Common Belief in Rationality in Psychological Games

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Introduction

- So far preferences over choices only depended on first-order beliefs wrt opponent behavior.

- This lecture: What if players care about opponent behavior and beliefs?

- Two examples with second-order beliefs:
  - If aiming to **meet opponent's expectations** (aka guilt aversion) you prefer a choice to the extent that you believe the opponent expects you to make that choice.
  - If aiming to **surprise opponent** you prefer a choice to the extent that you believe the opponent expects you to *not* make that choice.

- **Notes:**
  - Here, guilt/surprise emerge as reflections wrt (not) matching expectations. Such insights make psychological game useful.
  - No new tools needed here. Instead, different notion of optimal choice leads to more complex setting.

Introductory Example

Surprising Barbara, *baseline decision problem*

- *You* and *Barbara* are invited to a party. Each of you simultaneously choose from dress colors *blue, green, red*.
- Personally, you prefer *blue* to *green* to *red*. In addition, you seek to wear *different* color than Barbara.
- Same for Barbara with color preference *red* to *blue* to *green*.

You	*blue*	*green*	*red*
blue	0	3	3
green	2	0	2
red	1	1	0

Barbara	*blue*	*green*	*red*
blue	0	2	2
green	1	0	1
red	3	3	0

Introductory Example

Surprising Barbara, *surprise utilities*

- Additionally, you seek to **surprise** Barbara, deriving additional utility for surprising choices proportional to your color preference. Same is true for Barbara.

	Barbara expects		
You	*blue*	*green*	*red*
blue	0	3	3
green	2	0	2
red	1	1	0

	You expect		
Barbara	*blue*	*green*	*red*
blue	0	2	2
green	1	0	1
red	3	3	0

Introductory Example

Surprising Barbara, *full decision problem*

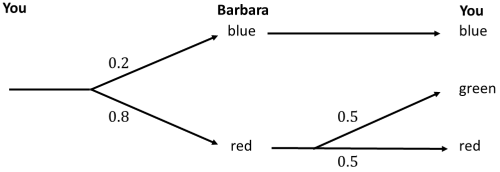
- Finally, suppose your overall utility is the sum of your baseline and surprise utilities.
- This yields decision problem with **choice-belief combinations** replacing choices for opponent.

You	$(b; b)$	$(b; g)$	$(b; r)$	$(g; b)$	$(g; g)$	$(g; r)$	$(r; b)$	$(r; g)$	$(r; r)$
blue	0	3	3	3	6	6	3	6	6
green	4	2	4	2	0	2	4	2	4
red	2	2	1	2	2	1	1	1	0

Barbara	$(b; b)$	$(b; g)$	$(b; r)$	$(g; b)$	$(g; g)$	$(g; r)$	$(r; b)$	$(r; g)$	$(r; r)$
blue	0	2	2	2	4	4	2	4	4
green	2	1	2	1	0	1	2	1	2
red	6	6	3	6	6	3	3	3	0

Introductory Example: Expected Utility

How to calculate **utility at a second-order belief**? Take following example:



- You believe w. 0.2: Barbara chooses *blue* and believes you choose *blue*. $\Rightarrow$ State (b, b) in decision problem.
- Similarly, you assign $0.8 \cdot 0.5 = 0.4$ each to states (r, g) and (r, r).
- Then, for example, choosing *blue* yields expected utility $0.2 \cdot 0 + 0.4 \cdot 6 + 0.4 \cdot 6 = 4.8$.

Introductory Example: Rationality

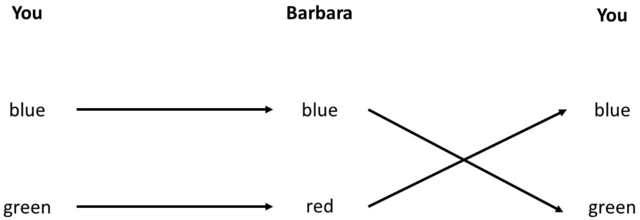
You	$(b; b)$	$(b; g)$	$(b; r)$	$(g; b)$	$(g; g)$	$(g; r)$	$(r; b)$	$(r; g)$	$(r; r)$
blue	0	3	3	3	6	6	3	6	6
green	4	2	4	2	0	2	4	2	4
red	2	2	1	2	2	1	1	1	0

Barbara	$(b; b)$	$(b; g)$	$(b; r)$	$(g; b)$	$(g; g)$	$(g; r)$	$(r; b)$	$(r; g)$	$(r; r)$
blue	0	2	2	2	4	4	2	4	4
green	2	1	2	1	0	1	2	1	2
red	6	6	3	6	6	3	3	3	0

- Your choice *red* is strictly dominated by (e.g.) $0{:}4$ *blue* + $0{:}6$ *green*.Similarly, *green* strictly dominated for Barbara by (e.g.) $0{:}4$ *red* + $0{:}6$ *blue*.
- Hence, no *second-order belief* makes these choices optimal for you and Barbara.) irrational

Introductory Example: Rationality

Remaining choices *blue* and *green* rational for you:



- *blue* strictly optimal if you believe Barbara chooses *blue* and believes you choose *green* (state ($b; g$)). Similar for *green* at ($r; b$).
- Also, *blue* is optimal for Barbara at ($g; r$) and *red* is optimal for her at ($b; b$).

