} \quad \forall j \in I \setminus \{i\} \\ 0 & \text{otherwise} \end{cases}$$

Consider an arbitrary type $t^{c_i} \in T_i$ from this epistemic model. When player i has this type, her marginal belief is $\operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t^{c_i}) = p_i(c_i)$. From equation (5) it follows that $c_i \succsim_i^{p_i(c_i)} c_i' \quad \forall c_i' \in C_i$. This means that c_i must be optimal for type t^{c_i}, for any $i \in I$ and $c_i \in C_i^1$.

Next, we will show that if we have $c_i \in C_i^{k+1}$ for some $i \in I$ and $k \in \mathbb{N}_+$, then t^{c_i} must have up to $k$fold belief in rationality. We prove this by induction on k. For $k = 1$, this is straightforward since $b_i(t^{c_i})$ only assigns positive probabilities to $(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i})$ pairs where $t_j = t^{c_j} \quad \forall j \in I \setminus \{i\}$, and we have just shown that c_j is optimal for each t^{c_j}. This means that t^{c_i} has onefold belief in rationality. Next, we should prove that $c_i \in C_i^{k+1}$ implies $t^{c_i} \in \varrho_i^k(\boldsymbol{T}, \boldsymbol{b})$, using the inductive hypothesis that $t^{c_j'} \in \varrho_j^{k-1}(\boldsymbol{T}, \boldsymbol{b}) \quad \forall j \in I, c_j' \in C_j^k$. We can see in equation (5) that for any $c_i \in C_i^{k+1}$ we have $p_i(c_i) \in \Delta(\boldsymbol{C}_{-i}^k)$. Therefore, $p_i(c_i)$ only assigns positive probabilities to a profile $\boldsymbol{c}_{-i}$ if $\boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}^k$. But then from the definition of b_i it follows that $b_i(t^{c_i})$ only assigns positive probabilities to a type t^{c_j} if $c_j \in C_j^k$, and our inductive hypothesis suggests that these types are in $\varrho_j^{k-1}(\boldsymbol{T}, \boldsymbol{b})$. This implies by definition that t^{c_i} has $k$fold belief in rationality. This ends the induction.

To sum up, we have shown that for any $c_i \in C_i^{k+1}$ there exists a type t^{c_i} in our epistemic model such that c_i is optimal for t^{c_i} and t^{c_i} has up to $k$fold belief in rationality. By definition, this means that $c_i \in R_i^k$, and thus $C_i^{k+1} \subseteq R_i^k$. □

Proposition 4. *For a player $i \in I$, choice $c_i \in C_i$ is rational under common belief in rationality if and only if $c_i \in IENC_i$.*

Proof. We first prove the 'if' part, and then the 'only if' part.

Proof of the 'if' part: Consider the epistemic model we have constructed in the proof of Lemma 2. We have shown that any choice $c_i \in IENC_i$ is optimal for the type t^{c_i}. By definition, we have $IENC_i = \bigcap_{k \in \mathbb{N}_0} C_i^k$, meaning that $c_i \in C_i^k \quad \forall k \in \mathbb{N}_0$. However, while proving Lemma 2, we also established that $c_i \in C_i^k$ implies $t^{c_i} \in \varrho_i^{k-1}(\boldsymbol{T}, \boldsymbol{b})$. Together, these findings imply that $t^{c_i} \in \bigcap_{k=1}^{\infty} \varrho_i^k(\boldsymbol{T}, \boldsymbol{b})$, and thus t^{c_i} has common belief in rationality by definition.

To sum up, we have managed to construct an epistemic model where for each $c_i \in IENC_i$ there exists a type t^{c_i} with common belief in rationality such that c_i is optimal for t^{c_i}. By definition, this means that c_i is rational under common belief in rationality.

