Epistemic Game Theory: Incomplete Information

Part I: Common Belief in Rationality

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Games with Uncertainty About Payoffs

- In games with incomplete information, players face uncertainty about their opponents' payoffs.
- More formally speaking, player *i* might be unsure whether the utility function of his opponent *j* is u_j or u'_j.
- There are various occasions in the real-world where incomplete information applies: *firms unsure about competitors' profits, employers doubful about their employees' qualities, etc.*

Interactive Reasoning about Rationality in the Presence of Incomplete Information

- A choice of player *i* is said to be **rational**, if it is optimal for some conjecture β_i AND some utility function u_i.

- Infusing such a notion of **rationality** into the interactive reasoning of a player enables the formulation of common belief in rationality also under incomplete information in the usual way:
  - *player i believes his opponents to choose **rationally**,*
  - *player i believes his respective opponents to believe their respective opponents to choose **rationally**,*
  - *player i believes his respective opponents to believe their respective opponents to believe their respective opponents to choose **rationally**,*
  - *etc.*

Example: What is Barbara's favourite Colour?

Story:

- *Barbara* and *you* are going together to another party.
- *You* wonder what colour you should wear.
- *You* prefer *blue* (4) to *green*, *green* (3) to *red*, *red* (2) to *yellow* (1), and dislike most to wear the same colour (0) as *Barbara*.
- However, *you* drank so much at the last party, that you forgot *Barbara*'s colour preferences.
- *You* are still certain about *Barbara* also disliking most to wear the same colour (0) as you.
- Also, *you* remember that *Barbara* either prefers *red* (4) to *yellow*, *yellow* (3) to *blue*, *blue* (2) to *green* (1); **or** *blue* (4) to *yellow*, *yellow* (3) to *green*, *green* (2) to *red* (1).
- Question: Which colours can *you* **rationally** choose for tonight's party under **common belief in rationality**?

Example: What is Barbara's favourite Colour?

- *Yellow* is not optimal for any conjecture about *Barbara*'s choice.
- Thus, if *Barbara* believes in your **rationality**, then she assigns probability 0 to *yellow* for you.
- *Barbara*'s 1st preferences: *blue* and *green* are not optimal for any conjecture about *your* choice that exludes *yellow*.
- *Barbara*'s 2nd preferences: *green* and *red* are not optimal for any conjecture about *your* choice that exludes *yellow*.
- Suppose that *you* believe in *Barbara*'s **rationality** and that she believes in your **rationality**.
- Then, *you* believe *Barbara* not to choose *green* independent of her preferences and hence *green* is better than *red* for *you*.

Example: What is Barbara's favourite Colour?

- The following belief hierarchy expresses common belief in rationality and supports *your* choice of *blue*:
  - *You* believe *Barbara* to choose *red* and entertain *"red-peaked"-preferences*.
  - *You* believe *Barbara* to believe *you* to choose *blue* and entertain *your preferences*.
  - *You* believe *Barbara* to believe *you* to believe *Barbara* to choose *red* and entertain *"red-peaked"-preferences*.
  - etc.
- Hence, *you* can **rationally** choose *blue* under **common belief in rationality**.

Example: What is Barbara's favourite Colour?

- The following belief hierarchy expresses common belief in rationality and supports *your* choice of *green*:
  - *You* believe *Barbara* to choose *blue* and entertain *"blue-peaked"-preferences*.
  - *You* believe *Barbara* to believe *you* to choose *green* and entertain *your preferences*.
  - *You* believe *Barbara* to believe *you* to believe *Barbara* to choose *blue* and entertain *"blue-peaked"-preferences*.
  - etc.
- Hence, *you* can **rationally** choose *green* under **common belief in rationality**.

Outline

- One-Person Perspective Form
- Epistemic Model
- Common Belief in Rationality
- Generalized Iterated Strict Dominance
- Charaterization

One-Person Perspective Form

Static Games with Incomplete Information

Definition 1

A *game with incomplete information* is a tuple

$$\Gamma = \big(I, (C_i, U_i)_{i \in I}\big)$$

where

- I denotes a finite set of *players*,
- C_i denotes a finite set of *choices* of player i,
- U_i denotes a finite set of *utility functions* of player i, with $u_i : \times_{j \in I} C_j \rightarrow \mathbb{R}$ for all $u_i \in U_i$.