) Common belief in rationality.

- **Note:** Both can choose at least 2 colors, so surprise possible at CBR.

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Agenda

- Psychological Games and Common Belief in Rationality
- Procedural Characterization
- Possibility
- Variants of the Procedure

Second-Order Expectations

Definition

A ***second-order expectation*** for player i is a probability distribution $e_i \in \Delta(C_i \times C_j)$.

- Second-order expectations concern events of form "player j chooses c_j and believes player i chooses c_i" ($\triangleq e_i(c_j; c_i)$).
- Formally, every second-order belief $b_i^2 \in \Delta(C_j \times \Delta(C_i))$ induces a second-order expectation e_i via

$$e_i(c_j; c_i) = b_i^1(c_j) \int_{\Delta(C_i)} b_j^1(c_i^0) \, db_i^2(\cdot|c_j);$$

where $b_i^2(E|c_j) = b_i^2(\{c_j\} \times E)/b_i^2(\{c_j\} \times \Delta(C_i))$ for every $E \subseteq \Delta(C_i)$.

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(Linear) Psychological Games (of Order 2)

Definition

A ***psychological game*** with two players specifies

a) finite set of choices C_i for both players i,

b) utility function $u_i : C_i \times (C_j \times C_i) \to \mathbb{R}$ for both players i,

where
$$u_i(c_i, e_i) = \sum_{(c_j, c_i^0) \in C_j \times C_i} e_i(c_j, c_i) u_i(c_i, (c_j, c_i^0)).$$

Notes:

- u_i generalizes standard expected utility using expectations.
- Assumptions: (i) u_i depends on **second-order beliefs** only,
  (ii) u_i is **linear in up to level-2 uncertainty**.

  $\Rightarrow$ Decision problems with set of states $C_j \times C_i$ iso C_j.
- General psychological games: u_i *non-linear* in *full belief hierarchy*.

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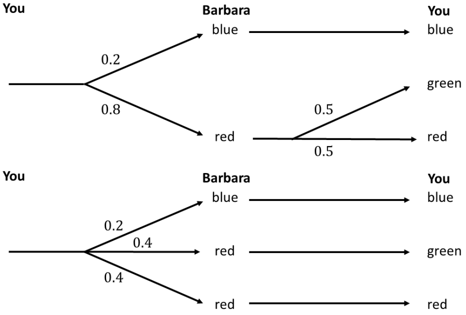
Linearity in up to Second-Order Uncertainty

Reconsider introductory example:



- Both second-order beliefs above induce the same expectation $e_i = 0{:}2 \ \ (b; b) + 0{:}4 \ \ (r; g) + 0{:}4 \ \ (r; r)$.
- Intuitively, it does not matter whether uncertainty emanates at level 1 (other's behavior) or level 2 (other's beliefs about behavior).

Epistemic Model for Introductory Example

- **Types:** $T_1 = \{t_1^{blue}, t_1^{green}\}$, $T_2 = \{t_2^{blue}, t_2^{red}\}$
- **Beliefs for You:** $b_1(t_1^{blue}) = 0.8 \cdot (blue, t_2^{blue}) + 0.2 \cdot (red, t_2^{red})$,

  $b_1(t_1^{green}) = (red, t_2^{red})$.
- **Beliefs for Barbara:** $b_2(t_2^{blue}) = (green, t_1^{green})$,

  $b_2(t_2^{red}) = 0.9 \cdot (blue, t_1^{blue}) + 0.1 \cdot (green, t_1^{green})$.

Types, Optimal and Rational Choices

- Consider epistemic models like in Chapter 3, but now possibly with **infinitely** many types.
- Main change in psychological games: optimality is wrt exectations.

Definition

Take type t_i with expectation e_i. Choice $c_i \in C_i$ is *optimal* for t_i if

$$u_i(c_i, t_i) = u_i(c_i, e_i) = \sum_{(c_j, c_i') \in C_j \times C_i} e_i(c_j, c_i') u_i(c_i, (c_j, c_i')) \geq u_i(c_i'', e_i)$$

for all $c_i'' \in C_i$.

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(Common) Belief in Rationality

Up to k-fold/common belief in rationality now defined like in standard game:

Definition

Type t_i,

- *believes in the opponents' rationality* if $b_i(t_i)$ only deems possible $(c_j; t_j)$ where c_j is optimal for t_j,
- *expresses up to k-fold belief in rationality* for k 1 if $b_i(t_i)$ only deems possible $(c_j; t_j)$ where c_j is optimal for t_j expressing up to $(k$ $1)$-fold belief in rationality,
- *expresses common belief in rationality* if $b_i(t_i)$ expresses up to k-fold belief in rationality for all k 1.

