EPICENTER Summer Course on Epistemic Game Theory
Chapter 8: Belief in the Opponents' Future Rationality

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Belief Revision

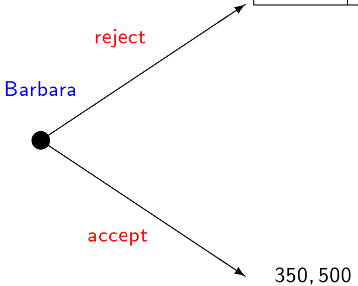
- In a dynamic game, players may choose one after the other.
- Before you make a choice, you may (partially) observe what your opponents have chosen so far.
- It may happen that your initial belief about the opponents' choices will be contradicted later on.
- Then you must revise your belief about the opponents' choices. But how?
- There may be several plausible ways to revise your belief.

Example: Painting Chris' house

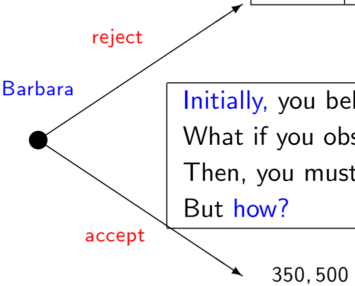
Story

- Chris is planning to paint his house tomorrow, and needs someone to help him.
- You and Barbara are both interested. This evening, both of you must come to Chris' house, and whisper a price in his ear. Price must be either 200, 300, 400 or 500 euros.
- Person with lowest price will get the job. In case of a tie, Chris will toss a coin.
- Before you leave for Chris' house, Barbara gets a phone call from a colleague, who asks her to repair his car tomorrow at a price of 350 euros.
- Barbara must decide whether or not to accept the colleague's offer.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Initially, you believe that Barbara accepts the offer.
What if you observe that she has rejected the offer?
Then, you must revise your belief.
But how?

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250

Barbara

reject

accept

350, 500

Scenario 1: You believe that ...

... rejecting offer was a mistake by Barbara,

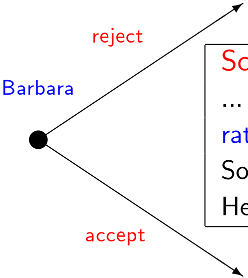
... Barbara will choose rationally from now on

... Barbara believes that you choose rationally.

So, you believe that Barbara chooses 200 or 300.

Hence, you will choose price 200.

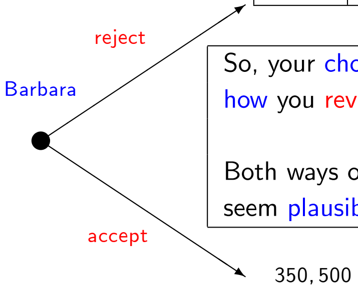
	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



350, 500

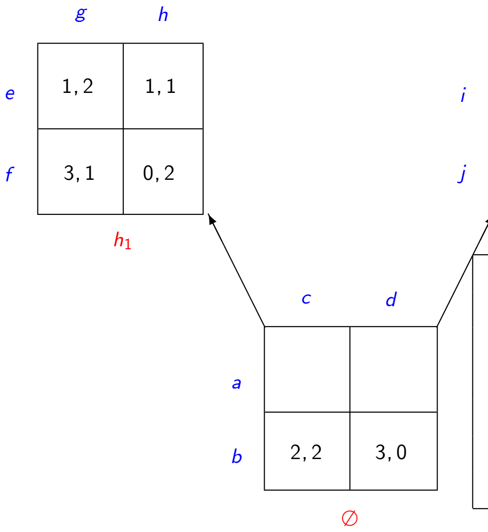
Scenario 2: You believe that ...
... rejecting colleague's offer was a rational choice for Barbara.
So, you believe that Barbara chooses price 400.
Hence, you will choose price 300.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



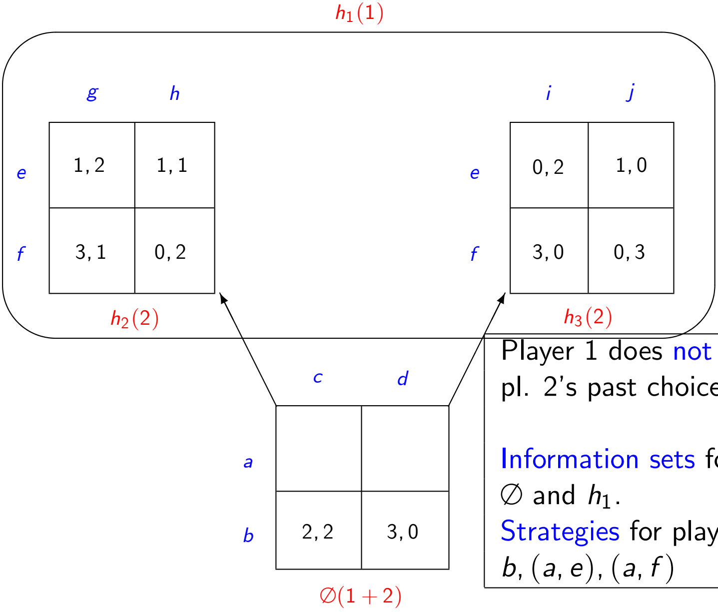
So, your choice crucially depends on how you revise your belief about Barbara.

