

Lexicographic Beliefs

Part I: Primary Belief in Rationality

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Introduction

- Thus far, a player's **belief** about his opponents' choices has been modelled by a **probability distribution**.
- Ways of reasoning have been described in which some choices are **completely discarded** by receiving probability 0.
- Now, **cautious reasoning** is considered: some choices can be deemed **much more likely** than others, while at the same time **no choice is completely discarded**.
- Tool used to model **cautious reasoning** in **Epistemic GT**:

lexicographic beliefs

Introduction

- In **Classical GT** the idea of **cautious reasoning** is modelled by **converging sequences of (full support) mixed choices**.
- **Example**: suppose a game where some player i chooses between three choices a , b , and c .

- **Caution** modelled **classically**:

$$\left(\left(1 - \frac{1}{n} - \frac{1}{n^2} \right) \cdot a + \frac{1}{n} \cdot b + \frac{1}{n^2} \cdot c \right)_{n \in \mathbb{N}}$$

- **Caution** modelled **epistemically**:

$$(a; b : c)$$

- Intuitively, the **epistemic** model of **caution** could be seen as a **one shot representation** of the **classical** model of **caution**.
- For details of how to go from “**epistemic caution**” to “**classical caution**” and vice versa: Blume et al. (1991a) and (1991b).

Introduction

Three ways of **cautious reasoning** based on **lexicographic beliefs** are presented in this part of the course:

1 Common Primary Belief in (Caution & Rationality)

(Brandenburger, 1992; Börgers, 1994)

- **Classical Analogue:** [Dekel-Fudenberg-Procedure](#) (Dekel & Fudenberg, 1990)
- **Related Equilibrium Concept:** [Perfect Equilibrium](#) (Selten, 1975)

2 Common Full Belief in (Caution & Respect of Preferences)

(Schuhmacher, 1999; Asheim, 2001)

- **Classical Analogue:** [Iterated Addition of Preference Restrictions](#) (Perea, 2011)
- **Related Equilibrium Concept:** [Proper Equilibrium](#) (Myerson, 1978)

3 Common Assumption of Rationality

(Brandenburger et al., 2008)

- **Classical Analogue:** [Iterated Weak Dominance](#) (Luce & Raiffa, 1957)
- **Related Equilibrium Concept:** none in the literature

Agenda

- Lexicographic Beliefs
- Lexicographic Epistemic Models
- Common Primary Belief in (Caution & Rationality)
- Existence
- Algorithm

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Example: Should I call or not?

Story

- Tonight *Barbara* will go to the cinema.
- *You* can join if you wish, but *Barbara* decides on the movie.
- There is the choice between *The Godfather* and *Casablanca*.
- *You* prefer *The Godfather* (utility 1) to *Casablanca* (utility 0).
- *Barbara*'s movie preferences are inverse to yours.
- Staying at home yields *you* utility 0.
- *Barbara* goes to the cinema in any case.
- **Question:** Should *you* call *Barbara* or not?

Example: Should I call or not?

		<i>Barbara</i>	
		<i>Godfather</i>	<i>Casablanca</i>
<i>You</i>	<i>call</i>	1, 0	0, 1
	<i>not call</i>	0, 0	0, 1

Example: Should I call or not?

		<i>Barbara</i>	
		<i>Godfather</i>	<i>Casablanca</i>
<i>You</i>	<i>call</i>	1, 0	0, 1
	<i>not call</i>	0, 0	0, 1

- Intuitively, the **unique best choice** for *you* is *call*!
 - **standard beliefs**
 - However, if *you* believe in *Barbara*'s rationality with **standard beliefs**, then *you* must assign **probability 0** to her choice *Godfather*.
 - Consequently, both of your choices would be optimal for *you*.
 - **lexicographic beliefs**
 - A state of mind can be modelled in which *you* deem *Barbara* choosing *Casablanca* **infinitely more likely** than her picking *Godfather*.
 - Yet, the possibility of *Barbara* choosing *Godfather* is not completely discarded.

Example: Should I call or not?

		Barbara	
		Godfather	Casablanca
You	call	1, 0	0, 1
	not call	0, 0	0, 1

- Suppose *you* hold the following **lexicographic belief** on *Barbara's* choice:
 - **primary belief**: *you* believe *Barbara* to choose *Casablanca*.
 - **secondary belief**: *you* believe *Barbara* to choose *Godfather*.
- *You* then deem the event that *Barbara* chooses *Casablanca* **infinitely more likely** than the event that she picks *Godfather*.
 - Yet, given this lexicographic belief, the **unique optimal choice** for *you* is then *call*!

Lexicographic Beliefs

Definition

A **lexicographic belief** on some set S is a finite sequence

$$b^{lex} = (b^1, b^2, \dots, b^k)$$

of distinct probability measures on S , where

- b^1 is called *level-1 belief*,
- b^2 is called *level-2 belief*,
- ...
- b^k is called *level- k belief*.

Remark.

Some authors require the probability measures in b^{lex} to have disjoint supports.

Intuition

- An event can be deemed **infinitely more likely** than another event, without completely discarding the latter!
- **Example:** lexicographic beliefs about the solar system
 - **primary belief:** the earth rotates around the sun
 - **secondary belief:** the sun rotates around the earth
 - **tertiary belief:** the sun and the earth both rotate around a hidden star
- A player i is said to deem an opponent j 's choice c_j **infinitely more likely** than some choice c'_j for j , if c_j receives positive probability at an earlier lexicographic level than c'_j under his lexicographic belief b_i^{lex} .

Example: Where to read my book?

Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- **Question:** Which pub should *you* go to?

Example: Where to read my book?

		<i>Barbara</i>		
		<i>Pub A</i>	<i>Pub B</i>	<i>Pub C</i>
<i>You</i>	<i>Pub A</i>	0,3	1,2	1,1
	<i>Pub B</i>	1,3	0,2	1,1
	<i>Pub C</i>	1,3	1,2	0,1

Example: Where to read my book?

		<i>Barbara</i>		
		<i>Pub A</i>	<i>Pub B</i>	<i>Pub C</i>
<i>You</i>	<i>Pub A</i>	0, 3	1, 2	1, 1
	<i>Pub B</i>	1, 3	0, 2	1, 1
	<i>Pub C</i>	1, 3	1, 2	0, 1

- Intuitively, the **unique best choice** for *you* is *Pub C*, since it is the least preferred pub for *Barbara*!
- However, if *you* believe in *Barbara's* rationality with **standard beliefs**, then *you* must assign **probability 0** to her choosing *Pub B* and *Pub C*.
- Consequently, both *Pub B* and *Pub C* are optimal for *you*.
- Indeed, with **standard beliefs** *you* cannot believe in *Barbara's* rationality, while at the same time deeming her choice *Pub C* less likely than *Pub B*.

