Rationalizable Behavior in the Hotelling-Downs Model of Spatial Competition *

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Abstract

We consider three scenarios of the Hotelling-Downs model of spatial competition. The first scenario is a static setting with fixed prices and an arbitrary number of agents. This setting has typically been explored using Nash equilibrium, but this paper uses rationalizability instead. These findings will be compared to the results of Eaton and Lipsey (1975) and Shaked (1982). We show that as the number of agents increases, the set of point rationalizable choices increases as well. The second variation consists of a sequential Hotelling-Downs model with three agents, which will be solved by backward induction. The third variation is the static case when agents have limited attraction intervals. In this variation, we show that the set of rationalizable choices does not depend on the number of agents, apart from the number of agents being odd or even. It does depend on the size of the attraction interval. More precisely, the set of rationalizable choices shrinks as the attraction interval gets larger.

Keywords: Hotelling Games, Rationalizability, Spatial Competition

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1 Introduction

Hotelling's (1929) paper on the duopoly location model was published almost a century ago, yet the contents of the paper are still taught today in economics courses all over the world. The paper contains a rich motivation and intuitive examples that explain the results from the model. Hotelling was the first to include the location of the firm as a feature in his model. The location model demonstrates the relationship between the pricing and location of a firm. Consumers are uniformly distributed on a line segment, and incur a transportation cost for the distance traveled to one of the firms. This allows for the feature that when a firm increases its price, it will gradually lose demand instead of instantaneously, which is the case for the Bertrand model.

Downs (1957) gave a different interpretation to Hotelling's model, in the case where each firm charges an identical price. The firms could then be interpreted as political agents in a society that can be ordered from left to right. These agents simultaneously choose a political position on the line. Instead of consumers, Downs' model consists of voters that will choose the agent whose position is closest to their preferred political position.

Most papers on the Hotelling-Downs model use the Nash equilibrium concept. Eaton and Lipsey (1975) analyzed the Hotelling-Downs model with an arbitrary number of firms, where all agents simultaneously choose their position. For three agents, they mention that it is impossible to satisfy the equilibrium conditions, and as a result there is no Nash equilibrium. For any other number of agents they are able to find one or multiple equilibria.

Pearce (1984) and Bernheim (1984) criticized the Nash equilibrium concept and independently developed a different solution concept, called rationalizability. Their main critique was that the Nash equilibrium is too restrictive in its assumptions, partially explaining why no Nash equilibrium was established in Eaton and Lipsey's model with three firms. Both Nash equilibrium and rationalizability require that each player believes that his opponents play rationally, believes that his opponents believe that the other players play rationally, and so on. Additionally, Nash equilibrium also imposes a correct beliefs assumption, while rationalizability does not. The correct beliefs assumption states that each player must believe that his opponents are correct about his own beliefs, and that his opponents share his own beliefs about other players. Because rationalizability does not impose this constraint, it is more permissive than Nash equilibrium.

In this paper, we will characterize the point rationalizable choices of the Hotelling-Downs model with an arbitrary number of agents. The main difference between rationalizability and point rationalizability is that point rationalizability only considers point beliefs, while rationalizability considers mixed beliefs as well. Point beliefs assign probability 1 to exactly one of each of the opponents' choices.

In most of the literature regarding the Hotelling-Downs model, the emphasis has been on finding the pure Nash equilibria. Mixed beliefs are not considered in a pure Nash equilibrium. Because we want to compare our results with pure Nash equilibria, point rationalizability is the preferred solution concept over rationalizability.

Our first result is a characterization of the point rationalizable choices in the static Hotelling-Downs model with fixed prices and an arbitrary number of agents. As the number of agents increases, the set of point rationalizable choices for each agent increases as well. When the number of agents gets very large, almost any position is point rationalizable, except the extreme positions on the line. We will compare our characterisation to the results of Eaton and Lipsey (1975). We find that the positions chosen by the agents in a pure Nash equilibrium are further from the edge than the point rationalizable choices closest to the edges.

Next, we characterize the backward induction solution for the dynamic Hotelling-Downs model with three agents. We find that the first two agents choose exactly the extreme choices contained in the set of point rationalizable choices that were found in the static Hotelling model with three agents. These extreme choices are $\frac{1}{4}$ and $\frac{3}{4}$. Our assumption about what the last agent does whenever he is indifferent was crucial for arriving at this conclusion. In Teitz' (1968) dynamic model, a leader can choose $N-1$ positions on the line, and after this the follower can choose one position on the line. If the leader can choose two positions, then he will choose the positions $\frac{1}{4}$ and $\frac{3}{4}$. These are exactly the positions that agent 1 and 2 will choose in our dynamic Hotelling-Downs model. This suggests that even if agent 1 and 2 would work together agents agent 3, they still choose the same positions on the line.

Lastly, we characterize the point rationalizable choices in a more recent variation of the Hotelling-Downs model, introduced by Feldman et al (2016). In this variation, clients are attracted to all agents within their attraction interval. We consider attraction intervals ranging from 0 to 1. We find that as the attraction interval increases, the set of point rationalizable choices decreases. The set of point rationalizable choices is not decreasing in the number of agents, but we do find different results depending on whether the number of agents is odd or even. When the attraction interval approaches 1, only the middle positions on the line are point rationalizable.

Section 2 introduces the Hotelling-Downs model and defines point rationalizability. Section 3 contains the results of the static Hotelling-Downs model with fixed prices. Section 4 solves the dynamic Hotelling-Downs model with three agents by backward induction. Section 5 introduces the Hotelling-Downs model with limited attraction and characterizes the point rationalizable choices in relation to the size of the attraction interval. Section 6 contains a literature review. Section 7 contains a discussion and conclusion. All proofs are collected in the

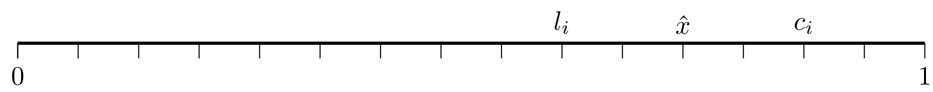

Figure 1: Rightmost choice

appendix.

2 Hotelling-Downs Model and Rationalizability

Let $I = \{1, ..., N\}$ denote the set of agents. Each agent $i \in I$ simultaneously selects a choice $c_i \in C_i$, where $C_i = \{0, \delta, 2\delta, ..., 1-\delta, 1\}$ denotes the set of positions that agent i can choose, where $\delta = \frac{1}{m}$ for some strictly positive integer m. At the end of this section, we provide an explanation why we use a finite choice set for each agent instead of the choice set $[0, 1]$. Clients are distributed uniformly on the interval $[0, 1]$. Clients will support an agent whose position is closest to the client. In the case that the client is equally close to multiple agents, he will randomly select one of these agents to support.

Agents are assumed to be support maximizers, meaning that their objective is to attract as many clients as possible. Hence, the utility for each agent i will be denoted by the fraction of clients that support agent i. Let $C_{-i} = C_1 \times ... \times C_{i-1} \times C_{i+1} \times ... \times C_N$ denote the set that contains all the choice combinations of the opponents of agent i, where $c_{-i} = (c_1, ..., c_{i-1}, c_{i+1}, ..., c_N) \in C_{-i}$.

Given a choice $c_i \in C_i$ and a choice combination of the opponents $c_{-i} \in C_{-i}$, let $l_i(c_i, c_{-i})$ denote the position of the closest opponent to the left of agent i, in case it exists. Let $r_i(c_i, c_{-i})$ denote the position of the closest opponent to the right of agent i, in case it exists. Let $D_i(c_i, c_{-i})$ denote the number of agents that occupy position c_i. A choice is called leftmost if it is (one of) the position(s) closest to 0. Similarly, a choice is called rightmost if it is (one of) the position(s) closest to 1. A choice is called middle if some choices of the opponents are closer to 0 and 1.

Figure 1 shows the position of the relevant indifferent client when agent i's choice c_i is rightmost. Because c_i is rightmost, there are no other agents occupying a position to the right of c_i. The indifferent client $\hat{x}$ is located in the middle between c_i and $l_i(c_i, c_{-i})$. Every client to the right of $\hat{x}$ will support the position c_i. If agent i is the only agent located at c_i, his utility is equal to $1 - \hat{x} = 1 - \frac{c_i + l_i(c_i, c_{-i})}{2} = \frac{2 - c_i - l_i(c_i, c_{-i})}{2}$. If the number of agents located at c_i is equal to $D_i(c_i, c_{-i})$, then the utility of agent i is $\frac{2 - c_i - l_i(c_i, c_{-i})}{2D_i(c_i, c_{-i})}$. A similar procedure can be followed for leftmost and middle choices. We can now write down the utility function for each agent i:

$$u_i(c_i, c_{-i}) = \begin{cases} \frac{2-c_i-l_i(c_i,c_{-i})}{2D_i(c_i,c_{-i})} & \text{if } c_i \text{ is rightmost} \\ \frac{r_i(c_i,c_{-i})-l_i(c_i,c_{-i})}{2D_i(c_i,c_{-i})} & \text{if } c_i \text{ is middle} \\ \frac{c_i+r_i(c_i,c_{-i})}{2D_i(c_i,c_{-i})} & \text{if } c_i \text{ is leftmost} \\ \frac{1}{N} & \text{if all agents are positioned at } c_i \end{cases}$$

Each agent i can motivate his choice by forming a belief about the opponents' choice combinations C_{-i}. A belief for agent i is a probability distribution b_i over the set C_{-i}. For every choice combination of the opponents c_{-i} , the belief $b_i(c_{-i})$ denotes the probability that agent i assigns to the event that this particular choice combination is indeed chosen by the opponents. We only consider point beliefs. That is, we only consider beliefs where $b_i(c_{-i}) = 1$, for some opponents' choice combination c_{-i}. Expected utility can be denoted as $u_i(c_i, b_i) = \sum_{c_{-i} \in C_{-i}} b_i(c_{-i}) u_i(c_i, c_{-i})$. A choice is optimal for an agent if it maximizes his utility for some belief.

Definition 1. *A choice $c_i \in C_i$ is optimal for agent i given a belief b_i if $\forall\ c_i^* \in C_i$,*

$$u_i(c_i, b_i) \geq u_i(c_i^*, b_i).$$

If $c_i \in A_i \subseteq C_i$ and the above inequality holds for every $c_i^* \in A_i$, then c_i is optimal in A_i for the belief b_i

We will now explain why we assume a finite set of choices for the agents. The utility functions of the agents are not continuous. If the choice set of each agent i would be infinite, such as $C_i = [0, 1]$, we would run into problems. We would then be unable to answer even the most basic questions, such as which choice or choices are optimal for a belief. For example, assume there are two agents and consider the belief b_1 where $b_1(0) = 1$. First consider the case where C_1 and C_2 are finite sets. Intuitively, choosing the closest position to the right of 0 will yield the largest utility to agent 1. Hence, the choice $c_1 = \delta$ would be the optimal choice for this belief. Now consider the case where $C_1 = C_2 = [0, 1]$. The closest choice to the right of 0 does not exists, as for any choice c_1 close to 0, there exists a choice c_1' closer to 0. As a result, there exists no optimal choice for this belief.

The following inductive procedure resembles Pearce's (1984) procedure to find the rationalizable choices, adapted for point beliefs.

Definition 2. *Let $P_i(0) = C_i$ for all $i \in I$. Then $P_i(k)$ is inductively defined for $k = 1, 2, ...$ by $P_i(k) = \{c_i \in P_i(k-1):$ there exists a point belief b_i over the set $P_{-i}(k-1)$ such that c_i is optimal in $P_i(k-1)$ given $b_i\}$. The set of point rationalizable choices for agent i is then $P_i = \bigcap_{k=1}^{\infty} P_i(k)$.*

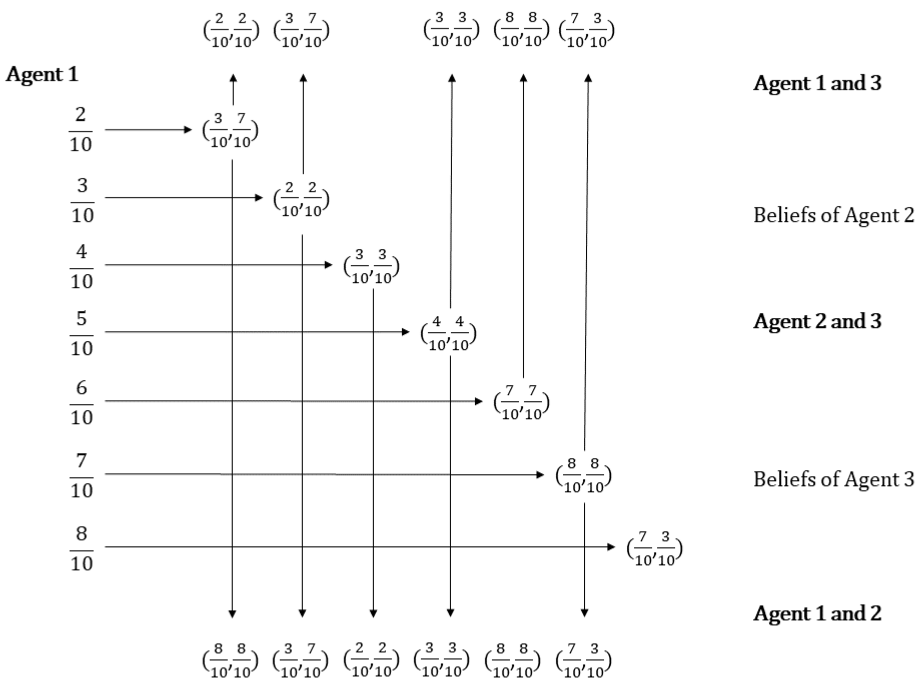

Figure 2: Beliefs diagram

3 Results for the Static Hotelling-Downs Model

The next theorem uses a ceiling function. The ceiling function $\lceil a \rceil$ returns the smallest value b bigger or equal to a such that b is a multiple of δ.

Theorem 1. *Suppose there are N agents and let $\delta \leq \frac{1}{3(N-3)+6}$. Then $\forall i \in I$,*

$$P_i = \{ \left\lceil \frac{1-(N-1)\delta}{2N-2} \right\rceil, ..., 1 - \left\lceil \frac{1-(N-1)\delta}{2N-2} \right\rceil \} \text{ if } N \in \{2, 3\}$$

$$P_i = \{ \left\lceil \frac{1-(N-2)\delta}{2N-2} \right\rceil, ..., 1 - \left\lceil \frac{1-(N-2)\delta}{2N-2} \right\rceil \} \text{ if } N \geq 4$$

An immediate result from this theorm is that the set of point rationalizable choices grows as the number of agents increases. With 2 agents, we are able to use the fact that as long as $k \leq \frac{1}{\delta} \cdot \left\lceil \frac{1-\delta}{2} \right\rceil$, in each round $P_i(k)$, the choice $k\delta$ is strictly dominated by the choice $(k+1)\delta$. A similar result is true at the other side of the line. As a result, only the middle choice(s) survive(s) the iterative procedure. With 3 agents or more however, this is no longer true.

For example, consider 3 agents and $\delta = 0.1$. The point rationalizable choices for

Number of agents	set of point rationalizable choices
2	$\{\frac{1}{2}\}$
3	$[\frac{1}{4}, \frac{3}{4}]$
4	$[\frac{1}{6}, \frac{5}{6}]$
5	$[\frac{1}{8}, \frac{7}{8}]$
6	$[\frac{1}{10}, \frac{9}{10}]$
7	$[\frac{1}{12}, \frac{11}{12}]$
8	$[\frac{1}{14}, \frac{13}{14}]$
N	$[\frac{1}{2N-2}, \frac{2N-3}{2N-2}]$

Table 1: Point rationalizable choices when δ approaches 0

each agent are then given by $\{\lceil\frac{1-2\delta}{4}\rceil, ..., 1-\lceil\frac{1-2\delta}{4}\rceil\} = \{\frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}\}$. A beliefs diagram (Perea (2012)) helps to visually show the reasoning of each agent in a game. The arrows represent the beliefs of an agent. For example, consider the arrow from agent 1 with the choice $\frac{2}{10}$ going to the choice pair $(\frac{3}{10}, \frac{7}{10})$ of agent 2 and 3. This arrow represents that choice $\frac{2}{10}$ of agent 1 is supported by the belief that agent 2 chooses $\frac{3}{10}$ and agent 3 chooses $\frac{7}{10}$. This is a first order belief of agent 1 that supports his choice of $\frac{2}{10}$. By following the arrows, we can also find the higher order beliefs. The choice $\frac{2}{10}$ of agent 1 is supported by the second-order belief that agent 2 believes that agent 1 and 3 choose $\frac{2}{10}$ and that agent 3 believes that agent 1 and 2 choose $\frac{8}{10}$. Continuing this way ad infinitum would give us the belief hierarchy of agent 1 that supports his choice $\frac{2}{10}$. Similarly, we can construct belief hierarchies that support the choices $\frac{3}{10}, ..., \frac{8}{10}$ of agent 1 or of a different agent. All these belief hierachies express common belief in rationality, because this belief diagram only consists of the choices in $P_i = \{\frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}\}$. Hence, these belief hierarchies support the point rationalizable choices.

