Lexicographic Beliefs in Static Games

Part II: Respect of Preferences

Stephan Jagau

Institute for Mathematical Behavioral Sciences

EPICENTER Research Center for Epistemic Game Theory

July 7, 2022

Introduction

- Cautious reasoning = deem any event possible, yet consider some events infinitely more likely than others
- Modelling tool: lexicographic beliefs
- Last lecture: full belief in caution + primary belief in rationality
  – players fully believe that any event is possible.
  – players believe in rationality only at first lexicographic level
- This lecture: full belief in caution + respect of preferences
  – *one version* of ordering events on *all* lexicographic levels in a rationality-driven way

Agenda

- Introductory Example
- Common Full Belief in Caution & Respect of Preferences
- Possibility
- Procedural Characterization

Agenda

- **Introductory Example**
- Common Full Belief in Caution & Respect of Preferences
- Possibility
- Procedural Characterization

Introductory Example

Where to read my book?

- *You* choose a pub to go read your book there.
- *Barbara* is also going to a pub, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.

Introductory Example

Barbara

		A	B	C
	A	$0, 3$	$1, 2$	$1, 1$
You	B	$1, 3$	$0, 2$	$1, 1$
	C	$1, 3$	$1, 2$	$0, 1$

Introductory Example

- **Type Spaces:** $T_Y = \{t_Y\}$ and $T_B = \{t_B\}$
- **Beliefs for** You: $b_Y(t_Y) = ((A, t_B); (C, t_B); (B, t_B))$
- **Beliefs for** Barbara: $b_B(t_B) = ((B, t_Y); (C, t_Y); (A, t_Y))$

		Barbara		
		A	B	C
You	A	0, 3	1, 2	1, 1
	B	1, 3	0, 2	1, 1
	C	1, 3	1, 2	0, 1

- Your type t_Y primarily believes in Barbara's rationality.
- However, t_Y's secondary and tertiary belief seem counter-intuitive.
- For Barbara, *B* is always strictly preferred to *C*, hence it seems reasonable to deem Barbara choosing *B* infinitely more likely than her choosing *C*.

Respecting the Opponent's Preferences

Intuition: Whenever a cautious player believes his opponent prefers some choice over another choice, he should deem the former infinitely more likely than the latter.

Definition

A cautious type t_i of player i ***respects the opponent's preferences*** if for all of the opponent's types t_j deemed possible by t_i, and for every two choices $c_j, \ c_j'$ where t_j prefers c_j to c_j', type t_i deems (c_j, t_j) infinitely more likely than (c_j', t_j).

Example: Where to read my book?

- **Type Spaces:** $T_Y = \{t_Y, t'_Y\}$ and $T_B = \{t_B\}$
- **Beliefs for You:** $b_Y(t_Y) = ((A, t_B); (C, t_B); (B, t_B))$
  $b_Y(t'_Y) = ((A, t_B); (B, t_B); (C, t_B))$
- **Beliefs for Barbara:** $b_B(t_B) = ((B, t_Y); (C, t_Y); (A, t_Y))$

		Barbara		
		A	B	C
You	A	0, 3	1, 2	1, 1
	B	1, 3	0, 2	1, 1
	C	1, 3	1, 2	0, 1

- Your type t_Y does not respect Barbara's preferences as previously seen.
- Your type t'_Y does respect Barbara's preferences since it deems A infinitely more likely than B and B infinitely more likely than C.
- Hence, if you respect Barbara's preferences, then you *must* rationally choose C.

Agenda

- Respecting the Opponent's Preferences
- **Common Full Belief in Caution & Respect of Preferences**
- Possibility
- Procedural Characterization

Common Full Belief in (Caution & Respect of Preferences)

Definition

A cautious type t_i of player i expresses ***common full belief in caution & respect of preferences***, if

- t_i only deems possible cautious types of opponent j **and** if t_i respects j's preferences, (1-fold full belief in caution and respect of preferences)
- t_i only deems possible types of opponent j that express 1-fold full belief in caution and respect of preferences, (2-fold full belief in caution and respect of preferences)
- ...

Introductory Example

		Barbara		
		A	B	C
You	A	$0, 3$	$1, 2$	$1, 1$
	B	$1, 3$	$0, 2$	$1, 1$
	C	$1, 3$	$1, 2$	$0, 1$

- **Type Spaces:** $T_Y = \{t_Y\}$ and $T_B = \{t_B\}$
- **Beliefs for You:** $b_Y(t_Y) = ((A, t_B); (B, t_B); (C, t_B))$
- **Beliefs for Barbara:** $b_B(t_B) = ((C, t_Y); (B, t_Y); (A, t_Y))$

- Your type t_Y is cautious, and respects Barbara's preferences.
- Barbara's type t_B is cautious, and respects your preferences.
- Thus, both types express common full belief in caution and respect of preferences.
- Hence, choice *C* is (cautiously) rational for You under common full belief in caution & respect of preferences and the same is true for Barbara's choice A.
- In fact, those choices are *uniquely* (cautiously) rational under common full belief in caution and respect of preferences. (Why?)
- And what about the rational choices under common full belief in caution & primary belief in rationality?

Respect of Preferences and Primary Belief in Rat.

Observation. If i is cautious and respects j's preferences, then i also primarily believes in j's rationality.

- Let type t_i be cautious and respect j's preferences and assume t_i deems possible some (c_j, t_j) such that c_j is not optimal for t_j.
- Then, there exists a choice c_j^* that t_j prefers to c_j. With respect of preferences, t_i must deem (c_j^*, t_j) infinitely more likely than (c_j, t_j).
- Hence, t_i's primary belief assigns probability 0 to (c_j, t_j).

By induction, we now find:

Proposition

For any $k \geq 1$, if a cautious type t_i expresses up to k-fold (common) full belief in caution & respect of preferences, then t_i also expresses up to k-fold (common) primary belief in rationality.

Example: Dividing a Pizza

- *You* have ordered a four-slice pizza with *Barbara*.
- Both of you simultaneously write down a number of slices you want to claim or simply ”rest”.
- Writing ”rest” means you claim all slices that the other person did not claim. If both write ”rest”, then the pizza is divided equally.
- If the sum of your claims exceeds four, an argument will ensue and no one gets any slice. If the sum of claims is at most four, each gets what they asked for.
- Both You and Barbara seek to claim the maximum number of slices given the other person’s claim.