Proof of the 'only if' part: The statement that $c_i \in C_i$ is rational under common belief in rationality means that there exists an epistemic model $(\boldsymbol{T}, \boldsymbol{b})$ with a type $t_i \in T_i$ such that c_i is optimal for t_i, and t_i has common belief in rationality. Using our notations from Lemma 2, we can rephrase this as:

$$\exists(\boldsymbol{T}, \boldsymbol{b}), t_i \in \bigcap_{\ell=1}^{\infty} \varrho_i^{\ell}(\boldsymbol{T}, \boldsymbol{b}) \text{ such that } c_i \text{ is optimal for } t_i$$

However, note that $\bigcap_{\ell=1}^{\infty} \varrho_i^{\ell}(\boldsymbol{T}, \boldsymbol{b}) \subseteq \bigcap_{\ell=1}^{k} \varrho_i^{\ell}(\boldsymbol{T}, \boldsymbol{b}) \quad \forall k \in \mathbb{N}_+$, and thus we must have $c_i \in R_i^k \quad \forall k \in \mathbb{N}_+$. We can conclude that $c_i \in \bigcap_{k=1}^{\infty} R_i^k$. Together with Lemma 2, we get:

$$c_i \in \bigcap_{k=1}^{\infty} R_i^k = \bigcap_{k=2}^{\infty} C_i^k = \bigcap_{k \in \mathbb{N}_0} C_i^k = IENC_i$$

This concludes our proof. □

Proposition 5. *If the conditional preference relation* $\succsim_i$ *of each player* $i \in I$ *is strongly continuous, then a Nash equilibrium exists.*

Proof. It is easy to see that for any $\boldsymbol{q} \in \times_{i \in I} \Delta(C_i)$ we have:

$$\boldsymbol{q} \in \boldsymbol{B}(\boldsymbol{q}) \iff \boldsymbol{q} \text{ is a Nash equilibrium}$$

This means that our proposition is equivalent to the claim that $\boldsymbol{B}$ has a fixed point when each $\succsim_i$ is strongly continuous. We can prove this by applying Kakutani's (1941) fixed point theorem to $\boldsymbol{B}$.

From Proposition 2, we know that the strong continuity of the conditional preference relation is equivalent to the existence of a continuous utility representation. Let us consider an arbitrary continuous utility representation u_i for each $i \in I$. This enables us to give an equivalent definition to the best response correspondence $\boldsymbol{B}$:

$$B_i(\boldsymbol{q}) = \Delta\left(\operatorname*{argmax}_{c_i \in C_i} u_i\left(c_i, q_{-i}\right)\right)$$

Since each C_i is non-empty, each $\Delta(C_i)$ set is a simplex. This means that $\boldsymbol{B}$'s domain $\times_{i \in I} \Delta(C_i)$ must be non-empty, convex and compact. Similarly, since the argmax operator always returns a non-empty set, $B_i(\boldsymbol{q})$ must also be non-empty, convex and compact. This implies that $\boldsymbol{B}(\boldsymbol{q})$ is also non-empty and convex, as it is the cartesian product of non-empty and convex sets. For Kakutani's theorem to apply, it remains to show that $\boldsymbol{B}$ has a closed graph[14].

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[14] In Kakutani's original formulation, the last requirement of the theorem was the function's upper hemicontinuity. However, the two can be shown to be equivalent in this context, see the Corollary in Berge (1959/1963, p. 112).

By definition, the closed graph property requires that, if we have two convergent sequences $(\boldsymbol{q}^k)_{k\in\mathbb{N}_0}$ and $(\widehat{\boldsymbol{q}}^k)_{k\in\mathbb{N}_0}$ such that $\widehat{\boldsymbol{q}}^k \in \boldsymbol{B}(\boldsymbol{q}^k) \quad \forall k \in \mathbb{N}_0$, then we must also have:

$$\lim_{k\to\infty} \widehat{\boldsymbol{q}}^k \in \boldsymbol{B}\left(\lim_{k\to\infty} \boldsymbol{q}^k\right)$$