We are particularly interested in the point rationalizable choices if δ approaches 0. When $N \geq 2$ and δ approaches zero, the point rationalizable choices are given by $[\frac{1}{2N-2}, ..., \frac{2N-3}{2N-2}]$. Table 1 shows the point rationalizable strategies for the first 8 agents when δ approaches zero. It is immediately evident that as N increases, the range of point rationalizable choices also increases. When N becomes very large, the set of point rationalizable choices approaches the interval $(0, 1)$.

Eaton and Lipsey (1975) found the same result as we did for two agents. In the pure Nash equilibrium, both agents are located at the center of the line, equivalent to the minimal differentiation result of Hotelling. For three agents, Eaton and Lipsey do not find any equilibrium. However, Shaked (1982) found a mixed Nash equilibrium where each agent places equal probability on all the choices in $[\frac{1}{4}, \frac{3}{4}]$. This result is similar to the point rationalizable choices for three agents when δ approaches 0, where $P_i = [\frac{1}{4}, \frac{3}{4}]$. For four agents, Eaton and Lipsey find the unique equilibrium where agent 1 and 2 are located on $\frac{1}{4}$, and agent 3 and 4 are located at $\frac{3}{4}$. With four agents, the point rationalizable choices for each agent i are given by $[\frac{1}{6}, ..., \frac{5}{6}]$. For five agents, Eaton and Lipsey find the unique equilibrium where agent 1 and 2 are located at $\frac{1}{6}$, agent 3 is located at the middle of the line, and agent 4 and 5 are located at $\frac{5}{6}$.

For six agents onward the equilibrium is no longer unique. The equilibrium with minimum product differentiation is given by agent 1 and 2 locating at $\frac{1}{8}$, agent 3 and 4 locating at the middle of the line, and agent 5 and 6 locating at $\frac{7}{8}$. The equilibrium with maximum product differentiation maximizes the distance between agent 3 and 4 by placing agent 3 on $\frac{3}{8}$ and agent 4 at $\frac{5}{8}$. The point rationalizable choices for each agent i with 6 agents is given by $[\frac{1}{10}, ..., \frac{9}{10}]$. The main difference is that the position of the outer agents in the Nash equilibria of Eaton and Lipsey is further from the edge of the line than the point rationalizable choices closest to the edges of the line.

4 Sequential Hotelling-Downs Model with Three Agents

In this section, three agents will sequentially choose a position. Agent 1 will first choose a position, and the other agents observe the choice of agent 1. Next, agent 2 will choose a position, and lastly agent 3 observes this decision of agent 2 and chooses a position afterwards. Instead of a finite choice set, each player can now choose a position in $[0, 1]$. However, there must be a minimum distance δ between the positions. Because at every information set there is exactly one agent active and this agent knows the choices that have been made by the other agents in the past, this is a game with perfect information. Backwards induction is a reasonable concept to apply here.

However, if we have 3 agents in the model, a problem arises. We could start by taking the perspective of the last agent, and compute the optimal choice for each combination of c_1 and c_2. Assuming that $c_1 < c_2$, agent 3 can either choose the position just to the left of c_1, in between c_1 and c_2, or just to the right of agent 2. Whenever it is optimal for agent 3 to position in between c_1 and c_2, any choice $c_3 \in [c_1 + \delta, c_2 - \delta]$ will be optimal. However, this choice c_3 does have an impact on the utility of agent 1 and agent 2. Authors have dealt with this issue in different ways. In Prescott and Visscher (1987) and Palfrey (1984), agent 1 and 2 would assume that agent 3 will locate at $\frac{c_1+c_2}{2}$, whereas in Rothschild (1976) agent 1 assumes that $c_3 = c_1 + \delta$ and agent 2 assumes that $c_3 = c_2 - \delta$. That is, Rothschild assumes that an agent believes that each opponent will choose the worst possible position for him, in case he is indifferent.

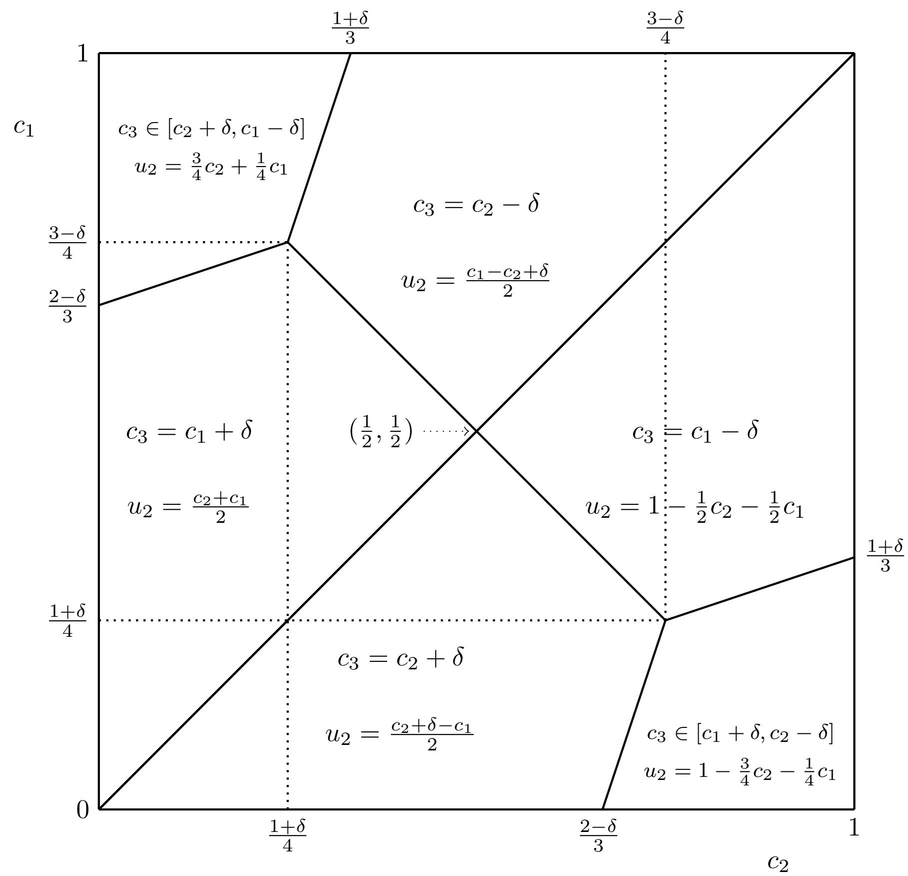

Figure 3: Optimal choice(s) for agent 3

We make an assumption similar to that of Prescott and Visscher.

When agent 3 is indifferent between some choices, we apply the principle of insufficient reason to determine the utility of agent 1 and agent 2. As an example, assume that $c_2 > c_1$ and that agent 3's optimal choice is to choose a middle location. Then any choice in $[c_1 + \delta, c_2 - \delta]$ is optimal for agent 3. With the principle of insufficient reason, we assume that each of these choices is equally likely to occur. The utility of agent 2 can then be calculated as $u_2 = \frac{1}{(c_2-\delta)-(c_1+\delta)} \int_{c_1+\delta}^{c_2-\delta} \frac{2-c_2-c_3}{2}\, dc_3 = 1 - \frac{3}{4}c_2 - \frac{1}{4}c_1$. Similarly, we have $u_1 = \frac{3}{4}c_1 + \frac{1}{4}c_2$. This is the same utility that agent 1 and 2 would receive when agent 3 would choose $c_3 = \frac{c_1+c_2}{2}$.

If agent 3 is indifferent between a leftmost and rightmost position, then $c_1 - \frac{\delta}{2} =$

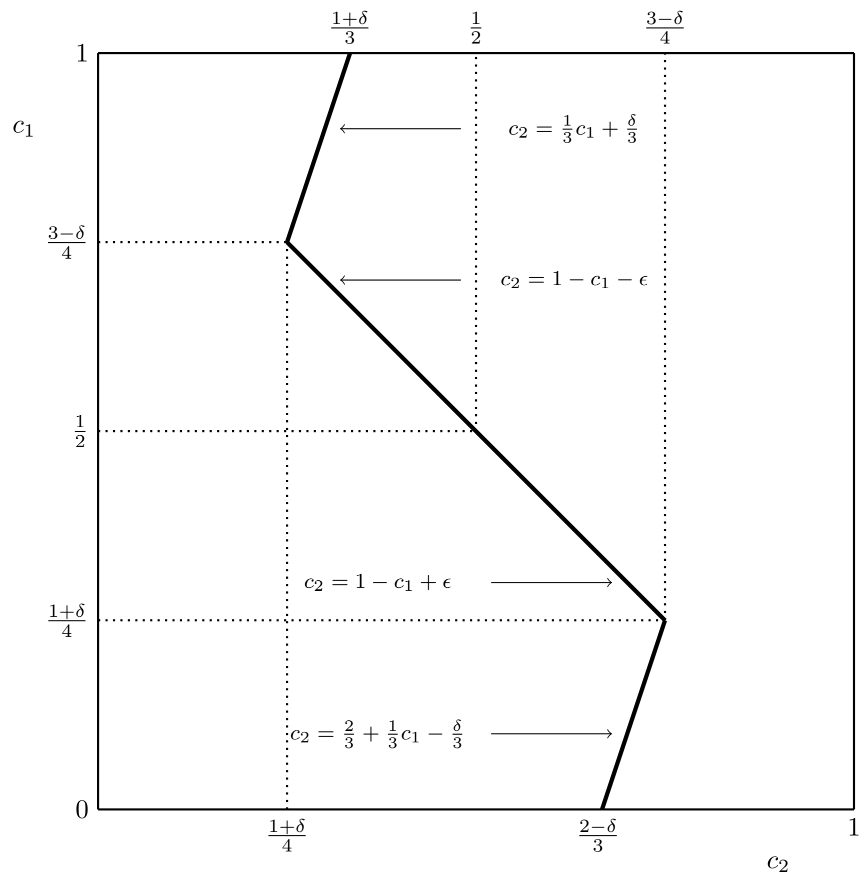

Figure 4: Best response function of agent 2

$1 - c_2 - \frac{\delta}{2}$, which implies $c_2 = 1 - c_1$ and $u_1 = \frac{1}{2}\frac{c_1+(1-c_1)}{2} + \frac{1}{2}\frac{1-c_1-c_1+\delta}{2} = \frac{1}{2} - \frac{1}{2}c_1 + \frac{\delta}{4}$ and $u_2 = \frac{1}{2}\frac{2-(1-c_1)-c_1}{2} + \frac{1}{2}\frac{1-c_1+\delta-c_1}{2} = \frac{1}{2} - \frac{1}{2}c_1 + \frac{\delta}{4}$. Note that if agent 3 is indifferent between choosing a middle location and a rightmost or leftmost position, then by the principle of insufficient reason, the utility of agent 1 and 2 can be calculated as $u_2 = 1 - \frac{3}{4}c_2 - \frac{1}{4}c_1$ and $u_1 = \frac{3}{4}c_1 + \frac{1}{4}c_2$, because there are an infinite number of positions between $c_1 + \delta$ and $c_2 - \delta$.

Figure 3 shows the optimal choice for agent 3 for any combination (c_1, c_2). The figure consist of 6 regions. For example, if $c_1 < c_2$ and both c_1 and c_2 are relatively small, then agent 3's optimal choice is then to position just a little to the right of agent 2, which is denoted by $c_2 + \delta$. Similarly, $c_3 = c_1 - \delta$ if c_1 and c_2 are relatively big. The optimal choice of agent 3 is not always unique. If c_1 is small and c_2 is large, then any choice $c_3 \in [c_1 + \delta, c_2 - \delta]$ is an optimal choice for agent 3. The other three regions are similar, but then $c_2 < c_1$ instead of $c_1 < c_2$.

Agent 3 can be indifferent between even more choices, for example, at the point $(\frac{1+\delta}{4}, \frac{3-\delta}{4})$, agent 3 is indifferent between $c_3 = c_1 - \delta, c_3 \in [c_1 + \delta, c_2 - \delta]$, and $c_3 = c_2 + \delta$.

Next, we can take the perspective of agent 2. In each region, we can calculate the utility of agent 2. This has also been depicted in figure 3. For example, if agent 1 chooses $0 < c_1 < \frac{1+\delta}{4}$, then choosing $c_2 = 0$ is clearly not optimal. As long as $c_2 < c_1$ agent 3 will choose $c_3 = c_1 + \delta$ and $u_2 = \frac{c_1+c_2}{2}$. Hence, the optimal choice for agent 2 in this region is $c_2 = c_1 - \delta$. Similarly, in the region where $c_3 = c_2 + \delta$, agent 2 should choose the maximum permitted value of c_2, which is a value slightly smaller than $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$. Lastly, in the region where $c_3 = [c_1 + \delta, c_2 - \delta]$, agent 2 should choose the minimum permitted value of c_2, which is $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$. Next, agent 2 will select the choice that gives him the highest utility given some c_1.

The main result can be summarized by figure 4. Let $0 \leq c_1 \leq \frac{1}{2}$. For low values of c_1, agent 3 will not choose to position to the left of agent 1, no matter what agent 2 chooses. Agent 2's best response is to choose $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$. This is the minimum c_2 such that choosing a middle position is optimal for agent 3. Agent 1's utility is then $u_1 = \frac{c_1 + \frac{c_1+c_2}{2}}{2} = \frac{1}{6} + \frac{5}{6}c_1 - \frac{\delta}{12}$. If c_1 is high enough, then agent 2 can choose a position $1 - c_1 + \epsilon$ such that agent 3 will position to the left of agent 1, which leaves agent 2 with half of the total market share. Here ϵ should be interpreted as a strictly positive and very small number, much smaller than δ. Agent 2 should not position exactly at $1 - c_1$, because then agent 3 is indifferent between positioning just to the left of agent 1 and just to the right of agent 2. Agent 1's utility can then be denoted as $u_1 = \frac{c_2-c_3}{2} = \frac{1-c_1+\epsilon-c_1+\delta}{2} = \frac{1}{2} - c_1 - \frac{\delta}{2} - \frac{\epsilon}{2}$. We have a similar response of agent 2 when $\frac{1}{2} \leq c_1 \leq 1$.

With the best response function of agent 2, we can calculate the utility of agent 1 for each c_1. This has been depicted in figure 5. The utility of agent 1 is initially increasing until $c_1 = \frac{1+\delta}{4}$. This is because for low values of c_1, $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$, and agent 3 will choose a middle position. However, for larger values closer to the middle of line, agent 2 is able to choose a position such that agent 3 will choose to position just to the left of agent 1. Agent 1 should avoid this from happening and select the largest c_1 such that this does not happen. This is the choice $c_1 = \frac{1+\delta}{4}$. By symmetry, the other optimal choice of agent 1 is the choice $c_1 = \frac{3-\delta}{4}$.

Theorem 2. *By backwards induction, agent 1 will choose* $c_1 \in \{\frac{1+\delta}{4}, \frac{3-\delta}{4}\}$, *agent 2 then chooses* $c_2 = 1 - c_1$ *and agent 3 is indifferent between choosing* $c_3 = \min(c_1, c_2) - \delta$, *any position* $c_3 \in [\min(c_1, c_2) + \delta, \max(c_1, c_2) - \delta]$, *and* $c_3 = \max(c_1, c_2) + \delta$

If δ approaches zero, Theorem 2 implies that $c_1 \in \{\frac{1}{4}, \frac{3}{4}\}$, $c_2 = 1 - c_1$, and firm 3 is indifferent between any position in between c_1 and c_2. The utility of

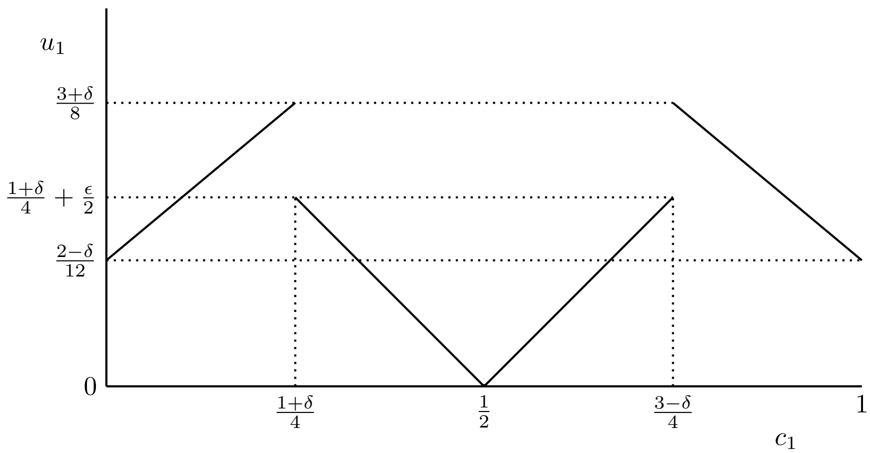

Figure 5: Utility of agent 1 for each c_1

agent 1 and 2 is $\frac{3}{8}$, whereas the utility of agent 3 is $\frac{1}{4}$. From the previous section, the set of point rationalizable choices for 3 agents when δ approaches zero is $[\frac{1}{4}, \frac{3}{4}] = [\min(c_1, c_2), \max(c_1, c_2)]$. Hence, in the dynamic model, agent 1 and 2 will choose the most extreme positions which were possible under point rationalizability in the static Hotelling-Downs model with three agents. Furthermore, from figure 4 we can observe that agent 2 will always choose a position in $[\frac{1}{4}, \frac{3}{4}]$, no matter what agent 1 chooses. Similarly, from figure 3 we can observe that agent 3 will always choose a position in $[\frac{1}{4}, \frac{3}{4}]$ if agent 1 and/or 2 chooses a position in $[\frac{1}{4}, \frac{3}{4}]$.

