EPICENTER Summer Course on Epistemic Game Theory
Chapter 4: Simple Belief Hierarchies

Andrés Perea

Maastricht University

July 5, 2022

Simple belief hierarchies

- Previously, we have discussed the idea of common belief in rationality.
- So, we focus on belief hierarchies in which you believe that
- your opponents choose rationally,
- your opponents believe that their opponents choose rationally,
- your opponents believe that their opponents believe that their opponents choose rationally,
- and so on.
- Can we still distinguish between such belief hierarchies?
- We will look at psychological factors beyond common belief in rationality.

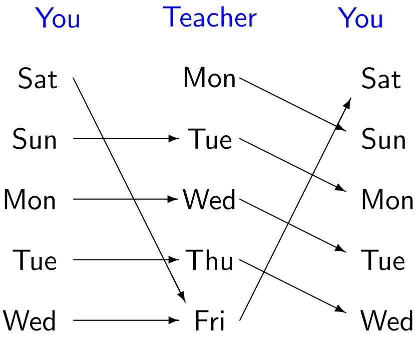
Example: Teaching a lesson

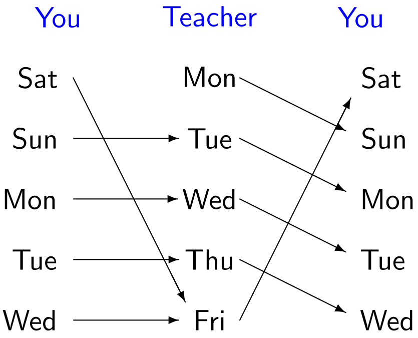
Story

- It is Friday, and your biology teacher tells you that he will give you a surprise exam next week.
- You must decide on what day you will start preparing for the exam.
- In order to pass the exam, you must study for at least two days.
- To write the perfect exam, you must study for at least six days. In that case, you will get a compliment by your father.
- Passing the exam increases your utility by 5.
- Failing the exam increases the teacher's utility by 5.
- Every day you study decreases your utility by 1, but increases the teacher's utility by 1.
- A compliment by your father increases your utility by 4.

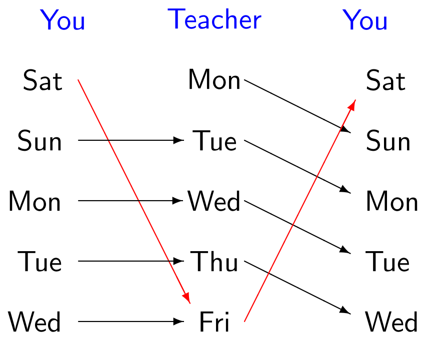
Teacher

You		Mon	Tue	Wed	Thu	Fri
	Sat	3, 2	2, 3	1, 4	0, 5	3, 6
	Sun	−1, 6	3, 2	2, 3	1, 4	0, 5
	Mon	0, 5	−1, 6	3, 2	2, 3	1, 4
	Tue	0, 5	0, 5	−1, 6	3, 2	2, 3
	Wed	0, 5	0, 5	0, 5	−1, 6	3, 2

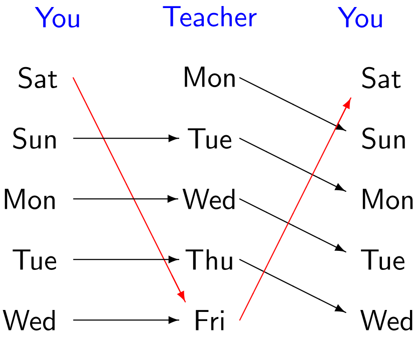




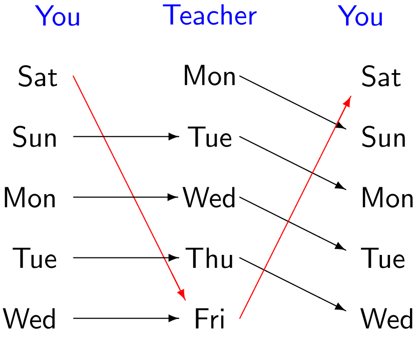
- Under common belief in rationality, you can rationally choose any day to start studying.
- Is there still a way to distinguish between your various choices?
- Yes! Some choices are supported by a simple belief hierarchy, whereas other choices are not.



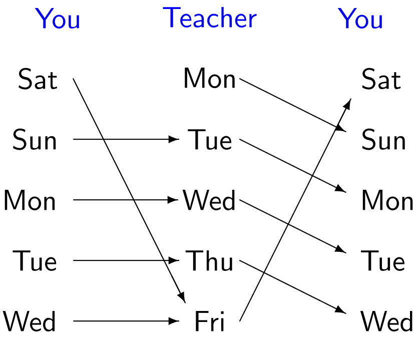
- Consider the belief hierarchy that supports your choices Saturday and Wednesday.
- This belief hierarchy is entirely generated by the belief σ_2 that the teacher puts the exam on Friday, and the belief σ_1 that you start studying on Saturday.