Let us denote the two limits above by $\boldsymbol{q}^\infty$ and $\widehat{\boldsymbol{q}}^\infty$ (respectively), and suppose for contradiction that $\widehat{\boldsymbol{q}}^\infty \notin \boldsymbol{B}(\boldsymbol{q}^\infty)$. By definition, this means that there exists a player $i \in I$ and a choice $c_i \in C_i$ such that $\widehat{q}_i^\infty(c_i) > 0$ but c_i does not maximise $u_i(\cdot, q_{-i}^\infty)$. However, the fact that $\widehat{q}_i^\infty$ is a limit means that for a large enough k, we must also have $\widehat{q}_i^k(c_i) > 0$. Moreover, the continuity of u_i also implies that for a large enough k, c_i cannot maximise $u_i(\cdot, q_{-i}^k)$ either. Together, these two statements yield that there exists a $k \in \mathbb{N}_0$ such that $\widehat{\boldsymbol{q}}^k \notin \boldsymbol{B}(\boldsymbol{q}^k)$. We have reached a contradiction, proving that $\widehat{\boldsymbol{q}}^\infty \in \boldsymbol{B}(\boldsymbol{q}^\infty)$. This means that $\boldsymbol{B}$ has a closed graph, so Kakutani's fixed point theorem can be applied to it. Therefore, $\boldsymbol{B}$ has a fixed point, and hence the game has a Nash equilibrium. ☐

Proposition 6. *For a player $i \in I$, choice $c_i \in C_i$ is rational under simple beliefs and common belief in rationality if and only if it is rational under Nash equilibrium.*

Proof. We first prove the 'if' part, and then the 'only if' part.

Proof of the 'if' part: Let $\boldsymbol{q}$ be a Nash equilibrium such that we have $c_i \succsim_i^{q_{-i}} c'_i \quad \forall c'_i \in C_i$. Consider the epistemic model $(\boldsymbol{T}, \boldsymbol{b})$ where for each $j \in I$ we have $T_j = \{t_j\}$ and $b_j(t_j)(\boldsymbol{c}_{-j}, \boldsymbol{t}_{-j}) = q_{-j}(\boldsymbol{c}_{-j}) \quad \forall \boldsymbol{c}_{-j} \in \boldsymbol{C}_{-j}$. Since we have $q_{-j}(\boldsymbol{c}_{-j}) = \prod_{k\in I\setminus\{j\}} q_k(c_k)$, it is easy to see that this epistemic model is simple by definition. Note also that each T_j is a singleton, so the marginals of the beliefs become:

$$\operatorname*{marg}_{C_{-j}} b_j(t_j) = q_{-j} \quad \forall j \in I \tag{6}$$

This must also be true for player i. Together with our assumption that $c_i \succsim_i^{q_{-i}} c'_i \quad \forall c'_i \in C_i$, we can conclude that choice c_i is optimal for type t_i. We have already argued that this t_i is simple, so it remains to show that t_i also has common belief in rationality.

Let us consider an arbitrary player[15] $j \in I$ and a truncated choice profile $\boldsymbol{c}^*_{-j} \in \boldsymbol{C}_{-j}$ such that $b_j(t_j)(\boldsymbol{c}^*_{-j}, \boldsymbol{t}_{-j}) = \prod_{k\in I\setminus\{i\}} q_k(c^*_k) > 0$. For this inequality to hold, we need $q_k(c^*_k) > 0 \quad \forall k \in I\setminus\{j\}$. From here, the definition of Nash equilibrium implies $c^*_k \succsim_k^{q_{-k}} c'_k \quad \forall c'_k \in C_k$. Together with our result from (6), we can conclude that c^*_k is optimal for t_k. This is true for any $k \in I \setminus \{j\}$, so type t_j has onefold belief in rationality (by definition). Moreover, we defined j as an arbitrary player, so each type in our epistemic model has onefold belief in rationality. By Lemma 1, this also means that these types all have common belief in rationality.

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[15] Note that we may have $j = i$ in this case.