Example: Dividing a Pizza

					Barbara		
		0	1	2	3	4	*rest*
	0	0, 0	0, 1	0, 2	0, 3	0, 4	0, 4
	1	1, 0	1, 1	1, 2	1, 3	0, 0	1, 3
You	2	2, 0	2, 1	2, 2	0, 0	0, 0	2, 2
	3	3, 0	3, 1	0, 0	0, 0	0, 0	3, 1
	4	4, 0	0, 0	0, 0	0, 0	0, 0	4, 0
	rest	4, 0	3, 1	2, 2	1, 3	0, 4	2, 2

Example: Dividing a Pizza

- Which choices are (cautiously) rational for You under common full belief in caution & respect of preferences?
- Your choices 0, 1, and 2 are weakly dominated by claiming the *rest*.
- Hence, if you are cautious, then *rest* is better for you than 0, 1, or 2.
- By symmetry, if you believe Barbara is cautious, then you believe she prefers *rest* over 0, 1, or 2.
- With respect of preferences, you must deem Barbara's choice *rest* infinitely more likely than 0, 1, and 2.

		Barbara					
		0	1	2	3	4	*rest*
You	0	0, 0	0, 1	0, 2	0, 3	0, 4	0, 4
	1	1, 0	1, 1	1, 2	1, 3	0, 0	1, 3
	2	2, 0	2, 1	2, 2	0, 0	0, 0	2, 2
	3	3, 0	3, 1	0, 0	0, 0	0, 0	3, 1
	4	4, 0	0, 0	0, 0	0, 0	0, 0	4, 0
	rest	4, 0	3, 1	2, 2	1, 3	0, 4	2, 2

Example: Dividing a Pizza

1-Fold Full Belief in Caution & Respect of Preferences

- Now suppose you deem Barbara's choice *rest* infinitely more likely than 0, 1, and 2.
- Then your belief about Barbara's choice follows one of these four possible likelihood orderings:

  $(\{rest\}; \{0, 1, 2, 3, 4\})$

  $(\{4, rest\}; \{0, 1, 2, 3\})$

  $(\{3, rest\}; \{0, 1, 2, 4\})$

  $(\{3, 4, rest\}; \{0, 1, 2\})$

  Here "$(\{rest\}; \{0, 1, 2, 3, 4\})$" means that you deem $rest$ infinitely more likely than $0, 1, 2, 3, 4$, etc.
- With each ordering, You prefer choice 4 over choice 3. (Why?) So under 1-fold full belief in caution and respect of preferences, you (cautiously) prefer 4 over 3 and $rest$ over $0, 1, 2$.

		Barbara					
		0	1	2	3	4	*rest*
You	0	0, 0	0, 1	0, 2	0, 3	0, 4	0, 4
	1	1, 0	1, 1	1, 2	1, 3	0, 0	1, 3
	2	2, 0	2, 1	2, 2	0, 0	0, 0	2, 2
	3	3, 0	3, 1	0, 0	0, 0	0, 0	3, 1
	4	4, 0	0, 0	0, 0	0, 0	0, 0	4, 0
	rest	4, 0	3, 1	2, 2	1, 3	0, 4	2, 2

Example: Dividing a Pizza

Common Full Belief in Caution & Respect of Preferences

- So far, we know that you must prefer 4 over 3 and *rest* over 0, 1, 2 under 1-fold full belief in caution & respect of preferences.
- We show that both 4 and *rest* are (cautiously) rational for you under common full belief in caution & respect of preferences.
- Consider the following **epistemic model**:

  Type Spaces:

  $T_Y = \{t_Y^4, t_Y^r\}$ and $T_B = \{t_B^4, t_B^r\}$

  Beliefs for You:

  $b_Y(t_Y^4) = ((rest, t_B^r); (1, t_B^r); (4, t_B^r); (3, t_B^r); (2, t_B^r); (0, t_B^r))$

  $b_Y(t_Y^r) = ((4, t_B^4); (3, t_B^4); (rest, t_B^4); (2, t_B^4); (1, t_B^4); (0, t_B^4))$

  Beliefs for Barbara:

  $b_B(t_B^4) = ((rest, t_Y^r); (1, t_Y^r); (4, t_Y^r); (3, t_Y^r); (2, t_Y^r); (0, t_Y^r))$

  $b_B(t_B^r) = ((4, t_Y^4); (3, t_Y^4); (rest, t_Y^4); (2, t_Y^4); (1, t_Y^4); (0, t_Y^4))$

		Barbara					
		0	1	2	3	4	*rest*
You	0	0, 0	0, 1	0, 2	0, 3	0, 4	0, 4
	1	1, 0	1, 1	1, 2	1, 3	0, 0	1, 3
	2	2, 0	2, 1	2, 2	0, 0	0, 0	2, 2
	3	3, 0	3, 1	0, 0	0, 0	0, 0	3, 1
	4	4, 0	0, 0	0, 0	0, 0	0, 0	4, 0
	rest	4, 0	3, 1	2, 2	1, 3	0, 4	2, 2

- All types are cautious and express 1-fold full belief in caution & respect of preferences. Hence, all types express common full belief in caution & respect of preferences.
- 4 is optimal for t_Y^4 and *rest* is optimal for t_Y^r. So 4 and *rest* are (cautiously) rational for You under common full belief in caution & respect of preferences.

Agenda

- Respecting the Opponent’s Preferences
- Common Full Belief in Caution & Respect of Preferences
- **Possibility**
- Procedural Characterization

An Important Question

Is it always possible for players to reason in line with common full belief in caution & respect of preferences?

Formally: For any given game, can we always find a lexicographic epistemic model such that all types in that model express common full belief in caution & respect of preferences?

Example: Hide and Seek

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*, and she would also like to talk to *you* (2 additional utils for her).
- Question: Which pub should *you* go to?

Example: Hide and Seek

$$\begin{array}{ccccc}
 & & \multicolumn{3}{c}{\text{Barbara}} \\
 & & A_B & B_B & C_B \\
 & A_y & 0,5 & 1,2 & 1,1 \\
\text{You} & B_y & 1,3 & 0,4 & 1,1 \\
 & C_y & 1,3 & 1,2 & 0,3
\end{array}$$

Example: Hide and Seek

Is common full belief in caution & respect of preferences possible?