Example: Where to read my book?

		Barbara		
		Pub A	Pub B	Pub C
You	Pub A	0, 3	1, 2	1, 1
	Pub B	1, 3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

- **Scenario 1:** Consider the **lexicographic belief** (*Pub A*; *Pub B*; *Pub C*) for *you* about *Barbara's* choice
 - **primary belief:** *you* believe *Barbara* to choose *Pub A*.
 - **secondary belief:** *you* believe *Barbara* to choose *Pub B*.
 - **tertiary belief:** *you* believe *Barbara* to choose *Pub C*.
 - **Interpretation:** *you* deem *Barbara's* choice *Pub A* infinitely more likely than *Pub B* and *Pub B* infinitely more likely than *Pub C*, yet *you* consider all her choices possible.
- Given this lexicographic belief, the **unique optimal choice** for *you* is *Pub C*!

Example: Where to read my book?

		Barbara		
		Pub A	Pub B	Pub C
You	Pub A	0, 3	1, 2	1, 1
	Pub B	1, 3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

- Scenario 2:** Consider the **lexicographic belief** ($Pub A; \frac{1}{3}Pub B + \frac{2}{3}Pub C$) for for *you* about *Barbara's* choice
 - primary belief:** *you* believe *Barbara* to choose *Pub A*.
 - secondary belief:** *you* believe with probability $\frac{1}{3}$ *Barbara* to choose *Pub B* and with probability $\frac{2}{3}$ her to choose *Pub C*.
- Given this lexicographic belief, the **unique optimal choice** for *you* is *Pub B!*

Example: Where to read my book?

		Barbara		
		Pub A	Pub B	Pub C
You	Pub A	0, 3	1, 2	1, 1
	Pub B	1, 3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

- **Scenario 3:** Consider the **lexicographic belief** (*Pub C*; *Pub B*; *Pub A*) for *you* about *Barbara's* choice
 - **primary belief:** *you* believe *Barbara* to choose *Pub C*.
 - **secondary belief:** *you* believe *Barbara* to choose *Pub B*.
 - **tertiary belief:** *you* believe *Barbara* to choose *Pub A*.
- Given this lexicographic belief, the **unique optimal choice** for *you* is *Pub A!*

Expected Utility under Lexicographic Beliefs

- Let $\Gamma = (\{i, j\}, (C_i, C_j), (U_i, U_j))$ be a two player game.
- Suppose that player i entertains a **lexicographic belief** $b_i^{lex} = (b_i^1, b_i^2, \dots, b_i^K)$ about j 's choice.
- For every level $k \in \{1, 2, \dots, K\}$ and for every choice $c_i \in C_i$ the **k -level expected utility** for player i of picking c_i is given by

$$u_i^k(c_i, b_i^{lex}) = \sum_{c_j \in C_j} (b_i^k(c_j) \cdot U_i(c_i, c_j))$$

- Hence, every choice $c_i \in C_i$ for player i induces a **sequence of expected utilities**: **lexicographic expected utility**

$$u_i^{lex}(c_i, b_i^{lex}) = (u_i^1(c_i, b_i^{lex}), u_i^2(c_i, b_i^{lex}), \dots, u_i^K(c_i, b_i^{lex}))$$

Preferences Induced by Lexicographic Beliefs

Definition

A player i with lexicographic belief b_i^{lex} **prefers** some choice c_i to c'_i , if there exists some lexicographic level k such that

- 1 $u_i^k(c_i, b_i^{lex}) > u_i^k(c'_i, b_i^{lex})$ and
- 2 $u_i^l(c_i, b_i^{lex}) = u_i^l(c'_i, b_i^{lex})$ for all lexicographic levels $l < k$.

Useful Fact: Note that the binary relation **prefer** is **transitive** on the respective agent's choice set!

Definition

Given a lexicographic belief b_i^{lex} a choice c_i is called **optimal**, if there exists no choice $c_i^* \in C_i$ such that i prefers c_i^* to c_i .

Rationality under Lexicographic Beliefs

Definition

A choice c_i is called *rational*, if there exists some lexicographic belief b_i^{lex} such that c_i is optimal.

Example: Where to read my book?

		<i>Barbara</i>		
		<i>Pub A</i>	<i>Pub B</i>	<i>Pub C</i>
<i>You</i>	<i>Pub A</i>	0, 3	1, 2	1, 1
	<i>Pub B</i>	1, 3	0, 2	1, 1
	<i>Pub C</i>	1, 3	1, 2	0, 1

- Consider **lexicographic belief** $b_{you}^{lex} = (Pub A; Pub B; Pub C)$
 - under the **primary belief**:
 $u_{you}^1(Pub A, b_{you}^{lex}) = 0$, $u_{you}^1(Pub B, b_{you}^{lex}) = 1$, $u_{you}^1(Pub C, b_{you}^{lex}) = 1$
 - under the **secondary belief**:
 $u_{you}^2(Pub B, b_{you}^{lex}) = 0$, $u_{you}^2(Pub C, b_{you}^{lex}) = 1$
- Hence, *you* prefer *Pub C* to *Pub B*, and *Pub B* to *Pub A*.
- Given b_{you}^{lex} the **unique optimal choice** is *Pub C* for *you*!

Example: Where to read my book?

		Barbara		
		Pub A	Pub B	Pub C
You	Pub A	0, 3	1, 2	1, 1
	Pub B	1, 3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

- Consider **lexicographic belief** $b_{you}^{lex'} = (Pub A; \frac{1}{3}Pub B + \frac{2}{3}Pub C)$
 - under the **primary belief**:
 $u_{you}^1(Pub A, b_{you}^{lex'}) = 0$, $u_{you}^1(Pub B, b_{you}^{lex'}) = 1$, $u_{you}^1(Pub C, b_{you}^{lex'}) = 1$
 - under the **secondary belief**:
 $u_{you}^2(Pub B, b_{you}^{lex'}) = \frac{2}{3}$, $u_{you}^2(Pub C, b_{you}^{lex'}) = \frac{1}{3}$
- Hence, *you* prefer *Pub B* to *Pub C*, and *Pub C* to *Pub A*.
- Given $b_{you}^{lex'}$, the **unique optimal choice** is *Pub B* for *you*!

Example: Where to read my book?