Both ways of revising your belief seem plausible.



Information sets: $\emptyset$, h_1 and h_2.
Situations where one or more players have to choose.

Strategies for player 1:
b, (a, e, i), (a, e, j), (a, f, i), (a, f, j)
Complete choice plan.



Player 1 does not observe pl. 2's past choice.

Information sets for pl.1: $\varnothing$ and h_1.

Strategies for player 1: $b, (a, e), (a, f)$

- An information set for player i
- is a situation where player i must make a choice,
- describes the information that player i has about the opponents' past choices.

- H_i: collection of information sets for player i.
- $C_i(h)$: set of available choices at information set h.
- At an information set h, more than one player can make a choice.

Definition (Strategy)

A strategy for player i is a function s_i that assigns to each of his information sets $h \in H_i$ some available choice $s_i(h)$, unless h cannot be reached due to some choice $s_i(h')$ at an earlier information set $h' \in H_i$.

In the latter case, no choice needs to be specified at h.

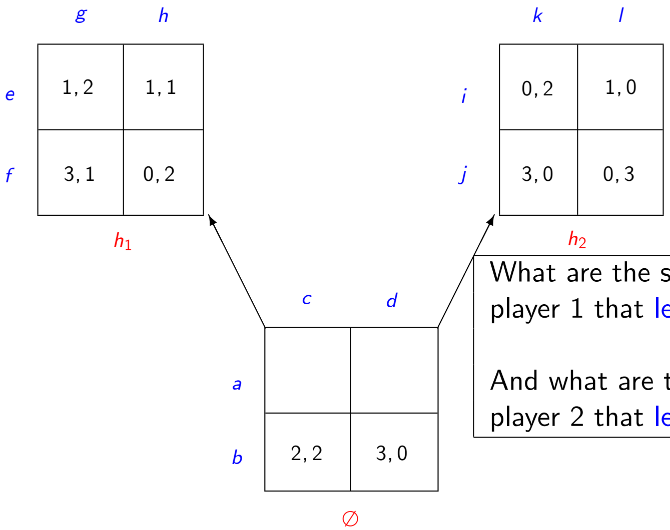
- This is different from the classical definition of a strategy!
- Rubinstein (1991) calls this a plan of action.

Conditional beliefs

- In a dynamic game, a player holds at each of his information sets a conditional belief about the opponents' strategy choices.
- $S_{-i}(h)$: set of opponents' strategy combinations that lead to an information set $h \in H_i$.

Definition (Conditional belief)

A conditional belief vector b_i for player i about the opponents' strategies assigns to every information set $h \in H_i$ some probability distribution $b_i(h) \in \Delta(S_{-i}(h))$ on the opponents' strategy combinations that lead to h.



What are the strategies for player 1 that lead to h_1 ?

And what are the strategies for player 2 that lead to h_1 ?

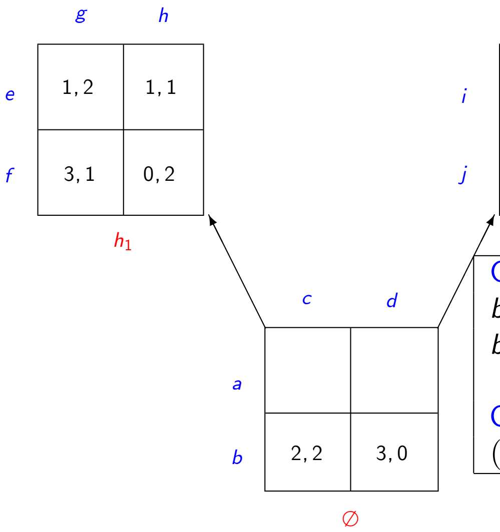
- A strategy s_i is optimal at an information set $h \in H_i$ that s_i leads to, for the belief $b_i(h)$, if

$$u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h))$$

for all strategies s_i' that lead to h.

Definition (Optimal strategy)

A strategy s_i is optimal for the conditional belief vector b_i, if at every information set $h \in H_i$ that s_i leads to, the strategy s_i is optimal for the belief $b_i(h)$.



Conditional belief for pl. 2:
$b_2(\emptyset) = \frac{1}{2}(a, f, j) + \frac{1}{2} b$
$b_2(h_1) = b_2(h_2) = (a, f, j).$

Optimal strategy for pl. 2:
(c, h).