Therefore, we were able to construct an epistemic model with a type (t_i) that is simple and has common belief in rationality, and we have also shown that c_i is optimal for this type. This means by definition that c_i is rational under simple beliefs and common belief in rationality.

Proof of the 'only if' part: Suppose that c_i is rational under simple beliefs and common belief in rationality. This means that there exists an epistemic model $(\boldsymbol{T}, \boldsymbol{b})$ with a type $t_i \in T_i$ such that c_i is optimal for t_i and t_i has both properties. The simpleness of t_i means that there is a truncated type profile $\boldsymbol{t}_{-i} \in \boldsymbol{T}_{-i}$ and a profile of distributions $\boldsymbol{q} \in \times_{j \in I} \Delta(C_j)$ such that $b_j(t_j)(\boldsymbol{c}'_{-j}, \boldsymbol{t}_{-j}) = \prod_{k \in I \setminus \{j\}} q_k(c'_k) \quad \forall j \in I, \boldsymbol{c}'_{-j} \in \boldsymbol{C}_{-j}$.

Consider an arbitrary choice profile $\boldsymbol{c}^* \in \boldsymbol{C}$ such that $q_j(c^*_j) > 0$ for each $j \in I$. For such profiles we must have that $b_i(t_i)(\boldsymbol{c}^*_{-i}, \boldsymbol{t}_{-i}) = \prod_{j \in I \setminus \{i\}} q_j(c^*_j) > 0$. Since we assumed that t_i has common (and hence also onefold) belief in rationality, it must be the case that c^*_j is optimal for t_j, for each $j \in I \setminus \{i\}$. Moreover, we also know that t_i has twofold belief in rationality, and hence each t_j has onefold belief in rationality. By a similar argument as above, $b_j(t_j)(\boldsymbol{c}^*_{-j}, \boldsymbol{t}_{-j}) = \prod_{k \in I \setminus \{j\}} q_k(c^*_k) > 0$ implies that c^*_i must be optimal for t_i as well. And since we defined $\boldsymbol{c}^*$ as an arbitrary choice profile that receives positive probabilities, we can write this result generally:

$$q_j(c^*_j) > 0 \implies c^*_j \text{ is optimal for } t_j \quad \forall j \in I, c^*_j \in C_j$$

The claim that c^*_j is optimal for t_j means that it is optimal under the marginal belief $\text{marg}_{\boldsymbol{C}_{-i}} b_j(t_j)$. And note that we have $(\text{marg}_{\boldsymbol{C}_{-j}} b_j(t_j))(\boldsymbol{c}_{-j}) = \prod_{k \in I \setminus \{j\}} q_k(c_k) = q_{-j}(\boldsymbol{c}_{-j})$. Hence our previous finding can be rewritten as:

$$q_j(c^*_j) > 0 \implies c^*_j \succsim^{q_{-j}}_j c'_j \quad \forall j \in I, c^*_j, c'_j \in C_j$$

This result shows us that the profile of distributions $\boldsymbol{q}$ is by definition a Nash equilibrium. We also assumed that c_i is optimal for t_i, so from $\text{marg}_{\boldsymbol{C}_{-i}} b_i(t_i) = q_{-i}$ it follows that $c_i \succsim^{q_{-i}}_i c'_i \quad \forall c'_i \in C_i$. Therefore, we were able to construct a Nash equilibrium in which c_i is rational. This concludes our proof. $\square$

Proposition 7. *If the conditional preference relation $\succsim_i$ of each player $i \in I$ is strongly continuous, then a correlated equilibrium exists.*