- Two possible sources for uncertainty for a player i:
  - opponents' choice combinations $c_{-i} \in C_{-i}$ *("strategic uncertainty")*
  - opponents' utility function combinations $u_{-i} \in U_{-i}$ *("payoff uncertainty")*

- Complete information as a special case, if $| U_i | = 1$ for all $i \in I$

- FRAMEWORK: belief-free games with private values

Decision Problems

- Given a game with incomplete information $\Gamma = \big(I, (C_i, U_i)_{i \in I}\big)$, a decision problem for player i is a tuple

$$\Gamma_i(u_i) = (D_i, D_{-i}, u_i|_{D_i \times D_{-i}})$$

where $D_i \subseteq C_i$ is a subset of i's choice set, $D_{-i} \subseteq C_{-i}$ is a subset of the set of opponents' choice combinations, and $u_i \in U_i$ is a concrete utility function of player i.

- A decision problem can be viewed as a one-person perspective model of a game-theoretic choice problem.

- A decision problem is called full, if $D_i = C_i$ and $D_{-i} = C_{-i}$, and reduced, if it is not the case that $D_i = C_i$ and $D_{-i} = C_{-i}$.

Illustration

- A game with incomplete information:

Alice (u_A), Bob (u_B):

	d	e	f
a	3, 3	2, 2	1, 0
b	2, 2	1, 1	3, 0
c	0, 1	0, 3	0, 0

Alice (u_A), Bob (u'_B):

	d	e	f
a	3, 1	2, 2	1, 0
b	2, 3	1, 1	3, 0
c	0, 1	0, 1	0, 0

Alice (u'_A), Bob (u_B):

	d	e	f
a	1, 3	3, 2	1, 0
b	2, 2	1, 1	1, 0
c	0, 1	0, 3	0, 0

Alice (u'_A), Bob (u'_B):

	d	e	f
a	1, 1	3, 2	1, 0
b	2, 3	1, 1	1, 0
c	0, 1	0, 1	0, 0

- All full decision problems of the game:

$\Gamma_A(u_A)$

	d	e	f
a	3	2	1
b	2	1	3
c	0	0	0

$\Gamma_A(u'_A)$

	d	e	f
a	1	3	1
b	2	1	1
c	0	0	0

$\Gamma_B(u_B)$

	a	b	c
d	3	2	1
e	2	1	3
f	0	0	0

$\Gamma_B(u'_B)$

	a	b	c
d	1	3	1
e	2	1	1
f	0	0	0

- Example of a reduced decision problem for *Alice*:

$\hat{\Gamma}_A(u'_A)$

	d	f
a	1	1
b	2	1

Representation of a Game in terms of Full Decision Problems

Definition 2

Let Γ be a game with incomplete information. The tuple

$$\mathcal{O}^{\Gamma} := \left(\cup_{u_i \in U_i} \{\Gamma_i(u_i)\} \right)_{i \in I}$$

is called the *one-person perspective form* of Γ, where

$$\Gamma_i(u_i) := (C_i, C_{-i}, u_i)$$

for all $u_i \in U_i$ and for all $i \in I$.

- $\mathcal{O}^{\Gamma}$ assembles all full decision problems per player.
- The subsequent analysis of games with incomplete information makes use of $\mathcal{O}^{\Gamma}$ – to formulate a solution concept generalizing ISD but also for the purposes of lucidity in exposition.

Illustration

- A game with incomplete information Γ:

Bob (u_B):

Alice (u_A)	c	d
a	$1,1$	$0,0$
b	$0,1$	$1,0$

Bob (u'_B):

Alice (u_A)	c	d
a	$1,0$	$0,2$
b	$0,0$	$1,2$

- One-person perspective form $\mathcal{O}^\Gamma$ of Γ:

$\Gamma_A(u_A)$	c	d
a	1	0
b	0	1

$\Gamma_B(u_B)$	a	b
c	1	1
d	0	0

$\Gamma_B(u'_B)$	a	b
c	0	0
d	2	2

Epistemic Model

Reasoning About the Game: Belief Hierarchies

- **With** incomplete information a belief hierarchy for player i consists of
  - *what i believes about his opponents' choices and payoffs*
    (FIRST-ORDER BELIEF),
  - *what i believes about what his opponents believe about their opponents' choices and payoffs*
    (SECOND-ORDER BELIEF),
  - *what i believes about what his opponents believe about what their opponents believe about their opponents' choices and payoffs*
    (THIRD-ORDER BELIEF),
  - *etc.*
- Thus with incomplete information – in a belief hierarchy – player i holds a belief
  - about his opponents' choices and payoffs,
  - as well as about his opponents' belief hierarchies.