		Barbara		
		A_B	B_B	C_B
	A_y	0, 5	1, 2	1, 1
You	B_y	1, 3	0, 4	1, 1
	C_y	1, 3	1, 2	0, 3

- Take an arbitrary cautious lexicographic belief about Barbara's choice, e.g. $(A_B; B_B; C_B)$.
- Given this belief, you prefer C_y to B_y to A_y.
- Take a cautious lexicographic belief for Barbara that respects these preferences, e.g. $(C_y; B_y; A_y)$.
- Given this belief, Barbara prefers A_B to C_B to B_B.
- Take a cautious lexicographic belief for you that respects these preferences, e.g. $(A_B; C_B; B_B)$.
- Given this belief, you prefer B_y to C_y to A_y.
- Take the cautious lexicographic belief $(B_y; C_y; A_y)$ for Barbara. Given this belief, she prefers B_B to A_B to C_B.
- Take the cautious lexicographic belief $(B_B; A_B; C_B)$ for you. Given this belief, you prefer C_y to A_y to B_y.
- Take the cautious lexicographic belief $(C_y; A_y; B_y)$ for Barbara. Given this belief, she A_B to C_B to B_B.
- Take the cautious lexicographic belief $(A_B; C_B; B_B)$ for you.
  This belief is the same as in the fourth bullet above!

Example: Hide and Seek

		Barbara		
		A_B	B_B	C_B
	A_y	$0, 5$	$1, 2$	$1, 1$
You	B_y	$1, 3$	$0, 4$	$1, 1$
	C_y	$1, 3$	$1, 2$	$0, 3$

- Here is the **sequence of lexicographic beliefs** we found:

  $(A_B; B_B; C_B) \rightarrow (C_y; B_y; A_y) \rightarrow (A_B; C_B; B_B)$
  $\rightarrow (B_y; C_y; A_y) \rightarrow (B_B; A_B; C_B) \rightarrow (C_y; A_y; B_y)$
  $\rightarrow (A_B; C_B; B_B)$

- Consider the sub-cycle:

  $(A_B; C_B; B_B) \rightarrow (B_y; C_y; A_y) \rightarrow (B_B; A_B; C_B)$
  $\rightarrow (C_y; A_y; B_y) \rightarrow (A_B; C_B; B_B)$

- Transform this cycle into an epistemic model.

  Type Spaces: $T_{you} = \{t_y, t_y'\}$ and $T_{Barbara} = \{t_B, t_B'\}$

  Beliefs for *You***:** $b_y^{lex}(t_y) = ((A_B, t_B); (C_B, t_B); (B_B, t_B))$ and $b_y^{lex}(t_y') = ((B_B, t_B'); (A_B, t_B'); (C_B, t_B'))$

  Beliefs for *Barbara***:** $b_B^{lex}(t_B) = ((C_y, t_y'); (A_y, t_y'); (B_y, t_y'))$ and $b_B^{lex}(t_B') = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$

- All types in this model are cautious and express 1-fold full belief in caution and respect of preferences.

- Hence, we have found an epistemic model in which all types express common full belief in caution & respect of preferences.

General Construction: Finding a Cycle

- Take a finite game and consider an arbitrary cautious lexicographic belief $b_i(1)$ for player i about j's choice.
- Let $R_i(1)$ be the induced preference relation on C_i for player i given this belief.
- Consider some cautious lexicographic belief $b_j(2)$ for player j about i's choice that respects the preference $R_i(1)$.
- Let $R_j(2)$ be the induced preference relation on C_j for player j given this belief.
- Consider some cautious lexicographic belief $b_i(3)$ for player i about j's choice that respects the preference $R_i(2)$.
- Let $R_i(3)$ be the induced preference relation on C_i for player i given this belief.
- ...
- Every belief in this sequence respects the preference induced by its predecessor.
- Moreover, since there are only finitely many preference relations over each player's choices, and since we can associate any recurring preference with the same lexicographic belief, we can always find a cycle.

General Construction: From Cycles to Types

- Take some cycle of lexicographic beliefs as derived on the previous slide, e.g. $b_i(1) \rightarrow b_j(2) \rightarrow b_i(3) \rightarrow \ldots \rightarrow b_j(K) \rightarrow b_i(1)$
- Transform the cycle into an epistemic model:

  $b_i(t_i^1) = (b_i(1), t_j^K)$

  $b_j(t_j^2) = (b_j(2), t_i^1)$

  $b_i(t_i^3) = (b_i(3), t_j^2)$

  $\ldots$

  $b_j(t_j^K) = (b_j(K), t_i^{K-1})$
- In such an epistemic model, all types are cautious and express 1-fold full belief in caution and respect of preferences.
- Hence, all types express common full belief in (caution & respect of preferences).

Possibility of Common Full Belief in Caution & Respect of Preferences

Theorem

Let Γ be some **finite** *two player game. Then, there exists a lexicographic epistemic model such that every type in the model is cautious and expresses common full belief in caution & respect of preferences.*

Furthermore, we can always construct the model such that every type deems possible only one opponent type, and assigns probability 1 to one of the opponent's choices at each lexicographic level.

Agenda

- Respecting the Opponent’s Preferences
- Common Full Belief in Caution & Respect of Preferences
- Possibility
- **Procedural Characterization**

Elimination of Choices?

- We seek a procedure to determine the choices that are (cautiously) rational under common full belief in caution & respect of preferences.
- So far all procedures we have seen proceed by iteratively eliminating choices from the game.
- We now show that this kind of procedure cannot work for common full belief in (caution & respect of preferences).

Example: Spy Game

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects *you* to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- Question: Which pub should *you* go to?

Example: Spy Game

Barbara

You

$$\begin{array}{cccc} & A_B & B_B & C_B \\ A_y & 0,3 & 1,2 & 1,4 \\ B_y & 1,3 & 0,2 & 1,1 \\ C_y & 1,6 & 1,2 & 0,1 \end{array}$$

Example: Spy Game

- Which choices are (cautiously) rational for you under common full belief in caution & respect of preferences?
- Barbara prefers A_B to B_B.
- Hence, you must deem A_B infinitely more likely than B_B.
- Then, you prefer B_y to A_y.
- Hence, you believe that Barbara deems B_y infinitely more likely than A_y.
- Then, you must believe that Barbara prefers B_B to C_B.
- Consequently, you must deem Barbara's choice B_B infinitely more likely than C_B.
- As you deem A_B infinitely more likely than B_B and B_B infinitely more likely than C_B, you can only rationally choose C_y.

		Barbara		
		A_B	B_B	C_B
	A_y	0, 3	1, 2	1, 4
You	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

Example: Spy Game

- But choice C_y cannot be selected by iterative elimination of dominated choices!

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- B_B is strictly dominated for Barbara. So any version of eliminating dominated choices would discard B_B.