		Barbara		
		Pub A	Pub B	Pub C
You	Pub A	0, 3	1, 2	1, 1
	Pub B	1, 3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

- Consider **lexicographic belief**

$$b_{you}^{lex''} = \left(\frac{1}{2} Pub A + \frac{1}{2} Pub B; \frac{1}{3} Pub B + \frac{2}{3} Pub C \right)$$

- under the **primary belief**:

$$u_{you}^1(Pub A, b_{you}^{lex''}) = \frac{1}{2}, \quad u_{you}^1(Pub B, b_{you}^{lex''}) = \frac{1}{2}, \quad u_{you}^1(Pub C, b_{you}^{lex''}) = 1$$

- under the **secondary belief**:

$$u_{you}^2(Pub A, b_{you}^{lex''}) = 1, \quad u_{you}^2(Pub B, b_{you}^{lex''}) = \frac{2}{3}$$

- Hence, **you** prefer **Pub C** to **Pub A**, and **Pub A** to **Pub B**.
- Given $b_{you}^{lex''}$, the **unique optimal choice** is **Pub C** for **you**!

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Reasoning with Lexicographic Beliefs

- When **reasoning** about his opponents a player does not only entertain a belief about his opponents' choices but also about their beliefs, their beliefs about their opponents' beliefs, etc., i.e. a **full belief hierarchy**.
- A full belief hierarchy with standard beliefs is modelled by types in an **epistemic model**: a **type** induces a **standard belief** about his opponents' **choice-type** combinations.
- Analogously, a full belief hierarchy with lexicographic beliefs is now modelled by types in a **lexicographic epistemic model**: a **type** induces a **lexicographic belief** about his opponents' **choice-type** combinations.

Epistemic Model with Lexicographic Beliefs

Definition

A **lexicographic epistemic model** is a tuple $\mathcal{M}_I = \langle (T_i)_{i \in I}, (b_i^{lex})_{i \in I} \rangle$ such that

- T_i is a set of **types** for player i ,
- every type $t_i \in T_i$ induces a **lexicographic belief** $b_i^{lex}(t_i)$ on the opponents' choice-type combinations $\times_{j \in I \setminus \{i\}} (C_j \times T_j)$.

Formalizing Caution

- **Intuition:** No opponent's choice is excluded from consideration, yet some opponent's choice can be deemed infinitely more likely than some other choice of his.
- A type t_i is said to **deem possible** an opponent's type t_j , whenever there exists some lexicographic level k such that t_j receives positive probability under b_i^k .

Definition

A type t_i is **cautious**, whenever, if t_i deems possible some opponent's type t_j , then t_i also deems possible the choice-type pair (c_j, t_j) for all $c_j \in C_j$.

Interpretation

- Agent i is cautious, if for every mental set-up (“type”) that i deems possible for j to entertain, i does not exclude any feasible act.

Example: Where to read my book?

		<i>Barbara</i>		
		<i>Pub A</i>	<i>Pub B</i>	<i>Pub C</i>
<i>You</i>	<i>Pub A</i>	0, 3	1, 2	1, 1
	<i>Pub B</i>	1, 3	0, 2	1, 1
	<i>Pub C</i>	1, 3	1, 2	0, 1

- Consider the following **lexicographic epistemic model**:

- Type Spaces:**

$$T_{you} = \{t_y, t'_y\}$$

$$T_{Barbara} = \{t_B, t'_B\}$$

- Beliefs for *You*:**

$$b_{you}^{lex}(t_y) = ((Pub\ A, t_B); \frac{1}{3}(Pub\ B, t'_B) + \frac{2}{3}(Pub\ C, t'_B))$$

$$b_{you}^{lex}(t'_y) = (\frac{1}{2}(Pub\ A, t_B) + \frac{1}{2}(Pub\ B, t'_B); (Pub\ C, t'_B))$$

- Beliefs for *Barbara*:**

$$b_{Barbara}^{lex}(t_B) = ((Pub\ A, t_y); \frac{3}{4}(Pub\ A, t'_y) + \frac{1}{4}(Pub\ C, t_y))$$

$$b_{Barbara}^{lex}(t'_B) = ((Pub\ A, t'_y); (Pub\ B, t_y); (Pub\ C, t'_y))$$

- No type in this lexicographic epistemic model is cautious!

Example: Where to read my book?

- A **lexicographic epistemic model** with a cautious type for *you*:

- **Type Spaces:**

$$T_{you} = \{t_y, t'_y, t''_y\}$$

$$T_{Barbara} = \{t_B, t'_B\}$$

- **Beliefs for *You*:**

$$b_{you}^{lex}(t_y) = ((Pub\ A, t_B); \frac{1}{3}(Pub\ B, t'_B) + \frac{2}{3}(Pub\ C, t'_B))$$

$$b_{you}^{lex}(t'_y) = (\frac{1}{2}(Pub\ A, t_B) + \frac{1}{2}(Pub\ B, t'_B); (Pub\ C, t'_B))$$

$$b_{you}^{lex}(t''_y) = ((Pub\ A, t_B); (Pub\ A, t'_B); \frac{1}{3}(Pub\ B, t_B) + \frac{2}{3}(Pub\ C, t'_B); \frac{1}{3}(Pub\ B, t'_B) + \frac{2}{3}(Pub\ C, t_B))$$

- **Beliefs for *Barbara*:**

$$b_{Barbara}^{lex}(t_B) = ((Pub\ A, t_y); \frac{3}{4}(Pub\ A, t'_y) + \frac{1}{4}(Pub\ C, t_y))$$

$$b_{Barbara}^{lex}(t'_B) = ((Pub\ A, t'_y); (Pub\ B, t_y); (Pub\ C, t'_y))$$

- Your type t''_y is cautious!

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Rationality in Lexicographic Epistemic Models

Definition

A choice c_i is called *rational*, if there exists some lexicographic epistemic model \mathcal{M}_l with a type t_i such that c_i is optimal for the induced lexicographic first-order belief of t_i .

Being Cautious and Believing in Rationality

- Caution and belief in the opponents' rationality at all lexicographic levels is generally impossible!
- Indeed, caution requires every choice – including non-rational ones (i.e. choices that are not optimal for any belief) – to receive positive probability at some lexicographic level.

Primary Belief in Rationality

- A type t_i is said to primarily believe in some property, if t_i 's primary belief only assigns positive probability to j 's choice-type pairs that satisfy this property.

Definition

A type t_i **primarily believes in rationality**, whenever t_i 's level-1 belief only assigns positive probability to opponent choice-type pairs (c_j, t_j) such that c_j is optimal for t_j .

- **Remark.**

Note that no conditions are put on any lexicographic level deeper than the primary one!

Example: Where to read my book?