Epistemic model

- We would like to model hierarchies of conditional beliefs.
- That is, we want to model
- the conditional belief that player i has, at every information set $h \in H_i$, about his opponents' strategy choices,
- the conditional belief that player i has, at every information set $h \in H_i$, about the conditional belief that opponent j has, at every information set $h' \in H_j$, about the opponents' strategy choices,
- and so on.

- Hence, in a conditional belief hierarchy you hold, at each of your information sets, a conditional belief about
- the opponents' strategy choices, and
- the opponents' conditional belief hierarchies.
- Like before, call a (conditional) belief hierarchy a type.
- Then, a type for you holds, at each of your information sets, a conditional belief about
- the opponents' strategy choices, and
- the opponents' types.
- This leads to an epistemic model.

Definition (Epistemic model)

An epistemic model for a dynamic game specifies for every player i a set T_i of possible types.

Moreover, every type t_i for player i specifies at every information set $h \in H_i$ a probabilistic belief $b_i(t_i, h)$ over the set $S_{-i}(h) \times T_{-i}$ of opponents' strategy-type combinations.

- Based on Ben-Porath (1997) and Battigalli and Siniscalchi (1999).
- Here, $b_i(t_i, h)$ represents the conditional belief that type t_i holds at information set $h \in H_i$ about the opponents' strategy-type combinations.
- From the epistemic model, we can deduce the complete belief hierarchy for every type.
- A type may revise his belief about the opponents' strategies during the game.
- A type may also revise his beliefs about the opponents' beliefs during the game.

	g	h
e	1, 2	1, 1
f	3, 1	0, 2

h_1

	k	l
i	0, 2	1, 0
j	3, 0	0, 3

h_2

	c	d
a		
b	2, 2	3, 0

$\varnothing$

Types	$T_1 = \{t_1, \hat{t}_1\},\ T_2 = \{t_2, \hat{t}_2\}$
Beliefs for player 1	$$\begin{array}{rcl} b_1(t_1, \varnothing) & = & ((c,h), t_2) \\ b_1(t_1, h_1) & = & ((c,h), t_2) \\ b_1(t_1, h_2) & = & ((d,k), \hat{t}_2) \\ & & \\ b_1(\hat{t}_1, \varnothing) & = & (0.3) \cdot ((c,g), t_2) + (0.7) \cdot ((d,l), \hat{t}_2) \\ b_1(\hat{t}_1, h_1) & = & ((c,g), t_2) \\ b_1(\hat{t}_1, h_2) & = & ((d,l), \hat{t}_2) \end{array}$$
Beliefs for player 2	$$\begin{array}{rcl} b_2(t_2, \varnothing) & = & (b, t_1) \\ b_2(t_2, h_1) & = & ((a,f,i), t_1) \\ b_2(t_2, h_2) & = & ((a,f,i), t_1) \\ & & \\ b_2(\hat{t}_2, \varnothing) & = & ((a,e,j), \hat{t}_1) \\ b_2(\hat{t}_2, h_1) & = & ((a,e,j), \hat{t}_1) \\ b_2(\hat{t}_2, h_2) & = & ((a,e,j), \hat{t}_1) \end{array}$$

Common belief in future rationality

- We would like to extend the idea of common belief in rationality to dynamic games.
- Problem: At certain information sets, it may not be possible to believe that
- opponent has chosen rationally in the past, or
- opponent has chosen rationally in the past, and that the opponent believes that you choose rationally.
- Hence, common belief in rationality at all information sets is in general not possible.
- We must therefore look for a weaker definition of common belief in rationality.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250

reject

Barbara

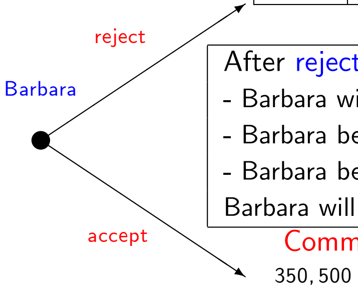
accept

350, 500

After reject, you cannot believe that

- Barbara has chosen rationally in the past,
- Barbara believes that you choose rationally.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



After reject, you can believe that
- Barbara will choose rationally in the future,
- Barbara believes that you will choose rationally,
- Barbara believes that you believe that
Barbara will choose rationally in the future, etc.

Common belief in future rationality.

- You believe in the opponents' future rationality if you always believe that your opponents will make optimal choices at every present and future information set.

Definition (Belief in the opponents' rationality)

Type t_i believes at h that opponent j chooses rationally at h' if his conditional belief $b_i(t_i, h)$ only assigns positive probability to strategy-type pairs (s_j, t_j) for player j where strategy s_j is optimal for type t_j at information set h'.

Definition (Belief in the opponents' future rationality)

Type t_i believes at h in opponent j's future rationality if t_i believes at h that j chooses rationally at every information set h' for player j that weakly follows h.

Type t_i believes in the opponents' future rationality if t_i believes, at every information set h for player i, in every opponent's future rationality.