- Also A_B and C_B are both strictly optimal for Barbara for some beliefs. So no version of eliminating dominated choices would discard A_B and C_B. (The same is true for each of Your choices.)

- After eliminating B_B, choice B_y is always at least as good as A_y and C_y. So no version of eliminating dominated choices would discard B_y.

- However, B_y would need to be discarded to arrive at C_y as your unique (cautiously) rational choice under common full belief in caution & respect of preferences in this game!

Example: Spy Game

			Barbara	
		A_B	B_B	C_B
	A_y	$0, 3$	$1, 2$	$1, 4$
You	B_y	$1, 3$	$0, 2$	$1, 1$
	C_y	$1, 6$	$1, 2$	$0, 1$

- What is “wrong” with trying to characterize respect of preferences via choice elimination?

- Eliminating the strictly dominated B_B would mean that A_B and C_B are deemed infinitely more likely than B_B.

- However, only A_B dominates B_B. Comparing C_B to B_B gives us no reason to think Barbara would prefer the former over the latter.

- And in fact, we saw that common full belief in caution & respect of preferences yields the opposite conclusion, requiring you to deem B_B infinitely more likely than C_B.

- How can we avoid “spurious” attributions of preference like in this example?

- We will seek a procedure that ranks choices only based on “direct evidence” from pairwise dominance relationships.

Example: Spy Game

- Barbara prefers A_B to B_B.
- Instead of eliminating choice B_B, introduce a preference restriction $(B_B, \{A_B\})$.
- To respect $(B_B, \{A_B\})$, you must deem A_B infinitely more likely than B_B.

		Barbara		
		A_B	B_B	C_B
	A_y	0, 3	1, 2	1, 4
You	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- So your belief about Barbara's choice follows one of these likelihood orderings:

  $(\{A_B\}, \{B_B\}, \{C_B\})$ $(\{A_B\}, \{C_B\}, \{B_B\})$ $(\{A_B\}, \{B_B, C_B\})$

  $(\{C_B\}, \{A_B\}, \{B_B\})$ $(\{A_B, C_B\}, \{B_B\})$
- Three likelihood orderings assume Barbara's choice A_B, i.e. they deem A_B infinitely more likely than her other choices. The other two assume $\{A_B, C_B\}$.
- B_y weakly dominates A_y on both $\{A_B\}$ and $\{A_B, C_B\}$. So you prefer B_y over A_y for all possible likelihood orderings.
- Since (given Barbara's restriction $(B_B, \{A_B\})$) you prefer B_y over A_y, we can introduce an new preference restriction $(A_y, \{B_y\})$ for you.

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
	A_y	$0, 3$	$1, 2$	$1, 4$
You	B_y	$1, 3$	$0, 2$	$1, 1$
	C_y	$1, 6$	$1, 2$	$0, 1$

- So far there are two preference restrictions: $(A_y, \{B_y\})$ and $(B_B, \{A_B\})$.
- If Barbara respects $(A_y, \{B_y\})$, she must deem B_y infinitely more likely than A_y. So her likelihood ordering assumes either B_y or $\{B_y, C_y\}$.
- B_B weakly dominates C_B on both B_y and $\{B_y, C_y\}$.
- So we get the new preference restriction $(C_B, \{B_B\})$ for Barbara.
- Next, if you respect restrictions $(B_B, \{A_B\})$ and $(C_B, \{B_B\})$, your likelihood ordering must be $(A_B; B_B; C_B)$. In particular, you assume $\{A_B, B_B\}$.
- On $\{A_B, B_B\}$, your choice B_y is weakly dominated by C_y.
- Thus, we get the preference restriction $(B_y, \{C_y\})$.
- With $(A_y, \{B_y\})$ and $(B_y, \{C_y\})$, you must prefer C_y over B_y over A_y.
- So (as seen previously) only C_y is (cautiously) rational for you under common full belief in caution & respect of preferences.

Likelihood Orderings

Definition

A ***likelihood ordering*** for player i on j's choice set is a sequence $L_i = (L_i^1; L_i^2; \ldots; L_i^K)$ of subsets of C_j, such that $(L_i^1; L_i^2; \ldots; L_i^K)$ forms an ordered partition of C_j.

- Likelihood orderings capture all possible rankings over opponent choices induced by the "infinitely more likely than" relation.
- With L_i, Player i deems all choices in L_i^1 infinitely more likely than all choices in L_i^2, deems all choices in L_i^2 infinitely more likely than all choices in L_i^3, . . .
- Whenever L_i deems all choices in $D_j \subseteq C_j$ infinitely more likely than all choices in $C_j \backslash D_j$, we say that L_i assumes D_j.
- Specifically, any given $L_i = (L_i^1; L_i^2; \ldots; L_i^K)$ can be seen to assume $\bigcup_{k=1}^{\ell} L_i^k$ for every $1 \leq \ell \leq K$.

Preference Restrictions

Definition

A ***preference restriction*** for player i is a pair (c_i, A_i), where $c_i \in C_i$ and $A_i \subseteq C_i$.

- (c_i, A_i) means that Player i “prefers” at least one choice in A_i to c_i.
- A likelihood ordering L_i for player i **respects a preference restriction** (c_j, A_j) for opponent j, if L_i deems at least one choice in A_j infinitely more likely than c_j.
- How to formalize “prefer” above?

Lemma

Let $D_j \subseteq C_j$ be a set of choices for opponent j. If a choice c_i for player i is weakly dominated on D_j by a randomized choice r_i over $A_i \subset C_i$, then, for every lexicographic belief b_i that assumes the set of choices D_j, i prefers some choice $c_i' \in A_i$ to c_i.