		<i>Barbara</i>		
		<i>Pub A</i>	<i>Pub B</i>	<i>Pub C</i>
<i>You</i>	<i>Pub A</i>	0, 3	1, 2	1, 1
	<i>Pub B</i>	1, 3	0, 2	1, 1
	<i>Pub C</i>	1, 3	1, 2	0, 1

- **Type Spaces:**

$$T_{\text{you}} = \{t_y, t'_y\}$$

$$T_{\text{Barbara}} = \{t_B, t'_B\}$$

- **Beliefs for *You*:**

$$b_{\text{you}}(t_y) = ((\text{Pub A}, t_B); \frac{1}{3}(\text{Pub B}, t'_B) + \frac{2}{3}(\text{Pub C}, t'_B))$$

$$b_{\text{you}}(t'_y) = (\frac{1}{2}(\text{Pub A}, t_B) + \frac{1}{2}(\text{Pub B}, t'_B); (\text{Pub C}, t'_B))$$

- **Beliefs for *Barbara*:**

$$b_{\text{Barbara}}(t_B) = ((\text{Pub B}, t_y); \frac{3}{4}(\text{Pub A}, t'_y) + \frac{1}{4}(\text{Pub C}, t_y))$$

$$b_{\text{Barbara}}(t'_B) = ((\text{Pub A}, t'_y); (\text{Pub B}, t_y); (\text{Pub C}, t'_y))$$

- If *you* primarily believe in *Barbara's* rationality, then your primary belief must only assign positive probability to *Barbara's* choice *Pub A*.
- Type t_y primarily believes in *Barbara's* rationality and t'_y does not.
- Type t_B primarily believes in *your* rationality and t'_B does not.

Common Primary Belief in (Caution & Rationality)

Definition

A type t_i expresses **common primary belief in (caution & rationality)**, whenever

- t_i expresses **1-fold primary belief** in (caution & rationality), i.e. t_i primarily believes in j 's caution and rationality, i.e. primarily only deems possible choice type pairs (c_j, t_j) such that t_j is cautious and c_j is optimal for t_j ,
- t_i expresses **2-fold primary belief** in (caution & rationality), i.e. t_i primarily only deems possible types t_j that express 1-fold primary belief in (caution & rationality),
- t_i expresses **3-fold primary belief** in (caution & rationality), i.e. t_i primarily only deems possible types t_j that express 2-fold primary belief in (caution & rationality),
- etc.

Note that all restrictions on the belief hierarchies are put on the *first lexicographic level*.

Example: Should I call or not?

		Barbara	
		Godfather	Casablanca
You	call	1, 0	0, 1
	not call	0, 0	0, 1

- **Type Spaces:**

$$T_{\text{you}} = \{t_y\}$$

$$T_{\text{Barbara}} = \{t_B\}$$

- **Beliefs for You:**

$$b_{\text{you}}(t_y) = ((\text{Casablanca}, t_B); (\text{Godfather}, t_B))$$

- **Beliefs for Barbara:**

$$b_{\text{Barbara}}(t_B) = ((\text{call}, t_y); (\text{not call}, t_y))$$

- If you are **cautious** then your only optimal choice is *call*.
- Your type t_y is **cautious** – thus *call* is optimal for him – and expresses **common primary belief in (caution & rationality)**.
- Hence, you can **rationally** and **cautiously** choose *call* under **common primary belief in (caution & rationality)**.

Example: Where to read my book?

		<i>Barbara</i>		
		<i>Pub A</i>	<i>Pub B</i>	<i>Pub C</i>
<i>You</i>	<i>Pub A</i>	0, 3	1, 2	1, 1
	<i>Pub B</i>	1, 3	0, 2	1, 1
	<i>Pub C</i>	1, 3	1, 2	0, 1

- If you **primarily believe in Barbara's rationality**, then your primary belief must assign probability 1 to Barbara's choice *Pub A*.
- Hence, *Pub A* cannot be optimal for you.
- Which of your remaining choices – *Pub B* and *Pub C* – can you **rationally** choose under **caution** and **common primary belief in (caution & rationality)**?

Example: Where to read my book?

		Barbara		
		Pub A	Pub B	Pub C
You	Pub A	0, 3	1, 2	1, 1
	Pub B	1, 3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

- **Type Spaces:**

$$T_{\text{you}} = \{t_y\}$$

$$T_{\text{Barbara}} = \{t_B\}$$

- **Beliefs for You:**

$$b_{\text{you}}(t_y) = ((\text{Pub A}, t_B); \frac{1}{3}(\text{Pub B}, t_B) + \frac{2}{3}(\text{Pub C}, t_B))$$

- **Beliefs for Barbara:**

$$b_{\text{Barbara}}(t_B) = ((\text{Pub B}, t_y); \frac{1}{2}(\text{Pub A}, t_y) + \frac{1}{2}(\text{Pub C}, t_y))$$

- Your type t_y is **cautious** and expresses **common primary belief in (caution & rationality)**.
- Your choice *Pub B* is optimal for type t_y .
- Hence, you can **rationally** and **cautiously** choose *Pub B* under **common primary belief in (caution & rationality)**.

Example: Where to read my book?

		Barbara		
		Pub A	Pub B	Pub C
You	Pub A	0, 3	1, 2	1, 1
	Pub B	1, 3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

- **Type Spaces:**

$$T_{\text{you}} = \{t_y\}$$

$$T_{\text{Barbara}} = \{t_B\}$$

- **Beliefs for You:**

$$b_{\text{you}}(t_y) = ((\text{Pub A}, t_B); \frac{2}{3}(\text{Pub B}, t_B) + \frac{1}{3}(\text{Pub C}, t_B))$$

- **Beliefs for Barbara:**

$$b_{\text{Barbara}}(t_B) = ((\text{Pub C}, t_y); \frac{1}{2}(\text{Pub A}, t_y) + \frac{1}{2}(\text{Pub B}, t_y))$$

- Your type t_y is **cautious** and expresses **common primary belief in (caution & rationality)**.
- Your choice *Pub C* is optimal for type t_y .
- Hence, you can **rationally** and **cautiously** choose *Pub C* under **common primary belief in (caution & rationality)**.

Agenda

- Lexicographic Beliefs
- Lexicographic Epistemic Models
- Common Primary Belief in (Caution & Rationality)
- **Existence**
- Algorithm

A Way of Cautious Reasoning

- A lexicographic cautious way of reasoning – **Common Primary Belief in (Caution & Rationality)** – has been introduced.
- Accordingly, a type
 - **primarily only deems possible** choice type pairs such that the type is **cautious** and the choice is **optimal** for the type,
[= 1-fold primary belief in (caution & rationality)]
 - **primarily only deems possible** opponent types that **primarily only deem possible** choice type pairs such that the type is **cautious** and the choice is **optimal** for the type,
[= 2-fold primary belief in (caution & rationality)]
 - **only primarily deems possible** opponent types that **primarily only deem possible** opponent types that **primarily only deem possible** choice type pairs such that the type is **cautious** and the choice is **optimal** for the type,
[= 3-fold primary belief in (caution & rationality)]
 - etc.
- **Two remaining key questions:**
existence and **algorithmic characterization**

Example: Hide and Seek

Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but *you* forgot to ask her to which one.
- *You* would like to avoid *Barbara*, in order to enjoy reading your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*, and would also like to talk to *you*.
- **Question:** Which pub should *you* go to?