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Towards an Iterative Procedure

- To find all choices consistent with common belief in rationality, we generalize iterated strict dominance.
- As seen in following example, eliminating strictly dominated choices and corresponding (standard) states in decision problems is not enough.
- More surprisingly, also eliminating choices and full states (deterministic second-order expectations) is not enough.

Example: "Black and White Dinner with a Twist"

- *You* and *Barbara* go to a dinner an simultaneously choose from dress colors *black* and *white*.
- Personally, you prefer *white* to *black*. However, to the degree that you believe Barbara wears *white* and expects you to wear *white*, you slightly prefer *black*.
- Barbara's preferences are the same with *black* and *white* reversed.

You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; w_1)$
black	0	0	0	3
white	2	2	2	2

Barbara	$(b_1; b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	3	0	0	0

- Note that no choice is strictly dominated for you or Barbara!

"Black and White Dinner with a Twist": Rationality

You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; w_1)$
black	0	0	0	3
white	2	2	2	2

Barbara	$(b_1; b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	3	0	0	0

- Even though no strategy is dominated, we are not done yet.
- **Why?**
  - Utilities depend on **second-order expectations**.
  - Hence, need to track **choices and first-order beliefs**.
- *black* rational for you iff $e_1(w_2; w_1) \quad 2=3:$
- Similarly, *white* rational for Barbara iff $e_2(b_1; b_2) \quad 2=3:$

"Blk and Wt Dinner w Twist": Belief in Rationality

You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; w_1)$
black	0	0	0	3
white	2	2	2	2

Barbara	$(b_1; b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	3	0	0	0

- How does belief in rationality affect states you deem possible?
  - For Barbara to rationally play *white*, need $b_2^1(b_1) \quad \frac{2}{3}$.
    (If not, could never have $e_2(b_1; b_2) \quad \frac{2}{3}$.)
  - But then, using Bayes' rule, belief in Barbara's rationality implies
    $e_1(w_1 j w_2) = \frac{e_1(w_2;w_1)}{e_1(w_2;b_1)+e_1(w_2;w_1)} \quad \frac{1=3}{2=3+1=3} = 1=3.$
  -) Conditional on Barbara rationally choosing w_2, you must believe Barbara assigns at most 1=3 to your choice w_1.
- Similarly, belief in rationality implies $e_2(b_1 j b_2) \quad$ 1=3 for Barbara.

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"Blk and Wt Dinner w Twist": Belief in Rationality

You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; w_1)$
black	0	0	0	3
white	2	2	2	2

Barbara	$(b_1; b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	3	0	0	0

- But then, *black* is not rational for you under belief in rationality! **Why?**
  - Rationality of *black* for you requires $e_1(w_2; w_1) \quad 2{=}3$
  - Belief in Barbara's rationality requires $e_1(w_1 j w_2) \quad 1{=}3;$
  - The latter implies $e_1(w_2; w_1) = b_1^1(w_2)\,[e_1(w_1 j w_2)] \quad 1{=}3.$

    $) \quad ?.$
- Similarly, *white* is not rational for Barbara under belief in rationality.

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"Blk and Wt Dinner w Twist": Belief in Rationality

- Clearly, cannot capture reasoning using strict dominance and elimination of standard states.
- However, also no **full** state among $(b_2; b_1)$, $(b_2; w_1)$, $(w_2; w_1)$, $(w_1; w_2)$ can be eliminated here (and similarly for Barbara).
- **Why?** Barbara's rational choice *white* puts **probabilistic** upper bound 1=3 on her belief in w_1 (and analogously for you).
- Hence, correct decision problems for belief in rationality:

You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; 2=3\ \ b_1 + 1=3\ \ w_1)$
black	0	0	0	1
white	2	2	2	2

Barbara	$(b_1; 2=3\ \ w_2 + 1=3\ \ b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	1	0	0	0

"Blk and Wt Dinner w Twist": CBR

- Eliminating *black* for you and *white* for Barbara (and one more round of eliminating states) yields:

You	$(b_2; w_1)$
white	2

Barbara	$(w_1; b_2)$
black	2

$\Rightarrow$ *white* for you and *black* for Barbara uniquely rational under CBR.