Epistemic Model

Definition 3

Let Γ be a game with incomplete information. An *epistemic model*

$$\mathcal{M}^\Gamma = (T_i, b_i)_{i \in I}$$

of Γ provides for every player $i \in I$,

- a finite set T_i of *types*,
- a *description map*

$$b_i : T_i \to \Delta\left((C_j \times T_j \times U_j)_{j \in I \setminus \{i\}}\right)$$

  that assigns to every type $t_i \in T_i$ a probability distribution $b_i[t_i]$ over the opponents' choice type utility function combinations.

Note that – similarly to the case of games with complete information – the description map b_i of player i provides for every type $t_i \in T_i$

- a **conjecture** (i.e. belief about the opponents' choice combinations),
- a **belief** about the opponents' **types** (i.e. implicit belief hierarchies).

Example: What is Barbara's favourite Colour?

Consider the following epistemic model of the game:

(u_y represents *your* preferences / u_B *Barbara´s* "red-peaked" preferences / u'_B *her* "blue-peaked" preferences)

- Type Sets:

$$T_{you} = \{t_y, t'_y\}$$

$$T_{Barbara} = \{t_B, t'_B\},$$

- Beliefs for *you*:

$$b_{you}[t_y] = (blue, t_B, u_B)$$

$$b_{you}[t'_y] = \frac{1}{2} \cdot (red, t_B, u_B) + \frac{1}{4} \cdot (red, t_B, u'_B) + \frac{1}{4} \cdot (yellow, t'_B, u'_B)$$

- Beliefs for *Barbara*:

$$b_{Barbara}[t_B] = (blue, t_y, u_y)$$

$$b_{Barbara}[t'_B] = (green, t_y, u_y)$$

Some Basic Epistemic Notions

- For a given type $t_i \in T_i$, and a utility function $u_i \in U_i$, the expression

$$\mathbb{E}u_i(c_i, t_i) := \sum_{c_{-i} \in C_{-i}} b_i[t_i](c_{-i}) \cdot u_i(c_i, c_{-i})$$

denotes the *expected utility* for choosing $c_i \in C_i$.

- A choice $c_i \in C_i$ is *optimal* for the type utility function pair (t_i, u_i), if

$$\mathbb{E}u_i(c_i, t_i) \geq \mathbb{E}u_i(c_i', t_i)$$

holds for all $c_i' \in C_i$.

- A choice $c_i \in C_i$ is *rational* given some utility function $u_i \in U_i$, if there exists an epistemic model of the game with a type $t_i \in T_i$ such that c_i is optimal for (t_i, u_i).

Common Belief in Rationality

Iterating Belief in Rationality

- Infusing the idea of rationality into higher-order thinking of players in the presence of incomplete information imposes restrictions on belief hierarchies with extended basic uncertainty (choices and utility functions).

- A player can be said to believe in rationality, if he only assigns positive probability to CHOICE & TYPE & UTILITY FUNCTION triples s.t. the CHOICE is optimal for the TYPE (induced conjecture) & UTILITY FUNCTION.

- Full-fledged rationality reasoning then again gives rise to the key epistemic notion of

  common belief in rationality:

  - believing in rationality,
  - believing your opponents to believe in rationality,
  - believing your opponents to believe their opponents to believe in rationality,
  - etc.

- These ideas are now laid out formally within the framework of **epistemic models**.

Belief in Rationality

Definition 4

Let Γ be a game with incomplete information, $\mathcal{M}^{\Gamma}$ an epistemic model of Γ, $i \in I$ some player, and $t_i \in T_i$ some type of player i. The type t_i *believes in rationality*, if t_i only assigns positive probability to choice type utility function combinations

$$\big((c_1, t_1, u_1), \ldots, (c_{i-1}, t_{i-1}, u_{i-1}), (c_{i+1}, t_{i+1}, u_{i+1}), \ldots (c_n, t_n, u_n)\big)$$

such that c_j is optimal for (t_j, u_j) for all $j \in I \setminus \{i\}$.