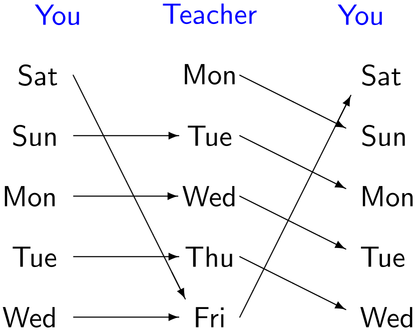
- Let σ_2 be the belief that the teacher chooses *Friday,* and let σ_1 be the belief that you choose *Saturday.*
- Then, in the belief hierarchy that supports your choices *Saturday* and *Wednesday,*
- your belief about the teacher's choice is σ_2,
- you believe, with probability 1, that the teacher's belief about your choice is σ_1,
- ...



- ... you believe, with prob. 1, that the teacher believes, with prob. 1, that your belief about the teacher's choice is indeed σ_2,
- you believe, with prob. 1, that the teacher believes, with prob. 1, that you believe, with prob. 1,that the teacher's belief about your choice is indeed σ_1,
- and so on.
- So, this belief hierarchy is completely generated by the beliefs σ_1 and σ_2. We call such a belief hierarchy simple.



- The belief hierarchies that support your choices Sunday, Monday and Tuesday are certainly not simple. Consider, for instance, the belief hierarchy that supports your choice Sunday. There,
- you believe that the teacher puts the exam on Tuesday,
- but you believe that the teacher believes that you believe that the teacher will put the exam on Wednesday.



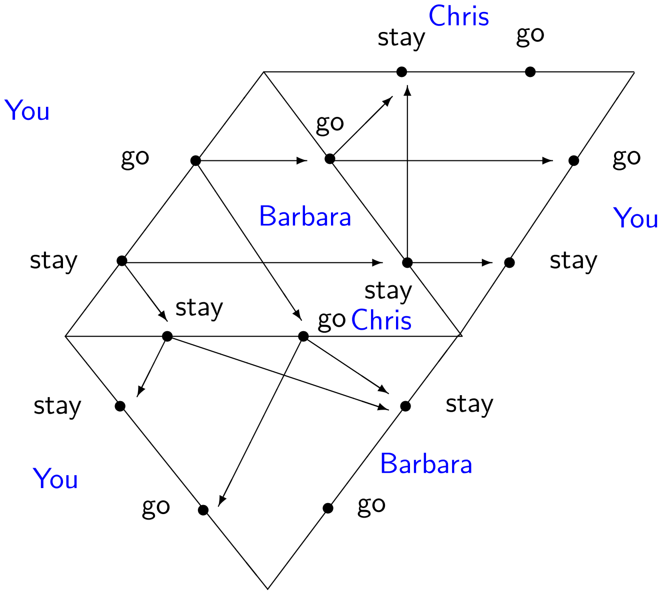
Summarizing

- Within this beliefs diagram:
- You can rationally make every choice under common belief in rationality.
- Your choices Saturday and Wednesday are supported by a simple belief hierarchy.
- Your other choices are supported by non-simple belief hierarchies.

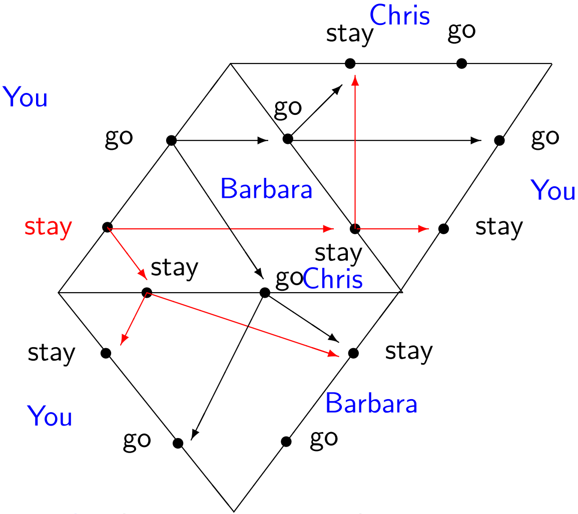
Example: Movie or party?

Story

- You have been invited to a party this evening, together with Barbara and Chris. But this evening, your favorite movie Once upon a time in America, starring Robert de Niro, will be on TV.
- Having a good time at the party gives you utility 3, watching the movie gives you utility 2, whereas having a bad time at the party gives you utility 0. Similarly for Barbara and Chris.
- You will only have a good time at the party if Barbara and Chris both join.
- Barbara and Chris had a fierce discussion yesterday. Barbara will only have a good time at the party if you join, but not Chris.
- Chris will only have a good time at the party if you join, but not Barbara.
- What should you do: Go to the party, or stay at home?

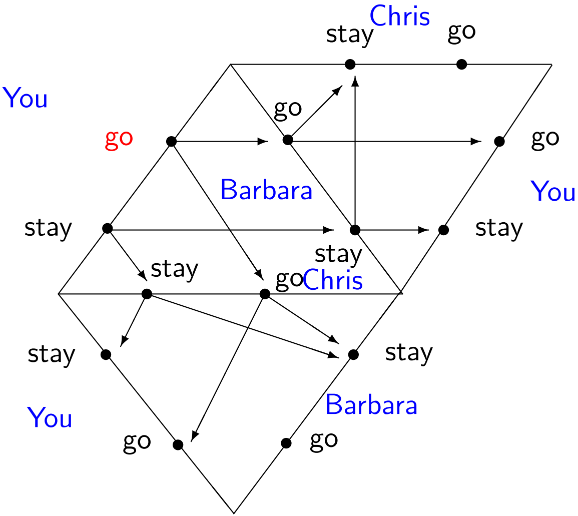


- Under common belief in rationality, you can go to the party or stay at home.