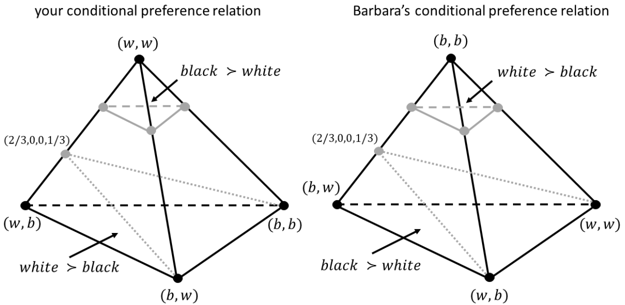
Elimination of Second-Order Expectations

- Crucial step in example: Eliminate e_i inconsistent w. j's rationality.
- More generally, following recipe:
  1) For every undominated c_j, find expectations $E_j(c_j)$ making c_j optimal.
  2) Let $B_j(c_j) = \{b_j \in \Delta(C_j) | b_j = \text{marg}_{C_j} e_j \text{ for some } e_j \in E_j(c_j)\}$ be corresponding first-order beliefs.
  3) Then, conditional on c_j, i must believe j's first-order belief is in $B_j(c_j)$. Formally, $e_i(\cdot|c_j) \in B_j(c_j)$, where

$$e_i(c_i|c_j) = \frac{e_i(c_j, c_i)}{\sum_{c_i' \in C_i} e_i(c_j, c_i')} \text{ for all } c_i \in C_i.$$

- **Notes:**
  - Let E_i be i's expectations satisfying (3). E_i is convex combination of finitely many extreme $e_i \in \Delta(C_j \times C_i)$.
  - Repeat steps above for e_i (in)consistent w. up to k-fold belief in rationality, $k > 1$.

"Blk and Wt Dinner w Twist": Eliminating Second-Order Expectations



- Tetrahedron: $(C_j \quad C_i)$-probability simplex.
- Solid triangle: Indifference hyperplane for choices *black* and *white*.
- Dotted triangle and below: Expectations consistent with belief in rationality.

It. Elim. of Choices and Second-Order Expectations

Definition

Round 1. For both players i, eliminate all strictly dominated choices. For all other c_i, let $E_i^1(c_i)$ be supporting expectations.

Round $k \geq$ **1.** For each player i and opp. choice c_j, let $B_j^{k-1}(c_j)$ be first-order beliefs induced by $E_j^{k-1}(c_j)$, and let E_i^k be i's expectations s.th. $e_i(\cdot|c_j) \in B_j^{k-1}(c_j)$ f. all c_j deemed possible by e_i. Eliminate all choices c_i that are not optimal for any $e_i \in E_i^k$. For all other c_i, let $E_i^k(c_i)$ be supporting expectations.

Proceed until no more choices/expectations can be eliminated.

Theorem

For any $k \geq 1$, choice c_i is rational for player i under up to k-fold (common) belief in rationality iff c_i survives $(k+1)$-fold (iterated) elimination of choices and expectations.

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Example: “Dinner w Strong Preference f Surprise”

- *You* and *Barbara* go to a dinner an simultaneously choose from dress colors *black* and *white*.
- Your preferences are the same as before, except each of you more strongly prefers your less liked choice if you mismatch with your opponent and surprise them as well.

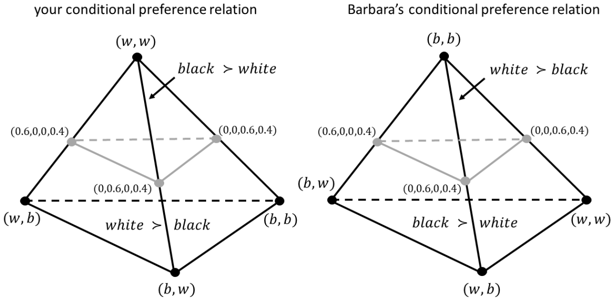
You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; w_1)$
black	0	0	0	5
white	2	2	2	2

Barbara	$(b_1; b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	5	0	0	0

- We use iterated elimination of choices and expectations to find choices consistent with common belief in rationality.

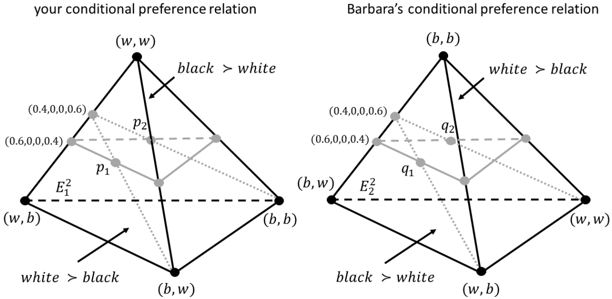
“Dinner w Str Pref f Surprise”: Rationality

- As before, no choices strictly dominated.
- *black* rational for you iff $e_1(w_2; w_1)$ 2=5 and *white* rational for Barbara iff $e_1(b_2; b_1)$ 2=5.



"Dinner w Str Pref f Surprise": Belief in Rationality

- With belief in rationality, must have $e_1(w_1 j w_2)$ 3=5. Hence, state $(w_2; w_1)$ in your decision problem replaced by 2=5 $(w_2; b_1)$ + 3=5 $(w_2; w_1)$.
- Similarly, state $(b_1; b_2)$ in Barbara's decision problem replaced by 2=5 $(b_1; w_2)$ + 3=5 $(b_1; b_2)$.



- As seen in the figure, no choices are eliminated at belief in rationality.