Proof of Lemma

- Let $c_i \in C_i$ be weakly dominated on $D_j \subseteq C_j$ by randomized choice r_i on $A_i \subset C_i$. Take a lexicographic belief $b_i = (b_i^1; \ldots; b_i^K)$ on C_j such that b_i assumes D_j.
- By definition, b_i deems all choices inside D_j infinitely more likely than all choices outside D_j. So there is a level ℓ such that

  1) $b_i^k(c_j) = 0$ for every $c_j \in C_j \backslash D_j$ and every $k \leq \ell$,

  2) for every $c_j \in D_j$ there exists $k \leq \ell$ such that $b_i^k(c_j) > 0$.
- Since r_i weakly dominates c_i on D_j, we have $u_i^k(c_i, b_i) \leq u_i^k(r_i, b_i)$ for all $k \leq \ell$, and since every $c_j \in D_j$ gets positive probability at some level $k \leq \ell$, there is $\ell^* \leq \ell$ such that $u_i^{\ell^*}(c_i, b_i) < u_i^{\ell^*}(r_i, b_i)$.
- Now by definition, $u_i^k(r_i, b_i) = \sum_{a_i \in A_i} r_i(a_i) u_i^k(a_i, b_i)$.
- Hence, since $u_i^k(c_i, b_i) \leq u_i^k(r_i, b_i)$ for every $k < \ell^*$, we must either have $u_i^k(c_i, b_i) = u_i^k(a_i, b_i)$ for all $a_i \in A_i$ or $u_i^k(c_i, b_i) < u_i^k(a_i, b_i)$ for at least one $a_i \in A_i$.
- And since $u_i^{\ell^*}(c_i, b_i) < u_i^{\ell^*}(r_i, b_i)$, we must have $u_i^{\ell^*}(c_i, b_i) < u_i^{\ell^*}(a_i, b_i)$ for at least one $a_i \in A_i$.
- Let $k^* \leq \ell^*$ be the smallest index with $u_i^{k^*}(c_i, b_i) < u_i^{k^*}(a_i^*, b_i)$ for some $a_i^* \in A_i$.
- Then, by construction, we have $u_i^k(c_i, b_i) = u_i^k(a_i^*, b_i)$ for all $k < k^*$ and $u_i^{k^*}(c_i, b_i) < u_i^{k^*}(a_i^*, b_i)$. Hence, i prefers a_i^* over c_i given belief b_i.

Example: Runaway Bride

- *You* are attending *Barbara's* wedding.
- However, when *Barbara* was supposed to say ”yes“, she suddenly changed her mind and ran away.
- *You* would like to find her, and you know that she is hiding in one of the following houses:

$$a \quad \rightleftharpoons \quad b \quad \rightleftharpoons \quad c \quad \rightleftharpoons \quad d \quad \rightleftharpoons \quad e$$

- *Barbara's* mother and grandmother live at a and e, respectively, and they will definitely not open the door for you.
- *Your* utility is 1 if you find her, and 0 otherwise.
- *Barbara's* utility is equal to the distance between the place you look for her and the place she is hiding.

Example: Runaway Bride

		Barbara				
		a_B	b_B	c_B	d_B	e_B
	a_Y	$0, 0$	$0, 1$	$0, 2$	$0, 3$	$0, 4$
	b_Y	$0, 1$	$1, 0$	$0, 1$	$0, 2$	$0, 3$
You	c_Y	$0, 2$	$0, 1$	$1, 0$	$0, 1$	$0, 2$
	d_Y	$0, 3$	$0, 2$	$0, 1$	$1, 0$	$0, 1$
	e_Y	$0, 4$	$0, 3$	$0, 2$	$0, 1$	$0, 0$

Example: Runaway Bride

		Barbara				
		a_B	b_B	c_B	d_B	e_B
	a_Y	$0, 0$	$0, 1$	$0, 2$	$0, 3$	$0, 4$
	b_Y	$0, 1$	$1, 0$	$0, 1$	$0, 2$	$0, 3$
You	c_Y	$0, 2$	$0, 1$	$1, 0$	$0, 1$	$0, 2$
	d_Y	$0, 3$	$0, 2$	$0, 1$	$1, 0$	$0, 1$
	e_Y	$0, 4$	$0, 3$	$0, 2$	$0, 1$	$0, 0$

- Which locations are (cautiously) rational for you under common full belief in caution & respect of preferences?
- $\frac{1}{2}b_B + \frac{1}{2}d_B$ weakly dominates c_B on C_Y.
- So Barbara prefers some choice out of $\{b_B, d_B\}$ over c_B, and we get the preference restriction $(c_B, \{b_B, d_B\})$.
- Similarly, $\frac{3}{4}a_B + \frac{1}{4}e_B$ weakly dominates b_B and $\frac{1}{4}a_B + \frac{3}{4}e_B$ weakly dominates d_B on C_Y, yielding restrictions $(b_B, \{a_B, e_B\})$, $(d_B, \{a_B, e_B\})$.
- For You, b_Y, c_Y, d_Y each dominate a_Y, e_Y. So we get the preference restrictions $(a_Y, \{b_Y\}), (a_Y, \{c_Y\}), (a_Y, \{d_Y\})$ and $(e_Y, \{b_Y\}), (e_Y, \{c_Y\}), (e_Y, \{d_Y\})$.
- Next, if you respect $(c_B, \{b_B, d_B\})$, you must deem b_B or d_B infinitely more likely than c_B. So you assume a set $D_B \subseteq C_B$ that contains b_B or d_B but not c_B.
- Since $r_Y = \frac{1}{2}b_Y + \frac{1}{2}d_Y$ is equally good as c_Y given a_B or e_B and strictly better than c_Y given b_B or d_B, r_Y weakly dominates c_Y on every set D_B that You may assume while respecting $(c_B, \{b_B, d_B\})$.
- Thus, we get the preference restriction $(c_Y, \{b_Y, d_Y\})$.

Example: Runaway Bride

- You prefer $\{b_Y, d_Y\}$ over c_Y and c_Y over a_Y and e_Y. So only b_Y and d_Y can be (cautiously) rational for You under common full belief in caution & respect of preferences.
- Similarly, Barbara prefers $\{a_B, e_B\}$ over b_B and d_B and $\{b_B, d_B\}$ over c_B. So only a_B and e_B can be (cautiously) rational for Barbara under common full belief in caution & respect of preferences.

Barbara

		a_B	b_B	c_B	d_B	e_B
	a_Y	$0, 0$	$0, 1$	$0, 2$	$0, 3$	$0, 4$
	b_Y	$0, 1$	$1, 0$	$0, 1$	$0, 2$	$0, 3$
You	c_Y	$0, 2$	$0, 1$	$1, 0$	$0, 1$	$0, 2$
	d_Y	$0, 3$	$0, 2$	$0, 1$	$1, 0$	$0, 1$
	e_Y	$0, 4$	$0, 3$	$0, 2$	$0, 1$	$0, 0$

- Here is an epistemic model that rationalizes these choices for You and Barbara under common full belief in caution and respect of preferences:

  Type Spaces:
  $T_Y = \{t_y^b, t_y^d\}$ and $T_B = \{t_B^a, t_B^e\}$

  Beliefs for *You*:
  $b_Y(t_Y^b) = ((a_B, t_B^a); (b_B, t_B^a); (e_B, t_B^a); (c_B, t_B^a); (d_B, t_B^a))$
  $b_Y(t_Y^d) = ((e_B, t_B^e); (d_B, t_B^e); (a_B, t_B^e); (c_B, t_B^e); (b_B, t_B^e))$

  Beliefs for *Barbara*:
  $b_B(t_B^a) = ((d_Y, t_Y^d); (c_Y, t_Y^d); (b_Y, t_Y^d); (a_Y, t_Y^d); (e_Y, t_Y^d))$
  $b_B(t_B^e) = ((b_Y, t_Y^b); (c_Y, t_Y^b); (d_Y, t_Y^b); (a_Y, t_Y^b); (e_Y, t_Y^b))$

The Procedure

Basic Idea: Add preference restrictions based on weak dominance relation. Eliminate likelihood orderings for each player according to preference restrictions.