Higher-order Beliefs in Rationality

Definition 5

Let Γ be a game with incomplete information, $\mathcal{M}^{\Gamma}$ an epistemic model of Γ, $i \in I$ some player, and $t_i \in T_i$ some type of player i.

- The type t_i expresses *1-fold belief in rationality*, if t_i believes in rationality.
- Let $k > 1$. The type t_i expresses *k-fold belief in rationality*, if t_i only assigns positive probability to opponents' types that express *(k-1)*-fold belief in rationality.

Let $l \geq 1$. The type t_i expresses *up to l-fold belief in rationality*, if t_i expresses *k*-fold belief in rationality for all $k \leq l$.

Common Belief in Rationality

Definition 6

Let Γ be a game with incomplete information, $\mathcal{M}^\Gamma$ an epistemic model of Γ, $i \in I$ some player, and $t_i \in T_i$ some type of player i. The type t_i expresses *common belief in rationality*, if t_i expresses *k*-fold belief in rationality for all $k \geq 1$.

Remark:

If all types in an epistemic model express belief in rationality, then all tyes express common belief in rationality.

This also holds under **incomplete information**!

Rational Choice under CBR

Definition 7

Let Γ be a game with incomplete information, $i \in I$ some player, $c_i \in C_i$ some choice of player i, and $u_i \in U_i$ some utility function of player i. The choice c_i is *rational under common belief in rationality* given u_i, if there exists an epistemic model $\mathcal{M}^\Gamma$ of Γ with some type $t_i \in T_i$ of player i such that

- t_i expresses common belief in rationality,
- c_i is optimal for (t_i, u_i).

Illustration

- Consider the following game in one-person perspective form:

$\Gamma_A(u_A)$

	d	e	f
a	3	2	1
b	2	1	3
c	0	0	0

$\Gamma_A(u'_A)$

	d	e	f
a	1	3	1
b	2	1	1
c	0	0	0

$\Gamma_B(u_B)$

	a	b	c
d	3	2	1
e	2	1	3
f	0	0	0

$\Gamma_B(u'_B)$

	a	b	c
d	1	3	1
e	2	1	1
f	0	0	0

- Suppose the following **epistemic model** of this game:
  - $T_{Alice} = \{t_A, t'_A\}$ and $T_{Bob} = \{t_B, t'_B\}$,
  - $b_{Alice}[t_A] = (d, t_B, u_B)$ and $b_{Alice}[t'_A] = (e, t'_B, u'_B)$,
  - $b_{Bob}[t_B] = (a, t_A, u_A)$ and $b_{Bob}[t'_B] = \frac{1}{2}(b, t_A, u'_A) + \frac{1}{2}(c, t'_A, u_A)$.
- Types t_A and t_B believe in rationality.
- Types t_A and t_B also express common belief in rationality.

Example: What is Barbara's favourite Colour?

- The game in one-person perspective form:

$\Gamma_y^0(u_y)$

	blue	green	red	yellow
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2
yellow	1	1	1	0

$\Gamma_B^0(u_B)$

	blue	green	red	yellow
blue	0	2	2	2
green	1	0	1	1
red	4	4	0	4
yellow	3	3	3	0

$\Gamma_B^0(u_B')$

	blue	green	red	yellow
blue	0	4	4	4
green	1	0	1	1
red	2	2	0	2
yellow	3	3	3	0

- Suppose the following **epistemic model** of this game:
  - $T_{you} = \{t_y, \hat{t}_y\}$ and $T_{Barbara} = \{t_B, \hat{t}_B\}$,
  - $b_{you}[t_y] = (red, \hat{t}_B, u_B)$ and $b_{you}[\hat{t}_y] = (blue, t_B, u_B')$,
  - $b_{Barbara}[t_B] = (green, \hat{t}_y, u_y)$ and $b_{Barbara}[\hat{t}_B] = (blue, t_y, u_y)$.
- All types believe in rationality, and thus – by the "SHORTCUT" – also express common belief in rationality.
- Consequently, *you* can **rationally** choose *blue* as well as *green* under **common belief in rationality** given utility function u_y.