- Based on Perea (2014). Similar ideas appear in Baltag, Smets and Zvesper (2009), Bonanno (2014) and Penta (2015).
- Common belief in future rationality means that you always believe that
- your opponents will choose rationally now and in the future,
- your opponents always believe that their opponents will choose rationally now and in the future,
- and so on.

Definition (Common belief in future rationality)

(1) Type t_i expresses 1-fold belief in future rationality if t_i believes in the opponents' future rationality.

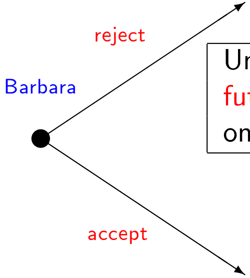
(2) Type t_i expresses 2-fold belief in future rationality if t_i assigns, at every information set $h \in H_i$, only positive probability to opponents' types that express 1-fold belief in future rationality.

And so on.

Type t_i expresses common belief in future rationality if t_i expresses k-fold belief in future rationality for every k.

- Based on Perea (2014).
- Similar concepts can be found in Baltag, Smets and Zvesper (2009), Bonanno (2014), Penta (2015), Dekel, Fudenberg and Levine (1999, 2002) and Asheim and Perea (2005).

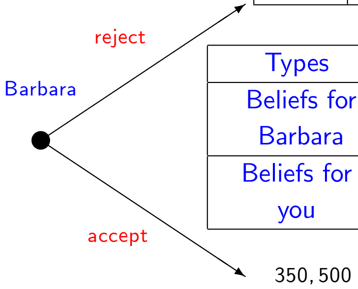
	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



350, 500

Under common belief in future rationality you can only rationally choose 200.

200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Types	$T_1 = \{t_1\}, \ T_2 = \{t_2\}$
Beliefs for Barbara	$b_1(t_1, \varnothing) = (200, t_2)$ $b_1(t_1, h_1) = (200, t_2)$
Beliefs for you	$b_2(t_2, h_1) = ((reject, 200), t_1)$

Both types express common belief in future rationality.

Relation with subgame perfect equilibrium and sequential equilibrium

- In the traditional analysis of dynamic games, subgame perfect equilibrium (Selten (1965)) and sequential equilibrium (Kreps and Wilson (1982)) play a dominant role.
- Subgame perfect equilibrium is defined in terms of behavioral strategies.
- Behavioral strategy σ_i assigns to every information set $h \in H_i$ a probability distribution over the available choices.
- Epistemic interpretation: σ_i represents what others believe about i's future choices in the game.
- Implicitly makes a correct beliefs assumption: You always believe that every opponent is always correct about your beliefs about the opponents' future choices.
- Optimality of behavioral strategies translates to belief in the opponents' future rationality.

- Consider a two-player dynamic game with observed past choices.
- Impose the following correct beliefs assumption: You always believe that your opponent is correct about your beliefs, and you always believe that the opponent always believes that you are correct about his beliefs.
- Then, common belief in future rationality, together with the correct beliefs assumption and Bayesian updating, leads exactly to subgame perfect equilibrium (and sequential equilibrium). See Perea and Predtetchinski (2019) for a proof.
- Research question: Epistemic characterization of subgame perfect equilibrium for more than two players?
- Research question: Epistemic characterization of sequential equilibrium in dynamic games with unobserved past choices?
- Research question: Applications of common belief in future rationality to models in economics, or games with infinite horizon?

Algorithm

- We wish to find those strategies that you can rationally choose under common belief in future rationality.
- Can we construct an algorithm that helps us find these strategies?
- Yes! It will proceed by iteratedly removing strategies at the various information sets in the game.

Step 1: 1-fold belief in future rationality.

- Which strategies can player i rationally choose if he expresses 1-fold belief in future rationality? That is, if he believes in the opponents' future rationality?
- Consider a type t_i that believes in the opponents' future rationality. Then, t_i believes at every information set $h \in H_i$ that opponent j chooses optimally at every information set $h' \in H_j$ that weakly follows h.
- A strategy s_j for player j is optimal at h' for some conditional belief at h', if and only if, s_j is not strictly dominated within the full decision problem $\Gamma^0(h') = (S_j(h'), S_{-j}(h'))$ at h'.
- So, t_i assigns at h only positive probability to j's strategies s_j that are not strictly dominated within any full decision problem $\Gamma^0(h')$ that weakly follows h, and at which j is active.

Step 1: 1-fold belief in future rationality.

- So, t_i assigns at h only positive probability to j's strategies s_j that are not strictly dominated within any full decision problem $\Gamma^0(h')$ that weakly follows h, and at which j is active.
- At every information set $h \in H_i$, delete from the full decision problem $\Gamma^0(h)$ those strategies s_j that are strictly dominated within some full decision problem $\Gamma^0(h')$ that weakly follows h, and at which j is active. This gives the reduced decision problem $\Gamma^1(h)$.
- Hence, type t_i assigns at every information set $h \in H_i$ only positive probability to opponents' strategies in $\Gamma^1(h)$.
- So, every strategy that is optimal for t_i at h, must not be strictly dominated within the reduced decision problem $\Gamma^1(h)$.