Example: Hide and Seek

		<i>Barbara</i>		
		<i>Pub A</i>	<i>Pub B</i>	<i>Pub C</i>
<i>You</i>	<i>Pub A</i>	0,5	1,2	1,1
	<i>Pub B</i>	1,3	0,4	1,1
	<i>Pub C</i>	1,3	1,2	0,3

Example: Hide and Seek

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 5	1, 2	1, 1
	B_y	1, 3	0, 4	1, 1
	C_y	1, 3	1, 2	0, 3

Is **common primary belief in (caution & rationality)** possible in this game?

- Consider some arbitrary cautious lexicographic belief for *you* about Barbara's choice, e.g. $(A_B; B_B; C_B)$.
- Given this belief, the choice C_y is optimal for *you*.
- Consider the belief $(C_y; A_y; B_y)$ for *Barbara* about your choice.
- Given this belief, the choice A_B is optimal for *Barbara*.
- Consider the belief $(A_B; B_B; C_B)$ for *you* about Barbara's choice.
- A **chain of lexicographic beliefs** has thus been formed which has entered in a cycle:
 $(A_B; B_B; C_B) \rightarrow (C_y; A_y; B_y) \rightarrow (A_B; B_B; C_B)$

Example: Hide and Seek

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 5	1, 2	1, 1
	B_y	1, 3	0, 4	1, 1
	C_y	1, 3	1, 2	0, 3

- The cycle $(A_B; B_B; C_B) \rightarrow (C_y; A_y; B_y) \rightarrow (A_B; B_B; C_B)$ is now transformed into a **lexicographic epistemic model**.
- **Type Spaces:** $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B\}$
- **Beliefs for You:** $b_{you}^{lex}(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$
- **Beliefs for Barbara:** $b_{Barbara}^{lex}(t_B) = ((C_y, t_y); (A_y, t_y); (B_y, t_y))$
- Both types in the epistemic model t_y and t_B are **cautious** and **primarily believe in rationality**.
- Hence, both types t_y and t_B express **common primary belief in (caution & rationality)**.
- Concluding, **common primary belief in (caution & rationality)** is indeed **possible** in the *Hide and Seek* game.

Generalizing the Construction for Existence

- Fix some finite game and consider an arbitrary **cautious lexicographic belief** b_i^{lex1} for player i about j 's choice.
- Let c_i^1 be **optimal** given this belief.
- Consider some **cautious lexicographic belief** b_j^{lex2} for player j about i 's choice such that the **primary belief assigns probability 1 to c_i^1** and also probability 1 to some choice at all deeper levels.
- Let c_j^2 be **optimal** given this belief.
- Consider some **cautious lexicographic belief** b_i^{lex3} for player i about j 's choice such that the **primary belief assigns probability 1 to c_j^2** and also probability 1 to some choice at all deeper levels..
- Let c_i^3 be **optimal** given this belief.
- etc.
- The sequence of lexicographic beliefs thus constructed bears the following property:
The unique **choice** in the support of the **primary belief** of any element of the sequence is **optimal** given the **immediate predecessor lexicographic belief** in the sequence.
- Since there are only **finitely many choices** and the same choices can always be specified for the support of all belief levels beyond level 1, respectively, the **sequence of lexicographic beliefs** must eventually enter into a **cycle of lexicographic beliefs**.

From Lexicographic Beliefs to Types

- Suppose some **cycle of lexicographic beliefs**:

$$b_i^{lex1} \rightarrow b_j^{lex2} \rightarrow b_i^{lex3} \rightarrow \dots \rightarrow b_j^{lexK} \rightarrow b_i^{lex1}$$

- This cycle can be transformed into an **lexicographic epistemic model**:

- $b_i(t_i^1) = (b_i^{lex1}, t_j^K)$, where $b_i^{lex1} = (c_j^K; \dots)$

- $b_j(t_j^2) = (b_j^{lex2}, t_i^1)$, where $b_j^{lex2} = (c_i^1; \dots)$

- $b_i(t_i^3) = (b_i^{lex3}, t_j^2)$, where $b_i^{lex3} = (c_j^2; \dots)$

- $b_j(t_j^4) = (b_j^{lex4}, t_i^3)$, where $b_j^{lex4} = (c_i^3; \dots)$

- etc.

- In such an epistemic model, every type is **cautious** and **primarily believes in rationality**.
- Hence, all types express **common primary belief in (caution & rationality)**!

Existence

Theorem

Let Γ be some finite two player game. Then, *there exists a lexicographic epistemic model* such that

- every type in the model is *cautious* and expresses *common primary belief in (caution & rationality)*,
- every type in the model *deems possible only one opponent's type*, and assigns at each lexicographic level *probability 1 to one of the opponent's choices*.

Agenda

- Lexicographic Beliefs
- Lexicographic Epistemic Models
- Common Full Belief in (Caution & Primary Belief in Rationality)
- Existence
- **Algorithm**

Towards Characterizing Cautious Reasoning

Definition

A choice c_i of player i is **weakly dominated** by some randomized choice $r_i \in \Delta(C_i)$, whenever

- $U_i(c_i, c_j) \leq V_i(r_i, c_j)$ for all $c_j \in C_j$,
- there exists $c_j^* \in C_j$ such that $U_i(c_i, c_j^*) < V_i(r_i, c_j^*)$.

Characterizing Cautious Reasoning

An analogy to **Pearce's Lemma** for lexicographic beliefs:

Theorem

*A choice c_i of player i can **optimally** be chosen under a **cautious lexicographic belief** if and only if c_i is **not weakly dominated** by some randomized choice r_i .*

Randomized Choices and Lexicographic Expected Utility

The k -level expected utility $v_i^k(r_i, b_i^{lex})$ of a randomized choice $r_i \in \Delta(C_i)$ is defined as

$$v_i^k(r_i, b_i^{lex}) := \sum_{c_j \in C_j} b_i^k(c_j) \left(\sum_{c_i \in C_i} (r_i(c_i) \cdot U_i(c_i, c_j)) \right)$$

A basic lemma

Basic-Lemma I

Let I be some index set, $0 \leq \alpha_i \leq 1$ for all $i \in I$ such that $\sum_{i \in I} \alpha_i = 1$, $x \in \mathbb{R}$, and $y_i \in \mathbb{R}$ for all $i \in I$. If $x < \sum_{i \in I} \alpha_i y_i$, then there exists $i^* \in I$ such that $x < y_{i^*}$.