Iterated Addition of Preference Restrictions (Perea, 2011)

- **Step 1.** For every player i, add a preference restriction (c_i, A_i), if c_i is weakly dominated by some randomized choice r_i on A_i.
- **Step k > 1.** For every player i, restrict to likelihood orderings L_i that respect all preference restrictions for the opponent derived up to Step $k-1$. If every such likelihood ordering L_i assumes a set of opponent choices D_j on which c_i is weakly dominated by some randomized choice r_i on A_i, then add a preference restriction (c_i, A_i) for player i.
- A choice $c_i \in C_i$ survives iterated addition of preference restrictions, if it remains unrestricted for all Steps $k \geq 1$.

Note: Since there are finitely many preference restrictions in a finite game, the procedure converges in finitely many steps.

Procedural Characterization

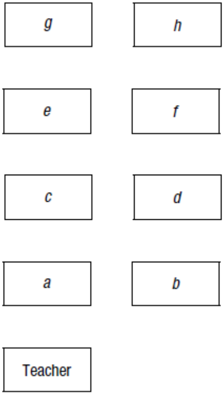
Theorem

For every $k \geq 1$, the choices that can rationally be made by a cautious type that expresses up to k-fold (common) full belief in caution and respect of preferences are exactly the choices that survive $k+1$-fold (iterated) addition of preference restrictions

Note: The procedure is order-independent. That is, any procedure that adds preference restrictions more slowly or in a different order will *eventually* yield the same output. However, only addition of preference restrictions like above will get *all intermediate steps* right.

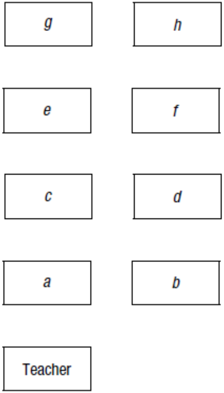
Example: Take a Seat

- *Barbara* and *you* are the only ones to take an exam.
- Both must choose a seat.
- If both choose the same seat, then with probability 0.5 you get the seat you want, and with probability 0.5 you get the one horizontally next to it.
- In order to pass the exam *you* must copy from *Barbara*. The same applies to her.
- A person can only copy from the other if seated horizontally next or diagonally behind them.



Example: Take a Seat

- If You (or Barbara) are able to copy, the probabilities (in %) of not getting caught for the respective seats are $a = 0$, $b = 10$, $c = d = 20$, $e = f = 45$, $g = h = 95$.
- The only objective for You and Barbara is to maximize the chance of successfully copying from the other player.
- Which seats are (cautiously) rational for you under common full belief in caution & respect of preferences?



Example: Take a Seat

		Barbara							
		a_B	b_B	c_B	d_B	e_B	f_B	g_B	h_B
You	a_Y	5, 5	0, 10	0, 0	0, 20	0, 0	0, 0	0, 0	0, 0
	b_Y	10, 0	5, 5	0, 20	0, 0	0, 0	0, 0	0, 0	0, 0
	c_Y	0, 0	20, 0	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
	d_Y	20, 0	0, 0	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
	e_Y	0, 0	0, 0	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
	f_Y	0, 0	0, 0	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
	g_Y	0, 0	0, 0	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
	h_Y	0, 0	0, 0	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

Example: Take a Seat

Barbara

You

	a_B	b_B	c_B	d_B	e_B	f_B	g_B	h_B
a_Y	5, 5	0, 10	0, 0	0, 20	0, 0	0, 0	0, 0	0, 0
b_Y	10, 0	5, 5	0, 20	0, 0	0, 0	0, 0	0, 0	0, 0
c_Y	0, 0	20, 0	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
d_Y	20, 0	0, 0	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
e_Y	0, 0	0, 0	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
f_Y	0, 0	0, 0	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
g_Y	0, 0	0, 0	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
h_Y	0, 0	0, 0	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

Step 1. Rationality & Caution

- a_Y is weakly dominated by b_Y and b_Y is weakly dominated by $\frac{1}{2}c_Y + \frac{1}{2}d_Y$ on C_B. With symmetry, the same is true for Barbara.
- So we get the preference restrictions $(a_Y, \{b_Y\})$ and $(b_Y, \{c_Y, d_Y\})$, as well as $(a_B, \{b_B\})$ and $(b_B, \{c_B, d_B\})$.

Example: Take a Seat

		Barbara					
		c_B	d_B	e_B	f_B	g_B	h_B
You	a_Y	0, 0	0, 20	0, 0	0, 0	0, 0	0, 0
	b_Y	0, 20	0, 0	0, 0	0, 0	0, 0	0, 0
	c_Y	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
	d_Y	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
	e_Y	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
	f_Y	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
	g_Y	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
	h_Y	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

Round 2. 1-Fold Full Belief in Caution and Respect of Preferences

- If you respect $(a_B, \{b_B\})$ and $(b_B, \{c_B, d_B\})$, then (using transitivity) you must assume a set $D_B \subset C_B$ containing c_B or d_b but neither b_B nor a_B.
- $\frac{1}{2}e_Y + \frac{1}{2}f_Y$ strictly dominates $c_Y,\ d_Y$ on $\{c_B, d_B\}$ and $\frac{1}{2}e_Y + \frac{1}{2}f_Y$ weakly dominates $c_Y,\ d_y$ on $\{e_B, f_B, g_B, h_B\}$. So you prefer some choice in $\{e_Y, f_Y\}$ to c_Y on all of the sets D_B, yielding restrictions $(c_Y, \{e_Y, f_Y\})$, $(d_Y, \{e_Y, f_Y\})$ and (with symmetry) $(c_B, \{e_B, f_B\})$, $(d_B, \{e_B, f_B\})$.
- Further new restrictions? $(b_Y, \{c_Y\})$, $(b_Y, \{d_Y\})$, $(b_B, \{c_B\})$, $(b_B, \{d_B\})$.