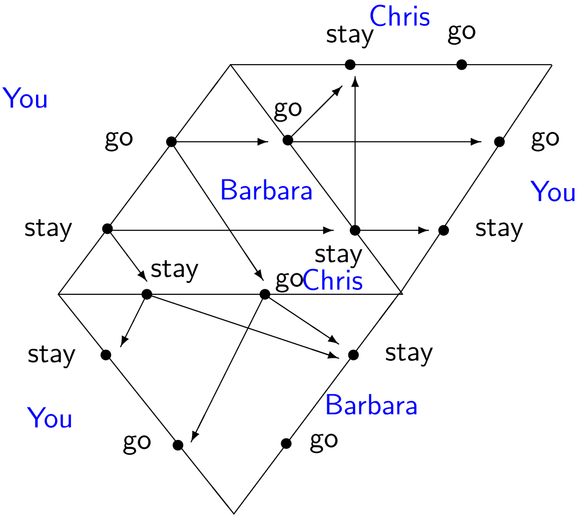


- The belief hierarchy that supports your choice stay is simple: It is completely generated by the beliefs

$$\sigma_1 = \text{ You stay, } \sigma_2 = \text{ Barbara stays, } \sigma_3 = \text{ Chris stays.}$$



- The belief hierarchy that supports your choice go is not simple:
- You believe that Chris will go to the party.
- You believe that Barbara believes that Chris will stay at home.



- Summarizing: Under common belief in rationality, you can rationally choose go or stay.
- In this beliefs diagram, stay is supported by a simple belief hierarchy, but go is not.

- In general, a belief hierarchy is called simple if it is generated by a combination of beliefs $\sigma_1, ..., \sigma_n$.

Definition (Belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$)

For every player i, let σ_i be a probabilistic belief about i's choice.

The belief hierarchy for player i that is generated by $(\sigma_1, ..., \sigma_n)$ states that

(1) player i has belief σ_j about player j's choice,

(2) player i believes that player j has belief σ_k about player k's choice,

(3) player i believes that player j believes that player k has belief σ_l about player l's choice,

and so on.

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- Observation 1: A type with a simple belief hierarchy always believes that his opponents are correct about his entire belief hierarchy.
- Proof. Take a type t_i with a simple belief hierarchy. Then, its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.
- Fix an opponent j. Then, t_i has belief σ_j about j's choice. But also, t_i believes that every opponent believes that he (player i) has indeed belief σ_j about j's choice.
- Fix an opponent j, and some player $k \neq j$. Then, t_i believes that player j has belief σ_k about k's choice. But also, t_i believes that every opponent believes that he (player i) indeed believes that player j has belief σ_k about k's choice.
- And so on. ■

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- Observation 2: In a game with three players or more, a type t_i with a simple belief hierarchy believes that his opponents share his beliefs about other players.

- Proof. Suppose that t_i's belief hierarchy is generated by $(\sigma_1, ..., \sigma_n)$.
- Fix two different opponents j and k. Then, t_i's belief about k's choice is σ_k. But t_i also believes that j has belief σ_k about k's choice.
- Take some player $l \neq k$. Then, t_i believes that k's belief about l's choice is σ_l. But t_i also believes that j believes that k's belief about l's choice is σ_l.
- And so on. ■

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- Observation 3: In a game with three players or more, consider a type t_i with a simple belief hierarchy.

  Then, player i's belief about j's choice is independent from i's belief about k's choice.

- Indeed, the probability that i assigns to j choosing c_j and k choosing c_k is given by the product

$$\sigma_j(c_j) \cdot \sigma_k(c_k).$$

- In the example "Movie or party?", for instance, the belief

$$(0.5) \cdot (stay,\ stay) + (0.5) \cdot (go,\ go)$$

  is not possible in a simple belief hierarchy.

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- Observation 4: Consider a type t_i with a simple belief hierarchy, which believes in j's rationality.

  Suppose that t_i assigns a positive probability to j's choices a and b.

  Then, t_i must believe that j is indifferent between a and b.
- Proof. Type t_i only deems possible one belief hierarchy for player j – the simple belief hierarchy for j generated by $(\sigma_1, ..., \sigma_n)$.
- Hence, if t_i assigns positive probability to a and b, and believes in j's rationality, then t_i must believe that both a and b are optimal for j's simple belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$.
- Thus, t_i must believe that j is indifferent between a and b. ■
- This is not true for non-simple belief hierarchies.

Nash equilibrium

- Previously we have focused on belief hierarchies that express common belief in rationality.
- So far in this chapter, we have focused on belief hierarchies that are simple.
- Can we characterize, in an easy way, those belief hierarchies that express common belief in rationality and are simple?