Generalized Iterated Strict Dominance

Strict Dominance and Incomplete Information

- The notion of strict dominance is first adapted to decision problems.
- Via decision problems it is then carried into the setting of games with incomplete information in one-person perspective form.
- Subsequently, an iteration of strict dominance in the one-person perspective form gives rise to:

  GENERALIZED ITERATED STRICT DOMINANCE (GISD)

- GISD constitutes a **solution concept** for incomplete information.

Strict Dominance In Decision Problems

Definition 8

Let Γ be a game with incomplete information, $\mathcal{O}^{\Gamma}$ the one-person perspective form of Γ, $i \in I$ some player, $\Gamma_i(u_i)$ some decision problem of player i, and $c_i \in D_i$ some choice of player i. The choice c_i is *strictly dominated*, if there exists some mixed choice $r_i \in \Delta(D_i)$ of player i such that

$$u_i(c_i, c_{-i}) < \sum_{c_i' \in D_i} r_i(c_i') \cdot u_i(c_i', c_{-i})$$

for all $c_{-i} \in D_{-i}$.

Generalized Iterated Strict Dominance

Definition 9

Let Γ be a game with incomplete information and $\mathcal{O}^\Gamma$ the one-person perspective form of Γ.

- **Round 1:** For all $i \in I$ and for all $u_i \in U_i$ consider the initial decision problem $\Gamma_i^0(u_i) := (C_i^0(u_i), C_{-i}^0(u_i), u_i)$ from $\mathcal{O}$ where $C_i^0(u_i) := C_i$ and $C_{-i}^0(u_i) := C_{-i}$.
  - Step 1.1: Set $C_{-i}^1(u_i) := C_{-i}^0(u_i)$.
  - Step 1.2: Form the reduced decision problem $\Gamma_i^1(u_i) := (C_i^1(u_i), C_{-i}^1(u_i), u_i)$, where $C_i^1(u_i) \subseteq C_i^0(u_i)$ only contains choices of i that are not stric. dom. in $(C_i^0(u_i), C_{-i}^1(u_i), u_i)$.
- **Round $k > 1$:** For all $i \in I$ and for all $u_i \in U_i$ consider the reduced decision problem $\Gamma_i^{k-1}(u_i) := (C_i^{k-1}(u_i), C_{-i}^{k-1}(u_i), u_i)$ from the previous round $k-1$.
  - Step $k.1$: Form the set $C_{-i}^k(u_i)$ by eliminating from $C_{-i}^{k-1}(u_i)$ every opponents' choice combination that contains for some opponent $j \in I \setminus \{i\}$ a choice which is strictly dominated for all $u_j \in U_j$ in $\Gamma_j^{k-1}(u_j)$ of the previous round $k-1$.
  - Step $k.2$: Form the reduced decision problem $\Gamma_i^k(u_i) := (C_i^k(u_i), C_{-i}^k(u_i), u_i)$, where $C_i^k(u_i) \subseteq C_i^{k-1}(u_i)$ only contains choices of i not strict. dom. in $(C_i^{k-1}(u_i), C_{-i}^k(u_i), u_i)$.
- **Output:** The set

$$GISD := \times_{i \in I} GISD_i \subseteq \times_{i \in I}(C_i \times U_i)$$

is called *Generalized Iterated Strict Dominance*, where for every player $i \in I$ the set $GISD_i \subseteq C_i \times U_i$ only contains choice utility function pairs $(c_i, u_i) \in C_i \times U_i$ such that $c_i \in C_i^k(u_i)$ for all $k \geq 0$.

Illustration

One-person perspective form = initial decision problems of GENERALIZED ITERATED STRICT DOMINANCE:

$\Gamma^0_A(u_A)$

	d	e	f
a	3	2	1
b	2	1	3
c	0	0	0

$\Gamma^0_A(u'_A)$

	d	e	f
a	1	3	1
b	2	1	1
c	0	0	0

$\Gamma^0_B(u_B)$

	a	b	c
d	3	2	1
e	2	1	3
f	0	0	0

$\Gamma^0_B(u'_B)$

	a	b	c
d	1	3	1
e	2	1	1
f	0	0	0

Round 1:

- For *Alice*, choice c is strictly dominated in the initial decision problems $\Gamma^0_A(u_A)$ and $\Gamma^0_A(u'_A)$.
- Eliminate choice c from $\Gamma^0_A(u_A)$ and from $\Gamma^0_A(u'_A)$.
- Similarly, eliminate choice f from *Bob*'s initial decision problems $\Gamma^0_B(u_B)$ and $\Gamma^0_B(u'_B)$.