The minimum degree of product differentiation in the dynamic Hotelling-Downs model is also higher than the static Hotelling-Downs model. Two out of the three agents will be positioned at $\frac{1}{4}$ and $\frac{3}{4}$, and one agent will be positioned at some position between $\frac{1}{4}$ and $\frac{3}{4}$. In the static model, the minimum degree of product differentiation is much lower, because it is possible that all three agents choose the same point rationalizable choice.

Teitz (1968) considered a model with 2 agents, consisting of a leader and a follower. The leader first chooses $N-1$ positions on the line, and then the follower can choose one position on the line. If the leader is allowed to choose 2 positions on the line, he will choose the positions $\frac{1}{4}$ and $\frac{3}{4}$. This suggests that in our dynamic model, even if agent 1 and 2 would work together, they would end up making the same choice, which is one agent at $\frac{1}{4}$ and the other agent at $\frac{3}{4}$.

5 Hotelling-Downs Model with Limited Attraction

Feldman and Fiat (2016) altered the standard Hotelling-Downs model. In this variation, clients do not necessarily choose the closest agent. Each agent i has an attraction region, given by ω. Given his position on the line c_i, he attracts clients in between $c_i - \frac{\omega}{2}$ and $c_i + \frac{\omega}{2}$. A client positioned at x on the line will equally divide his support among the agents within $x-\frac{\omega}{2}$ and $x+\frac{\omega}{2}$. If no agent is located within $x-\frac{\omega}{2}$ and $x+\frac{\omega}{2}$, then the client will not support any agent. If we take the perspective of some agent i, then the set of opponent agents that attract client x can be denoted by $I_x(c_{-i}) = \{j \in \{1, ..., i-i, i+1, ..., N\} | x \in [c_j - \frac{\omega}{2}, c_j + \frac{\omega}{2}]\}$. If agent i chooses a position c_i such that $x \in [c_i - \frac{\omega}{2}, c_i + \frac{\omega}{2}]$, then we can denote agent i's share of client x as

$$a_{x,i}(c_{-i}) = \frac{1}{|I_x(c_{-i})| + 1}$$

Assuming a uniform distribution of the clients with density $f(x) = 1 \ \forall x \in [0, 1]$, the utility function of each agent i is given by

$$u_i(c_i, c_{-i}) = \int_{c_i-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}(c_{-i}) f(x)\, dx$$

If agent i chooses in $[\frac{\omega}{2}, 1-\frac{\omega}{2}]$, then we are able to write the utility of agent i as $\int_{c_i-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}\, dx$ instead of $\int_{c_i-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i} f(x)\, dx$. However, if agent i chooses $c_i = 0$, then agent i's utility is $\int_{0-\frac{\omega}{2}}^{0+\frac{\omega}{2}} a_{x,i} f(x)\, dx = \int_0^{\frac{\omega}{2}} a_{x,i}\, dx$. If agent i chooses $c_i \in [0, \frac{\omega}{2})$ we cannot omit $f(x)$. We characterize the point rationalizable choices, where $0 \leq \omega < 1$. Note that for any $\omega \geq 1$, an agent can simply locate at the middle of the line, attracting all clients.

Theorem 3. *Consider a Hotelling-Downs model with limited attraction and let $0 < \omega \leq 1$. If the number of agents is odd, then $\forall\ i \in I$,*

$$P_i = [\frac{\omega}{2}, 1 - \frac{\omega}{2}].$$

If the number of agents is even, then

$$\textit{if } 0 < \omega < \frac{1}{3},\ \textit{then}, P_i = [\frac{\omega}{2}, 1 - \frac{\omega}{2}],$$

$$\textit{if } \frac{1}{3} < \omega < \frac{1}{2},\ \textit{then}, P_i = \{[\frac{\omega}{2}, 1 - 1.5\omega], [1.5\omega, 1 - \frac{\omega}{2}]\},$$

$$\textit{if } \frac{1}{2} < \omega < 1,\ \textit{then}, P_i = \{\frac{\omega}{2}, 1 - \frac{\omega}{2}\}.$$

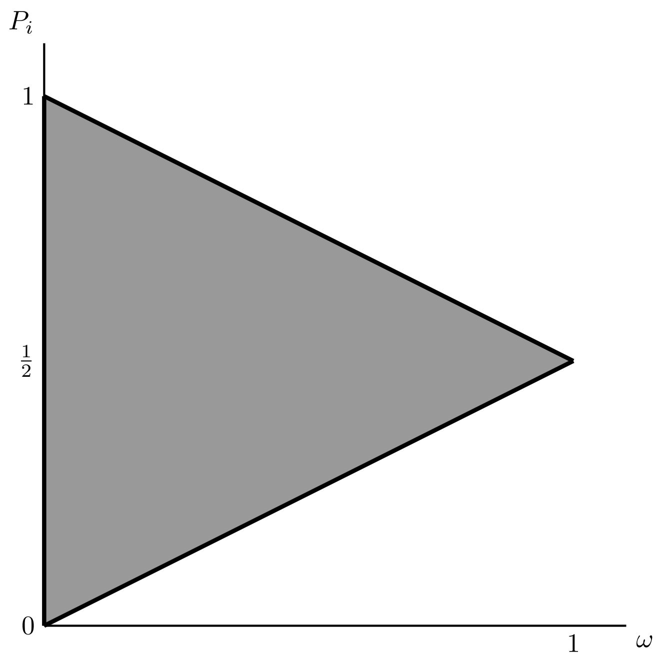

Figure 6: Point rationalizable choices for an odd number of agents

Figure 6 and 7 graphically show the point rationalizable choices for an odd number of agents and an even number of agents respectively. Let us first consider the case for an odd number of agents. Intuitively, a rational agent should not position too close to the end of the line. If it does, it can receive a higher utility by positioning a little bit more in the direction of the center of the line, no matter what the other agents choose. In particular, for any ω, the positions $[0, \frac{\omega}{2})$ and $(1 - \frac{\omega}{2}, 1]$ are too close to the end of the line for a rational agent.

Furthermore, if $\omega \leq \frac{1}{3}$, then any choice in $[\frac{\omega}{2}, \frac{1}{2}]$ is optimal for an agent for the point belief that all the opponents are located at $1 - \frac{\omega}{2}$. Figure 8 shows a visual example. A similar result is true for the choices of $[\frac{1}{2}, 1 - \frac{\omega}{2}]$. Hence, the set of point rationalizable choices is given by $[\frac{\omega}{2}, 1 - \frac{\omega}{2}]$.

Similarly, if $\omega > \frac{1}{3}$, we can show that all the choices in $[\frac{\omega}{2}, 1 - \frac{\omega}{2}]$ are optimal for agent i for some point belief. Some or all of these choices are motivated by the point belief that half of his opponents are located at $\frac{\omega}{2}$, and half of his opponents are located at $1 - \frac{\omega}{2}$. Hence, each choice of agent i that is not irrational is rationalizable.

For an odd number of agents, the set of point rationalizable choices is given by $P_i = [\frac{\omega}{2}, 1 - \frac{\omega}{2}]$. We showed that the set of irrational choices expands as

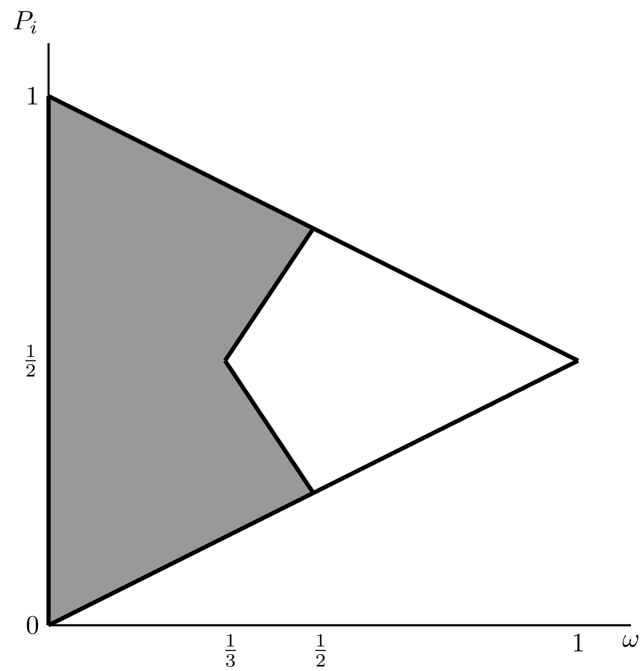

Figure 7: Point rationalizable choices for an even number of agents

ω increases, and that each choice that is not irrational, is rationalizable. As a result, the set of point rationalizable choices shrinks as ω increases.

For an even number of agents and some agent i, the point belief that half of his opponents are located at $\frac{\omega}{2}$, and half of his opponents are located at $1-\frac{\omega}{2}$ does not exist. If $\frac{1}{3} < \omega < \frac{1}{2}$, then there does not exist a point belief of agent i in $[\frac{\omega}{2}, 1-\frac{\omega}{2}] \times ... \times [\frac{\omega}{2}, 1-\frac{\omega}{2}]$, such that his choice $c_i \in (1-1.5\omega, 1.5\omega)$ are optimal. Hence, then $P_i = \{[\frac{\omega}{2}, 1-1.5\omega], [1.5\omega, 1-\frac{\omega}{2}\}$. In this model, maximizing utility is equivalent to choosing the position where you share as little clients as possible with other agents. If the attraction interval is large enough, then a rational agent will always attract agents near the middle of the line, no matter what he chooses. Hence, an agent will always be better off not choosing a position located near the middle of the line.

Lastly, for an even number of agents and $\frac{1}{2} < \omega < 1$, there does not exist a point belief of agent i in $[\frac{\omega}{2}, 1-\frac{\omega}{2}] \times ... \times [\frac{\omega}{2}, 1-\frac{\omega}{2}]$, such that his choice $c_i \in (\frac{\omega}{2}, 1-\frac{\omega}{2})$ is optimal. As a result, his only rationalizable choices are $\frac{\omega}{2}$ and $1-\frac{\omega}{2}$.

Furthermore, the point rationalizable procedure only runs for 1 iteration when we have an odd number of agents and at most 2 rounds for an even number

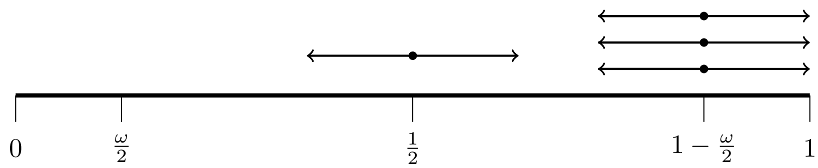

Figure 8: Rationalizable choices when $\omega = \frac{4}{15} < \frac{1}{3}$

of agents. For both an odd and an even number of agents, the set of point rationalizable choices of an agent i is decreasing in ω, but is not increasing in the number of agents, which was the case for model in Section 3.

6 Literature Review

6.1 Rationalizable Choices and Best Response Sets

Pearce (1984) and Bernheim (1984) independently developed the solution concept of rationalizable choices. The main reason for the development of this concept was that they realized Nash equilibrium can be too restrictive, because it rules out perfectly reasonable choice combinations. Pearce reached his solution by defining an iterative procedure that at each stage eliminates the choices that are not optimal for any independent belief about the opponents choices. The choices that survive this procedure are the rationalizable choices in the game. In a game with two players, beliefs will always be independent. For three or more players, we have independent beliefs if the belief of player i about player j's choice is independent from i's belief about player k's choice.

Pearce also defined best response sets, which he used to characterize the rationalizable choices in the game. He showed that the rationalizable choices are exactly those choices that are part of a best response set. We will be using this definition for the static Hotelling model in the appendix. Furthermore, Pearce showed that if a set of choices constitutes a Nash equilibrium, then this set of choices is also rationalizable. The converse is not true, i.e. a set of choices can be rationalizable but does not constitute a Nash equilibrium. Bernheim used the notion of a consistent system of beliefs to define the rationalizable choices in a game. Just like Pearce, he used the assumption that beliefs are independent. He showed that the choices that can rationally be chosen under a consistent system of beliefs are rationalizable. Bernheim introduced the concept of point rationalizability, which restricts to beliefs which assign probability 1 to exactly one choice of the opponents. These probability 1 beliefs are called point beliefs. Point beliefs are always independent, because they place all probability weight on only one choice combination.

Closely related to rationalizability is common knowledge of rationality. This concept was introduced by Tan and Werlang (1988). The main difference between rationalizability and common knowledge of rationality is the assumption about the beliefs of the opponents. While rationalizability assumes independent beliefs, common knowledge of rationality allows for correlated beliefs. Just like Pearce, they defined their own iterative procedure. The procedure of Tan and Werlang is called the iterated elimination of strictly dominated choices. In each round, the choices that are not optimal for a correlated belief are eliminated from the game. With the assumption of correlated beliefs, Tan and Werlang (1988) showed that the choices that can rationally be made under common knowledge of rationality are exactly those choices that survive the iterated elimination of strictly dominated choices.

Brandenburger and Dekel (1987) found similar results. They defined the best response sets when beliefs are allowed to be correlated. They used this to define correlated rationalizability, which is different from rationalizability from Pearce and Bernheim. The choices that are selected by correlated rationalizability are equivalent to the choices that are selected by common knowledge of rationality.

6.2 Static Hotelling-Downs Models

Eaton and Lipsey (1975) investigated how robust the minimum differentiation result from the Hotelling model with identical prices is to changes in the model. The main differences between their model and our model are the choice sets and how many agents can occupy a location. In Eaton and Lipsey's model, agents can choose any location in $[0, 1]$ and only one agent can occupy a given location. Between each agent there must be a distance of at least δ, which is very small relative to the line segment and market. They find at least one Nash equilibrium for any number of agents, except 3 agents. The Nash equilibria up to 5 agents are unique, and from 6 agents onwards there are an infinite number of Nash equilibria. In our model, the choice set is finite and agents are allowed to occupy the same location. Furthermore, we do find a set of point rationalizable choices for any number of agents.

Shaked (1982) wrote a short note about the existence of a mixed strategy Nash equilibrium for the 3 agent Hotelling model with identical prices. He demonstrated that the mixed strategy Nash equilibrium is given by each agent avoiding the extreme quartiles of the line and choosing the remaining locations with equal probability. These consist exactly of all the point rationalizable choices that we find for 3 agents. Hence, in the mixed Nash equilibrium an agent chooses each point rationalizable choice with equal probability.

6.3 Sequential Hotelling-Downs Models

Teitz (1968) introduced a dynamic component of the Hotelling model by introducing a leader and a follower in the model. The leader first chooses N positions on the line, the follower observes this and will then choose $M \leq N$ positions on the line. A useful property for the leader is that he can maximize his utility by minimizing the utility of the follower. By backward induction, the leader should choose the positions $(\frac{1}{2N-2}, \frac{3}{2N-2}, ..., \frac{2N-3}{2N-2})$. The first and last position are exactly the extreme choices in the set of point rationalizable choices in our static Hotelling-Downs model. Additionally, the point belief in our model that supports the choice $\frac{1}{2N-2}$ is given by b_i where $b_i(\frac{1}{2N-2}, \frac{3}{2N-2}, ..., \frac{2N-3}{2N-2}) = 1$.

A dynamic example of Presscot and Visscher (1977) consists of 3 agent. First agent 1 chooses a position, then agent 2, and then agent 3. The backward induction positions that are found is that agent 1 locates at $\frac{1}{4}$, agent 2 locates at $\frac{3}{4}$, and agent 3 locates between agent 1 and 2. Strictly speaking, agent 3 is indifferent between all the positions between agent 1 and agent 2, just to the left of agent 1, and just to the right of agent 2. Presscot and Visscher assume that agent 3 will position exactly in the middle between agent 1 and 2. This leads to a higher utility to agent 1 and 2 compared to agent 3. They mention that for more than three agents, the method they use for analyzing the 3-agent model becomes impractical. Instead, they focus on a model where the number of agents in the model is endogenous, and there are entry costs for each agent entering the market. In our model, we put more focus on the decision process of each agent, and provide an intuitive reasoning to arrive at the backward induction solution. Dewatripont (1987) showed that the indifference of the last agent plays a big role in the solution for the sequential model of Presscot and Visscher. Dewatripont shows that the last agent can use his indifference to obtain a higher payoff by announcing to the previous agents how he will act when he is indifferent.