Proof:

- Towards a contradiction suppose that $x \geq y_i$ for all $i \in I$.
- Then, $\alpha_i x \geq \alpha_i y_i$ holds for all $i \in I$.
- It directly follows that $1 \cdot x = \sum_{i \in I} \alpha_i x \geq \sum_{i \in I} \alpha_i y_i$, a contradiction.

A second basic lemma

Basic-Lemma II

Let I be some index set, $0 < \alpha_i < 1$ for all $i \in I$ such that $\sum_{i \in I} \alpha_i = 1$, $x \in \mathbb{R}$, and $y_i \in \mathbb{R}$ for all $i \in I$. If $x \leq \sum_{i \in I} \alpha_i y_i$, then (there exists $i^* \in I$ such that $x < y_{i^*}$) or ($x = y_i$ for all $i \in I$).

Proof:

- By contraposition, suppose that $x \geq y_i$ for all $i \in I$ and that there exists $i' \in I$ such that $x \neq y_{i'}$.
- Then, $x > y_{i'}$.
- As $0 < \alpha_i < 1$ holds for all $i \in I$, it is the case that $\alpha_{i'} x > \alpha_{i'} y_{i'}$ and $\alpha_i x \geq \alpha_i y_i$ for all $i \in I \setminus \{i'\}$.
- It follows that $x = \sum_{i \in I} \alpha_i x > \sum_{i \in I} \alpha_i y_i$.

Proof of the *only if* (\Rightarrow) Direction of the Theorem

- The proof proceeds by contraposition.
- Let $c_i \in C_i$ be weakly dominated by some randomized choice $r_i \in \Delta(C_i)$.
- Thus, $U_i(c_i, c_j) \leq \sum_{c_i \in C_i} (r_i(c_i) \cdot U_i(c_i, c_j))$ for all $c_j \in C_j$ and there exists some choice $c_j^* \in C_j$ such that $U_i(c_i, c_j^*) < \sum_{c_i \in C_i} (r_i(c_i) \cdot U_i(c_i, c_j^*))$.
- Suppose that player i holds some cautious lexicographic belief $b_i^{lex} = (b_i^1, b_i^2, \dots, b_i^K)$.
- Then, for all levels k

$$\sum_{c_j \in C_j} (b_i^k(c_j) \cdot U_i(c_i, c_j)) \leq \sum_{c_j \in C_j} (b_i^k(c_j) \sum_{c_i \in C_i} (r_i(c_i) \cdot U_i(c_i, c_j)))$$

i.e.

$$u_i^k(c_i, b_i^{lex}) \leq \sum_{c_i' \in C_i} r_i(c_i') u_i^k(c_i', b_i^{lex}) = v_i^k(r_i, b_i^{lex}),$$

and, by caution there exists a level k^* such that $c_j^* \in \text{supp}(b_i^{k^*})$ and thus

$$\sum_{c_j \in C_j} (b_i^{k^*}(c_j) \cdot U_i(c_i, c_j)) < \sum_{c_j \in C_j} (b_i^{k^*}(c_j) \sum_{c_i \in C_i} (r_i(c_i) \cdot U_i(c_i, c_j)))$$

i.e.

$$u_i^{k^*}(c_i, b_i^{lex}) < \sum_{c_i' \in C_i} r_i(c_i') u_i^{k^*}(c_i', b_i^{lex}) = v_i^{k^*}(r_i, b_i^{lex}).$$

Proof of the *only if* (\Rightarrow) Direction of the Theorem (continued)

- Consider the set $\text{supp}(r_i) \subseteq C_i$ of i 's choices to which r_i assigns positive probability and level-1 belief b_i^1 .
- Then, by Basic-Lemma II, either **(a)** there exists some $c'_i \in \text{supp}(r_i)$ such that $u_i^1(c_i, b_i^{\text{lex}}) < u_i^1(c'_i, b_i^{\text{lex}})$, or **(b)** $u_i^1(c_i, b_i^{\text{lex}}) = u_i^1(c'_i, b_i^{\text{lex}})$ for all $c'_i \in \text{supp}(r_i)$.
- If case **(a)** holds, then player i prefers c'_i to c_i , and c_i is thus not optimal.
- If case **(b)** holds, i.e., $u_i^1(c_i, b_i^{\text{lex}}) = u_i^1(c'_i, b_i^{\text{lex}})$ for all $c'_i \in \text{supp}(r_i)$, then consider b_i^2 .
- Then, again by Basic-Lemma II, either **(a)** there exists some $c'_i \in \text{supp}(r_i)$ such that $u_i^2(c_i, b_i^{\text{lex}}) < u_i^2(c'_i, b_i^{\text{lex}})$, or **(b)** $u_i^2(c_i, b_i^{\text{lex}}) = u_i^2(c'_i, b_i^{\text{lex}})$ for all $c'_i \in \text{supp}(r_i)$.
- If case **(a)** holds, then $u_i^1(c_i, b_i^{\text{lex}}) = u_i^1(c'_i, b_i^{\text{lex}})$ and $u_i^2(c_i, b_i^{\text{lex}}) < u_i^2(c'_i, b_i^{\text{lex}})$, and consequently player i prefers c'_i to c_i , implying that c_i is not optimal.
- If case **(b)** holds, i.e., $u_i^1(c_i, b_i^{\text{lex}}) = u_i^1(c'_i, b_i^{\text{lex}})$ and $u_i^2(c_i, b_i^{\text{lex}}) = u_i^2(c'_i, b_i^{\text{lex}})$ for all $c'_i \in \text{supp}(r_i)$, then consider b_i^3 .
- etc.
- As $u_i^{k^*}(c_i, b_i^{\text{lex}}) < v_i^{k^*}(r_i, b_i^{\text{lex}})$ there must eventually be some level l' such that – by Basic-Lemma I – it is the case that $u_i^{l'}(c_i, b_i^{\text{lex}}) < u_i^{l'}(c'_i, b_i^{\text{lex}})$ for some $c'_i \in \text{supp}(r_i)$.
- Hence, there exists some choice $c'_i \in \text{supp}(r_i)$ that player i prefers to c_i , and therefore c_i is not optimal.

Towards an Algorithm

It is desirable to **algorithmically characterize** the **choices** under

- rationality (=optimality given the agent's lex. beliefs),
- caution,
- common primary belief in (caution & rationality).

Lexicographic Optimality and Standard Optimality

Lemma

If a choice c_i is lexicographically-optimal given a lexicographic belief b_i^{lex} , then c_i is standard-optimal given b_i^1 .