Illustration

Output from **Round 1** $=$ 1-fold reduced decision problems:

$\Gamma_A^1(u_A)$

	d	e	f
a	3	2	1
b	2	1	3

$\Gamma_A^1(u_A')$

	d	e	f
a	1	3	1
b	2	1	1

$\Gamma_B^1(u_B)$

	a	b	c
d	3	2	1
e	2	1	3

$\Gamma_B^1(u_B')$

	a	b	c
d	1	3	1
e	2	1	1

Round 2:

- For *Bob*, choice f is not in any of his decision problems $\Gamma_B^1(u_B)$ and $\Gamma_B^1(u_B')$.
- Eliminate choice f from $\Gamma_A^1(u_A)$ and from $\Gamma_A^1(u_A')$.
- Similarly, eliminate choice c from *Bob*'s 1-fold reduced decision problems $\Gamma_B^1(u_B)$ and $\Gamma_B^1(u_B')$.

Illustration

$(C_A^1(u_A), C_B^2(u_A), u_A)$

	d	e
a	3	2
b	2	1

$(C_A^1(u'_A), C_B^2(u'_A), u'_A)$

	d	e
a	1	3
b	2	1

$(C_B^1(u_B), C_A^2(u_B), u_B)$

	a	b
d	3	2
e	2	1

$(C_B^1(u'_B), C_A^2(u'_B), u'_B)$

	a	b
d	1	3
e	2	1

Round 2 (continued):

- Then, choice b becomes strictly dominated for *Alice* in the decision problem $\left(C_A^1(u_A), C_B^2(u_A), u_A\right)$.
- Eliminate choice b from the decision problem $\left(C_A^1(u_A), C_B^2(u_A), u_A\right)$.
- Similarly, eliminate choice e from *Bob*'s decision problem $\left(C_B^1(u_B), C_A^2(u_B), u_B\right)$.

Illustration

Output from **Round 2** $=$ 2-fold reduced decision problems:

$\Gamma_A^2(u_A)$

	d	e
a	3	2

$\Gamma_A^2(u_A')$

	d	e
a	1	3
b	2	1

$\Gamma_B^2(u_B)$

	a	b
d	3	2

$\Gamma_B^2(u_B')$

	a	b
d	1	3
e	2	1

Round 3:

- The algorithm stops.

- Output:

$$GISD_{Alice} = \{(a, u_A), (a, u_A'), (b, u_A')\}$$

and

$$GISD_{Bob} = \{(d, u_B), (d, u_B'), (e, u_B')\}.$$

Example: What is Barbara's favourite Colour?

One-person perspective form = initial decision problems of GENERALIZED ITERATED STRICT DOMINANCE:

$\Gamma_y^0(u_y)$

	blue	green	red	yellow
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2
yellow	1	1	1	0

$\Gamma_B^0(u_B)$

	blue	green	red	yellow
blue	0	2	2	2
green	1	0	1	1
red	4	4	0	4
yellow	3	3	3	0

$\Gamma_B^0(u_B')$

	blue	green	red	yellow
blue	0	4	4	4
green	2	0	2	2
red	1	1	0	1
yellow	3	3	3	0

Round 1:

- For *you*, choice *yellow* is strictly dominated in the initial decision problem $\Gamma_y^0(u_y)$ by $\frac{1}{2}$ *blue* $+ \frac{1}{2}$ *green*.
- Eliminate choice *yellow* from $\Gamma_y^0(u_y)$.
- For *Barbara*, choice *green* is strictly dominated in the initial decision problem $\Gamma_B^0(u_B)$ by $\frac{1}{2}$ *red* $+ \frac{1}{2}$ *yellow*.
- Eliminate choice *green* from $\Gamma_B^0(u_B)$.
- For *Barbara*, choice *red* is strictly dominated in the initial decision problem $\Gamma_B^0(u_B')$ by $\frac{1}{2}$ *green* $+ \frac{1}{2}$ *yellow*.
- Eliminate choice *red* from $\Gamma_B^0(u_B)$.

Example: What is Barbara's favourite Colour?