Palfrey's (1984) model consists of N agents, where agent 1 until $N-1$ simultaneously have to choose a position, and agent N will choose a position afterwards. The solution concept that is used for this is called the limit equilibrium. For 3 agents, they find that agent 1 should position at $\frac{1}{4}$, agent 2 will position at $\frac{3}{4}$, and the assumption is made that agent 3 will then randomly choose a location in between $\frac{1}{4}$ and $\frac{3}{4}$. Then the expected payoff is the highest for the first two agents, and the lowest for agent 3. For the general N agent model, agent 1 until $N-1$ should position at $(\frac{1}{2N-2}, \frac{3}{2N-2}, ..., \frac{2N-3}{2N-2})$ and agent N will choose all available positions in between $\frac{1}{2N-2}$ and $\frac{2N-3}{2N-2}$ with equal probability. The expected payoff is then the highest for the 2 most extreme agents, and the lowest for the last agent N. From Teitz (1968), we know that if agent 1 until $N-1$ would work together, they would exactly choose the positions in $(\frac{1}{2N-2}, \frac{3}{2N-2}, ..., \frac{2N-3}{2N-2})$.

Rothschild's (1976) dynamic model consists of N agents that have to make a decision where to position on the line. First agent 1 makes a decision, then agent 2, then agent 3, and so on. To make the analysis easier, it is assumed that there already is an agent at the extreme positions of the line. Clients can then be interpreted as being positioned on a circle instead of on a line. Furthermore, it is assumed that each agent i assumes that the agents choosing after him will choose in the worst possible way for agent i whenever these agents are indifferent between positions on the line. In Rothschild's solution with N agents, the first $N-1$ agents choose a position in $\{\frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N}\}$ that has not been taken by the previous agents. The last agent is then indifferent between all the remaining positions on the circle. Rothschild mentions that the analysis for N agents on the line is the same as the analysis for $N+1$ agents on the circle. This would correspond to first $N-1$ agents choosing a position in $\{\frac{1}{2N-2}, \frac{3}{2N-2}, ..., \frac{2N-3}{2N-2}\}$ that has not been taken by the previous agents. Again similar to Teitz, even though agents act out of self interest, if agent 1 until $N-1$ would work together, they would choose exactly the same positions.

7 Discussion and Conclusion

This paper characterized the point rationalizable choices of the static Hotelling-Downs model with fixed prices, for any number of agents. We observed that as the number of agents increased, the set of point rationalizable choices also increased. Consider for example the classical Hotelling beach. The minimimum differentiation result only holds when there are 2 agents in the model, but is not necessary true when there are more than 2 agents. However, the socially optimal solution is also not possible if each agent makes a point rationalizable choice. From a social viewpoint, when there are 3 agents, it would be best if one agent positions at $\frac{1}{6}$, one agent at $\frac{1}{2}$, and one agents at $\frac{5}{6}$. However, the choices $\frac{1}{6}$ and $\frac{5}{6}$ are not point rationalizable. This result is also true for N agents. The socially optimal solution would be where each agent chooses an untaken position in $\{\frac{1}{2N}, \frac{3}{2N}, ..., \frac{2N-1}{2N}\}$. The set of point rationalizable choices is given by $[\frac{1}{2N-2}, \frac{2N-3}{2N-2}]$, so the choices $\frac{1}{2N}$ and $\frac{2N-1}{2N}$ are not point rationalizable.

Next, this paper found the backward induction solution for the sequential Hotelling Model with 3 agents. However, the assumption what the last agent will do when he is indifferent was crucial for arriving at this conclusion. The choices of agent 1 and 2 are exactly the extreme choices of the set of point rationalizable choices of an agent in the static Hotelling model with 3 agents. If agent 1 and 2 would work together, they would choose exactly the same positions on the line as they would if they would act out of self interest.

We also characterized the point rationalizable choices in the Hotelling-Downs model with limited attraction. The set of point rationalizable choices mainly depends on the size of the attraction interval and whether the number of agents in the game is odd or even. For any number of agents, as the size of the attraction interval increases, choosing positions towards the extremes of the line get less attractive.

One of the assumptions that has been made throughout this paper is that clients are uniformly distributed. For some applications, such as voter distributions in a country, this might not be a realistic assumption. It would be interesting to find a characterization of the point rationalizable choices in the static Hotelling model, but with a more arbitrary distribution of the clients. Similarly, in our Hotelling model with limited attraction, we assumed that each agent has an attraction region given by ω. It would be interesting to see how the results would generalize if each agent has a different attraction region. For our dynamic model, it remains an open problem to characterize the backward induction solution for any number of agents.

Appendix

Lemmas and Definitions Used in Theorem 1

With 2 agents, if $k \leq \frac{1}{\delta} \cdot \lceil \frac{1-\delta}{2} \rceil$ we are able to use the fact that in each round $P_i(k)$, the choice $k\delta$ is strictly dominated by the choice $(k+1)\delta$. With 3 agents or more however, this is no longer true. The lemma below does give us an important insight about the point rationalizable choices in relation to the iterative procedure.

Lemma 1. *Suppose there are $N \geq 2$ agents, then $\forall i \in I$, $P_i(k)$ has the following property:*

- $\{k\delta, ..., 1-k\delta\} \subseteq P_i(k)$ *if* $k \leq \frac{1}{\delta} \cdot \lceil \frac{1-\delta}{2} \rceil$

Proof. We will prove by induction. The base case is round $k = 1$. Without loss of generality we will prove that $\{\delta, ..., 1-\delta\} \subseteq P_1(1)$ by proving that these choices are optimal for agent 1 for a point belief. We will show that the choice $c_1 \in \{\delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for the point belief b_1 with $b_1(c_1-\delta, ..., c_1-\delta) = 1$. With this belief, c_1 is rightmost and $u_1(c_1, b_1) = \frac{2-2c_1+\delta}{2} = 1-c_1+\frac{\delta}{2} \geq 1-\frac{1}{2}+\frac{\delta}{2} > \frac{1}{2}$. If agent 1 chooses $c \in \{c_1+\delta, ..., 1\}$, then c is rightmost and $u_1(c, b_1) = \frac{2-c-c_1+\delta}{2} < \frac{2-2c_1+\delta}{2} = u_1(c_1, b_1)$. If agent 1 chooses $c_1-\delta$, then all agents chose $c_1-\delta$ and $u_1(c_1-\delta, b_1) = \frac{1}{N} < u_1(c_1, b_1)$. If agent 1 chooses $c \in \{0, ..., c_1-2\delta\}$, then c is leftmost and $u_1(c, b_1) = \frac{c+c_1-\delta}{2} \leq \frac{2c_1-3\delta}{2} \leq \frac{1-3\delta}{2} < \frac{1}{2} < u_1(c_1, b_1)$. Hence, the choice $c_1 \in \{\delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for the point belief b_1 with $b_1(c_1-\delta, ...c_1-\delta) = 1$. By symmetry, the choice $c_1 \in \{\lfloor \frac{1+\delta}{2} \rfloor, ..., 1-\delta\}$ is optimal for the point belief b_1 with $b_1(c_1+\delta, ..., c_1+\delta) = 1$. This proves that the choices $\{\delta, ..., 1-\delta\} \subseteq P_1(1)$. Hence, for all $i \in I$, $\{\delta, ..., 1-\delta\} \subseteq P_i(1)$.

Now assume that $1 \leq k < m \cdot \lceil \frac{m-1}{2m} \rceil$ and assume that $\forall i \in I$, $\{k\delta, ..., 1-k\delta\} \subseteq P_i(k)$. Without loss of generality we will prove that $\{(k+1)\delta, ..., 1-(k+1)\delta\} \subseteq P_1(k+1)$. The choice $c_1 \in \{(k+1)\delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for the point belief b_1 with $b_1(c_1-\delta, ..., c_1-\delta) = 1$. With this belief, c_1 is rightmost and $u_1(c_1, b_1) = \frac{2-2c_1+\delta}{2} = 1-c_1+\frac{\delta}{2} \geq 1-\frac{1}{2}+\frac{\delta}{2} > \frac{1}{2}$. If agent 1 chooses $c \in \{c_1+\delta, ..., 1\}$, then c is rightmost and $u_1(c, b_1) = \frac{2-c-c_1+\delta}{2} < \frac{2-2c_1+\delta}{2} = u_1(c_1, b_1)$. If agent 1 chooses $c_1-\delta$, then all agents are positioned at $c_1-\delta$ and $u_1(c_1-\delta, b_1) = \frac{1}{N} < u_1(c_1, b_1)$. If agent 1 chooses $c \in \{0, ..., c_1-2\delta\}$, then c is leftmost and $u_1(c, b_1) = \frac{c+c_1-\delta}{2} \leq \frac{2c_1-3\delta}{2} \leq \frac{1-3\delta}{2} < \frac{1}{2} < u_1(c_1, b_1)$. Hence, the choice $c_1 \in \{(k+1)\delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for the point belief b_1 with $b_1(c_1 - \delta, ..., c_1 - \delta) = 1$. By symmetry, the choice $c_1 \in \{\lfloor \frac{1+\delta}{2} \rfloor, ..., 1-(k+1)\delta\}$ is optimal for the point belief b_1 with $b_1(c_1 + \delta, ..., c_1 + \delta) = 1$. This proves that $\{(k+1)\delta, ..., 1-(k+1)\delta\} \subseteq P_1(k+1)$. □

Because there is a finite number of choices for each agent, there exists some integer k' such that $P_i(k) = P_i(k')$ for all $k \geq k'$, $i \in I$. The point rationalizable choices of agent i are exactly those choices in the minimum round $k+1$ such that all choices $c \in P_i(k)$ are optimal in $P_i(k)$ for a point belief over $P_{-i}(k)$. Now consider round 1 of the iterative procedure. Lemma 1 implies that the choices $\{\delta, ..., 1-\delta\}$ are optimal for some point belief. The remaining question is whether the choice 0 is optimal for some point belief, and because of symmetry, the answer for the choice 1 will be the same. Hence, if 0 is optimal in $P_i(0)$ for a point belief over $P_{-i}(0)$, then all choices in $P_i(0)$ are point rational. If 0 is not optimal for any point belief, then 0 and 1 will be eliminated and $P_i(1) = \{\delta, ..., 1-\delta\}$. The choices $\{2\delta, ..., 1-2\delta\}$ are optimal in $P_i(1)$ for some point belief over $P_{-i}(1)$. Hence, if $P_i(1) = \{\delta, ..., 1-\delta\}$, then the point rationalizable choices of agent i are $\{\delta, ..., 1-\delta\}$ if δ is optimal in $P_i(1)$ for a point belief over $P_{-i}(1)$. Lemma 1 implies that we can find the exact set of point rationalizable choices by finding the minimum choice x such that x is optimal in $P_i(\frac{1}{\delta} \cdot x)$ for a point belief over $P_{-i}(\frac{1}{\delta} \cdot x)$. The point rationalizable choices for each agent i are then given by $\{x, ..., 1-x\}$.

Definition 3. *Consider some sets $A_i \subseteq C_i$, for all $i \in I$.*

The tuple of sets $(A_1, ..., A_N)$ is a point best reponse set if $\forall i \in I$, $c_i \in A_i$ implies that there exist a point belief b_i over the set A_{-i} such that c_i is optimal for b_i.

Pearce (1984) and Bernheim (1984) proved that rationalizable choices are exactly those choices that are part of some point best response set. In the following lemma, we will do the same for point rationalizable choices.

Lemma 2. *For every agent i we have that $P_i = \{c_i \in C_i :$ there exists a point best response set $(A_1, ..., A_N)$ with $c_i \in A_i\}$*

Proof. We will first prove that all point rationalizable choices of agent i are part of a point best response set. It is sufficient to prove that the set $(P_1, ...P_N)$ is a point best response set. Consider a agent $i \in I$. By definition of the inductive procedure, $c_i \in P_i$ implies that there exists a point belief b_i over the set P_{-i} such that c_i is optimal in P_i given b_i. If we can prove that c_i is optimal among all choices in the original game, then $(P_1, ...P_N)$ is a point best response set. Assume by contradiction that c_i is not optimal given b_i. Then there is some $c_i' \in C_i$ such that $u_i(c_i', b_i) > u_i(c_i, b_i)$. Now let $c_i^* \in C_i$ be an optimal choice given b_i in the original game. Then $u_i(c_i^*, b_i) \geq u_i(c_i', b_i) > u_i(c_i, b_i)$. We have that c_i^* is optimal for the belief b_i in the original game, where b_i is over P_{-i}.

By definition of the iterative procedure, all choices contained in the choice combinations of the opponents $c_{-i} \in P_{-i}$ are not eliminated in any round. As c_i^* is optimal for b_i in the original game, and b_i is over P_{-i}, the choice c_i^* is not eliminated in any round of the iterative procedure and $c_i^* \in P_i$. We have now that $u_i(c_i^*, b_i) > u_i(c_i, b_i)$ and $c_i^* \in P_i$, which contradicts the original fact that c_i is optimal in P_i for the point belief b_i. Hence, c_i is optimal for the point belief b_i in the original game and as a result $(P_1, ...P_N)$ is a best response set.

The other direction is to prove that if a choice is part of a point best response set, it is point rationalizable. For all i, let $B_i = \{c_i \in C_i : \text{there exists a point best response set } A_1, ..., A_N, \text{ and } c_i \in A_i\}$. We will first prove that the set $(B_1, ..., B_N)$ is a point best response set. For agent $i \in I$, a choice $c_i \in B_i$ implies that there exists a set $(A_1, ..., A_N)$ such that c_i is optimal given a point belief b_i over A_{-i}. Because $A_i \subseteq B_i \ \forall i \in I$, we know that $A_{-i} \subseteq B_{-i}$. Hence, $(B_1, ..., B_N)$ is a point best response set. We will now show by induction that $B_i \subseteq P_i(k)$ for all k and i. It is immediate that $B_i \subseteq P_i(1) \ \forall i \in I$, because $(B_1, ..., B_N)$ is a point best response set. Now assume that $B_i \subseteq P_i(k) \ \forall i \in I$ and for some k . The choice $c_i \in B_i$ implies that c_i is optimal for a point belief b_i over the set $B_{-i} \subseteq P_{-i}(k)$, which implies that $c_i \in P_i(k+1)$. Hence, for all k and i we have that $B_i \subseteq P_i(k)$. This in turn implies that $B_i \subseteq P_i$, which completes the proof. □

Proof of Theorem 1 with Two Agents

Proof. We will prove that in each round $k \leq \frac{1}{\delta} \cdot \left\lceil \frac{1-\delta}{2} \right\rceil$ of the iterative procedure, we have $P_1(k) = P_2(k) = \{k\delta, ..., 1 - k\delta\}$. If $k > \frac{1}{\delta} \cdot \left\lceil \frac{1-\delta}{2} \right\rceil$, then $P_1(k) = P_2(k) = \{\left\lceil \frac{1-\delta}{2} \right\rceil, 1 - \left\lceil \frac{1-\delta}{2} \right\rceil\}$. As a result, the set of point rationalizable is given by $P_1(k) = P_2(k) = \{\left\lceil \frac{1-\delta}{2} \right\rceil, 1 - \left\lceil \frac{1-\delta}{2} \right\rceil\}$.

We start by proving the base case $P_1(1) = P_2(1) = \{\delta, ..., 1 - \delta\}$. Without loss of generality we take the perspective of agent 1. The choice δ yields a strictly higher utility than 0 for any point belief over $P_2(0) = C_2$. Consider the point belief b_1 with $b_1(0) = 1$. Then $u_1(0, b_1) = \frac{1}{2} < \frac{2-\delta-0}{2} = u_1(\delta, b_1)$. Now consider the point belief $b_1(\delta) = 1$. Then $u_1(0, b_1) = \frac{0+\delta}{2} < \frac{1}{2} = u_1(\delta, b_1)$. Lastly, consider the point belief b_1 with $b_1(c_2) = 1$, where $c_2 \in \{2\delta, ..., 1\}$. Then $u_1(0, b_1) = \frac{0+c_2}{2} < \frac{\delta+c_2}{2} = u_1(\delta, b_1)$. Hence, 0 is not optimal for any point belief and by symmetry, the choice 1 is also not optimal for any point belief.

We will show that the choice $c_1 \in \{\delta, ..., \left\lceil \frac{1-\delta}{2} \right\rceil\}$ is optimal for the point belief b_1 with $b_1(c_1 - \delta) = 1$. With this belief, c_1 is rightmost and $u_1(c_1, b_1) = \frac{2-2c_1+\delta}{2} = 1 - c_1 + \frac{\delta}{2} \geq 1 - \frac{1}{2} + \frac{\delta}{2} > \frac{1}{2}$. If agent 1 chooses $c \in \{c_1 + \delta, ..., 1\}$, then c is rightmost and $u_1(c, b_1) = \frac{2-c-c_1+\delta}{2} < \frac{2-2c_1+\delta}{2} = u_1(c_1, b_1)$. If agent 1 chooses $c_1 - \delta$, then both agents choose $c_1 - \delta$ and $u_1(c_1 - \delta, b_1) = \frac{1}{2} < u_1(c_1, b_1)$. If agent 1 chooses $c \in \{0, ..., c_1 - 2\delta\}$, then c is leftmost and $u_1(c, b_1) = \frac{c+c_1-\delta}{2} \leq \frac{2c_1-3\delta}{2} \leq \frac{1-3\delta}{2} < \frac{1}{2} < u_1(c_1, b_1)$. Hence, the choice $c_1 \in \{\delta, ..., \left\lceil \frac{1-\delta}{2} \right\rceil\}$ is optimal for the point belief b_1 with $b_1(c_1 - \delta) = 1$. By symmetry, the choice $c_1 \in \{\lfloor \frac{1+\delta}{2} \rfloor, ..., 1-\delta\}$ is optimal for the point belief b_1 with $b_1(c_1 + \delta) = 1$. This proves that $P_1(1) = P_2(1) = \{\delta, ..., 1-\delta\}$.