- Consider a type t_i with a simple belief hierarchy. Then, t_i's belief hierarchy is generated by some combination $(\sigma_1, ..., \sigma_n)$ of beliefs. Hence:

- t_i's belief about the opponents' choices is σ_{-i},

- t_i believes that player j's has belief σ_{-j} about his opponents' choices,

- t_i believes that player j believes that player k has belief σ_{-k} about his opponents' choices,

- and so on.

- Suppose that, in addition, type t_i expresses common belief in rationality.

- Take some opponent's choice c_j with $\sigma_j(c_j) > 0$.
- Then, t_i assigns positive probability to c_j.
- As t_i believes in j's rationality, choice c_j must be optimal for player j under the belief σ_{-j} about the opponents' choices.

- Now, take some own choice c_i with $\sigma_i(c_i) > 0$.
- Then, type t_i believes that every opponent j assigns positive probability to c_i.
- As t_i believes that j believes in i's rationality, choice c_i must be optimal for player i under the belief σ_{-i} about the opponents' choices.
- Conclusion: If t_i is a type that
- has a simple belief hierarchy, generated by the combination of beliefs $(\sigma_1, .., \sigma_n)$, and
- expresses common belief in rationality,
- then, for every player j, the belief σ_j only assigns positive probability to choices c_j that are optimal under the belief σ_{-j}.

Definition (Nash equilibrium)

The combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium if for every player j, the belief σ_j only assigns positive probability to choices c_j that are optimal under the belief σ_{-j}.

- Based on Nash (1950, 1951).

Theorem

Consider a type t_i which

(1) has a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs, and

(2) expresses common belief in rationality.

Then, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ must be a Nash equilibrium.

- We can show that also the opposite direction is true.

Theorem

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

If the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium, then type t_i expresses common belief in rationality.

- Proof. We first show that t_i believes in his opponents' rationality.
- Take an opponent j, and assume that t_i assigns positive probability to choice c_j.
- Then $\sigma_j(c_j) > 0$, and hence c_j must be optimal for player j under the belief σ_{-j}.
- Since t_i believes that j's belief about the opponents' choices is σ_{-j}, type t_i believes that c_j is optimal for player j.
- So, t_i only assigns positive probability to a choice c_j if he believes that c_j is optimal for player j.
- Hence, type t_i believes in his opponents' rationality.

- Proof continued. We next show that t_i believes that his opponents believe in their opponents' rationality.

- Take an opponent j, and some player $k \neq j$. Suppose, t_i believes that player j assigns positive probability to choice c_k.
- Then $\sigma_k(c_k) > 0$, and hence c_k must be optimal for player k under the belief σ_{-k}.
- Since t_i believes that player j believes that k's belief about his opponents' choices is σ_{-k}, type t_i believes that player j believes that c_k is optimal for player k.
- So, if t_i believes that player j assigns positive probability to choice c_k, then t_i believes that player j believes that c_k is optimal for player k.
- Hence, type t_i believes that player j believes in k's rationality.
- As such, type t_i believes that his opponents believe in their opponents' rationality.
- And so on. ■

- By combining the two theorems above, we obtain the following characterization.

Theorem (Simple belief hierarchies versus Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

Then, type t_i expresses common belief in rationality, if and only if, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

- Other epistemic foundations of Nash equilibrium can be found in Spohn (1982), Brandenburger and Dekel (1987, 1989), Tan and Werlang (1988), Aumann and Brandenburger (1995), Polak (1999), Asheim (2006), Perea (2007), Barelli (2009) and Bach and Tsakas (2014).
- All these foundations involve some correct beliefs assumption: You believe that your opponents are correct about your first-order belief.
- Not all layers of common belief in rationality are needed to obtain Nash equilibrium.

Theorem (Simple belief hierarchies versus Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

Then, type t_i expresses common belief in rationality, if and only if, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

- Important consequence:
- Suppose that in a given game, we wish to find the simple belief hierarchies that express common belief in rationality.
- Then, it is sufficient to find all the Nash equilibria $(\sigma_1, ..., \sigma_n)$ in the game.

- Question: Can we always find simple belief hierarchies that express common belief in rationality?
- The answer is given by John Nash, in his PhD dissertation.

Theorem (Nash equilibrium always exists)

For every game with finitely many choices there is at least one Nash equilibrium $(\sigma_1, ..., \sigma_n)$.

Theorem (Common belief in rationality with simple belief hierarchies is always possible)

Consider a game with finitely many choices. Then, for every player i *there is at least one simple belief hierarchy that expresses common belief in rationality.*

Computing Nash equilibria

- We wish to find those choices you can rationally make if you
- express common belief in rationality, and
- hold a simple belief hierarchy.
- Is there a method to find these choices?

- Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

- Remember: Type t_i expresses common belief in rationality, if and only if, the combination $(\sigma_1, ..., \sigma_n)$ of beliefs is a Nash equilibrium.

- Moreover, choice c_i is optimal for t_i if c_i is optimal under the belief σ_{-i} about the opponents' choices.

- Hence, choice c_i can rationally be made under common belief in rationality with a simple belief hierarchy, if and only if, there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal under σ_{-i}.