Output from **Round 1** $=$ 1-fold reduced decision problems:

$\Gamma_y^1(u_y)$

	blue	green	red	yellow
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2

$\Gamma_B^1(u_B)$

	blue	green	red	yellow
blue	0	2	2	2
red	4	4	0	4
yellow	3	3	3	0

$\Gamma_B^1(u_B')$

	blue	green	red	yellow
blue	0	4	4	4
green	2	2	0	2
yellow	3	3	3	0

Round 2:

- For *you*, choice *yellow* is not in any of your decision problems as *you* only have one decision problem $\Gamma_Y^1(u_Y)$ and it is not in there.
- Eliminate choice *yellow* from $\Gamma_B^1(u_B)$ and from $\Gamma_B^1(u_B')$.
- For every choice of *Barbara* there exists a 1-fold reduced decision problem that contains it.
- Consequently, nothing can be eliminated in terms of *Barbara*'s choices from $\Gamma_y^1(u_y)$.

Example: What is Barbara's favourite Colour?

$\Gamma^1_y(u_y)$

	blue	green	red	yellow
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2

$(C^1_B(u_B), C^2_y(u_B), u_B)$

	blue	green	red
blue	0	2	2
red	4	4	0
yellow	3	3	3

$(C^1_B(u'_B), C^2_y(u'_B), u'_B)$

	blue	green	red
blue	0	4	4
green	2	2	0
yellow	3	3	3

Round 2 (continued):

- Choice *blue* becomes strictly dominated by *yellow* for *Barbara* in the decision problem $\left(C^1_B(u_B), C^2_y(u_B), u_B\right)$.
- Eliminate choice *blue* from $\left(C^1_B(u_B), C^2_y(u_B), u_B\right)$.
- Choice *green* becomes strictly dominated by *yellow* for *Barbara* in the decision problem $\left(C^1_B(u'_B), C^2_y(u'_B), u'_B\right)$.
- Eliminate choice *green* from $\left(C^1_B(u'_B), C^2_y(u'_B), u'_B\right)$.

Example: What is Barbara's favourite Colour?

Output from **Round 2** = 2-fold reduced decision problems:

$\Gamma_y^2(u_y)$

	blue	*green*	*red*	*yellow*
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2

$\Gamma_B^2(u_B)$

	blue	*green*	*red*
red	4	4	0
yellow	3	3	3

$\Gamma_B^2(u_B')$

	blue	*green*	*red*
blue	0	4	4
yellow	3	3	3

Round 3:

- For *you*, the choices *blue*, *green*, and *red* are in your only decision problem $\Gamma_y^2(u_y)$.
- For *Barbara*, choice *green* is not in any of her decision problems $\Gamma_B^2(u_B)$ and $\Gamma_B^2(u_B')$.
- Eliminate choice *green* from $\Gamma_y^2(u_y)$.

Example: What is Barbara's favourite Colour?

$(C_y^2(u_y), C_B^3(u_y), u_y)$

	blue	*red*	*yellow*
blue	0	4	4
green	3	3	3
red	2	2	2

$\Gamma_B^2(u_B)$

	blue	*green*	*red*
red	4	4	0
yellow	3	3	3

$\Gamma_B^2(u'_B)$

	blue	*green*	*red*
blue	0	4	4
yellow	3	3	3

Round 3 (continued):

- Choice *red* becomes strictly dominated by *green* for *you* in the decision problem $(C_y^2(u_y), C_B^3(u_y), u_y)$.
- Eliminate choice *red* from $(C_y^2(u_y), C_B^3(u_y), u_y)$.

Example: What is Barbara's favourite Colour?

Output from **Round 3** $=$ 3-fold reduced decision problems:

$\Gamma_y^3(u_y)$

	blue	red	yellow
blue	0	4	4
green	3	3	3

$\Gamma_B^3(u_B)$

	blue	green	red
red	4	4	0
yellow	3	3	3

$\Gamma_B^3(u'_B)$

	blue	green	red
blue	0	4	4
yellow	3	3	3

Round 4:

- For *you*, choice *red* is no longer in any of your decision problems, i.e. it is not in $\Gamma_y^3(u_y)$.
- Eliminate choice *red* from $\Gamma_B^3(u_B)$ and from $\Gamma_B^3(u'_B)$.
- For *Barbara*, the choices *red*, *blue*, and *yellow* are in some 3-fold decision problem of hers.