Now we assume that $1 \leq k < \frac{1}{\delta} \cdot \lceil \frac{1-\delta}{2} \rceil$ and $P_1(k) = P_2(k) = \{k\delta, ..., 1-k\delta\}$ and prove that $P_1(k+1) = P_2(k+1) = \{(k+1)\delta, ..., 1-(k+1)\delta\}$. The choice $(k+1)\delta$ yields a strictly higher utility than $k\delta$ for any point belief over $P_2(k)$. Consider the point belief b_1 with $b_1(k\delta) = 1$. Then $u_1(k\delta, b_1) = \frac{1}{2}$ and $u_1((k+1)\delta, b_1) = \frac{2-(k+1)\delta-k\delta}{2} \geq \frac{2-\frac{1}{2}-\frac{1-2\delta}{2}}{2} > \frac{1}{2} = u_1(k\delta, b_1)$. Next, consider the point belief b_1 with $b_1((k+1)\delta) = 1$. Then $u_1(k\delta, b_1) = \frac{k\delta+(k+1)\delta}{2} \leq \frac{\frac{1-2\delta}{2}+\frac{1}{2}}{2} < \frac{1}{2} = u_1((k+1)\delta, b_1)$. Lastly, consider the point belief b_1 with $b_1(c_2)$, where $c_2 \in \{(k+2)\delta, ..., 1-k\delta\}$. Then $u_1(k\delta, b_1) = \frac{k\delta+c_1}{2} < \frac{(k+1)\delta+c_2}{2} = u_1((k+1)\delta, b_1)$. Hence, the choice $k\delta$ is not optimal for any point belief and because of symmetry, the choice $1-k\delta$ is also not optimal for any point belief.

The choice $c_1 \in \{(k+1)\delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for the point belief b_1 with $b_1(c_1 - \delta) = 1$. With this belief, c_1 is rightmost and $u_1(c_1, b_1) = \frac{2-2c_1+\delta}{2} = 1 - c_1 + \frac{\delta}{2} \geq 1 - \frac{1}{2} + \frac{\delta}{2} > \frac{1}{2}$. If agent 1 chooses $c \in \{c_1+\delta, ..., 1\}$, then c is rightmost and $u_1(c, b_1) = \frac{2-c-c_1+\delta}{2} < \frac{2-2c_1+\delta}{2} = u_1(c_1, b_1)$. If agent 1 chooses $c_1 - \delta$, then both agents choose $c_1 - \delta$ and $u_1(c_1 - \delta, b_1) = \frac{1}{2} < u_1(c_1, b_1)$. If agent 1 chooses $c \in \{0, ..., c_1 - 2\delta\}$, then c is leftmost and $u_1(c, b_1) = \frac{c+c_1-\delta}{2} \leq \frac{2c_1-3\delta}{2} \leq \frac{1-3\delta}{2} < \frac{1}{2} < u_1(c_1, b_1)$. Hence, the choice $c_1 \in \{(k+1)\delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for the point belief b_1 with $b_1(c_1 - \delta) = 1$. By symmetry, the choice $c_1 \in \{\lfloor \frac{1+\delta}{2} \rfloor, ..., 1-(k+1)\delta\}$ is optimal for the point belief b_1 with $b_1(c_1+\delta) = 1$. This proves that $P_1(k+1) = P_2(k+1) = \{(k+1)\delta, ..., 1-(k+1)\delta\}$.

Next let $k = \frac{1}{\delta} \cdot \lceil \frac{1-\delta}{2} \rceil$. Then $P_1(k) = P_2(k) = \{k\delta, ..., 1-k\delta\}$. If $\frac{1}{\delta}$ is even, then $k = \frac{1}{\delta} \cdot \frac{1}{2}$ and $P_1(k) = P_2(k) = \{\frac{1}{2}\}$. Because both agents only have one choice remaining, it must be that $P_1(k) = P_2(k) = P_1(k+1) = P_2(k+1) = \{\frac{1}{2}\}$. If $\frac{1}{\delta}$ is odd, then $k = \frac{1}{\delta} \cdot \frac{1-\delta}{2}$ and $P_1(k) = P_2(k) = \{\frac{1-\delta}{2}, 1 - \frac{1-\delta}{2}\} = \{\frac{1-\delta}{2}, \frac{1+\delta}{2}\}$. Consider the point belief b_1 with $b_1(\frac{1-\delta}{2}) = 1$. Then $u_1(\frac{1-\delta}{2}, b_1) = \frac{1}{2}$ and $u_1(\frac{1+\delta}{2}, b_1) = \frac{\frac{1-\delta}{2}+\frac{1+\delta}{2}}{2} = \frac{1}{2}$. Similarly for the point belief $b_1(\frac{1+\delta}{2}) = 1$ we have $u_1(\frac{1-\delta}{2}, b_1) = \frac{1}{2}$ and $u_1(\frac{1+\delta}{2}, b_1) = \frac{1}{2}$. Hence, $P_1(k) = P_2(k) = P_1(k+1) = P_2(k+1) = \{\frac{1-\delta}{2}, \frac{1+\delta}{2}\}$. Because no choices are eliminated for both agents in round k, we have that $P_1(k) = P_2(k) = \{\lceil \frac{1-\delta}{2} \rceil, 1 - \lceil \frac{1-\delta}{2} \rceil\}$ if $k \geq \frac{1}{\delta} \cdot \lceil \frac{1-\delta}{2} \rceil$. □

Proof of Theorem 1 with Three Agents

Proof. Let $x = \lceil \frac{1-2\delta}{4} \rceil$. We will show that the set $(\{x, ..., 1-x\}, \{x, ..., 1-x\}, \{x, ..., 1-x\})$ is a point best response set, and by lemma 1, the choices $\{x, ..., 1-x\}$ are point rationalizable for each agent. Without loss of generality we prove that all choices in $\{x, ..., 1-x\}$ are optimal for agent 1 for some point belief over the set $\{x, ..., 1-x\} \times \{x, ..., 1-x\}$. We assume that $\delta \leq \frac{1}{7}$.

This ensures that the proof can be read more intuitively. For example, in the proof we use the choice $c \in \{x + 2\delta, ..., 1 - x - 2\delta\}$. Intuitively, we want that $x + 2\delta < 1 - x - 2\delta$, so the left element in the set is smaller than the right element. We can ensure this by assuming that $\delta \leq \frac{1}{7}$.

Consider the belief b_1 with $b_1(x + \delta, 1 - x - \delta) = 1$. Then x is leftmost and $u_1(x, b_1) = \frac{x+x+\delta}{2} = x + \frac{\delta}{2} \geq \frac{1-2\delta}{4} + \frac{\delta}{2} = \frac{1}{4}$. The choice $c \in \{0, ..., x - \delta\}$ is leftmost and $u_1(c, b_1) = \frac{c+x+\delta}{2} \leq \frac{x-\delta+x+\delta}{2} = x < u_1(x, b_1)$. The choice $x + \delta$ is leftmost sharing with agent 2 and $u_1(x + \delta, b_1) = \frac{x+\delta+1-x-\delta}{4} = \frac{1}{4} \leq u_1(x, b_1)$. The choice $c \in \{x + 2\delta, ..., 1 - x - 2\delta\}$ is middle and $u_1(c, b_1) = \frac{1-x-\delta-x-\delta}{2} = \frac{1}{2} - x - \delta \leq \frac{1}{2} - \frac{1-2\delta}{4} - \delta = \frac{1-2\delta}{4} < u_1(x, b_1)$. The choice $1 - x - \delta$ is rightmost sharing with agent 3 and $u_1(1 - x - \delta, b_1) = \frac{2-(1-x-\delta)-(x+\delta)}{4} = \frac{1}{4} \leq u_1(x, b_1)$. The choice $c \in \{1 - x, ..., 1\}$ is rightmost and $u_1(c, b_1) = \frac{2-c-(1-x-\delta)}{2} \leq \frac{2-(1-x)-(1-x-\delta)}{2} = \frac{2x+\delta}{2} = u_1(x, b_1)$. Hence, the choice x is optimal for agent 1 for belief b_1 with $b_1(x + \delta, 1 - x - \delta) = 1$ and by symmetry the choice $1 - x$ is also optimal for this belief.

We will show that the choice $c_1 \in \{x + \delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for belief b_1 with $b_1 = (c_1 - \delta, c_1 - \delta) = 1$. With this belief, c_1 is rightmost and $u_1(c_1, b_1) = \frac{2-2c_1+\delta}{2} = 1 - c_1 + \frac{\delta}{2} \geq 1 - \frac{1}{2} + \frac{\delta}{2} > \frac{1}{2}$. If agent 1 chooses $c \in \{c_1 + \delta, ..., 1\}$, then c is rightmost and $u_1(c, b_1) = \frac{2-c-c_1+\delta}{2} < \frac{2-2c_1+\delta}{2} = u_1(c_1, b_1)$. If agent 1 chooses $c_1 - \delta$, then all agents are positioned at $c_1 - \delta$ and $u_1(c_1 - \delta, b_1) = \frac{1}{3} < u_1(c_1, b_1)$. If agent 1 chooses $c \in \{0, ..., c_1 - 2\delta\}$, then c is leftmost and $u_1(c, b_1) = \frac{c+c_1-\delta}{2} \leq \frac{2c_1-3\delta}{2} \leq \frac{1-3\delta}{2} < \frac{1}{2} < u_1(c_1, b_1)$. Hence, the choice $c_1 \in \{x+\delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for belief b_1 with $b_1 = (c_1-\delta, c_1-\delta) = 1$. By symmetry, the choice $c_1 \in \{\lfloor \frac{1+\delta}{2} \rfloor, ..., 1 - x - \delta\}$ is optimal for the belief $(c_1 + \delta, c_1 + \delta)$.

Let $y = x - \delta = \lceil \frac{1-6\delta}{4} \rceil$. Then $y > 0$, because $\delta \leq \frac{1}{7}$. We will prove that $(\{y, ..., 1-y\}, \{y, ..., 1-y\}, \{y, ..., 1-y\})$ does not have the point best response property. Consider the choice y of agent 1. We will prove that there does not exist a point belief over $\{y, ..., 1 - y\} \times \{y, ..., 1 - y\}$ such that y is optimal for agent 1. Consider the belief where agent 2 and 3 both choose y, then agent 1 is better off choosing $y + \delta$ rather than y. Now consider a belief where agent 2 chooses y as well. If agent 3 is positioned at $y + \delta$, then agent 1 is better off locating at $y + 2\delta$. Now assume that agent 3 is positioned at one of the remaining positions L. Agent 1's payoff can then be written as $\frac{y+L}{4}$. If agent 1 locates between y and L instead, he obtains $\frac{L-y}{2}$. For y to be optimal for agent 1 we need $\frac{y+L}{4} \geq \frac{L-y}{2}$ which implies that $L \leq 3y$. If agent 1 chooses L as well he obtains $\frac{2-L-y}{4}$. For y to be optimal we need $\frac{y+L}{4} \geq \frac{2-L-y}{4}$, which implies that $L \geq 1 - y$. This leads to $y \geq \frac{1}{4}$, but we know $y = \lceil \frac{1-6\delta}{4} \rceil < \frac{1}{4}$. As a result, there is no point belief that contains y such that y is optimal for agent 1.

From now on, we will assume that agent 1's belief does not contain the position y. If the belief does not contain at least one choice positioned at $y + \delta$, then agent 1 is better off locating to this choice. So without loss of generality, we assume that agent 1 believes that agent 2 is positioned at $y + \delta$. A belief where agent 3 also chooses $y + \delta$ does not work, because locating to $y + 2\delta$ is then optimal for agent 1. Hence, agent 1 should believe that agent 3 is positioned to the right of $y + \delta$. Agent 1's payoff for choosing y is then always $\frac{y+y+\delta}{2} = y + \frac{\delta}{2}$. Agent 3's choice cannot be too close to agent 1. If agent 1 believes that agent 3 is positioned to any position to the left of $1 - y - \delta$, agent 1 is better off locating to $1 - y - \delta$ and obtaining at least $\frac{2-(1-y-\delta)-(1-y-2\delta)}{2} = \frac{2y+3\delta}{2} = y + \frac{3}{2}\delta$. Consider the belief b_1 with $b_1(y+\delta, 1-y-\delta) = 1$. Then $u_1(y, b_1) = y + \frac{\delta}{2} = \left\lceil \frac{1-6\delta}{4} \right\rceil + \frac{1}{2}\delta \leq \frac{1-3\delta}{4} + \frac{1}{2}\delta = \frac{1-\delta}{4}$ and $u_1(y + \delta, b_1) = \frac{y+\delta+1-y-\delta}{4} = \frac{1}{4}$. Lastly, consider the belief b_1 where $b_1(y+\delta, 1-y) = 1$. Then $u_1(y, b_1) \leq \frac{1-\delta}{4}$ and any choice $c \in \{y+2\delta, ..., 1-y-\delta\}$ yields $u_1(c, b_1) = \frac{1-y-y-\delta}{2} = \frac{1}{2} - y - \frac{\delta}{2} = \frac{1}{2} - \left\lceil \frac{1-6\delta}{4} \right\rceil - \frac{1}{2}\delta = \left\lceil \frac{1+4\delta}{4} \right\rceil$. Hence, there does not exist a point belief over $\{y, ..., 1 - y\} \times \{y, ..., 1 - y\}$ such that y is optimal for agent 1. By symmetry, a similar result holds for the choice $1 - y$. □

Proof of Theorem 1 with Four or more Agents

Proof. Let $x = \left\lceil \frac{1-(N-2)\delta}{2N-2} \right\rceil$. We prove that the set $(\{x, ..., 1 - x\}, ..., \{x, ..., 1 - x\})$ is a point best response set. Hence, by lemma 1 we would know that the choices $\{x, ..., 1 - x\}$ are point rationalizable for each agent. Without loss of generality we prove that all choices in $\{x, ..., 1 - x\}$ are optimal for agent 1 for some point belief over the set $\{x, ..., 1 - x\} \times ... \times \{x, ..., 1 - x\}$. Consider the belief b_1 with $b_1(c_2, ..., c_N) = 1$ where $c_2 = x + \delta$, $c_N = 1 - x - \delta$ and for $j \in \{3, ..., N - 1\}$, $c_j = (2j - 3)x + (j - 2)\delta$. With this belief, x is leftmost and $u_1(x, b_1) = \frac{x+x+\delta}{2} = x + \frac{1}{2}\delta \geq \frac{1+(2-N)\delta}{(2N-2)} + \frac{1}{2}\delta = \frac{4+4\delta}{4(2N-2)}$. The choice $c \in \{0, ..., x - \delta\}$ is leftmost with payoff $u_1(c, b_1) = \frac{c+x+\delta}{2} \leq \frac{x-\delta+x+\delta}{2} = \frac{2x}{2} < x + \frac{1}{2}\delta = u_1(x, b_1)$. The choice $c_2 = x + \delta$ is leftmost sharing with agent 2. The payoff is $u_1(c_2, b_1) = \frac{x+\delta+3x+\delta}{4} = \frac{4x+2\delta}{4} = x + \frac{1}{2}\delta = u_1(x, b_1)$.

For all agents $j \in \{3, ..., N - 1\}$, the distance between c_j and c_{j-1} is $c_j - c_{j-1} \leq (2j - 3)x + (j - 2)\delta - (2j - 5)x - (j - 3)\delta \leq 2x + \delta$. An inequality sign is used because the distance between c_3 and c_2 is $c_3 - c_2 = 3x + \delta - x - \delta = 2x$. The distance between the remaining agents is $2x + \delta$. Consider the choices in $c \in \{c_2 + \delta, ..., c_{N-1} - \delta\}$. This set includes all choices in between c_2 and c_{N-1}, except the choice(s) c_k, where $k \in \{3, ..., N - 2\}$. This case has been dealt with seperately. The reason for this is that if agent 1 chooses c, he will be the only agent at this position, whereas if agent 1 would choose c_k he will have to share his position with another agent. The choice c is middle and in between c_j and c_{j-1}, where $j \in \{3, ..., N - 1\}$. The payoff is $u_1(c, b_1) = \frac{c_j - c_{j-1}}{2} \leq \frac{2x+\delta}{2} = x + \frac{1}{2}\delta = u_1(x, b_1)$. Now consider a choice positioned at $c_k = (2k-3)x+(k-2)\delta$, where $k \in \{3, ..., N-2\}$. Then c_k is middle sharing with agent k and $u_1(c_k, b_1) = \frac{c_{k+1}-c_{k-1}}{4} \leq \frac{(2k-1)x+(k-1)\delta-(2k-5)x-(k-3)\delta}{4} = \frac{4x+2\delta}{4} = x + \frac{1}{2}\delta = u_1(x, b_1)$.