Definition (Nash choice)

A choice c_i is a Nash choice if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

Definition (Nash choice)

A choice c_i is a Nash choice if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

- Observation 1: If there is a Nash equilibrium $(\sigma_1, ..., \sigma_n)$ with $\sigma_i(c_i) > 0$, then c_i is a Nash choice.
- Proof: Take some choice c_i with $\sigma_i(c_i) > 0$. Since $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium, c_i is optimal under the belief σ_{-i}.
- Hence, c_i is a Nash choice. ■

Definition (Nash choice)

A choice c_i is a Nash choice if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

- Observation 2: A Nash choice c_i need not always receive positive probability in a Nash equilibrium.
- Proof: Consider the game

	c	d
a	$2, 0$	$0, 1$
b	$1, 0$	$1, 0$

.

- Then, $(b, \frac{1}{2}c + \frac{1}{2}d)$ is a Nash equilibrium.
- Since a is optimal under the belief $\frac{1}{2}c + \frac{1}{2}d$, choice a is a Nash choice.
- However, there is no Nash equilibrium (σ_1, σ_2) with $\sigma_1(a) > 0$.
- Indeed, if $\sigma_1(a) > 0$, then only d is optimal for player 2, and hence $\sigma_2 = d$.
- But then, only b can be optimal for player 1, hence $\sigma_1 = b$. This is a contradiction. ∎

Theorem (Simple belief hierarchies versus Nash choices)

Player i can rationally make choice c_i under common belief in rationality with a simple belief hierarchy, if and only if, c_i is a Nash choice.

- Proof: (a) Suppose that player i can rationally make choice c_i under common belief in rationality with a simple belief hierarchy.
- Then, there is an epistemic model and a type t_i in it, such that t_i has a simple belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$, expresses common belief in rationality, and c_i is optimal for t_i.
- We have seen that $(\sigma_1, ..., \sigma_n)$ must be a Nash equilibrium.
- Since c_i is optimal for player i under the belief σ_{-i}, it follows that c_i is a Nash choice.

Theorem (Simple belief hierarchies versus Nash choices)

Player i can rationally make choice c_i under common belief in rationality with a simple belief hierarchy, if and only if, c_i is a Nash choice.

- Proof: (b) Suppose that c_i is a Nash choice.
- Then, there is a Nash equilibrium $(\sigma_1, ..., \sigma_n)$ such that c_i is optimal for player i under the belief σ_{-i}.
- Let t_i be the type with the simple belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$.
- We have seen that t_i expresses common belief in rationality.
- Hence, c_i is optimal for the type t_i that has a simple belief hierarchy and expresses common belief in rationality. ■

Theorem (Simple belief hierarchies versus Nash choices)

Player i can rationally make choice c_i under common belief in rationality with a simple belief hierarchy, if and only if, c_i is a Nash choice.

- Suppose we wish to find those choices that player i can make if
- he holds a simple belief hierarchy, and
- he expresses common belief in rationality.
- Then, it is sufficient to compute all Nash choices for player i in the game.
- Bad news: There is no simple algorithm for computing all Nash equilibria in a game.
- In some games, this is a difficult task.

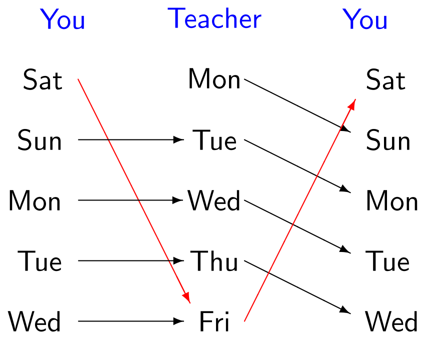
Example: Teaching a lesson

Teacher

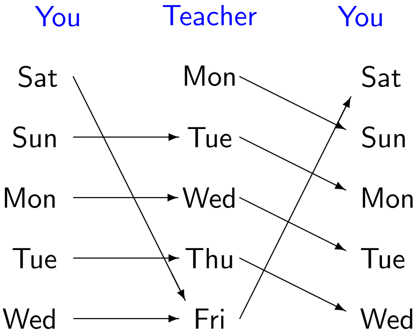
You

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

- On what days can you rationally start to study if you hold a simple belief hierarchy, and express common belief in rationality?



- We have seen:
- You can rationally choose Saturday or Wednesday under common belief in rationality with a simple belief hierarchy.
- Namely, the belief hierarchy that supports your choices Saturday and Wednesday is simple, as it is generated by the beliefs $\sigma_1 =$ Sat and $\sigma_2 =$ Fri.



- Are there any other choices you can rationally make under common belief in rationality with a simple belief hierarchy?
- The beliefs diagram does not help here.
- Compute all Nash equilibria (σ_1, σ_2) in the game.