"Dinner w Str Pref f Surprise": Belief in Rationality

- Decision problems after 2-fold elimination of choices and expectations:

You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; 2{=}5\ \ b_1 + 3{=}5\ \ w_1)$
black	0	0	0	3
white	2	2	2	2

Barbara	$(b_1; 2{=}5\ \ w_2 + 3{=}5\ \ b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	3	0	0	0

- As follows from the table, *black* rational for you under belief in rationality iff $e_1(w_2; 2{=}5\ \ b_1 + 3{=}5\ \ w_1)\ \ \ \ 2{=}3$.
- Analogously, *white* rational for Barbara under belief in rationality iff $e_2(b_1; 2{=}5\ \ w_2 + 3{=}5\ \ b_2)\ \ \ \ 2{=}3$.

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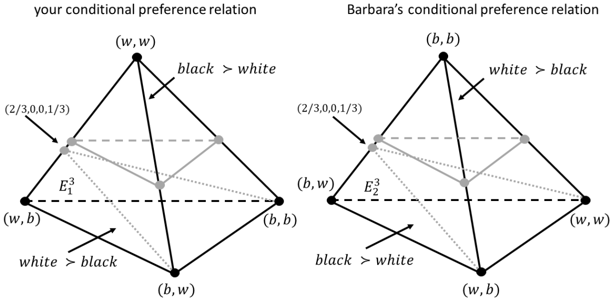
"Dinner w Str Pref f Surp": Up to 2-Fold Bel in Rat

- With up to 2-fold belief in rationality (given new extreme state), must now have $e_1(w_1 j w_2)$ 1=3. Hence, state 2=5 $(w_2; b_1)$ + 3=5 $(w_2; w_1)$ in your decision problem replaced by 2=3 $(w_2; b_1)$ + 1=3 $(w_2; w_1)$.
- Similarly, state 2=5 $(b_1; w_2)$ + 3=5 $(b_1; b_2)$ in Barbara's decision problem replaced by 2=3 $(b_1; w_2)$ + 1=3 $(b_1; b_2)$.



- As seen in figure, *black* eliminated for you and *white* for Barbara.

"Dinner w Str Pref f Surp": Common Belief in Rat

- Decision problems after 3-fold elimination of choices and expectations:

You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; 2{=}3\ \ b_1 + 1{=}3\ \ w_1)$
black	0	0	0	1
white	2	2	2	2

Barbara	$(b_1; 2{=}3\ \ w_2 + 1{=}3\ \ b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	1	0	0	0

- With 4-fold elimination of choices and expectations, states involving w_2 are eliminated for you and states involving b_1 are eliminated for Barbara.
- Then, with 5-fold elimination of choices and expectations, state $(b_2; b_1)$ is eliminated for you and state $(w_1; w_2)$ is eliminated for Barbara.
- Beliefs diagram for CBR:

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Example: “Dinner w Huge Preference f Surprise”

- Different from previous procedures, elimination of choices and expectations is **not** finite, even with finitely many choices for both players.
- This is seen in following variation of previous examples:

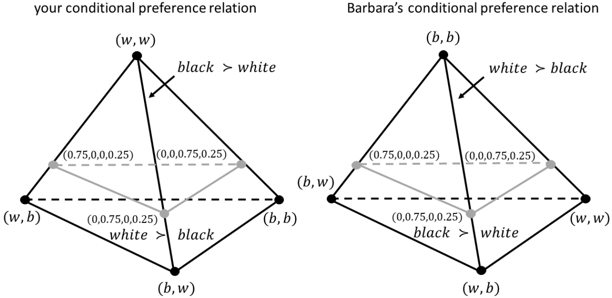
You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; w_1)$
black	0	0	0	8
white	2	2	2	2

Barbara	$(b_1; b_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	8	0	0	0

- We use iterated elimination of choices and expectations to find choices consistent with common belief in rationality.

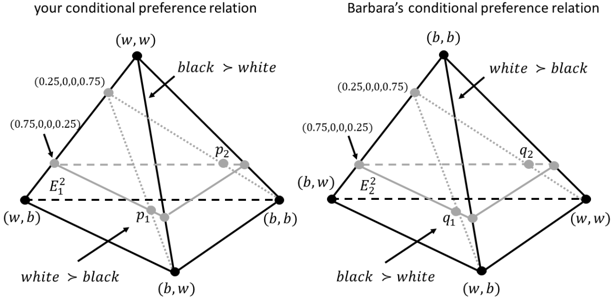
"Dinner w Huge Pref f Surprise": Rationality

- Again, no choices strictly dominated.
- *black* rational for you iff $e_1(w_2; w_1)$ 1=4 and *white* rational for Barbara iff $e_1(b_2; b_1)$ 1=4.



"Dinner w Huge Pref f Surp": Belief in Rationality

- With belief in rationality, must have $e_1(w_1 j w_2)$ 3=4. Hence, state $(w_2; w_1)$ in your decision problem replaced by 1=4 $(w_2; b_1)$ + 3=4 $(w_2; w_1)$.
- Similarly, 1=4 $(b_1; w_2)$ + 3=4 $(b_1; b_2)$ replaces $(b_1; b_2)$ for Barbara.