Step 1: 1-fold belief in future rationality.

- So, every strategy that is optimal for t_i at h, must not be strictly dominated within the reduced decision problem $\Gamma^1(h)$.
- Let $\Gamma^2(\emptyset)$ be reduced decision problem at $\emptyset$ which is obtained by eliminating, for every player i, those strategies that are strictly dominated within some reduced decision problem $\Gamma^1(h)$ at which i is active.
- Conclusion: Every strategy s_i that is optimal for some type t_i which expresses 1-fold belief in future rationality, must be in $\Gamma^2(\emptyset)$.

Step 2: Up to 2-fold belief in future rationality.

- Which strategies can player i rationally choose if he expresses up to 2-fold belief in future rationality?
- Consider a type t_i that expresses up to 2-fold belief in future rationality. Then, t_i assigns at every $h \in H_i$ only positive probability to opponents' strategy-type pairs (s_j, t_j) where s_j is optimal for t_j at every $h' \in H_j$ that weakly follows h, and t_j expresses 1-fold belief in future rationality.
- We know from Step 1 that every such type t_j assigns at every $h' \in H_j$ only positive probability to opponents' strategies in $\Gamma^1(h')$.
- So, every such strategy s_j above must at every $h' \in H_j$ weakly following h not be strictly dominated within $\Gamma^1(h')$.

Step 2: Up to 2-fold belief in future rationality.

- So, every such strategy s_j above must at every $h' \in H_j$ weakly following h not be strictly dominated within $\Gamma^1(h')$.
- Let $\Gamma^2(h)$ be the reduced decision problem at h which is obtained from $\Gamma^1(h)$ by removing all strategies s_j which are strictly dominated within some $\Gamma^1(h')$ weakly following h, at which j is active.
- Then, type t_i will assign at h only positive probability to strategies s_j in $\Gamma^2(h)$.
- So, every strategy s_i which is optimal for t_i at h must not be strictly dominated within $\Gamma^2(h)$.

Step 2: Up to 2-fold belief in future rationality.

- So, every strategy s_i which is optimal for t_i at h must not be strictly dominated within $\Gamma^2(h)$.
- Let $\Gamma^3(\varnothing)$ be reduced decision problem at $\varnothing$ which is obtained by eliminating, for every player i, those strategies that are strictly dominated within some reduced decision problem $\Gamma^2(h)$ at which i is active.
- Conclusion: Every strategy s_i that is optimal for some type t_i which expresses up to 2-fold belief in future rationality, must be in $\Gamma^3(\varnothing)$.

- Fix an information set h for player i.
- The full decision problem for player i at h is $\Gamma^0(h) = (S_i(h), S_{-i}(h))$, where $S_i(h)$ is the set of strategies for player i that lead to h, and $S_{-i}(h)$ is the set of opponents' strategy combinations that lead to h.
- A reduced decision problem for player i at h is $\Gamma(h) = (D_i(h), D_{-i}(h))$, where $D_i(h) \subseteq S_i(h)$ and $D_{-i}(h) \subseteq S_{-i}(h)$.

Algorithm (Backward dominance procedure)

Step 1. *At every full decision problem* $\Gamma^0(h)$, *eliminate for every player* i *those strategies that are strictly dominated at some full decision problem* $\Gamma^0(h')$ *that weakly follows* h *and at which player* i *is active. This leads to reduced decision problems* $\Gamma^1(h)$ *at every information set* h.

Step 2. *At every reduced decision problem* $\Gamma^1(h)$, *eliminate for every player* i *those strategies that are strictly dominated at some reduced decision problem* $\Gamma^1(h')$ *that weakly follows* h *and at which player* i *is active. This leads to new reduced decision problems* $\Gamma^2(h)$ *at every information set.*

And so on. Continue until no more strategies can be eliminated in this way.

- Based on Perea (2014).

Algorithm (Backward dominance procedure)

Step 1. *At every full decision problem $\Gamma^0(h)$, eliminate for every player i those strategies that are strictly dominated at some full decision problem $\Gamma^0(h')$ that weakly follows h and at which player i is active. This leads to reduced decision problems $\Gamma^1(h)$ at every information set h.*

Step 2. *At every reduced decision problem $\Gamma^1(h)$, eliminate for every player i those strategies that are strictly dominated at some reduced decision problem $\Gamma^1(h')$ that weakly follows h and at which player i is active. This leads to new reduced decision problems $\Gamma^2(h)$ at every information set.*

And so on. Continue until no more strategies can be eliminated in this way.

- The algorithm always stops within finitely many steps.
- At every information set, it yields a nonempty set of strategies for every player.
- The order in which we eliminate strategies – including the order in which we walk through the information sets – is not important for the final result!