	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

- Suppose that (σ_1, σ_2) is a Nash equilibrium.
- Step 1. Show that $\sigma_2(Thu) = 0$.
- Suppose that $\sigma_2(Thu) > 0$. Then, *Thu* must be optimal for the teacher under the belief σ_1 about your choice.
- This is only possible if $\sigma_1(Wed) > 0$.
- So, *Wed* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(Fri) = 1$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

- Step 2. Show that $\sigma_2(Wed) = 0$.
- Suppose that $\sigma_2(Wed) > 0$. Then, *Wed* must be optimal for the teacher under the belief σ_1.
- This is only possible if $\sigma_1(Tue) > 0$.
- Then, *Tue* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(Thu) > 0$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

- Step 3. Show that $\sigma_2(Tue) = 0$.
- Suppose that $\sigma_2(Tue) > 0$. Then, *Tue* must be optimal for the teacher under the belief σ_1.
- This is only possible if $\sigma_1(Mon) > 0$. Otherwise, *Tue* would be strictly dominated for the teacher by $(0.9) \cdot Wed + (0.1) \cdot Thu$.
- So, *Mon* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(Wed) > 0$ or $\sigma_2(Thu) > 0$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

- Step 4. Show that $\sigma_2(Mon) = 0$.
- Suppose that $\sigma_2(Mon) > 0$. Then, *Mon* must be optimal for the teacher under the belief σ_1.
- This is only possible if $\sigma_1(Sun) > 0$. Otherwise, *Mon* would be strictly dominated for the teacher by $(0.9) \cdot Tue + (0.09) \cdot Wed + (0.01) \cdot Thu$.
- So, *Sun* must be optimal for you under the belief σ_2.
- This is only possible if $\sigma_2(Tue) > 0$. Contradiction.

	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

- So, if (σ_1, σ_2) is a Nash equilibrium, then σ_2 must assign probability 0 to Mon, Tue, Wed and Thu. Hence, $\sigma_2 = Fri$.
- But then, your optimal choices under the belief σ_2 are Sat and Wed.
- Hence, your only Nash choices in this game are Sat and Wed.
- These are the only choices you can rationally make under common belief in rationality with a simple belief hierarchy.

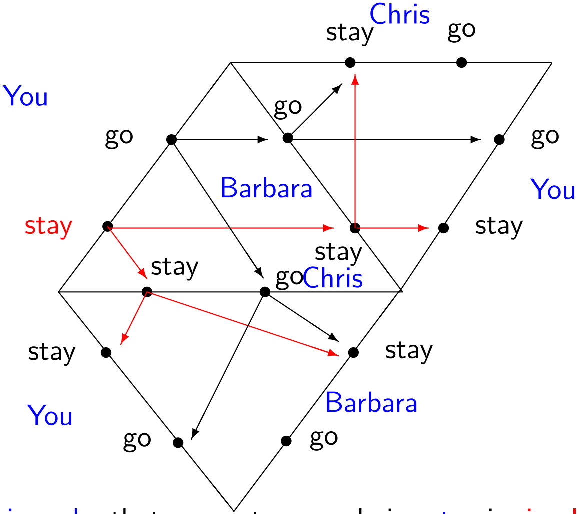
	Mon	Tue	Wed	Thu	Fri
Sat	$3,2$	$2,3$	$1,4$	$0,5$	$3,6$
Sun	$-1,6$	$3,2$	$2,3$	$1,4$	$0,5$
Mon	$0,5$	$-1,6$	$3,2$	$2,3$	$1,4$
Tue	$0,5$	$0,5$	$-1,6$	$3,2$	$2,3$
Wed	$0,5$	$0,5$	$0,5$	$-1,6$	$3,2$

Summarizing

- Under common belief in rationality, you can rationally start to study on any day between Saturday and Wednesday.
- However, if you hold a simple belief hierarchy in addition, then under common belief in rationality you can only rationally start to study on Saturday or Wednesday.
- Crucial difference: With a simple belief hierarchy, you believe that the teacher is correct about your beliefs.

Example: Movie or party?

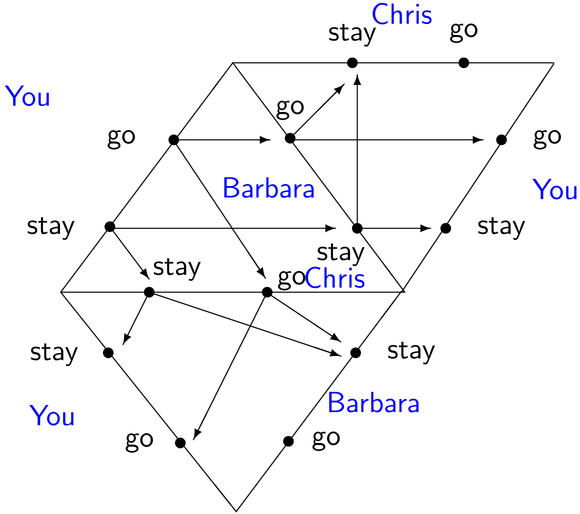
- Having a good time at the party gives you utility 3, watching the movie gives you utility 2, whereas having a bad time at the party gives you utility 0. Similarly for Barbara and Chris.
- You will only have a good time at the party if Barbara and Chris both join.
- Barbara will only have a good time at the party if you join, but not Chris.
- Chris will only have a good time at the party if you join, but not Barbara.
- What choice(s) can you rationally make if you hold a simple belief hierarchy, and express common belief in rationality?



- The belief hierarchy that supports your choice stay is simple: It is completely generated by the beliefs

$$\sigma_1 = \text{You stay}, \ \sigma_2 = \text{Barbara stays}, \ \sigma_3 = \text{Chris stays.}$$

- So, you can rationally stay at home under common belief in rationality with a simple belief hierarchy.