- As seen in figure, more expectations supporting *black* for you and *white* for Barbara survive initial restrictions.

“Dinner w Huge Pref f Surp”: Common Bel in Rat

- It turns out that some beliefs supporting *black* for you and *white* for Barbara are **never** eliminated.
- To see this write $(1-e^{k-1})$ for maximum weight on $(w_2, w_1)/(b_1, b_2)$ after round $k-1$ and consider reduced decision problems at round k:

You	(b_2, b_1)	(b_2, w_1)	(w_2, b_1)	$(w_2, (1-e^{k-1})w_1 + e^{k-1}b_1)$
black	0	0	0	$(1-e^{k-1})8$
white	2	2	2	2

Barbara	$(b_1, (1-e^{k-1})b_2 + e^{k-1}w_2)$	(b_1, w_2)	(w_1, b_2)	(w_1, w_2)
black	2	2	2	2
white	$(1-e^{k-1})8$	0	0	0

- New minimum weight e^k on $(w_2, w_1)/(b_1, b_2)$ solves $e^k \geq \frac{2}{8(1-e^{k-1})}$.
- $e^k \neq e^{k-1}$ for any finite k.
- Furthermore, at common belief in rationality/iterated elimination of choices and expectations, one has $e^k = e^{k-1} = 1/2$.

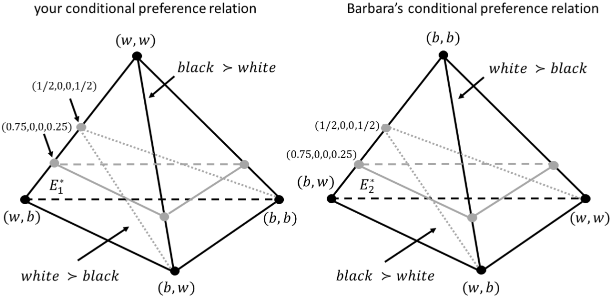
"Dinner w Huge Pref f Surp": Common Bel in Rat

- Reduced decision problems after countably many rounds:

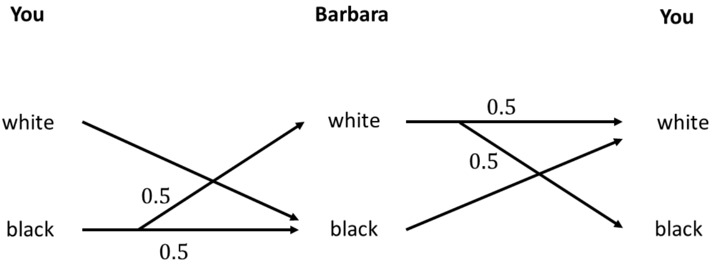
You	$(b_2; b_1)$	$(b_2; w_1)$	$(w_2; b_1)$	$(w_2; 1{=}2\ \ w_1 + 1{=}2\ \ b_1)$
black	0	0	0	4
white	2	2	2	2

Barbara	$(b_1; 1{=}2\ \ b_2 + 1{=}2\ \ w_2)$	$(b_1; w_2)$	$(w_1; b_2)$	$(w_1; w_2)$
black	2	2	2	2
white	4	0	0	0

- Expectations consistent with CBR:



"Dinner w Huge Pref f Surp": Beliefs Diagram



Agenda

- Psychological Games and Common Belief in Rationality
- Procedural Characterization
- Possibility
- Variants of the Procedure

Possibility of Common Belief in Rationality

- An important question is whether psychological games as defined here are always consistent with common belief in rationality.
- In other words, for any such game , can we find a model M such that some type t_i for every i expresses common belief in rationality?
- The answer is non-obvious in view of the procedure's countable length (see previous example).

Possibility of Common Belief in Rationality

- Using that E_i^k is a convex polytope for both players i and any k, standard techniques (Cantor's intersection theorem) imply that $\bigcap_{k \geq 1} E^k$ is non-empty for both players.

- For similar reasons, any choice elimination must occur within finite steps.

- However, between two consecutive choice eliminations, the procedure may take any finite number of steps.

- **Note:**
  - General psychological games can feature both **non-existence** and eliminations after **countable** steps.
  - Linearity ensures all choice eliminations are after finite steps. Dependence of u_i on finite orders of beliefs ensures existence. Both conditions can be weakened.

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Agenda

- Psychological Games and Common Belief in Rationality
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Order Independence

- Similar to standard iterated strict dominance, iterated elimination of choices and expectations is *order-independent*.

- Intuitively, this is true for two reasons:

  1) If a choice is strictly dominated in a decision problem, it is also strictly dominated in any reduced version of that problem.

  2) If an expectation is not eliminated a some step, it can still be eliminated at a later step.

- As a consequence, we can start off eliminating strictly dominated choices and probability-one second-order expectations and then apply the full procedure to the simplified problem.