Example: Take a Seat

Barbara

		b_B	c_B	d_B	e_B	f_B	g_B	h_B
You	a_Y	$0, 10$	$0, 0$	$0, 20$	$0, 0$	$0, 0$	$0, 0$	$0, 0$
	b_Y	$5, 5$	$0, 20$	$0, 0$	$0, 0$	$0, 0$	$0, 0$	$0, 0$
	c_Y	$20, 0$	$20, 20$	$20, 20$	$0, 0$	$0, 45$	$0, 0$	$0, 0$
	d_Y	$0, 0$	$20, 20$	$20, 20$	$0, 45$	$0, 0$	$0, 0$	$0, 0$
	e_Y	$0, 0$	$0, 0$	$45, 0$	$45, 45$	$45, 45$	$0, 0$	$0, 95$
	f_Y	$0, 0$	$45, 0$	$0, 0$	$45, 45$	$45, 45$	$0, 95$	$0, 0$
	g_Y	$0, 0$	$0, 0$	$0, 0$	$0, 0$	$95, 0$	$95, 95$	$95, 95$
	h_Y	$0, 0$	$0, 0$	$0, 0$	$95, 0$	$0, 0$	$95, 95$	$95, 95$

Round 2. 1-Fold Full Belief in Caution and Respect of Preferences

- If you respect preference restriction $(a_B, \{b_B\})$, you must assume a set $D_B \subset C_B$ which contains b_B but not a_B.
- Since c_Y strictly dominates d_Y given b_B and since c_Y weakly dominates d_Y given $\{c_B, d_B, e_B, f_B, g_B, h_B\}$, we get the preference restrictions $(d_Y, \{c_Y\})$ and (with symmetry) $(d_B, \{c_B\})$.

Example: Take a Seat

		Barbara			
		e_B	f_B	g_B	h_B
You	a_Y	$0, 0$	$0, 0$	$0, 0$	$0, 0$
	b_Y	$0, 0$	$0, 0$	$0, 0$	$0, 0$
	c_Y	$0, 0$	$0, 45$	$0, 0$	$0, 0$
	d_Y	$0, 45$	$0, 0$	$0, 0$	$0, 0$
	e_Y	$45, 45$	$45, 45$	$0, 0$	$0, 95$
	f_Y	$45, 45$	$45, 45$	$0, 95$	$0, 0$
	g_Y	$0, 0$	$95, 0$	$95, 95$	$95, 95$
	h_Y	$95, 0$	$0, 0$	$95, 95$	$95, 95$

Round 3. 2-Fold Full Belief in Caution and Respect of Preferences

- If you respect $(a_B, \{b_B\})$, $(b_B, \{d_B\})$,$(d_B, \{c_B\})$, and $(c_B, \{e_B,f_B\})$, then you must assume a set $D_B \subset C_B$ containing e_B or f_b but none of a_b, b_B, c_b, d_b.
- $\frac{1}{2}g_Y + \frac{1}{2}h_Y$ strictly dominates e_Y and f_Y given $\{e_B,f_B\}$ and $\frac{1}{2}g_Y + \frac{1}{2}h_Y$ weakly dominates e_Y and f_Y given $\{g_B, h_B\}$. So we get the restrictions $(e_Y, \{g_Y, h_Y\})$, $(f_Y, \{g_Y, h_Y\})$ and (with symmetry) $(e_B, \{g_B, h_B\})$, $(f_B, \{g_B, h_B\})$.
- Any more new restrictions? Yes: $(c_Y, \{e_Y\})$, $(c_Y, \{f_Y\})$, $(c_B, \{e_B\})$, $(c_B, \{f_B\})$.

Example: Take a Seat

		Barbara				
		c_B	e_B	f_B	g_B	h_B
	a_Y	0, 0	0, 0	0, 0	0, 0	0, 0
	b_Y	0, 20	0, 0	0, 0	0, 0	0, 0
	c_Y	20, 20	0, 0	0, 45	0, 0	0, 0
You	d_Y	20, 20	0, 45	0, 0	0, 0	0, 0
	e_Y	0, 0	45, 45	45, 45	0, 0	0, 95
	f_Y	45, 0	45, 45	45, 45	0, 95	0, 0
	g_Y	0, 0	0, 0	95, 0	95, 95	95, 95
	h_Y	0, 0	95, 0	0, 0	95, 95	95, 95

Round 3. 2-Fold Full Belief in Caution and Respect of Preferences

- If you respect $(a_B, \{b_B\})$, $(b_B, \{d_B\})$,$(d_B, \{c_B\})$, then (using transitivity) you must assume a set $D_B \subset C_B$ containing c_B but none of a_b, b_B, d_b.
- f_Y strictly dominates e_Y given c_B and f_Y weakly dominates e_Y given $\{e_b, f_B, g_B, h_B\}$. So we get the restrictions $(e_Y, \{f_Y\})$ and (with symmetry) $(e_B, \{f_B\})$.

Example: Take a Seat

Barbara

		g_B	h_B
You	a_Y	$0, 0$	$0, 0$
	b_Y	$0, 0$	$0, 0$
	c_Y	$0, 0$	$0, 0$
	d_Y	$0, 0$	$0, 0$
	e_Y	$0, 0$	$0, 95$
	f_Y	$0, 95$	$0, 0$
	g_Y	$95, 95$	$95, 95$
	h_Y	$95, 95$	$95, 95$

Round 4. 3-Fold Full Belief in Caution and Respect of Preferences

- If you respect $(a_B, \{b_B\})$, $(b_B, \{d_B\})$,$(d_B, \{c_B\})$,$(c_B, \{e_B\})$,$(e_B, \{f_B\})$, and $(f_B, \{g_B, h_B\})$, then you must assume a set $D_B \subset C_B$ containing g_b or h_B but none of Barbara's other choices.
- g_Y and h_y strictly dominate f_Y given $\{g_B, h_b\}$. So we get the restrictions $(f_Y, \{g_Y\})$,$(f_Y, \{h_Y\})$ and (with symmetry) $(f_B, \{g_B\})$,$(f_B, \{h_B\})$.