Proof:

- Towards a contradiction suppose that c_i is lexicographically-optimal given b_i^{lex} , but not standard-optimal given b_i^1 .
- Then, there exists a choice $c_i^* \in C_i$ such that $u_i^1(c_i, b_i^{lex}) = u_i(c_i, b_i^1) < u_i(c_i^*, b_i^1) = u_i^1(c_i^*, b_i^{lex})$.
- However, this contradicts lexicographic optimality of c_i according to which there exists no choice $c_i' \in C_i$ such that $u_i^k(c_i, b_i^{lex}) < u_i^k(c_i', b_i^{lex})$ for some level k and $u_i^k(c_i, b_i^{lex}) = u_i^k(c_i', b_i^{lex})$ for all levels $l < k$.

Step 1

1-fold primary belief in (caution & rationality)

- Which choices can **optimally** and **cautiously** be made under **1-fold primary belief in (caution & rationality)**?
- Suppose that type t_i is cautious and expresses 1-fold primary belief in (caution & rationality).
- Then, by the Theorem, t_i 's primary belief assigns **probability 0** to all **weakly dominated** choices for j .
- Note that due to t_i being **cautious**, t_i cannot optimally choose any **weakly dominated choice** himself.
- Let Γ^1 be the reduced game that remains after **eliminating all weakly dominated choices** from the game: t_i 's primary belief is concentrated on Γ^1 .
- Hence, every **optimal** choice for t_i must be **optimal** for some lexicographic belief with primary belief restricted to Γ^1 , i.e. standard-optimal given the primary belief.
- Thus, by Pearce's Lemma applied to Γ^1 , every **optimal** choice for t_i must **not be strictly dominated** on Γ^1 .
- Let Γ^2 be the reduced game that remains after **eliminating all strictly dominated choices** from Γ^1 .
- Then, every optimal choice for t_i must be in Γ^2 .
- **Conclusion:** If type t_i is **cautious** and expresses **1-fold primary belief in (caution & rationality)**, then every optimal choice for t_i must be in Γ^2 .
- Note that Γ^2 is obtained by **first eliminating all weakly dominated choices**, and then **eliminating all strictly dominated choices**.

Step 2

Up to 2-fold primary belief in (caution & rationality)

- Which choices can **optimally** and **cautiously** be made under **up to 2-fold primary belief in (caution & rationality)**?
- Suppose that type t_i is cautious and expresses up to 2-fold primary belief in (caution & rationality).
- Then, t_i 's primary belief **only assigns positive probability** to choice-type pairs (c_j, t_j) such that c_j is optimal for t_j , and t_j expresses 1-fold primary belief in (caution & rationality).
- From Step 1 it follows that all such choices c_j receiving positive probability by t_i 's primary belief are in Γ^2 .
- As t_i satisfies 1-fold primary belief in (caution & rationality), every optimal choice for t_i is in Γ^2 .
- Hence, every **optimal** choice for t_i must be **optimal** for some lexicographic belief with primary belief restricted to Γ^2 , i.e. standard-optimal given the primary belief.
- Thus, by Pearce's Lemma applied to Γ^2 , every **optimal** choice for t_i must **not be strictly dominated** in Γ^2 .
- Let Γ^3 be the reduced game that remains after **eliminating all strictly dominated choices** from Γ^2 .
- Then, every optimal choice for t_i must be in Γ^3 .
- **Conclusion:** If type t_i is **cautious** and expresses **up to 2-fold primary belief in (caution & rationality)**, then every optimal choice for t_i must be in Γ^3 .
- Note that Γ^3 is obtained by **first eliminating all weakly dominated choices**, and **then applying two-fold strict dominance**.

Algorithm

Definition (Dekel-Fudenberg-Procedure)

Step 1. Eliminate all choices that are weakly dominated in the game.

Step 2. Within the reduced game after Step 1, apply iterated strict dominance.

- The algorithm stops after finitely many steps.
- The algorithm returns a non-empty set.
- The order and speed in which choices are eliminated after Step 1 is not relevant for the set it returns.

Algorithmic Characterization

Theorem

For all $k \geq 1$, the choices that can rationally be made by a cautious type that expresses up to k -fold primary belief in (caution & rationality) are exactly those choices that survive the first $k + 1$ steps of the Dekel-Fudenberg-Procedure.

Corollary

The choices that can rationally be made by a cautious type that expresses common primary belief in (caution & rationality) are exactly those choices that survive the Dekel-Fudenberg-Procedure.

Example: Teaching a Lesson

Story

- It is Friday and your teacher announces a surprise exam for next week.
- You must decide on what day you will start preparing for the exam.
- In order to pass the exam you must study for at least two days.
- For a perfect exam and a subsequent compliment by your father you need to study for at least six days.
- Passing the exam increases your utility by 5.
- Failing the exam increases the teacher's utility by 5.
- Every day you study decreases your utility by 1, but increases the teacher's utility by 1.
- A compliment by your father increases your utility by 4.

Example: Teaching a Lesson

		<i>Teacher</i>				
		<i>Mon</i>	<i>Tue</i>	<i>Wed</i>	<i>Thu</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	2, 3	1, 4	0, 5	3, 6
	<i>Sun</i>	-1, 6	3, 2	2, 3	1, 4	0, 5
	<i>Mon</i>	0, 5	-1, 6	3, 2	2, 3	1, 4
	<i>Tue</i>	0, 5	0, 5	-1, 6	3, 2	2, 3
	<i>Wed</i>	0, 5	0, 5	0, 5	-1, 6	3, 2

Example: Teaching a Lesson

		Teacher				
		Mon	Tue	Wed	Thu	Fri
You	Sat	3, 2	2, 3	1, 4	0, 5	3, 6
	Sun	-1, 6	3, 2	2, 3	1, 4	0, 5
	Mon	0, 5	-1, 6	3, 2	2, 3	1, 4
	Tue	0, 5	0, 5	-1, 6	3, 2	2, 3
	Wed	0, 5	0, 5	0, 5	-1, 6	3, 2

- With standard beliefs under **common belief in rationality** you can rationally choose **any** day.
- With standard beliefs under **common belief in rationality** and a **simple belief hierarchy** you can only rationally pick *Saturday* or *Wednesday*.
- What days can you **rationally** and **cautiously** choose under **common primary belief in (caution & rationality)**?

Example: Teaching a Lesson

		<i>Teacher</i>				
		<i>Mon</i>	<i>Tue</i>	<i>Wed</i>	<i>Thu</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	2, 3	1, 4	0, 5	3, 6
	<i>Sun</i>	-1, 6	3, 2	2, 3	1, 4	0, 5
	<i>Mon</i>	0, 5	-1, 6	3, 2	2, 3	1, 4
	<i>Tue</i>	0, 5	0, 5	-1, 6	3, 2	2, 3
	<i>Wed</i>	0, 5	0, 5	0, 5	-1, 6	3, 2

Step 1.