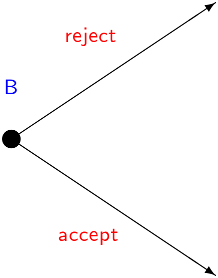
Theorem (Algorithm "works")

(1) For every $k \geq 1$, the strategies that can rationally be chosen by a type that expresses up to k-fold belief in future rationality are exactly the strategies that survive the first $k+1$ steps of the backward dominance procedure at $\emptyset$.

(2) The strategies that can rationally be chosen by a type that expresses common belief in future rationality are exactly the strategies that survive the full backward dominance procedure at $\emptyset$.

- Based on Perea (2014).
- A strategy survives the first $k+1$ steps of the backward dominance procedure at $\emptyset$ if it is in the reduced decision problem $\Gamma^{k+1}(\emptyset)$.
- A strategy survives the full backward dominance procedure at $\emptyset$ if it is in the reduced decision problem $\Gamma^{k}(\emptyset)$ for every k.

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250

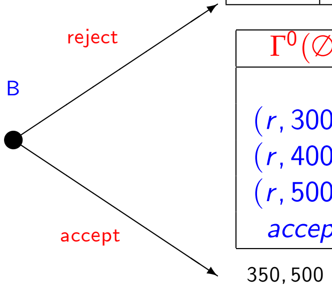


$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

350, 500

Step 1

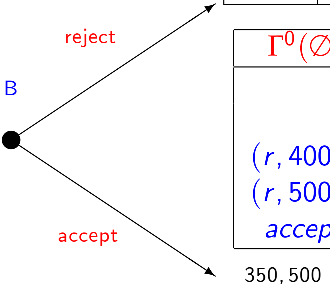
$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Step 1

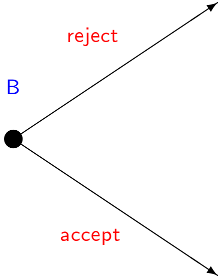
$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

Step 1

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250

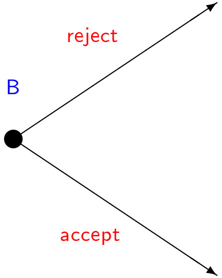


350, 500

$\Gamma^0(\varnothing)$	200	300	400	500
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
accept	350, 500	350, 500	350, 500	350, 500

Step 1

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$

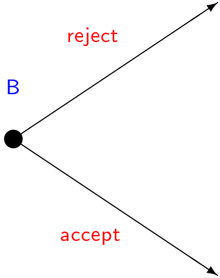


350, 500

$\Gamma^0(\varnothing)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Step 1

$\Gamma^0(h_1)$	200	300	400	
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	

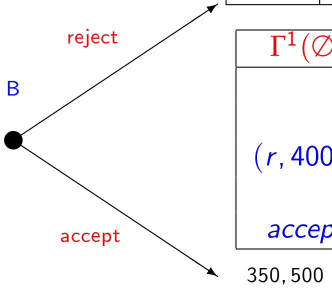


350, 500

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Step 1

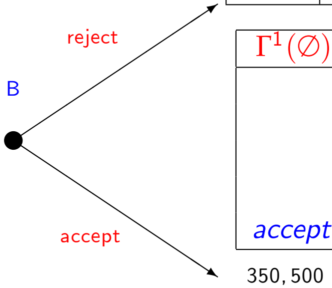
$\Gamma^1(h_1)$	200	300	400	
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	



$\Gamma^1(\varnothing)$	200	300	400
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$
accept	$350, 500$	$350, 500$	$350, 500$

End of Step 1

$\Gamma^1(h_1)$	200	300	400	
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	



$\Gamma^1(\varnothing)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

Step 2

$\Gamma^1(h_1)$	200	300	400	
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	

reject

B

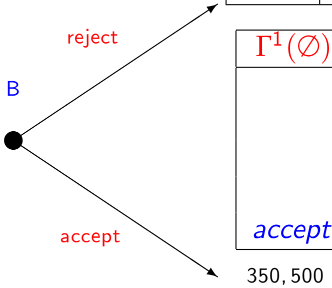
accept

$350, 500$

$\Gamma^1(\emptyset)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

Step 2

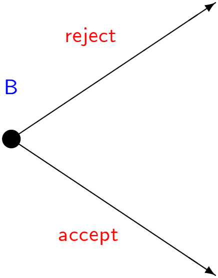
$\Gamma^1(h_1)$	200	300		
$(r, 200)$	$100, 100$	$200, 0$		
$(r, 300)$	$0, 200$	$150, 150$		



$\Gamma^1(\varnothing)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

Step 2

$\Gamma^2(h_1)$	200	300		
$(r, 200)$	$100, 100$	$200, 0$		
$(r, 300)$	$0, 200$	$150, 150$		