- In this beliefs diagram, your choice to go the party is not supported by a simple belief hierarchy.
- But can your choice go be supported by a simple belief hierarchy that expresses common belief in rationality?

- Let us try to find all Nash equilibria in this game, and see whether your choice go is a Nash choice.

You stay	C stays	C goes
B stays	2, 2, 2	2, 2, 0
B goes	2, 0, 2	2, 0, 0

You go	C stays	C goes
B stays	0, 2, 2	0, 2, 3
B goes	0, 3, 2	3, 0, 0

- Suppose that $(\sigma_1, \sigma_2, \sigma_3)$ is a Nash equilibrium in this game.
- We first show that $\sigma_1(go) = 0$.
- Assume that $\sigma_1(go) > 0$. Then, *go* must be optimal for you under the belief (σ_2, σ_3).
- For you, $u_1(go) = 3 \cdot \sigma_2(go) \cdot \sigma_3(go)$, whereas $u_1(stay) = 2$.
- Hence, $\sigma_2(go) \cdot \sigma_3(go) \geq 2/3$, which implies $\sigma_2(go) \geq 2/3$ and $\sigma_3(go) \geq 2/3$. This implies $\sigma_3(stay) \leq 1/3$.
- So, *go* must be optimal for Barbara under the belief (σ_1, σ_3).
- But for Barbara,

$$u_2(go) = 3 \cdot \sigma_1(go) \cdot \sigma_3(stay) \leq 1 < u_2(stay),$$

which means that *go* is not optimal for Barbara. Contradiction.

You stay	C stays	C goes
B stays	2, 2, 2	2, 2, 0
B goes	2, 0, 2	2, 0, 0

You go	C stays	C goes
B stays	0, 2, 2	0, 2, 3
B goes	0, 3, 2	3, 0, 0

- So we conclude that $\sigma_1(stay) = 1$.
- But then, for Barbara only stay can be optimal under the belief (σ_1, σ_3). Hence, $\sigma_2 = stay$.
- Similarly, for Chris only stay can be optimal under the belief (σ_1, σ_2). Consequently, $\sigma_3 = stay$.
- So, the only Nash equilibrium is

$$\sigma_1 = stay, \ \sigma_2 = stay, \ \sigma_3 = stay.$$

- Under the belief (σ_2, σ_3), your only optimal choice is to stay at home. Hence, your only Nash choice is to stay at home.

You stay	C stays	C goes
B stays	2, 2, 2	2, 2, 0
B goes	2, 0, 2	2, 0, 0

You go	C stays	C goes
B stays	0, 2, 2	0, 2, 3
B goes	0, 3, 2	3, 0, 0

Summarizing

- Under common belief in rationality you can either stay at home, or go to the party.
- However, if you hold a simple belief hierarchy, then under common belief in rationality your only rational choice is to stay at home.
- Crucial difference: With a simple belief hierarchy, you believe that Barbara has the same belief about Chris' choice as you do.

Other classes of games

- Simple belief hierarchies, and variants of Nash equilibrium, have also been defined for other classes of games:
- generalized Nash equilibrium in games with incomplete information: Bach and Perea (2020a, 2022)
- psychological Nash equilibrium in psychological games: Geanakoplos, Pearce and Stacchetti (1989)
- Research question: Other epistemic foundations for Nash equilibrium?
- Research question: Applications of generalized Nash equilibrium to models in economics?

Common prior

- Common prior is a condition on belief hierarchies that is weaker than simple belief hierarchies.
- Common belief in rationality together with a common prior leads to correlated equilibrium: Aumann (1974, 1987). See Bach and Perea (2020b) for a proof.
- Some years earlier, Harsanyi (1967–1968) defined Bayesian equilibrium in games with incomplete information, which is also based on common belief in rationality with a common prior (Bach and Perea (2022)).
- Correlated equilibrium is Bayesian equilibrium when applied to games with complete information.
- Research question: Other conditions on belief hierarchies, besides simple belief hierarchies and common prior?
- Research question: Epistemic foundation for common prior?

Conditions leading to simple belief hierarchies

- We have concentrated on simple belief hierarchies.
- But which epistemic conditions lead to a simple belief hierarchy?
- We focus on the case of two players only.

- In a two-player game, a simple belief hierarchy for player i is completely generated by a pair of beliefs (σ_i, σ_j). That is:

- player i holds belief σ_j about j's choice,
- player i believes that player j holds belief σ_i about i's choice,
- player i believes that player j believes that, indeed, player i holds belief σ_j about j's choice,
- player i believes that player j believes that player i believes that, indeed, player j holds belief σ_i about i's choice,
- and so on.

- So, if player i holds a simple belief hierarchy, then he believes that his opponent is correct about his belief hierarchy. We say that player i believes that player j holds correct beliefs.

- Moreover, if player i holds a simple belief hierarchy, he also believes that player j believes that i has correct beliefs.

Definition (Belief that opponents hold correct beliefs)

A type t_i believes that his opponent holds correct beliefs if he believes that his opponent believes that, indeed, his type is t_i.