Proof. In Proposition 5 we have shown that the strong continuity of the preferences implies the existence of a Nash equilibrium. We prove the statement by constructing a correlated equilibrium from an arbitrary Nash equilibrium $\boldsymbol{q} \in \times_{i \in I} \Delta(C_i)$. Consider an epistemic model $(\boldsymbol{T}, \boldsymbol{b})$ where $\boldsymbol{T} = \{\boldsymbol{t}\}$ is a singleton, and define $\pi \in \Delta(\boldsymbol{C} \times \boldsymbol{T})$ as $\pi(\boldsymbol{c}, \boldsymbol{t}) = \prod_{i \in I} q_i(c_i) \quad \forall \boldsymbol{c} \in \boldsymbol{C}$. Now consider a choice $c_i \in C_i$ of player $i \in I$ for which we have $\pi(c_i, t_i) > 0$. By definition, we have:

$$\pi(c_i, t_i) = \left(\operatorname*{marg}_{C_i \times T_i} \pi\right)(c_i, t_i) = \sum_{\boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}} \pi(c_i, \boldsymbol{c}_{-i}, \boldsymbol{t}) = \sum_{\boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}} \prod_{j \in I} q_j(c_j) =$$
$$= q_i(c_i) \cdot \prod_{j \in I \backslash \{i\}} \sum_{c_j \in C_j} q_j(c_j) = q_i(c_i)$$

Hence we must necessarily have $q_i(c_i) > 0$. However, since $\boldsymbol{q}$ is a Nash equilibrium, this implies that $c_i \succsim_i^{q_{-i}} c'_i \quad \forall c'_i \in C_i$. And note that for any $\boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}$, the value of q_{-i} can be expressed as:

$$q_{-i}(\boldsymbol{c}_{-i}) = \prod_{j \in I \backslash \{i\}} q_j(c_j) = \frac{\prod_{j \in I} q_j(c_j)}{q_i(c_i)} = \frac{\pi(\boldsymbol{c}, \boldsymbol{t})}{\pi(c_i, t_i)} = \frac{\pi(\boldsymbol{c}, t_i)}{\pi(c_i, t_i)}$$

Therefore, we have found that $\pi(c_i, t_i) > 0$ implies $c_i \succsim_i^{q_{-i}} c'_i \quad \forall c'_i \in C_i$, and $q_{-i}(\boldsymbol{c}_{-i}) = \frac{\pi(\boldsymbol{c}, t_i)}{\pi(c_i, t_i)}$. By definition, this means that π is a correlated equilibrium, so a correlated equilibrium exists. $\square$

Proposition 8. *For a player $i \in I$, choice $c_i \in C_i$ is rational under a common prior and common belief in rationality if and only if it is rational under correlated equilibrium.*

Proof. We first prove the 'if' part, and then the 'only if' part.

Proof of the 'if' part: Let $\pi \in \Delta(\boldsymbol{C} \times \boldsymbol{T})$ be a correlated equilibrium that satisfies $c_i \succsim_i^{p} c''_i \quad \forall c''_i \in C_i$, where $p(\boldsymbol{c}_{-i}) = \frac{\pi(c'_i, \boldsymbol{c}_{-i}, t_i)}{\pi(c'_i, t_i)} \quad \forall \boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}$ and $(c'_i, t_i) \in C_i \times T_i$, and let the corresponding epistemic model be $(\boldsymbol{T}, \boldsymbol{b})$. The definition of correlated equilibrium requires that π is a common prior in $(\boldsymbol{T}, \boldsymbol{b})$, and thus $\pi(c'_i, t_i) > 0$ implies $b_i(t_i)(\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) = \frac{\pi(c'_i, \boldsymbol{c}_{-i}, \boldsymbol{t})}{\pi(c'_i, t_i)} \quad \forall (\boldsymbol{c}_{-i}, \boldsymbol{t}_{-i}) \in \boldsymbol{C}_{-i} \times \boldsymbol{T}_{-i}$. It is easy to see that $\operatorname{marg}_{\boldsymbol{C}_{-i}} b_i(t_i) = p$, meaning that c_i is by definition optimal for type t_i. It is also trivial that t_i has a common prior – in fact, any type in T_i has a common prior since π is a common prior in $(\boldsymbol{T}, \boldsymbol{b})$. Therefore, c_i is rational under a common prior.