Example: Take a Seat

		Barbara		
		f_B	g_B	h_B
You	a_Y	$0, 0$	$0, 0$	$0, 0$
	b_Y	$0, 0$	$0, 0$	$0, 0$
	c_Y	$0, 45$	$0, 0$	$0, 0$
	d_Y	$0, 0$	$0, 0$	$0, 0$
	e_Y	$45, 45$	$0, 0$	$0, 95$
	f_Y	$45, 45$	$0, 95$	$0, 0$
	g_Y	$95, 0$	$95, 95$	$95, 95$
	h_Y	$0, 0$	$95, 95$	$95, 95$

Round 4. 3-Fold Full Belief in Caution and Respect of Preferences

- If you respect $(a_B, \{b_B\})$, $(b_B, \{d_B\})$,$(d_B, \{c_B\})$,$(c_B, \{e_B\})$, and $(e_B, \{f_B\})$, then you must assume a set $D_B \subset C_B$ containing f_B but none of a_b, b_B, c_b, d_b, e_b.
- g_Y strictly dominates h_Y given f_B and g_Y weakly dominates h_Y given $\{g_B, h_B\}$. So we get the restrictions $(h_Y, \{g_Y\})$ and (with symmetry) $(h_B, \{g_B\})$.

Example: Take a Seat

Common Full Belief in Caution and Respect of Preferences

- Combining all restrictions, You are committed to the likelihood ordering $(\{g_B\}; \{h_B\}; \{f_B\}; \{e_B\}; \{c_B\}; \{d_B\}; \{b_B\}; \{a_B\})$ and g_Y is uniquely (cautiously) rational for You. After relabeling, the same is true for Barbara.
- **Epistemic model**:

  Type Spaces: $T_Y = \{t_Y\}$ and $T_B = \{t_B\}$

  Beliefs for *You*:
  $b_Y(t_Y) = ((g_B, t_B); (h_B, t_B); (f_B, t_B); (e_B, t_B); (c_B, t_B); (d_B, t_B); (b_B, t_B); (a_B, t_B))$

  Beliefs for *Barbara*:
  $b_B(t_B) = ((g_Y, t_Y); (h_Y, t_Y); (f_Y, t_Y); (e_Y, t_Y); (c_Y, t_Y); (d_Y, t_Y); (b_Y, t_Y); (a_Y, t_Y))$

Properties of the Procedure: Finding Restrictions

1. **Implications from Primitive Preference Restrictions:**
   – A restriction (c_j, A_j) splits $C_j \backslash \{c_j\}$ into "active" component A_j and "background" component $C_j \backslash \{c_j\}$. Some A_j-choice must rank above c_j. $C_j \backslash \{c_j\}$-choices rank against c_j in any way.

   – (c_j, A_j) implies (c_i, A_i) iff there is a randomized choice r_i on A_i

   s.th. (i) r_i strictly dominates c_i on A_j,

   (ii) r_i weakly dominates c_i on $C_j \backslash \{c_j\}$.

2. **Loosening Restrictions:**
   – With (1), (c_j, A_j) implies (c_j, A'_j) for all $A_j \subseteq A'_j \subseteq C_j \backslash \{c_j\}$.

3. **Transitivity of Restrictions:**
   – Preference restrictions are transitive (like weak dominance):
   $\big((c_j, A'_j) \,\&\, \forall c'_j \in A'_j : (c'_j, A_j)\big) \Rightarrow (c_j, A_j)$.

   – For singleton A'_j, this reduces to regular transitivity.

Properties of the Procedure: Reduced Games

4. **Reduced Games and Combining Restrictions:**
   - Let $\mathcal{R}_j(k)$ be the set of restrictions for player j after step $k \geq 1$. For every $A_j \subset C_j$, define $\underline{\mathcal{R}}_j(A_j, k) = \{c_j | (c_j, A_j) \in \mathcal{R}_j(k)\}$.
   - Every nonempty $\underline{\mathcal{R}}_j(A_j, k)$ defines a reduced game $\Gamma^k_{A_j}$ where player j chooses from $C_j \backslash \underline{\mathcal{R}}_j(A_j, k)$.
   - $\{(c_j, A_j)\}_{c_j \in \underline{\mathcal{R}}_j(A_j,k)}$ jointly imply (c_i, A_i) iff there is a randomized choice r_i on A_i s.th. (i) r_i strictly dominates c_i on A_j,
     (ii) r_i weakly dominates c_i in $\Gamma^k_{A_j}$.
   - Using (ii), $\Gamma^k_{A_j}$ is maximally informative at k for every $A_j \subset C_j$. I.e., checking $\Gamma^k_{A_j}$ yields all restrictions we may derive from $\{(c_j, A_j)\}_{c_j \in \underline{\mathcal{R}}_j(A_j,k)}$.

Properties of the Procedure: Reduced Games

5. **Overlapping Restrictions:**
   - Note that $C_j \backslash (\underline{\mathcal{R}}_j(A_j, k) \cup \underline{\mathcal{R}}_j(A'_j, k))$ is non-empty for non-empty $\underline{\mathcal{R}}_j(A_j, k)$ and $\underline{\mathcal{R}}_j(A'_j, k)$. (If not, all $c_j \in C_j$ would be restricted, a contradiction).
   - Overlap limits what is learned from checking additional restrictions:
     Let (c_i, A_i) follow from $\Gamma^k_{A_j^{m+1}}$ but not from $\Gamma^k_{A_j^1}, \ldots, \Gamma^k_{A_j^m}$.
     Then there is randomized choice r_i on A_i s.th. r_i is at least indifferent to c_i on $C_j \backslash (\underline{\mathcal{R}}_j(A_j^{m+1}, k) \cup \bigcap_{\ell=1}^m \underline{\mathcal{R}}_j(A_j^\ell, k))$.

Related Concept: Proper Equilibrium (Myerson 78)

Proper Equilibrium with Lexicographic Beliefs (Brandenburger et al., 1991)

A pair of beliefs $(\sigma_1, \sigma_2) \in \Delta(C_1) \times \Delta(C_2)$ constitutes a proper equilibrium, if there exists a pair of cautious lexicographic beliefs (b_1, b_2) such that $b_1^1 = \sigma_2$ as well as $b_2^1 = \sigma_1$ and for all $c_i, c_i' \in C_i$, if $u_i(c_i, b_i) < u_i(c_i', b_i)$, then b_j deems c_i infinitely less likely than c_i'.

- **Epistemic Conditions** (Exercise 6.9):
  1) Common full belief in caution & respect of preferences
  2) “Full correct beliefs” (lexicographic version of simple belief hierarchies)
- This show that respect of preferences embodies Myerson’s original properness concepts after stripping away equilibrium conditions.
- Note that surviving iterated addition of preference restrictions is only necessary for a Nash equilibrium to be proper.