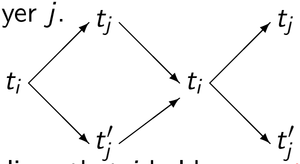
- Based on Perea (2007).
- We have seen that in a two-player game, a type with a simple belief hierarchy believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.
- In fact, the other direction is also true: If in a two-player game a type believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too, then this type has a simple belief hierarchy.

Theorem (Characterization of types with a simple belief hierarchy in two-player games)

Consider a game with two players.

A type t_i for player i has a simple belief hierarchy, if and only if, t_i believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.

- Proof. Based on Perea (2007). Suppose that type t_i believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.
- Show: Type t_i assigns probability 1 to a single type t_j for player j.
- Suppose that t_i would assign positive probability to two different types t_j and t_j' for player j.



- Then, t_j would not believe that i holds correct beliefs. Contradiction.

Theorem (Characterization of types with a simple belief hierarchy in two-player games)

Consider a game with two players.

A type t_i for player i has a simple belief hierarchy, if and only if, t_i believes that his opponent holds correct beliefs, and believes that his opponent believes that he himself holds correct beliefs too.

- So, we know that t_i assigns probability 1 to some type t_j for player j, and t_j assigns probability 1 to t_i.
- Let σ_j be the belief that t_i has about j's choice, and let σ_i be the belief that t_j has about i's choice.

$$t_i \xrightarrow[\sigma_j]{} t_j \xrightarrow[\sigma_i]{} t_i$$

- But then, t_i's belief hierarchy is generated by (σ_i, σ_j). So, t_i has a simple belief hierarchy. ■

- Be careful: If we have more than two players, then these conditions are no longer enough to induce simple belief hierarchies.
- In a game with more than two players, we need to impose the following extra conditions:
- you believe that player j has the same belief about player k as you do;
- your belief about player j's choice must be independent from your belief about player k's choice.

How reasonable is Nash equilibrium?

- We have seen that a Nash equilibrium makes the following assumptions:
- you believe that your opponents are correct about the beliefs that you hold;
- you believe that player j holds the same belief about player k as you do;
- your belief about player j's choice is independent from your belief about player k's choice.
- Each of these conditions is actually very questionable.
- Therefore, Nash equilibrium is perhaps not such a natural concept after all.

Asheim, G.B. (2006), *The consistent preferences approach to deductive reasoning in games,* Theory and Decision Library, Springer, Dordrecht, The Netherlands.

Aumann, R. J. (1974), Subjectivity and correlation in randomized strategies, *Journal of Mathematical Economics* **1**, 67–96.

Aumann, R. J. (1987): Correlated equilibrium as an expression of Bayesian rationality, *Econometrica* **55,** 1–18.

Aumann, R. and A. Brandenburger (1995), Epistemic conditions for Nash equilibrium, *Econometrica* **63,** 1161–1180.

Bach, C.W. and A. Perea (2020a), Generalized Nash equilibrium without common belief in rationality, *Economics Letters* **186,** 108526, 1–6.

Bach, C.W. and A. Perea (2020b), Two definitions of correlated equilibrium, *Journal of Mathematical Economics* **90,** 12–24.

Bach, C.W. and A. Perea (2022), Is Bayesian equilibrium the incomplete information counterpart to Nash equilibrium?, Manuscript.

Bach, C.W. and E. Tsakas (2014), Pairwise epistemic conditions for Nash equilibrium, *Games and Economic Behavior* **85,** 48–59.

Barelli, P. (2009), Consistency of beliefs and epistemic conditions for Nash and correlated equilibrium, *Games and Economic Behavior* **67,** 363–375.

Brandenburger, A. and E. Dekel (1987), Rationalizability and correlated equilibria, *Econometrica* **55,** 1391–1402.

Brandenburger, A. and E. Dekel (1989), The role of common knowledge assumptions in game theory, in *The Economics of Missing Markets, Information and Games,* ed. by Frank Hahn. Oxford: Oxford University Press, pp. 46–61.

Geanakoplos, J., Pearce, D. and E. Stacchetti (1989), Psychological games and sequential rationality, *Games and Economic Behavior* **1,** 60–79.

Harsanyi, J.C. (1967–68), Games with incomplete information played by "bayesian" players, I–III, *Management Science* **14**, 159–182, 320–334, 486–502.

- Nash, J.F. (1950), Equilibrium points in N-person games, *Proceedings of the National Academy of Sciences of the United States of America* **36,** 48–49.
- Nash, J.F. (1951), Non-cooperative games, *Annals of Mathematics* **54**, 286–295.
- Perea, A. (2007), A one-person doxastic characterization of Nash strategies, *Synthese* **158**, 251–271 (*Knowledge, Rationality and Action* 341–361).
- Polak, B. (1999), Epistemic conditions for Nash equilibrium, and common knowledge of rationality, *Econometrica* **67,** 673–676.
- Spohn, W. (1982), How to make sense of game theory, in W. Stegmüller, W. Balzer and W. Spohn (eds.), *Philosophy of Economics,* Springer Verlag, pp. 239–270.
- Tan, T. and S.R.C. Werlang (1988), The bayesian foundations of solution concepts of games, *Journal of Economic Theory* **45,** 370–391.