It remains to show that c_i is rational under common belief in rationality. Consider an arbitrary type $t^*_j \in T_j$ of an arbitrary player $j \in I$, and a $(\boldsymbol{c}^*_{-j}, \boldsymbol{t}^*_{-j}) \in \boldsymbol{C}_{-j} \times \boldsymbol{T}_{-j}$ pair for which $b_j(t^*_j)(\boldsymbol{c}^*_{-j}, \boldsymbol{t}^*_{-j}) > 0$ holds. Since π is a common prior, we must have $\pi(t^*_j) > 0$, which means that there exists a $c^*_j \in C_j$ such that $\pi(c^*_j, t^*_j) > 0$. But then the common prior property also implies $b_j(t^*_j)(\boldsymbol{c}^*_{-j}, \boldsymbol{t}^*_{-j}) = \frac{\pi(\boldsymbol{c}^*, \boldsymbol{t}^*)}{\pi(c^*_j, t^*_j)}$. We assumed that this is positive, which can only be the case if $\pi(\boldsymbol{c}^*, \boldsymbol{t}^*) > 0$, and thus $\pi(c^*_k, t^*_k) > 0 \quad \forall k \in I \backslash \{j\}$. Since π is a correlated equilibrium, we must have $c^*_k \succsim_k^{p_k} c''_k \quad \forall c''_k \in C_k$ whenever $p_k(\hat{\boldsymbol{c}}_{-k}) = \frac{\pi(c^*_k, \hat{\boldsymbol{c}}_{-k}, t^*_k)}{\pi(c^*_k, t^*_k)} \quad \forall \hat{\boldsymbol{c}}_{-k} \in \boldsymbol{C}_{-k}$. But π is also a common prior, so $\pi(c^*_k, t^*_k) > 0$ also implies $b_k(t^*_k)(\hat{\boldsymbol{c}}_{-k}, \hat{\boldsymbol{t}}_{-k}) = \frac{\pi(c^*_k, \hat{\boldsymbol{c}}_{-k}, t^*_k, \hat{\boldsymbol{t}}_{-k})}{\pi(c^*_k, t^*_k)} \quad \forall (\hat{\boldsymbol{c}}_{-k}, \hat{\boldsymbol{t}}_{-k}) \in \boldsymbol{C}_{-k} \times \boldsymbol{T}_{-k}$. Like above, we can see that $\operatorname{marg}_{\boldsymbol{C}_{-k}} b_k(t^*_k) = p_k$, and thus c^*_k is optimal for t^*_k Therefore, we have found that for any $k \in I \setminus \{j\}$, c^*_k must be optimal for t^*_k whenever $b_j(t^*_j)(\boldsymbol{c}^*_{-j}, \boldsymbol{t}^*_{-j}) > 0$. By definition, this means that t^*_j has onefold belief in rationality. But since we defined t^*_j as an arbitrary type of an arbitrary player, this must be true for any type in our epistemic model $(\boldsymbol{T}, \boldsymbol{b})$. By Lemma 1, we can conclude that every type has common belief in rationality, including t_i from above. And we have already established that c_i is optimal for t_i, so c_i is rational under common belief in rationality.

Proof of the 'only if' part: Consider an epistemic model $(\boldsymbol{T}, \boldsymbol{b})$ and a type $t_i \in T_i$ such that c_i is optimal for t_i, and t_i has common belief in rationality and a common prior. The latter means that $(\boldsymbol{T}, \boldsymbol{b})$ has a submodel $(\boldsymbol{T}', \boldsymbol{b}')$ with a common prior $\pi \in \Delta(\boldsymbol{C} \times \boldsymbol{T}')$, and $t_i \in T'_i$. Therefore, for any $j \in I$ and $t_j \in T'_j$ we must have $\pi(t_j) > 0$, and hence there exists a $c'_j \in C_j$ such that $\pi(c'_j, t_j) > 0$. From here the common prior property implies $b'_j(t_j)(\boldsymbol{c}_{-j}, \boldsymbol{t}_{-j}) = \frac{\pi(c'_j, \boldsymbol{c}_{-j}, \boldsymbol{t})}{\pi(c'_j, t_j)} \quad \forall (\boldsymbol{c}_{-j}, \boldsymbol{t}_{-j}) \in \boldsymbol{C}_{-j} \times \boldsymbol{T}'_{-j}$.