- **Caution:** Correct **intermediate** outputs (k-fold elim of chs and exps, k 1) only found when eliminating **full-speed** in the **original order**.

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States-First Procedure

The following procedure is output-equivalent to the original one:

Definition

Round 1. For both players i, eliminate all strictly dominated choices.

Round k **1.** For each player i's decision problem, eliminate all states $(c_j; c_i)$ such that either choice has been eliminated for the respective player at the previous round. In the reduced problem, eliminate all strictly dominated choices.

Proceed until no more choices/states can be eliminated. **Subsequently** *perform elimination of choices and expectations.*

Theorem

The states-first procedure always yields the same final output as iterated elimination of choices and expectations.

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Example: "Exceeding Barbara's Expectations"

- *You* and *Barbara* record a song together, each practicing 1, 3, 5, or 7 weeks.
- Investing w_i weeks costs w_i^2 for both players i.
- Direct benefits of practice are given by $w_i \quad w_j$ with own investment w_i and opponent investment w_j.
  Additionally, each of you wants to **exceed other's expectations** w_i^0, giving you added benefit of $(w_i \quad w_i^0)$ for $w_i > w_i^0$.
- Utility functions: $u_i(w_i; (w_j; w_i^0)) = \begin{cases} w_i \quad w_j \quad w_i^2 + (w_i \quad w_i^0); & \text{if } w_i > w_i^0; \\ w_i \quad w_j \quad w_i^2; & \text{otherwise.} \end{cases}$

You/Barbara	(1,1)	(1,3)	(1,5)	(1,7)	(3,1)	(3,3)	(3,5)	(3,7)	(5,1)	(5,3)	(5,5)	(5,7)	(7,1)	(7,3)	(7,5)	(7,7)
1	0	0	0	0	2	2	2	2	4	4	4	4	6	6	6	6
3	-4	-6	-6	-6	2	0	0	0	8	6	6	6	14	12	12	12
5	-16	-18	-20	-20	-6	-8	-10	-10	4	2	0	0	14	12	10	10
7	-36	-38	-40	-42	-22	-24	-26	-28	-8	-10	-12	-14	6	4	2	0

- We use states-first procedure to find choices consistent with common belief in rationality.

"Exceeding Barbara's Expectations": Rationality

You/Barbara	(1,1)	(1,3)	(1,5)	(1,7)	(3,1)	(3,3)	(3,5)	(3,7)	(5,1)	(5,3)	(5,5)	(5,7)	(7,1)	(7,3)	(7,5)	(7,7)
1	0	0	0	0	2	2	2	2	4	4	4	4	6	6	6	6
3	-4	-6	-6	-6	2	0	0	0	8	6	6	6	14	12	12	12
5	-16	-18	-20	-20	-6	-8	-10	-10	4	2	0	0	14	12	10	10
7	-36	-38	-40	-42	-22	-24	-26	-28	-8	-10	-12	-14	6	4	2	0

- 7 strictly dominated by 5 for you and Barbara.

"Exceeding Barbara's Exp": States-First Proc Rd 2

You/Barbara	(1,1)	(1,3)	(1,5)	(3,1)	(3,3)	(3,5)	(5,1)	(5,3)	(5,5)
1	0	0	0	2	2	2	4	4	4
3	-4	-6	-6	2	0	0	8	6	6
5	-16	-18	-20	-6	-8	-10	4	2	0

- All states of form $(7;\ )$ and $(\ ;7)$ eliminated.
- Then, 3 strictly dominates 5.

“Exceeding Barbara’s Exp”: States-First Proc Rd 3

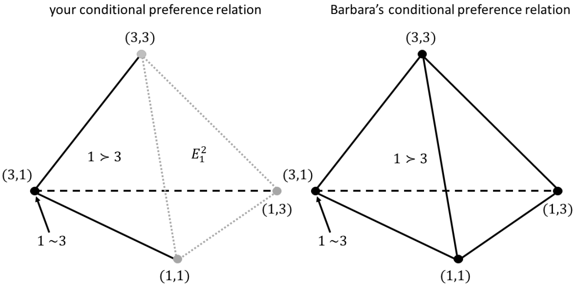
You/Barbara	(1,1)	(1,3)	(3,1)	(3,3)
1	0	0	2	2
3	-4	-6	2	0

- All states of form (5;) and (; 5) eliminated.
- No more choices strictly dominated.

) Switch to elimination of choices and expectations.
- 1 *weakly* dominates 3.
- Hence, 3 is optimal iff $e_i(3;1) = 1$ and 1 is optimal for any expectation.

"Exceeding Barb's Exp": States-First Proc Rd 4 ff

- Given 3-fold reduced decision problem, belief in Barbara's rationality requires that $e_1(3|3) = 1$.
- Hence, surviving states at rd 4 in $\text{Conv}\{(1,1),(1,3),(3,3)\}$.



- Since state $(3,1)$ is eliminated, choice 3 is also eliminated.
  - $\Rightarrow$ 1 uniquely rational under CBR for both players.