Example: What is Barbara's favourite Colour?

$\Gamma_y^3(u_y)$

	blue	red	yellow
blue	0	4	4
green	3	3	3

$(C_B^3(u_B), C_y^4(u_B), u_B)$

	blue	green
red	4	4
yellow	3	3

$(C_B^3(u_B), C_y^4(u_B), u_B)$

	blue	green
blue	0	4
yellow	3	3

Round 4 (continued):

- Choice *yellow* becomes strictly dominated by *red* for *Barbara* in the decision problem $(C_B^3(u_B), C_y^4(u_B), u_B)$.
- Eliminate choice *yellow* from $(C_B^3(u_B), C_y^4(u_B), u_B)$.

Example: What is Barbara's favourite Colour?

Output from **Round 4** = 4-fold reduced decision problems:

$\Gamma_y^4(u_y)$

	blue	green	yellow
blue	0	4	4
green	3	3	3

$\Gamma_B^4(u_B)$

	blue	green
green	4	4

$\Gamma_B^4(u_B')$

	blue	green
blue	0	4
yellow	3	3

Round 5:

- The algorithm stops.
- Output:

$$GISD_{you} = \{(blue, u_y), (green, u_y)\}$$

and

$$GISD_{Barbara} = \{(red, u_B), (blue, u_B'), (yellow, u_B')\}.$$

Characterization

- The epistemic condition of **common belief in rationality** and the solution concept of **Generalized Iterated Strict Dominance** are now related to each other in general.

- The two notions gave rise to the same choices in the two specific games considered.

- Indeed this relationship turns out to be true in general: CBR and GISD are equivalent in terms of their results.
  - Thus, the meaning of GISD in terms of interactive thinking is captured by CBR.
  - Alternatively phrased, CBR can be procedurally characterized by GISD.

A Generalization of Pearce's Lemma

Theorem 10

Let Γ be a game with incomplete information, $\mathcal{O}$ the one-person perspective form of Γ, i some player, $\Gamma_i(u_i)$ some decision problem of player i, and $c_i \in D_i$ some choice of player i. The choice c_i is strictly dominated in $\Gamma_i(u_i)$, if and only if, there exists some conjecture $\beta_i \in \Delta(D_{-i})$ such that c_i is optimal for (β_i, u_i).

Proof

- Define a two player game $\Gamma' = \big((i,j), \{C'_i, C'_j\}, \{u'_i, u'_j\}\big)$ with complete information, where
  - $C'_i := D_i$,
  - $C'_j := D_{-i}$,
  - $u'_i(d_i, d_{-i}) := u_i(d_i, d_{-i})$ for all $d_i \in C'_i$ and for all $d_{-i} \in C'_j$,
  - and $u'_j(d_{-i}, d_i) := 0$ for all $d_{-i} \in C'_j$ and for all $d_i \in C'_i$.
- Observe that the choice $c_i \in D_i$ is strictly dominated in the decision problem $\Gamma_i(u_i)$, if and only if, it is strictly dominated in the game Γ'.
- By PEARCE´S LEMMA applied to the game Γ', if follows that c_i is strictly dominated in Γ', if and only if, there exists no conjecture $\beta_i \in \Delta(C'_j)$ such that c_i is optimal for (β_i, u'_i).
- Since $C'_j = D_{-i}$ and $u'_i = u_i$, it follows that $c_i \in D_i$ is strictly dominated in the decision problem $\Gamma_i(u_i)$, if and only if, there exists no conjecture $\beta_i \in \Delta(D_{-i})$ such that c_i is optimal for (β_i, u_i).

Epistemic Characterization of Generalized Iterated Strict Dominance

Theorem 11

Let Γ be a game with incomplete information, $i \in I$ some player, $c_i \in C_i$ some choice of player i, and $u_i \in U_i$ some utility function of player i. The choice c_i is rational under common belief in rationality given u_i, if and only if, $(c_i, u_i) \in GISD_i$.

Summary

- The reasoning concept of **common belief in rationality** has been extended to static games with incomplete information.

- Its algorithmic characterization has brought to light a basic (non-equilibrium) solution concept:

  Generalized Iterated Strict Dominance (GISD)

- The ready-made algorithm of **GISD** could constitute a useful tool for economists when analyzing situations with payoff uncertainty.