Consider a $(\boldsymbol{c}^*, \boldsymbol{t}^*_{-i}) \in \boldsymbol{C} \times \boldsymbol{T}_{-i}$ pair which satisfies $\pi(\boldsymbol{c}^*, t_i, \boldsymbol{t}^*_{-i}) > 0$. We must necessarily have $\pi(c^*_i, t_i) > 0$, and thus $b_i(t_i)(\boldsymbol{c}^*_{-i}, \boldsymbol{t}^*_{-i}) = \frac{\pi(\boldsymbol{c}^*, t_i, \boldsymbol{t}^*_{-i})}{\pi(c^*_i, t_i)} > 0$. Together with t_i's common belief in rationality, this implies that for each $j \in I \setminus \{i\}$, the type t^*_j also has common belief in rationality. Then for each j, we can similarly consider a pair $(\boldsymbol{c}^{**}, \boldsymbol{t}^{**}_{-j}) \in \boldsymbol{C} \times \boldsymbol{T}_{-i}$ satisfying $\pi(\boldsymbol{c}^{**}, t^*_j, \boldsymbol{t}^{**}_{-j}) > 0$ and conclude that each t^{**}_k has common belief in rationality. If we continue recursively in this manner, we end up with a set of types (denoted by T^*_j) for each player, with each element having common belief in rationality. Formally, for each $j \in I$ we define the function $\tau_j : \bigcup_{k \in I \setminus \{j\}} T_k \to 2^{T_j}$ as:

$$\tau_j(t^*_k) = \{t^*_j \in T_j : \exists (\boldsymbol{c}^*, \boldsymbol{t}^*_{-k-j}) \in \boldsymbol{C} \times \boldsymbol{T}_{-k-j} \text{ with } \pi(\boldsymbol{c}^*, \boldsymbol{t}^*) > 0\}$$

Next, we define the following sequence of sets recursively:

$$\begin{aligned} T^0_i &= \{t_i\} \\ T^0_j &= \emptyset \quad \forall j \in I \setminus \{i\} \\ T^{\ell+1}_j &= T^\ell_j \cup \bigcup_{k \in I \setminus \{j\}} \bigcup_{t^*_k \in T^\ell_k} \tau_j(t^*_k) \quad \forall j \in I, \ell \in \mathbb{N}_0 \end{aligned}$$

Finally, the T^*_j sets are defined as the union of these sets:

$$T^*_j = \bigcup_{\ell \in \mathbb{N}_0} T^\ell_j \quad \forall j \in I$$

As we have argued, each element of T^*_j has common belief in rationality. Now consider the tuple $(\boldsymbol{T}^*, \boldsymbol{b}^*)$, where the function $b^*_j : T^*_j \to \Delta(\boldsymbol{C}_{-j} \times \boldsymbol{T}^*_{-j})$ is defined as $b^*_j(t_j) = b'_j(t_j) \quad \forall t_j \in T^*_j$. The way we have defined the T^*_j sets ensures that when $t_j \in T^*_j$, the $b'_j(t_j)$ distribution only assigns positive probabilities to truncated type profiles in $\boldsymbol{T}^*_{-j}$, meaning that this $(\boldsymbol{T}^*, \boldsymbol{b}^*)$ is in fact an epistemic model. But then it must be a submodel of $(\boldsymbol{T}', \boldsymbol{b}')$, and thus also of $(\boldsymbol{T}, \boldsymbol{b})$. We can also define a probability distribution $\pi^* \in \Delta(\boldsymbol{C} \times \boldsymbol{T}^*)$ in the following way:

$$\pi^*(\boldsymbol{c}, \boldsymbol{t}^*) = \frac{\pi(\boldsymbol{c}, \boldsymbol{t}^*)}{\sum_{\boldsymbol{t}' \in \boldsymbol{T}^*} \pi(\boldsymbol{c}, \boldsymbol{t}')} \quad \forall(\boldsymbol{c}, \boldsymbol{t}^*) \in \boldsymbol{C} \times \boldsymbol{T}^*$$