Consider the choice $c_{N-1} = (2N-5)x + (N-3)\delta$. This choice is middle sharing with agent $N-1$ and $u_1(c_{N-1}, b_1) = \frac{c_N - c_{N-2}}{4} = \frac{1-x-\delta-(2N-7)x-(N-4)\delta}{4} = \frac{1-(2N-6)x-(N-3)\delta}{4} \leq \frac{1+(6-2N)(\frac{1+(2-N)\delta}{2N-2})+(3-N)\delta}{4}$. This can be rewritten to $\frac{4-(2N-6)\delta}{4(2N-2)} < \frac{4+4\delta}{4(2N-2)} \leq u_1(x, b_1)$

Consider a choice $c \in \{c_{N-1}+\delta, ..., c_N - \delta\}$. This choice is middle and $u_1(c, b_1) = \frac{c_N - c_{N-1}}{2} = \frac{1-x-\delta-(2N-5)x-(N-3)\delta}{2} = \frac{1+(4-2N)x+(2-N)\delta}{2} =\leq \frac{1+(4-2N)(\frac{1-(N-2)\delta}{(2N-2)})+(2-N)\delta}{2}$. This can be rewritten to $\frac{4-(4N-8)\delta}{4} < \frac{4+4\delta}{4(2N-2)} \leq u_1(x, b_1)$

Consider the choice $c_N = 1-x-\delta$. This choice is rightmost sharing with agent N and $u_1(c_N, b_1) = \frac{2-(1-x-\delta)-c_{N-1}}{4} = \frac{1+x+\delta-(2N-5)x-(N-3)\delta}{4} = \frac{1+(6-2N)x+(4-N)\delta}{4} \leq \frac{1+(6-2N)(\frac{1-(N-2)\delta}{(2N-2)})+(4-N)\delta}{4}$. This can be rewritten to $\frac{4+4\delta}{4(2N-2)} \leq u_1(x, b_1)$

Lastly, consider the choice $c \in \{c_N+\delta, ..., 1\}$. Then c is rightmost and $u_1(c, b_1) = \frac{2-c-c_N}{2} \leq \frac{2-(c_N+\delta)-c_N}{2} = \frac{2-2c_N-\delta}{2} = \frac{2-2(1-x-\delta)-\delta}{2} = \frac{2x+\delta}{2} = u_1(x, b_1)$.

Hence, the choice x is optimal for agent 1 for belief b_1 with $b_1(c_2, ..., c_N) = 1$. Now consider the belief b_1 with $b_1(c_2, ..., c_N) = 1$ where $c_2 = 1 - x - \delta$, $c_N = x$ and for $j \in \{3, ..., N-1\}$, $c_j = 1-(2j-3)x-(j-2)\delta$. By symmetry, the choice $1 - x$ is optimal for this belief.

The choice $c_1 \in \{x + \delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for the belief b_1 where $b_1(c_1 - \delta, ..., c_1 - \delta) = 1$. With this belief, c_1 is rightmost and $u_1(c_1, b_1) = \frac{2-2c_1+\delta}{2} = 1-c_1+\frac{1}{2}\delta \geq 1-\frac{1}{2}+\frac{1}{2}\delta > \frac{1}{2}$. If agent 1 chooses $c \in \{c_1+\delta, ..., 1\}$, then c is rightmost and $u_1(c, b_1) = \frac{2-c-c_1+\delta}{2} < \frac{2-2c_1+\delta}{2} = u_1(c_1, b_1)$. If agent 1 chooses $c_1 - \delta$, then all agents are positioned at $c_1 - \delta$ and $u_1(c_1 - \delta, b_1) = \frac{1}{N} < u_1(c_1, b_1)$. If agent 1 chooses $c \in \{0, ..., c_1 - 2\delta\}$, then c is leftmost and $u_1 = (c, b_1) = \frac{c+c_1-\delta}{2} \leq \frac{2c_1-3\delta}{2} \leq \frac{1-3\delta}{2} < \frac{1}{2} < u_1(c_1, b_1)$. Hence, the choice $c_1 \in \{x + \delta, ..., \lceil \frac{1-\delta}{2} \rceil\}$ is optimal for belief b_1 with $b_1 = (c_1 - \delta, ..., c_1 - \delta) = 1$. By symmetry, the choice $c_1 \in \{\lfloor \frac{1+\delta}{2} \rfloor, ..., 1-x-\delta\}$ is optimal for the belief b_1 with $b_1(c_1+\delta, ..., c_1+\delta) = 1$. As a result, we have that $(\{x, ..., 1-x\}, ..., \{x, ..., 1-x\})$ is a point best response set.

Let $y = x - \delta = \left\lceil \frac{1-(N-2)\delta}{(2N-2)} \right\rceil - \frac{(2N-2)\delta}{(2N-2)} = \left\lceil \frac{1-(3N-4)\delta}{(2N-2)} \right\rceil$. The condition $\delta < \frac{1}{3(N-3)+6}$ is needed to ensure that $y > 0$. If $y = 0$, then the proof is no longer valid. Point beliefs about the choices of the opponents we will label such as $c_2 = y + \delta$ and $c_3 = 3y + \delta$ would then be the same choice, which is not what is meant in the proof.

We will prove that $(\{y,...,1-y\},...,\{y,...,1-y\})$ is not a point best response set by showing that y is not optimal for agent 1 for any point belief over $\{y,...,1-y\} \times ... \times \{y,...,1-y\}$. For the point belief b_1 where $b_1(y,...,y) = 1$, agent 1 is better off locating to $y+\delta$. Hence, for the remainder of the proof agent 1 assumes that at least one agent is not positioned at y. We will denote the belief of agent 1 about an individual opponent's choice as c_j, where $j \in \{2,...,N\}$. Because of symmetry, we can assume that $c_2 \leq c_3 \leq ... \leq c_N$. Let L denote the closest agent to agent 1. Because y is a leftmost choice for any point belief, the payoff of agent 1 can be denoted as $\frac{y+L}{2D_1(y,c_{-1})}$. Hence, for higher values of y the payoff for agent 1 increases, which makes deviating to a different choice less attractive. If the distance between some other agents is too large, agent 1 can deviate between these agents to obtain a higher payoff. Similarly, if all the agents are positioned too closely to each other, there will exist a "gap" on the line where agent 1 can deviate to. Between each agent there will be a maximum distance such that agent 1 does not want to deviate from his original choice y. We will show that even if we maximize the space between all agents given the constraints that we have, there is still too much space at the end of the line, which results in a higher payoff for agent 1 if he deviates to a certain position at the end of the line. We will also show that the main reason for this is because y is not large enough.

Assume that agent 1 believes that there is exactly one other agent positioned at y as well. This implies that $c_2 = y$. The payoff of agent 1 then depends on the position of the closest agent to y, which is c_3. The payoff of agent 1 is thus equal to $\frac{y+c_3}{4}$. If agent 1 would locate in between y and c_3, his payoff is equal to $\frac{c_3-y}{2}$. Hence, $\frac{y+c_3}{4} \geq \frac{c_3-y}{2}$ which implies $c_3 \leq 3y$. To maximize the distance between agent 1 and agent 3, agent 1 believes that agent 3 is positioned at $3y$. agent 1's payoff is then equal to $\frac{y+3y}{4} = y$. Now let L be the closest agent to the right of $3y$. agent 1's payoff for choosing $3y$ is equal to $\frac{L-y}{2D_1(3y,c_{-1})}$. Note that $D_1(3y,c_{-1}) = 2$ when only agent 1 and 3 are positioned on $3y$. If additional agents are also positioned on $3y$, then $D_1(3y,c_{-1}) > 2$. Hence, $y \geq \frac{L-y}{2D_1(3y,c_{-1})}$ which implies $L \leq (2D_1(3y,c_{-1})+1)y$. If $D_1(3y,c_{-1}) = 2$, this simplifies to $L \leq 5y$. If additional agents are positioned on $3y$, then the right term of the constraint will increase as well. For example, if agent 1 and 3 and 4 are positioned on $3y$, the constraint simplifies to $L \leq 7y$. However, even if additional agents are positioned on $3y$, when agent 1 chooses a position in between $3y$ and L, it leads to $y \geq \frac{L-3y}{2}$ which implies that $L \leq 5y$. Hence, if additional agents are positioned on $3y$, it leads to the same maximum choice where L can be positioned. Furthermore, additional agents on $3y$ implies that there are less agents in total to use to maximize the distance between the agents. Hence, $c_4 = 5y$.

This pattern continues and we find that for $j \in \{3,.....,N-1\}, c_j = (2j-3)y$. agent 3 until agent $N-1$ are spread out in a manner such that the distance is maximized between the agents and agent 1 has no incentive to deviate to any choice in $\{y+\delta,...,c_{N-1}-\delta\}$. If agent 1 would locate at c_N, his payoff is equal

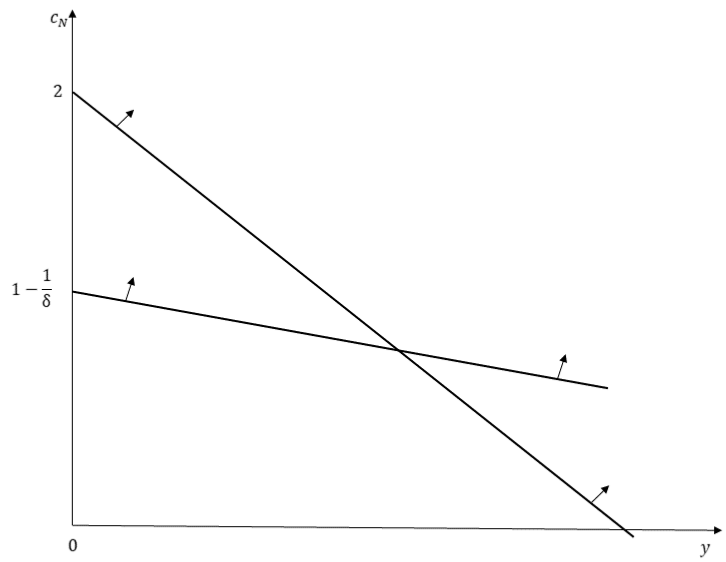

Figure 9: Feasible region when agent 2 is also positioned at y

to $\frac{2-c_N-c_{N-1}}{4} = \frac{2-c_N-(2N-5)y}{4}$. Hence, $y \geq \frac{2-c_N-(2N-5)y}{4}$ which implies that $c_N \geq 2-(2N-1)y$. Note that at least one agent should locate at $1-y-\delta$ or $1-y$, because otherwise agent 1 can deviate to $1-y-\delta$ and obtain a higher payoff of at least $\frac{2-(1-y-\delta)-(1-y-2\delta)}{2} = y + \frac{3}{2m} > y$. Hence, we have $c_N \geq 1-y-\delta$ and $c_N \geq 2-(2N-1)y$. We are interested in the minimum value of y such that both constraints are satisfied. The first constraint has a smaller slope and a smaller intercept than the second constraint. Figure 9 plots the two constraints. The arrows represent the direction in which each constraint is satisfied. From the figure it is clear that the minimum value of y for which both constraints are satisfied is exactly when the lines intersect. The lines intersect when $y = \frac{1+\delta}{2N-2}$. This might not be a position on the line, so the minimum permitted value is given by $y = \left\lceil \frac{1+\delta}{2N-2} \right\rceil$. However, we have that $y = \left\lceil \frac{1-(3N-4)\delta}{2N-2} \right\rceil < \left\lceil \frac{1+\delta}{2N-2} \right\rceil$. Hence, a point belief where one other agent is also positioned at y does not lead to y being optimal for agent 1. A point belief where additional agents are positioned at y does not lead to y being optimal for agent 1. The payoff of agent 1 is then $\frac{y+L}{2D_1(y,c_{-1})} < \frac{y+L}{4}$, which means that deviating from y is more attractive and leads to the condition that agents need to be positioned even closer to each other. Furthermore, there is at least one less agent to the right of y, which cannot be used for maximizing the space between the agents. As a result, a point belief of agent 1 that contains the choice y does not lead to y being optimal for agent 1.

From now on, we assume that the point belief of agent 1 does not contain y. If none of the agents locate at $y + \delta$, agent 1 is better off locating to $y + \delta$, so without loss of generality agent 1 believes that $c_2 = y + \delta$. Because the remaining agents are positioned at a choice to the right of c_2 or exactly on c_2, the payoff of agent 1 is equal to $\frac{y+c_2}{2} = \frac{y+y+\delta}{2} = y + \frac{1}{2}\delta$.

If agent 1 would choose to locate at c_2 instead, the payoff it will receive depends on how many agents are positioned at c_2 and the position of the closest agent to the right of c_2. We will call this position L. Deviating to c_2 leads to a payoff of $\frac{y+\delta+L}{2D_1(c_2,c_{-1})}$ for agent 1. We have that $D_1(c_2, c_{-1}) = 2$ when only agent 1 and 2 are positioned on c_2. If additional agents are also positioned on c_2, then $D_1(c_2, c_{-1}) > 2$. Hence, $y + \frac{1}{2}\delta \geq \frac{y+\delta+L}{2D_1(c_2,c_{-1})}$ which implies $L \leq 2D_1(c_2, c_{-1})(y + \frac{1}{2}\delta) - y - \delta$. If $D_1(c_2, c_{-1}) = 2$, this simplifies to $L \leq 3y + \delta$. If $D_1(c_2, c_{-1}) > 2$, then the right term of the constraint will increase with $2y + \delta$ for each additional agent. If agent 1 would deviate to $c \in \{y + 2\delta, ..., L - \delta\}$ it leads to a payoff of $\frac{L-c_2}{2} = \frac{L-y-\delta}{2}$ for agent 1. Hence, $y + \frac{1}{2}\delta \geq \frac{L-y-\delta}{2}$ which implies $L \leq 3y + 2\delta$. We thus have the two constraints $L \leq 2D_1(c_2, c_{-1})(y+\frac{1}{2}\delta)-y-\delta$ and $L \leq 3y+2\delta$. Which constraint is the stricter one depends on the value of $D_1(c_2, c_{-1})$. If $D_1(c_2, c_{-1}) = 2$, then only agent 1 and 2 are positioned at c_2 and the first constraint simplifies to $L \leq 3y + \delta$. This is a stricter constraint than the other constraint of $L \leq 3y + 2\delta$. To maximize the distance between the agents, we have $c_3 = 3y + \delta$.

For now, assume that $D_1(c_2, c_{-1}) = 2$. The agent closest to the right of $c_3 = 3y + \delta$ is now labeled L. The payoff of agent 1 for locating to c_3 is then equal to $\frac{L-c_2}{2D_1(c_3,c_{-1})} = \frac{L-y-\delta}{2D_1(c_3,c_{-1})}$. Hence, $y + \frac{1}{2}\delta \geq \frac{L-y-\delta}{2D_1(c_3,c_{-1})}$ which implies $L \leq 2D_1(c_3, c_{-1})(y+\frac{1}{2}\delta)+y+\delta$. If $D_1(c_3, c_{-1}) = 2$, then only agent 1 and 3 are positioned at c_3 and the constraint simplifies to $L \leq 5y+3\delta$. If $D_1(c_3, c_{-1}) > 2$, then the right term of the constraint will increase with $2y+\delta$ for each additional agent. The payoff for agent 1 of locating to $c \in \{c_3 + \delta, ..., L - \delta\}$ is $\frac{L-3y-\delta}{2}$. Hence, $y + \frac{1}{2}\delta \geq \frac{L-3y-\delta}{2}$ which implies $L \leq 5y + 2\delta$. Regardless of how many agents are positioned at $3y+\delta$, the strictest constraint will always be $L \leq 5y+2\delta$ and as a result $c_4 = 5y + 2\delta$.

If $D_1(c_2, c_{-1}) > 2$, then there is at least one additional agent positioned at c_2, which implies $c_3 = y + \delta = c_2$. This also means that the position L is no longer about agent 3. The right term of the first constraint $L \leq 2D_1(c_2, c_{-1})(y+\frac{1}{2}\delta) - y - \delta$ will be at least $5y + 2\delta$, because $D_1(c_2, c_{-1}) > 2$. Now the second constraint $L \leq 3y + 2\delta$ is the strictest. Note that if there is more than one additional agent, the strictest constraint is still $L \leq 3y + 2\delta$. Hence, we assume that L is the position of agent 4. To maximize the distance between the agents, we have that $c_4 = 3y + 2\delta$.