- Your choice *Wednesday* is weakly dominated by your choice *Saturday*.
- Eliminate your choice *Wednesday* from the original game.

Example: Teaching a Lesson

		<i>Teacher</i>				
		<i>Mon</i>	<i>Tue</i>	<i>Wed</i>	<i>Thu</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	2, 3	1, 4	0, 5	3, 6
	<i>Sun</i>	-1, 6	3, 2	2, 3	1, 4	0, 5
	<i>Mon</i>	0, 5	-1, 6	3, 2	2, 3	1, 4
	<i>Tue</i>	0, 5	0, 5	-1, 6	3, 2	2, 3

Step 2.

- The *teacher's* choice *Thursday* is strictly dominated by *Friday*.
- Eliminate the *teacher's* choice *Friday* from the reduced game after Step 1.

Example: Teaching a Lesson

		<i>Teacher</i>			
		<i>Mon</i>	<i>Tue</i>	<i>Wed</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	2, 3	1, 4	3, 6
	<i>Sun</i>	-1, 6	3, 2	2, 3	0, 5
	<i>Mon</i>	0, 5	-1, 6	3, 2	1, 4
	<i>Tue</i>	0, 5	0, 5	-1, 6	2, 3

Step 3.

- Your choice *Tuesday* is strictly dominated by *Saturday*.
- Eliminate the *your* choice *Tuesday* from the reduced game after Step 2.

Example: Teaching a Lesson

		<i>Teacher</i>			
		<i>Mon</i>	<i>Tue</i>	<i>Wed</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	2, 3	1, 4	3, 6
	<i>Sun</i>	-1, 6	3, 2	2, 3	0, 5
	<i>Mon</i>	0, 5	-1, 6	3, 2	1, 4

Step 4.

- The *teacher's* choice *Wednesday* is strictly dominated by *Friday*.
- Eliminate the *teacher's* choice *Wednesday* from the reduced game after Step 3.

Example: Teaching a Lesson

		<i>Teacher</i>		
		<i>Mon</i>	<i>Tue</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	2, 3	3, 6
	<i>Sun</i>	-1, 6	3, 2	0, 5
	<i>Mon</i>	0, 5	-1, 6	1, 4

Step 5.

- Your choice *Monday* is strictly dominated by *Saturday*.
- Eliminate your choice *Monday* from the reduced game after Step 4.

Example: Teaching a Lesson

		<i>Teacher</i>		
		<i>Mon</i>	<i>Tue</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	2, 3	3, 6
	<i>Sun</i>	-1, 6	3, 2	0, 5

Step 6.

- The *teacher's* choice *Tuesday* is strictly dominated by *Friday*.
- Eliminate the *teacher's* choice *Tuesday* from the reduced game after Step 5.

Example: Teaching a Lesson

		<i>Teacher</i>	
		<i>Mon</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	3, 6
	<i>Sun</i>	-1, 6	0, 5

Step 7.

- Your choice *Sunday* is strictly dominated by *Saturday*.

- Eliminate your choice *Sunday* from the reduced game after Step 6.

Example: Teaching a Lesson

		<i>Teacher</i>	
		<i>Mon</i>	<i>Fri</i>
<i>You</i>	<i>Sat</i>	3, 2	3, 6

Step 8.

- The *teacher's* choice *Monday* is strictly dominated by *Friday*.
- Eliminate the *teacher's* choice *Monday* from the reduced game after Step 7.

The algorithm stops.

		<i>Teacher</i>	
		<i>Fri</i>	
<i>You</i>	<i>Sat</i>	3, 6	

Example: Teaching a Lesson

		Teacher				
		Mon	Tue	Wed	Thu	Fri
You	Sat	3, 2	2, 3	1, 4	0, 5	3, 6
	Sun	-1, 6	3, 2	2, 3	1, 4	0, 5
	Mon	0, 5	-1, 6	3, 2	2, 3	1, 4
	Tue	0, 5	-1, 6	3, 2	2, 3	1, 4
	Wed	0, 5	0, 5	0, 5	-1, 6	3, 2

- **Type Spaces:**

$$T_{\text{you}} = \{t_y\}$$

$$T_{\text{Teacher}} = \{t_B\}$$

- **Beliefs for You:**

$$b_{\text{you}}(t_y) = ((\text{Fri}, t_T); \frac{1}{4}(\text{Mon}, t_T) + \frac{1}{4}(\text{Tue}, t_T) + \frac{1}{4}(\text{Wed}, t_T) + \frac{1}{4}(\text{Thu}, t_T))$$

- **Beliefs for Teacher:**

$$b_{\text{Teacher}}(t_T) = ((\text{Sat}, t_y); \frac{1}{4}(\text{Sun}, t_y) + \frac{1}{4}(\text{Mon}, t_y) + \frac{1}{4}(\text{Tue}, t_y) + \frac{1}{4}(\text{Wed}, t_y))$$

- Your type t_y is **cautious** and expresses **common primary belief in (caution & rationality)**.
- Your choice *Saturday* is optimal for type t_y .
- Hence, you can indeed **cautiously** and **rationally** choose *Saturday* under **common primary belief in (caution & rationality)**.

Related Solution Concept of Perfect Equilibrium (Selten, 1975)

Classical Definition

A pair of mixed choices $(\sigma_i, \sigma_j) \in \Delta(C_i) \times \Delta(C_j)$ constitutes a **perfect equilibrium**, if there exists a converging sequence (σ_i^n, σ_j^n) of **full support** mixed choices with limit (σ_i, σ_j) such that $\sigma_i(c_i) > 0$ implies that c_i is optimal for σ_i^n as well as $\sigma_j(c_j) > 0$ implies that c_j is optimal for σ_i^n for all $n \in \mathbb{N}$.

Epistemic Definition

A pair of beliefs $(\sigma_i, \sigma_j) \in \Delta(C_i) \times \Delta(C_j)$ constitutes a **perfect equilibrium**, if there exists a pair of **cautious** lexicographic beliefs (b_i^{lex}, b_j^{lex}) such that $b_i^1 = \sigma_j$ as well as $b_j^1 = \sigma_i$ and $b_j^1(c_i) > 0$ implies that c_i is optimal for b_i^{lex} as well as $b_i^1(c_j) > 0$ implies that c_j is optimal for b_j^{lex} .

- **Epistemic Conditions** for the **two player case**:

common primary belief in (caution & rationality) + "simple belief hierarchies"

- **Epistemic Conditions** for the general **n player case** (Bach & Cabessa, 2022):

primary belief in caution + knowledge of rationality + common knowledge of lexicographic conjectures
+ conjectural compatibility