$\Gamma^2(\varnothing)$	200	300
accept	$350, 500$	$350, 500$

End of Step 2

$\Gamma^2(h_1)$	200	300		
$(r, 200)$	$100, 100$	$200, 0$		

reject

B

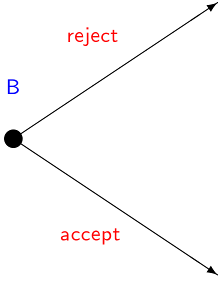
accept

$350, 500$

$\Gamma^2(\varnothing)$	200	300
accept	$350, 500$	$350, 500$

Step 3

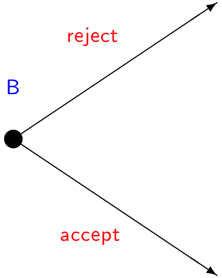
$\Gamma^2(h_1)$	200			
$(r, 200)$	$100, 100$			



$\Gamma^2(\varnothing)$	200	300
accept	$350, 500$	$350, 500$

Step 3

$\Gamma^3(h_1)$	200			
$(r, 200)$	$100, 100$			



$350, 500$

$\Gamma^3(\emptyset)$	200
accept	$350, 500$

End of algorithm

Belief in restricted past rationality

- In general, it may not be possible to always believe that your opponent has chosen rationally in the past.
- But we could impose the following additional condition: If you are at information set h, then you must believe that in the past, the opponent has always chosen rationally among the strategies that lead to h.
- Belief in restricted past rationality: Becerril and Perea (2020).
- Leads to common belief in future and restricted past rationality.
- Becerril and Perea (2020) also develop an algorithm for this concept, similar to the backward dominance procedure.
- Becerril and Perea (2020) show that common full belief in caution and respect of preferences, when applied to the normal form, implies common belief in future and restricted past rationality.
- Research question: Other relationships between cautious reasoning in the normal form, and concepts for dynamic games?

Backwards order of elimination

- When we use the backward dominance procedure, the order in which we eliminate strategies is not important for the eventual result.
- In particular, it does not matter in which order we walk through the information sets when eliminating strategies.
- In many games, there is a very convenient order of elimination: backwards order of elimination.
- First, consider the ultimate information sets, and apply iterated elimination of strategies there.
- Then, consider penultimate information sets, and apply iterated elimination of strategies there.
- And so on, until we reach the beginning of the game.

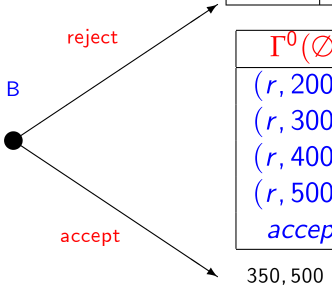
- The backwards order of elimination works whenever the game is with observed past choices.

Definition (Game with observed past choices)

A dynamic game is with observed past choices if at every information set, the active players know precisely the choices made by the opponents in the past.

- However, the backwards order of elimination may not be possible if there are unobserved past choices in the game!

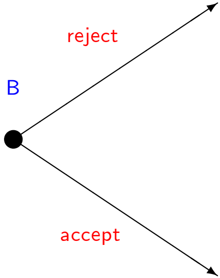
$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

First, do iterated elimination of strictly dominated strategies at h_1.

$\Gamma^0(h_1)$	200			
$(r, 200)$	100, 100			

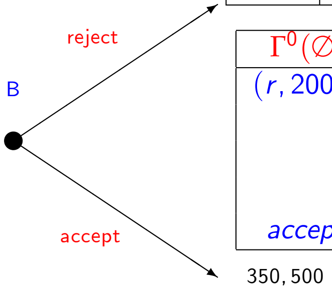


$\Gamma^0(\varnothing)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

350, 500

Then, eliminate these strategies also at $\varnothing$.

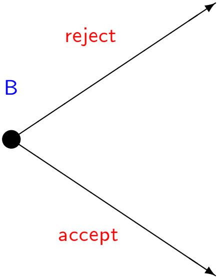
$\Gamma^0(h_1)$	200			
$(r, 200)$	100, 100			



$\Gamma^0(\emptyset)$	200
$(r, 200)$	100, 100
accept	350, 500

Finally, do elimination of strictly dominated strategies at $\emptyset$.

$\Gamma^0(h_1)$	200			
$(r, 200)$	$100, 100$			



$\Gamma^0(\emptyset)$	200
accept	$350, 500$

$350, 500$

End of algorithm

Backward induction

- For dynamic games with perfect information, the backward dominance procedure reduces to a very simple procedure called backward induction.

Definition (Game with perfect information)

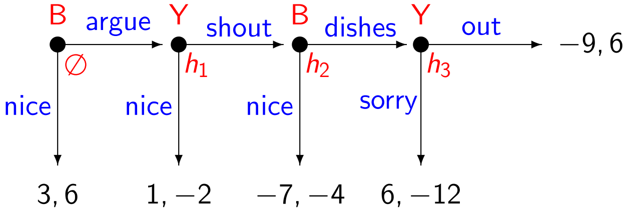
A dynamic game is with perfect information if at every information set there is only one active player, and this player always knows exactly what choices have been made by his opponents in the past.