The assumption that $D_1(c_2, c_{-1}) = 2$ results in $c_3 = 3y + \delta$ and $c_4 = 5y + 2\delta$,

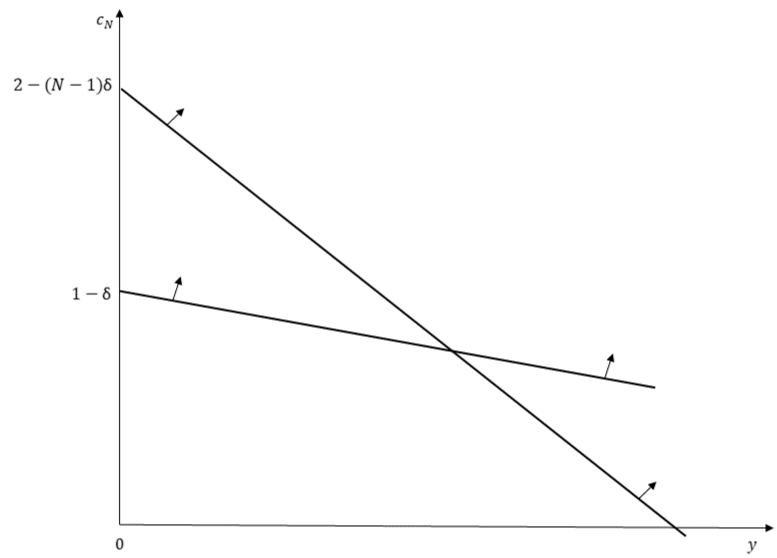

Figure 10: Feasible region when no other agent chooses y

while $D_1(c_2, c_{-1}) > 2$ results in $c_2 = c_3 = y + \delta$ and $c_4 = 3y + 2\delta$. Because $D_1(c_2, c_{-1}) = 2$ is better at maximizing the space between the agents, we will further explore this assumption.

From agent 3 on wards a pattern emerges. The maximum distance between each agent will be equal to $2y + \delta$. for example, $c_4 - c_3 = 5y + 2\delta - 3y - \delta = 2y + \delta$. This pattern continues until the second to last agent $N - 1$. We can construct this as the point belief b_1 with $b_1(c_2, ..., c_N) = 1$ where $c_2 = y + \delta$ and for $j \in \{3, ..., N-1\}$, $c_j = (2j-3)y + (j-2)\delta$. If agent 1 would locate at some c_j, then his payoff is equal to $\frac{c_{j+1}-c_{j-1}}{4} = \frac{(2j-1)y+(j-1)\delta-(2j-5)y-(j-3)\delta}{4} = \frac{4y+2\delta}{4} = y + \frac{1}{2}\delta = u_1(y, b_1)$. Similarly, if agent 1 would locate in between some c_j and c_{j+1}, his payoff is equal to $\frac{c_j - c_{j+1}}{2} = \frac{(2j-3)y+(j-2)\delta-(2j-1)y-(j-1)\delta}{2} = \frac{2y+\delta}{2} = y + \frac{1}{2}\delta = u_1(y, b_1)$. Hence, this way the space between the agents is maximized and we can now check if there is too much space at the end of the line.

If agent 1 would locate at c_N, his payoff is equal to $\frac{2-c_N-c_{N-1}}{4} = \frac{2-c_N-(2N-5)y-(N-3)\delta}{4}$. Hence, $y + \frac{1}{2}\delta \geq \frac{2-c_N-(2N-5)y-(N-3)\delta}{4}$ which implies that $c_N \geq 2-(2N-1)y-(N-1)\delta$. If agent 1 believes that agent N is positioned to any position to the left of $1-y-\delta$, agent 1 is better of locating to $1-y-\delta$ and obtaining at least $\frac{2-(1-y-\delta)-(1-y-2\delta)}{2} = \frac{2y+3\delta}{2} = y + \frac{3}{2}\delta$, which is greater than $y + \frac{1}{2}\delta$. Hence, we have that $c_N \geq 1-y-\delta$ and $c_N \geq 2-(2N-1)y-(N-1)\delta$. We are interested in the minimum value of y such that both these constraints are satisfied. The first constraint has a smaller slope and a smaller intercept than the second constraint. Figure 10 plots the two constraints. The difference between figure 9 and figure 10 is the intercept of the second constraint. The arrows represent the direction in which each constraint is satisfied. From the figure it is clear that the minimum value of y for which both constraints are satisfied is exactly when the lines intersect. The lines intersect when $y = \frac{1-(N-2)\delta}{2N-2}$. This might not be a position on the line, so the minimum permitted value is given by $y = \left\lceil \frac{1-(N-2)\delta}{2N-2} \right\rceil = x$. However, we have $y = x - \delta$ so this is not possible. Hence, a point belief where no agent is positioned at y also does not lead to y being optimal for agent 1. To conclude, there is no point belief over $\{y, ..., 1-y\} \times ... \times \{y, ..., 1-y\}$ such that y is optimal. By symmetry, a similar result holds for the choice $1-y$. □

Proof of Theorem 2

Proof. We will prove Theorem 2 by verifying Figures 3, 4 and 5. Let $c_1 \leq c_2$. If we take the perspective of the last agent, the he will either choose $c_3 = c_1 - \delta$, $c_3 \in [c_1 + \delta, c_2 - \delta]$, or $c_3 = c_2 + \delta$. Agent 3's choice $c_3 = c_1 - \delta$ is optimal if $c_1 - \frac{\delta}{2} \geq \frac{c_2 - c_1}{2}$ and $c_1 - \frac{\delta}{2} \geq 1 - c_2 - \frac{\delta}{2}$. This can be rewritten to $c_1 \geq \frac{1}{3}c_2 + \frac{\delta}{3}$ and $c_1 \geq 1 - c_2$. Hence, the region in Figure 3 that corresponds to $c_3 = c_1 - \delta$ being optimal for agent 3 corresponds to the region where $c_1 \leq c_2$, $c_1 \geq \frac{1}{3}c_2 + \frac{\delta}{3}$, and $c_1 \geq 1 - c_2$. If agent 3 chooses $c_3 = c_1 - \delta$, then agent 2 is a rightmost firm with a corresponding utility of $\frac{2-c_2-c_1}{2} = 1 - \frac{1}{2}c_2 - \frac{1}{2}c_1$. We applied this procedure at every region to end up with Figure 3.

Next, we take the perspective of agent 2 and let $c_1 = [0, \frac{1}{2}]$. If $c_1 < \frac{1+\delta}{4}$, then no matter what agent 2 chooses, agent 3 will never position to the left of agent 1. Positioning to the left of agent 1 is also not optimal for agent 2, as this yields agent 2 a utility of $\frac{c_2+c_1}{2} \leq \frac{c_1-\delta+c_1}{2} = c_1 - \frac{\delta}{2} < \frac{1-\delta}{4}$. If agent 2 positions to the right of agent 1, his utility is increasing in c_2 in the first region, and decreasing in the last region. Hence agent 2's optimal choice is positioned near $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$. Positioning exactly at $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$ yields agent 2 a utility of $1 - \frac{3}{4}c_2 - \frac{1}{4}c_1 = 1 - \frac{3}{4}(\frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}) - \frac{1}{4}c_1 = \frac{1}{2} - \frac{1}{2}c_1 + \frac{\delta}{4} > \frac{3+\delta}{8}$. If agent 2 positions just before $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$, then $u_2 = \frac{c_2+\delta-c_1}{2} < \frac{1}{3} - \frac{1}{3}c_1 + \frac{\delta}{3} < \frac{3+\delta}{8}$. Hence, if $c_1 < \frac{1+\delta}{4}$, then $c_2 = \frac{2}{3} + \frac{1}{3}c_1 - \frac{\delta}{3}$ and agent 3 is indifferent between any position in between agent 1 and 2. Agent 1's utility can then be calculated as $u_1 = \frac{c_1+c_3}{2}$, and by the principle of insufficient reason this is equivalent to $\frac{3}{4}c_1 + \frac{1}{4}c_2 = \frac{1}{6} + \frac{5}{6}c_1 - \frac{\delta}{12} < \frac{3+\delta}{8}$.

Next, let $c_1 = \frac{1+\delta}{4}$ and consider agent 2. If $c_2 = c_2^* = \frac{3-\delta}{4}$, then agent 3 is indifferent between choosing $c_3 = c_1 - \delta$, any position $c_3 \in [c_1 + \delta, c_2 - \delta]$, and $c_3 = c_2 + \delta$. By the principle of insufficient reason, for agent 1 and 2, this can be interpreted as agent 3 choosing $c_3 = \frac{c_1+c_2}{2} = \frac{1}{2}$ and $u_2 = \frac{2 - \frac{3-\delta}{4} - \frac{1}{2}}{2} = \frac{3+\delta}{8}$. If agent 2 would choose $c_2 \in (\frac{3-\delta}{4}, 1]$, then it is optimal for agent 3 to choose a middle position and by the principle of insufficient reason, $c_3 = \frac{c_1+c_2}{2} > \frac{1}{2}$. Then $u_2 = \frac{2-c_2-c_3}{2} < \frac{3+\delta}{8}$ as $c_2 > \frac{3-\delta}{4}$ and $c_3 > \frac{1}{2}$. If $c_2 \in [c_1+\delta, \frac{3-\delta}{4})$, then it is optimal for agent 3 to choose $c_3 = c_2 + \delta$ and $u_2 = \frac{c_2+\delta-c_1}{2} < \frac{\frac{3-\delta}{4}+\delta-\frac{1+\delta}{4}}{2} = \frac{1+\delta}{4} < \frac{3+\delta}{8}$. Lastly, if $c_2 \in [0, c_1-\delta]$, then agent 3 chooses $c_3 = c_1^* + \delta$ and $u_2 \leq c_1 - \frac{\delta}{2} = \frac{1-\delta}{4}$. Hence, if $c_1 = \frac{1+\delta}{4}$, the optimal choice for agent 2 is $c_2 = \frac{3-\delta}{4} = 1 - c_1$ and $u_1 = u_2 = \frac{3+\delta}{8}$.

Next, let $\frac{1+\delta}{4} \leq c_1 \leq \frac{1-\delta}{2}$. If agent 2 chooses $c_2 = [0, c_1-\delta]$, then agent 3 will position to the right of agent 1 and agent 2's utility is $\frac{c_2+c_1}{2} \leq \frac{c_1-\delta+c_1}{2} = c_1 - \frac{\delta}{2} < \frac{1}{2} - \frac{\delta}{2}$. If agent 2 chooses a position in $[c_1+\delta, 1-c_1)$, then his utility is $\frac{c_2+\delta-c_1}{2} < \frac{1-c_1+\delta-c_1}{2} = \frac{1}{2} - c_1 + \frac{\delta}{2}$. If agent 2 chooses $c_2 \in (1-c_1, \min(3c_1-\delta, 1)]$, then agent 2 can ensure a utility of $\frac{1-\epsilon}{2}$ by choosing the position $1 - c_1 + \epsilon$. Agent 3 will then position to the left of agent 1, leaving agent 2 with approximately half of the clients. If agent 2 would choose $c_2 = 1 - c_1$, then agent 3 is indifferent between positioning to the left of agent 1 and the right of agent 2, which would lead to a lower expected utility of agent 2. Lastly, if $c_1 \leq \frac{1+\delta}{3}$, then choosing $c_2 \in (3c_1 - \delta, 1]$ leads to agent 3 choosing a middle position, and $u_2 = 1 - \frac{3}{4}c_2 - \frac{1}{4}c_1 < 1 - 2.5c_1 + \frac{3\delta}{4} < \frac{1-\epsilon}{2}$. Hence, if agent 1 chooses $\frac{1+\delta}{4} \leq c_1 < \frac{1}{2}$, then agent 2 chooses $c_2 = 1 - c_1 + \epsilon$, $c_3 = c_1 - \delta$ and $u_1 = \frac{c_2-c_3}{2} = \frac{1-c_1+\epsilon-c_1+\delta}{2} = \frac{1}{2} - c_1 + \frac{\delta}{2} + \frac{\epsilon}{2} < \frac{3+\delta}{8}$.

If agent 1 chooses $\frac{1-\delta}{2} < c_1 < \frac{1}{2}$, then agent 2 can position just to the right of agent 1, which leads agent 3 to positioning just to the left of agent 1. Then $u_1 = \frac{c_2-c_3}{2} = \frac{c_1+\delta-c_1+\delta}{2} = \delta < \frac{3+\delta}{8}$.

If agent 1 chooses $c_1 = \frac{1}{2}$. Then agent 2 can choose either $c_2 = c_1 + \delta$ or $c_2 = c_1 - \delta$ and $c_3 = 1 - c_2$. Then $u_1 = \delta < \frac{3+\delta}{8}$.

These results lead to figure 4 and 5. If $c_1 \in [\frac{1}{2}, 1]$, we find very similar results. □

Lemmas used for Theorem 3

Lemma 3. *Consider agent i and two choices c_i and c_i', where $c_i < c_i'$ and $c_i' - c_i < \omega$. Then choosing c_i yields a higher utility to agent i than choosing c_i' if and only if*

$$\int_{c_i-\frac{\omega}{2}}^{c_i'-\frac{\omega}{2}} a_{x,i}\,dx > \int_{c_i+\frac{\omega}{2}}^{c_i'+\frac{\omega}{2}} a_{x,i}\,dx$$

Similarly, choosing c_i yields a lower utility than c_i' if and only if

$$\int_{c_i-\frac{\omega}{2}}^{c_i'-\frac{\omega}{2}} a_{x,i}\,dx < \int_{c_i+\frac{\omega}{2}}^{c_i'+\frac{\omega}{2}} a_{x,i}\,dx$$

Proof. Choosing c_i yields a higher utility to agent i then choosing c_i' if and only if

$$\int_{c_i-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx > \int_{c_i'-\frac{\omega}{2}}^{c_i'+\frac{\omega}{2}} a_{x,i}\,dx \ ,$$

which means that

$$\int_{c_i-\frac{\omega}{2}}^{c_i'-\frac{\omega}{2}} a_{x,i}\,dx + \int_{c_i'-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx > \int_{c_i'-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx \int_{c_i+\frac{\omega}{2}}^{c_i'+\frac{\omega}{2}} a_{x,i}\,dx \ ,$$

which is equivalent to

$$\int_{c_i-\frac{\omega}{2}}^{c_i'-\frac{\omega}{2}} a_{x,i}\,dx > \int_{c_i+\frac{\omega}{2}}^{c_i'+\frac{\omega}{2}} a_{x,i}\,dx$$

□

Observation 1. *Consider an agent i and $0 < \omega < 1$. If $x, x' \in [0, \frac{\omega}{2}]$ and $x < x'$, then $|I_x| \leq |I_{x'}|$. Similarly, if $x, x' \in [1-\frac{\omega}{2}, 1]$ and and $x < x'$, then $|I_x| \geq |I_{x'}|$.*

Proof. Let $x, x' \in [0, \frac{\omega}{2}]$ and $x < x'$. A client at x will be attracted by agents in between 0 and $x + \frac{\omega}{2}$. A client at x' will be attracted by agents in between 0 and $x' + \frac{\omega}{2}$. Hence $|I_x| \leq |I_{x'}|$. Similarly, if $x, x' \in [1-\frac{\omega}{2}, 1]$ and $x < x'$ then $|I_x| \geq |I_{x'}|$. □

Observation 2. *Consider agent i and let each opponent agent $j \neq i$ choose $c_j \in [\frac{\omega}{2}, 1-\frac{\omega}{2}]$ and $0 < \omega \leq 1$. If $x, x' \in [0, \omega]$ and and $x < x'$, then $|I_x| \leq |I_{x'}|$. Similarly, if $x, x' \in [1-\omega, 1]$ and and $x < x'$, then $|I_x| \geq |I_{x'}|$.*

Proof. Let $x, x' \in [0, \omega]$ and $x < x'$. A client at x will be attracted by agents in between $\frac{\omega}{2}$ and $x + \frac{\omega}{2}$. A client at x' will be attracted by agents in between $\frac{\omega}{2}$ and $x' + \frac{\omega}{2}$. Hence $|I_x| \leq |I_{x'}|$. Similarly, if $x, x' \in [1-\omega, 1]$ and $x < x'$ then $|I_x| \geq |I_{x'}|$. □

Lemma 4. *Consider agent i and let $0 < \omega < 1$. Let $a, b, c \in [0, \frac{\omega}{2}]$ and $a < b < c$. Then*

$$\frac{1}{b-a} \cdot \int_a^b a_{x,i}\,dx \geq \frac{1}{c-b} \cdot \int_b^c a_{x,i}\,dx$$

Similarly, if we let $a, b, c \in [1-\frac{\omega}{2}, 1]$ and $a < b < c$, then

$$\frac{1}{b-a} \cdot \int_a^b a_{x,i}\,dx \leq \frac{1}{c-b} \cdot \int_b^c a_{x,i}\,dx$$

Proof. We will prove the case when $a, b, c \in [0, \frac{\omega}{2}]$ and $a < b < c$. By Observation 1, we know that $|I_x| \leq |I_{x'}|\ \forall x \in [a, b]$ and $x' \in [b, c]$. If $|I_x| = |I_{x'}|\ \forall x, x' \in [a, c]$, then there are exactly $l \in \{1, ..., N-1\}$ opponent agents that attract clients in $[a, c]$. This implies that $\int_a^b a_{x,i}\, dx = (b-a)\frac{1}{l+1}$ and $\int_b^c a_{x,i}\, dx = (c-b)\frac{1}{l+1}$ and hence $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx = \frac{1}{b-a} \cdot (b-a)\frac{1}{l+1} = \frac{1}{c-b} \cdot (c-b)\frac{1}{j+1} = \frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx$.