It is easy to see that this π^* is a common prior in $(\boldsymbol{T}^*, \boldsymbol{b}^*)$. Now for some $j \in I$, consider an arbitrary pair $(c_j^*, t_j^*) \in C_j \times T_j^*$ which satisfies $\pi^*(c_j^*, t_j^*) > 0$. This means that there exists a $(\boldsymbol{c}_{-j}^*, \boldsymbol{t}_{-j}^*) \in \boldsymbol{C}_{-j} \times \boldsymbol{T}_{-j}^*$ pair such that $\pi^*(\boldsymbol{c}^*, \boldsymbol{t}^*) > 0$, and thus $\pi^*(c_k^*, t_k^*) > 0 \quad \forall k \in I \setminus \{j\}$. Together, these imply $b_k^*(t_k^*)(\boldsymbol{c}_{-k}^*, \boldsymbol{t}_{-k}^*) = \frac{\pi^*(\boldsymbol{c}^*, \boldsymbol{t}^*)}{\pi^*(c_k^*, t_k^*)} > 0$, so the type t_k^* assigns a positive probability to (c_j^*, t_j^*). However, since $t_k^* \in T_k^*$, and any type in T_k^* has common (in particular, onefold) belief in rationality, c_j^* must necessarily be optimal for t_j^*. The belief of t_j^* is $b_j^*(t_j^*)(\boldsymbol{c}'_{-j}, \boldsymbol{t}'_{-j}) = \frac{\pi^*(c_j^*, \boldsymbol{c}'_{-j}, t_j^*, \boldsymbol{t}'_{-j})}{\pi^*(c_j^*, t_j^*)} \quad \forall(\boldsymbol{c}'_{-j}, \boldsymbol{t}'_{-j}) \in \boldsymbol{C}_{-j} \times \boldsymbol{T}_{-j}^*$, whose marginal is $(\mathrm{marg}_{\boldsymbol{C}_{-j}}\, b_j^*(t_j^*))(\boldsymbol{c}'_{-j}) = \frac{\pi^*(c_j^*, \boldsymbol{c}'_{-j}, t_j^*)}{\pi^*(c_j^*, t_j^*)} \quad \forall \boldsymbol{c}'_{-j} \in \boldsymbol{C}_{-j}$. Denoting this $\mathrm{marg}_{\boldsymbol{C}_{-j}}\, b_j^*(t_j^*)$ distribution by p_j, we have that $c_j^* \succsim_j^{p_j} c_j' \quad \forall c_j' \in C_j$ holds whenever $\pi^*(c_j^*, t_j^*) > 0$, and p_j is defined as above. This is precisely the requirement of a correlated equilibrium, so we can conclude that π^* is a correlated equilibrium of the game.

We have also assumed that c_i is optimal for t_i, meaning that $c_i \succsim_i^p c_i'' \quad \forall c_i'' \in C_i$, where $p = \mathrm{marg}_{\boldsymbol{C}_{-i}}\, b_i(t_i)$. From $t_i \in T_i^*$ it also follows that $p = \mathrm{marg}_{\boldsymbol{C}_{-i}}\, b_i^*(t_i)$. Then the common prior property implies that $p(\boldsymbol{c}_{-i}) = \frac{\pi^*(c_i', \boldsymbol{c}_{-i}, t_i)}{\pi^*(c_i', t_i)}$ for any $\boldsymbol{c}_{-i} \in \boldsymbol{C}_{-i}$, and any $c_i' \in C_i$ with $\pi^*(c_i', t_i) > 0$. Together with the fact that π^* is a correlated equilibrium, this by definition means that c_i is rational under correlated equilibrium. $\square$

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