Example: The heat of the fight.

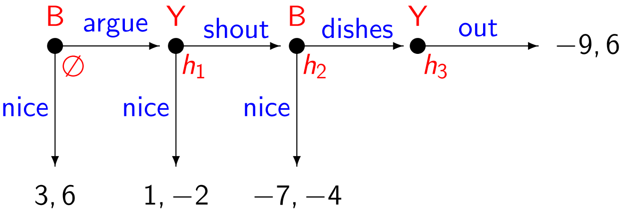
Story

- Barbara and you must decide with TV program to watch: Blackadder or Dallas.
- You prefer Blackadder (utility 6) to Dallas (utility 3).
- Barbara prefers Dallas (utility 6) to Blackadder (utility 3).

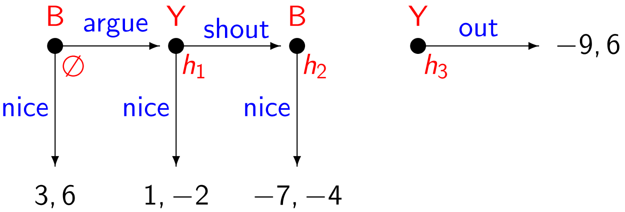
- At the beginning, Barbara can either be nice to you (let you watch your favorite program), or can start to argue with you.
- If she starts arguing, you can either be nice to her (let her watch her favorite program), or you can start shouting at her.
- If you start shouting, then Barbara can either be nice to you (let you watch your favorite program), or she can throw dishes on the floor, as a sign of her anger.
- If she starts throwing dishes on the floor, you can either apologize to her, and let her watch her favorite program, or you can walk out the door and watch Blackadder at Chris' freshly painted house.
- The utility for you and Barbara decreases by 5 every time the conflict escalates.
- If you apologize to Barbara, her utility would increase by 15.
- If you watch Blackadder at Chris' house, your utility would increase by 15.



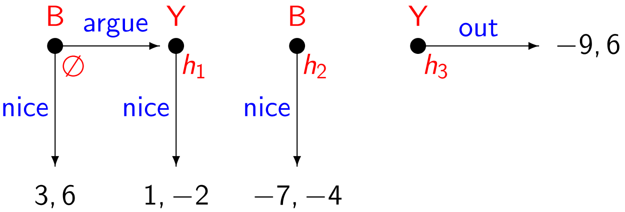
- Backward dominance procedure with backwards order of elimination:
- At h_3, select your optimal choice out.



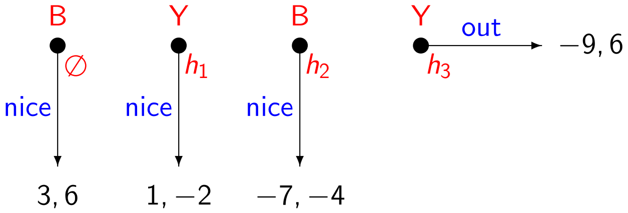
- At h_2, select Barbara's optimal choice nice.



- At h_1, select your optimal choice nice.



- At $\varnothing$, select Barbara's optimal choice nice.



- This is backward induction.
- Hence, the backward dominance procedure uniquely selects your strategy nice.

Definition (Backward induction procedure)

Consider a dynamic game with perfect information. At the beginning, we select at every ultimate information set all choices for the active player that are optimal at this information set. These are called the backward induction choices at the ultimate information sets.

We then select, at every penultimate information set, all choices for the active player that are optimal for some configuration of opponents' backward induction choices at the ultimate information sets. These are called the backward induction choices at the penultimate information sets.

And so on, until we reach the beginning of the game.

- A strategy is called a backward induction strategy if it consists of backward induction choices.
- For games with perfect information, the backward dominance procedure with the backwards order of elimination is equivalent to the backward induction procedure.

Theorem (Common belief in future rationality leads to backward induction)

Consider a dynamic game with perfect information.

Then, the strategies that can rationally be chosen under common belief in future rationality are exactly the backward induction strategies.

- If the game with perfect information is generic – that is, all utilities at the terminal histories are different – then there is a unique backward induction strategy for every player.
- In non-generic games with perfect information, there may be more than one backward induction strategy for a player.

Theorem (Common belief in future rationality leads to backward induction)

Consider a dynamic game with perfect information.

Then, the strategies that can rationally be chosen under common belief in future rationality are exactly the backward induction strategies.

- Hence, common belief in future rationality can be viewed as an epistemic foundation for backward induction.
- Other epistemic foundations for backward induction: Aumann (1995), Samet (1996), Stalnaker (1996, 1998), Balkenborg and Winter (1997), Asheim (2002), Quesada (2002, 2003), Clausing (2003, 2004), Feinberg (2005), Bach and Heilmann (2011).
- See Perea (2007) for an overview.
- Research question: Other epistemic foundations for backward induction?

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