Interacting Belief Restrictions & Strict Dominance

- In “Black and White Dinner with a Twist” and other examples, standard iterated strict dominance is **insufficient** for CBR.
- This is due to interacting belief restrictions.
- E.g., in “Dinner w twist” your choosing *black* requires sufficiently high expectation of $(w_2; w_1)$.
- But any such expectation for you goes beyond Barbara’s maximum belief in w_1 while rationally choosing w_2.
- Hence, belief in Barbara’s rationality eliminates these expectations and your choice *black*.

Interacting Belief Restrictions & Strict Dominance

- Interacting belief restrictions are the reason why iterated strict dominance does not work in psychological games.
- Conversely, special psychological games may exclude such interactions, allowing us to use strict dominance.
- In psychological games as studied here, this will be true for player i if:
  - i cares only about j's behavior and j only cares about i's first-order beliefs.
  - i cares only about j's first-order beliefs.
- In particular, iterated strict dominance works for **both** players if one player only cares about behavior and the other only cares about first-order beliefs.

Example: "Barbara's Birthday"

- *You* choose to buy a *necklace*, *ring*, or *bracelet* as a gift for *Barbara*.
- You personally prefer *necklace* over *ring* over *bracelet*. In addition, you seek to surprise Barbara with your gift. Meanwhile, Barbara seeks to guess which gift you bought her.

You	(; *n*)	(; *r*)	(; *b*)
necklace	0	3	3
ring	2	0	2
bracelet	1	1	0

Barbara	(*n*;)	(*r*;)	(*b*;)
necklace	1	0	0
ring	0	1	0
bracelet	0	0	1

- Your behavior matters for Barbara but not vice versa. Similarly, you care what Barbara expect you to do but not vice versa.
- Hence, no belief restrictions for you and Barbara interact in this game.

) Iterated strict dominance finds choices consistent with CBR.

"Barbara's Birthday": Rationality

You	$(\,;n)$	$(\,;r)$	$(\,;b)$
necklace	0	3	3
ring	2	0	2
bracelet	1	1	0

Barbara	$(n;\,)$	$(r;\,)$	$(b;\,)$
necklace	1	0	0
ring	0	1	0
bracelet	0	0	1

- *bracelet* strictly dominated for you by (e.g.) $0{:}4$ *necklace* + $0{:}6$ *ring*.
- No choice dominated for Barbara.

"Barbara's Birthday": Belief in Rationality

You	$(\, ; n)$	$(\, ; r)$	$(\, ; b)$
necklace	0	3	3
ring	2	0	2

Barbara	$(n; \,)$	$(r; \,)$
necklace	1	0
ring	0	1
bracelet	0	0

- Under belief in rationality, Barbara discards all states of form $(b; \,)$.
- Then, *bracelet* strictly dominated by (e.g.) $0{:}5 \quad$ *necklace* + $0{:}5 \quad$ *ring*.
- No choice or state eliminated for you.
  Caution: $(\, ; b)$ eliminated for you at up to 2-fold belief in rationality!

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"Barbara's Bday": Common Belief in Rationality

You	$(\,; n)$	$(\,; r)$
necklace	0	3
ring	2	0

Barbara	$(n; \,)$	$(r; \,)$
necklace	1	0
ring	0	1

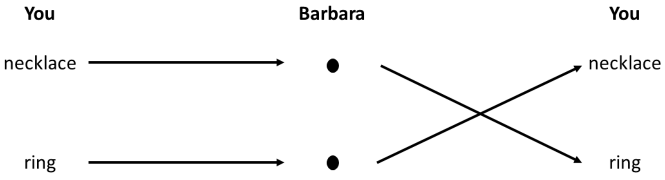
- Under up to 2-fold belief in rationality, you discard $(\,; b)$ as well as $(b; n)$ and $(b; r)$.
- Finally, under up to 3-fold belief in rationality, Barbara discards $(n; b)$ and $(r; b)$.
- No further choices are eliminated, so the procedure stops.
- Reduced decision problems:

You	$(n; n)$	$(n; r)$	$(r; n)$	$(r; r)$
necklace	0	3	0	3
ring	2	0	2	0

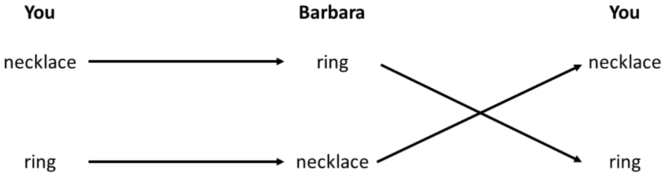
Barbara	$(n; n)$	$(n; r)$	$(r; n)$	$(r; r)$
necklace	1	1	0	0
ring	0	0	1	1

"Barbara's Birthday": Beliefs Diagram

- To support your choices, only need **partial** beliefs diagram, omitting beliefs about Barbara's behavior:



- Now complete diagram to also support Barbara's choices:



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