Suppose that $|I_x| < |I_{x'}|$ for some $x \in [a, b]$ and $x' \in [b, c]$, Then $|I_a| < |I_c|$. Let $|I_a| = N_1$. Then there exists a point $\beta \in (a, c)$ such that $|I_x| = N_1\ \forall x \in [a, \beta]$ and $|I_x| > N_1\ \forall x \in (\beta, c]$. We will now consider 2 cases.

Case 1: Let $\beta \in (a, b)$. Then $|I_b| = N_2 > N_1 = |I_a|$, and $|I_x| \leq N_2\ \forall x \in (\beta, b]$ and $|I_x| \geq N_2\ \forall [b, c]$. Then $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx \geq \frac{1}{b-a} \cdot ((\beta - a) \cdot \frac{1}{N_1+1} + (b-\beta) \cdot \frac{1}{N_2+1}) > \frac{1}{b-a} \cdot ((\beta - a) \cdot \frac{1}{N_2+1} + (b - \beta) \cdot \frac{1}{N_2+1}) = \frac{1}{N_2+1}$, whereas $\frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx \leq \frac{1}{c-b}((c-b)\frac{1}{N_2+1}) = \frac{1}{N_2+1}$.

Case 2: Let $\beta \in [b, c)$. Then $|I_b| = |I_a| = N_1$, $|I_x| = N_1\ \forall x \in [b, \beta]$ and $|I_x| > N_1\ \forall x \in (\beta, c]$. Then $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx = \frac{1}{b-a} \cdot ((b-a) \cdot \frac{1}{N_1+1}) = \frac{1}{N_1+1}$, wheras $\frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx < \frac{1}{c-b}((\beta - b) \cdot \frac{1}{N_1+1} + (c - \beta) \cdot \frac{1}{N_1+1}) = \frac{1}{N_1+1}$.

Finally, we can conclude that if $|I_x| < |I_{x'}|$ for some $x \in [a, b]$ and $x' \in [b, c]$, then $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx > \frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx$. □

Lemma 5. *Consider agent i and let each opponent agent $j \neq i$ choose $c_j \in [\frac{\omega}{2}, 1 - \frac{\omega}{2}]$ and $0 < \omega \leq 1$. Let $a, b, c \in [0, \omega]$ and $a < b < c$. Then*

$$\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx \geq \frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx$$

Similarly, if we let $a, b, c \in [1 - \omega, 1]$ and $a < b < c$, then

$$\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx \leq \frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx$$

Proof. We will prove the case when $a, b, c \in [0, \omega]$ and $a < b < c$. By Observation 2, we know that $|I_x| \leq |I_{x'}|\ \forall x \in [a, b]$ and $x' \in [b, c]$. If $|I_x| = |I_{x'}|\ \forall x, x' \in [a, c]$, then there are exactly $l \in \{1, ..., N-1\}$ opponent agents that attract clients in $[a, c]$. This implies that $\int_a^b a_{x,i}\, dx = (b-a)\frac{1}{l+1}$ and $\int_b^c a_{x,i}\, dx = (c-b)\frac{1}{l+1}$ and hence $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx = \frac{1}{b-a} \cdot (b-a)\frac{1}{l+1} = \frac{1}{c-b} \cdot (c-b)\frac{1}{j+1} = \frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx$.

Suppose that $|I_x| < |I_{x'}|$ for some $x \in [a, b]$ and $x' \in [b, c]$, Then $|I_a| < |I_c|$. Let $|I_a| = N_1$. Then there exists a point $\beta \in (a, c)$ such that $|I_x| = N_1\ \forall x \in [a, \beta]$ and $|I_x| > N_1\ \forall x \in (\beta, c]$. We will now consider 2 cases.

Case 1: Let $\beta \in (a, b)$. Then $|I_b| = N_2 > N_1 = |I_a|$, and $|I_x| \leq N_2\ \forall x \in (\beta, b]$ and $|I_x| \geq N_2\ \forall [b, c]$. Then $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx \geq \frac{1}{b-a} \cdot ((\beta - a) \cdot \frac{1}{N_1+1} + (b-\beta) \cdot \frac{1}{N_2+1}) > \frac{1}{b-a} \cdot ((\beta - a) \cdot \frac{1}{N_2+1} + (b - \beta) \cdot \frac{1}{N_2+1}) = \frac{1}{N_2+1}$, whereas $\frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx \leq \frac{1}{c-b}((c-b)\frac{1}{N_2+1}) = \frac{1}{N_2+1}$.

Case 2: Let $\beta \in [b, c)$. Then $|I_b| = |I_a| = N_1$, $|I_x| = N_1\ \forall x \in [b, \beta]$ and $|I_x| > N_1\ \forall x \in (\beta, c]$. Then $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx = \frac{1}{b-a} \cdot ((b-a) \cdot \frac{1}{N_1+1}) = \frac{1}{N_1+1}$, wheras $\frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx < \frac{1}{c-b}((\beta - b) \cdot \frac{1}{N_1+1} + (c-\beta) \cdot \frac{1}{N_1+1}) = \frac{1}{N_1+1}$.

Finally, we can conclude that if $|I_x| < |I_{x'}|$ for some $x \in [a, b]$ and $x' \in [b, c]$, then $\frac{1}{b-a} \cdot \int_a^b a_{x,i}\, dx > \frac{1}{c-b} \cdot \int_b^c a_{x,i}\, dx$. □

Proof of Theorem 3 for an Odd Number of Agents

Proof. Consider an agent $i \in I$. We start by showing that $c_i \in [0, \frac{\omega}{2})$ is irrational. Choosing $\frac{\omega}{2}$ yields agent i $\int_0^\omega a_{x,i} f(x)\, dx = \int_0^\omega a_{x,i}\, dx$, whereas choosing $c_i \in [0, \frac{\omega}{2})$ yields agent i $\int_{c_i - \frac{\omega}{2}}^{c_i + \frac{\omega}{2}} a_{x,i} f(x)\, dx = \int_0^{c_i + \frac{\omega}{2}} a_{x,i}\, dx < \int_0^{c_i + \frac{\omega}{2}} a_{x,i}\, dx + \int_{c_i + \frac{\omega}{2}}^{\omega} a_{x,i}\, dx = \int_0^\omega a_{x,i}\, dx$. Hence, $\frac{\omega}{2}$ strictly dominates the choices in $[0, \frac{\omega}{2})$. Similarly, $1 - \frac{\omega}{2}$ strictly dominates the choices in $(1 - \frac{\omega}{2}, 1]$. Hence, $c_i \in (1 - \frac{\omega}{2}, 1]$ is irrational.

First, let $0 < \omega \leq \frac{1}{3}$ and consider an agent i. The choice $c_i \in [\frac{\omega}{2}, \frac{1}{2}]$ is optimal for the belief b_i with $b_i(1 - \frac{\omega}{2}, ..., 1 - \frac{\omega}{2}) = 1$. Since $\omega \leq \frac{1}{3}$, we have that $c_i + \frac{\omega}{2} \leq \frac{1}{2} + \frac{\omega}{2} \leq 1 - \omega$. Hence, all clients in between $c_i - \frac{\omega}{2}$ and $c_i + \frac{\omega}{2}$ are only attracted to agent i. Thus, c_i is optimal. Similarly, the choice $c_i \in [\frac{1}{2}, 1 - \frac{\omega}{2}]$ is optimal for the belief b_i with $b_i(\frac{\omega}{2}, ..., \frac{\omega}{2}) = 1$.

Next, let $\frac{1}{3} < \omega \leq \frac{1}{2}$. Consider the choice $c_i \in [\frac{\omega}{2}, 1 - 1.5\omega]$ and the belief b_i with $b_i(1 - \frac{\omega}{2}, .., 1 - \frac{\omega}{2}) = 1$. Since $c_i + \frac{\omega}{2} \leq 1 - \omega$, all clients in between $c_i - \frac{\omega}{2}$ and $c_i + \frac{\omega}{2}$ are only attracted to agent i. Hence, c_i is optimal. Note that $\omega < \frac{1}{2}$ ensures that $\frac{\omega}{2} < 1 - 1.5\omega$. Similarly, the choice $c_i \in [1.5\omega, 1 - \frac{\omega}{2}]$ is optimal for the belief b_i with $b_i(\frac{\omega}{2}, .., \frac{\omega}{2}) = 1$. The choice $c_i \in [1 - 1.5\omega, 1.5\omega]$ is optimal for the point belief where $k = \frac{N-1}{2}$ agents are positioned at $\frac{\omega}{2}$ and k agents are positioned at $1 - \frac{\omega}{2}$. For this point belief, $|I_x| = k\ \forall x \in [0, \omega]$, $|I_x| = 0\ \forall x \in [\omega, 1 - \omega]$, and $|I_x| = k\ \forall x \in [1 - \omega, 1]$. Then c_i is optimal if it attracts all clients in between ω and $1 - \omega$. As $c_i \in (1 - 1.5\omega, 1.5\omega)$, we have that $c_i - \frac{\omega}{2} < \omega$ and $c_i + \frac{\omega}{2} > 1 - \omega$. Hence, c_i is optimal for the point belief above.

Lastly, let $\frac{1}{2} < \omega \leq 1$. Then the choice $c_i \in [\frac{\omega}{2}, 1 - \frac{\omega}{2}]$ is optimal for the point belief that $k = \frac{N-1}{2}$ opponents are positioned at $\frac{\omega}{2}$ and k opponents are positioned at $1 - \frac{\omega}{2}$. Note that for this point belief, $|I_x| = k\ \forall x, \in [0, 1 - \omega]$,$|I_x| = 2k\ \forall x \in [1 - \omega, \omega]$, and $|I_x| = k\ \forall x \in [\omega, 1]$. Hence,

$$u_i(c_i, c_{-i}) = ((1 - \omega) - (c_i - \frac{\omega}{2})) \cdot \frac{1}{k+1} + (\omega - (1 - \omega)) \cdot \frac{1}{2k+1} + (c_i + \frac{\omega}{2} - \omega) \cdot \frac{1}{k+1},$$

which simplifies to

$$(1 - \omega) \cdot \frac{1}{k+1} + (2\omega - 1) \cdot \frac{1}{2k+1}.$$

Hence, the choice c_i is optimal for the point belief described above.

We have shown that $P_i(1) = [\frac{\omega}{2}, 1-\frac{\omega}{2}]\ \forall i \in \{1, 2, ..., N\}$. Because for each agent i, the choice $c_i \in P_i(1)$ is optimal for a point belief in $P_{-i}(1)$, the algorithm terminates after round 1 and we conclude that $P_i = [\frac{\omega}{2}, 1-\frac{\omega}{2}]\ \forall i \in \{1, 2, ..., N\}$. $\square$

Proof of Theorem 3 for an Even Number of Agents

Proof. Similarly to the case with an odd number of agents and $0 < \omega < 1$, we can show that $c_i \in [0, \frac{\omega}{2})$ is irrational and that $c_i \in (1 - \frac{\omega}{2}, 1]$ is irrational.

Let $0 < \omega \leq \frac{1}{3}$. Also similarly to the proof of the case with an odd number of agents, the choice $c_i \in [\frac{\omega}{2}, \frac{1}{2}]$ is optimal for the belief b_i with $b_i(1 - \frac{\omega}{2}, ..., 1 - \frac{\omega}{2}) = 1$. Similarly, the choice $c_i \in [\frac{1}{2}, 1 - \frac{\omega}{2}]$ is optimal for the belief b_i with $b_i(\frac{\omega}{2}, ..., \frac{\omega}{2}) = 1$. Hence, $P_i(1) = [\frac{\omega}{2}, 1 - \frac{\omega}{2}]$. Because for each agent i, the choice $c_i \in P_i(1)$ is optimal for a point belief in $P_{-i}(1)$, the algorithm terminates after round 1 and we conclude that $P_i = [\frac{\omega}{2}, 1-\frac{\omega}{2}]\ \forall i \in \{1, 2, ..., N\}$.

Next, consider $\frac{1}{3} < \omega \leq \frac{1}{2}$. Consider the choice $c_i \in [\frac{\omega}{2}, \frac{1}{2}]$ and the belief b_i with $b_i(1, ..., 1) = 1$. Since $\omega < \frac{1}{2}$, we have that $c_i + \frac{\omega}{2} \leq \frac{1}{2} + \frac{\omega}{2} \leq 1 - \frac{\omega}{2}$. Hence, all clients in between $c_i - \frac{\omega}{2}$ and $c_i + \frac{\omega}{2}$ are only attracted to agent i. Thus, c_i is optimal. Similarly, the choice $c_i \in [\frac{1}{2}, 1 - \frac{\omega}{2}]$ is optimal for the belief b_i with $b_i(0, ..., 0) = 1$. Hence, $P_i(1) = [\frac{\omega}{2}, 1 - \frac{\omega}{2}]\ \forall i \in \{1, ..., N\}$.

Consider agent i in the next round. Consider the choice $c_i \in [\frac{\omega}{2}, 1 - 1.5\omega]$ and the point belief b_i with $b_i = (1 - \frac{\omega}{2}, .., 1 - \frac{\omega}{2}) = 1$. Since $\omega < \frac{1}{2}$, we have that $c_i + \frac{\omega}{2} \leq 1 - 1.5\omega + \frac{\omega}{2} \leq 1 - \omega$. Hence, all clients in between $c_i - \frac{\omega}{2}$ and $c_i + \frac{\omega}{2}$ are only attracted to agent i. Thus, c_i is optimal. Similarly, the choice $c_i \in [1.5\omega, 1 - \frac{\omega}{2}]$ is optimal for the belief b_i with $b_i = (\frac{\omega}{2}, .., \frac{\omega}{2}) = 1$. Suppose by contradiction that $c_i \in (1 - 1.5\omega, 1.5\omega)$ is optimal for some point belief. Choosing c_i should yield at least as much as choosing $1 - 1.5\omega$. Lemma 3 then implies

$$\int_{1-\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx \geq \int_{1-2\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx \tag{1}$$

Choosing c_i should also yield at least as much as choosing 1.5ω. Lemma 3 then implies

$$\int_{c_i-\frac{\omega}{2}}^{\omega} a_{x,i}\,dx \geq \int_{c_i+\frac{\omega}{2}}^{2\omega} a_{x,i}\,dx \tag{2}$$

Furthermore, lemma 5 implies

$$\frac{1}{c_i + 1.5\omega - 1} \int_{1-2\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx \geq \frac{1}{1.5\omega - c_i} \int_{c_i-\frac{\omega}{2}}^{\omega} a_{x,i}\,dx \tag{3}$$

Similarly, lemma 5 implies

$$\frac{1}{c_i+1.5\omega-1}\int_{1-\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx \le \frac{1}{1.5\omega-c_i}\int_{c_i+\frac{\omega}{2}}^{2\omega} a_{x,i}\,dx \tag{4}$$

Combining (2),(3) and (4) implies that

$$\begin{aligned}&\frac{1}{c_i+1.5\omega-1}\int_{1-2\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx \ge \frac{1}{1.5\omega-c_i}\int_{c_i-\frac{\omega}{2}}^{\omega} a_{x,i}\,dx\\ \ge&\frac{1}{1.5\omega-c_i}\int_{c_i+\frac{\omega}{2}}^{2\omega} a_{x,i}\,dx \ge \frac{1}{c_i+1.5\omega-1}\int_{1-\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx\end{aligned} \tag{5}$$

Then (1) and (5) imply

$$\begin{aligned}&\frac{1}{c_i+1.5\omega-1}\int_{1-2\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx = \frac{1}{1.5\omega-c_i}\int_{c_i-\frac{\omega}{2}}^{\omega} a_{x,i}\,dx\\ =&\frac{1}{1.5\omega-c_i}\int_{c_i+\frac{\omega}{2}}^{2\omega} a_{x,i}\,dx = \frac{1}{c_i+1.5\omega-1}\int_{1-\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx\end{aligned} \tag{6}$$

The proof of Lemma 5 and the first equality in (6) imply that

$$|I_x| = |I_{x'}|\ \forall x, x' \in [1-2\omega, \omega]. \tag{7}$$

Similarly, the proof of Lemma 5 and the last equality in (6) implies that

$$|I_x| = |I_{x'}|\ \forall x, x' \in [1-\omega, 2\omega]. \tag{8}$$

Furthermore, (6) implies that

$$|I_x| = |I_{1-x}|\ \forall x \in [1-2\omega, \omega] \text{ and } |I_x| = |I_{1-x}|\ \forall x \in [1-\omega, 2\omega]. \tag{9}$$

By contradiction, assume that $c_j \in (1-1.5\omega, 1.5\omega)$ for some opponent j. Then this opponent attracts clients in between $c_j - \frac{\omega}{2}$ and $c_j + \frac{\omega}{2}$, where $1-2\omega < c_j - \frac{\omega}{2} < \omega$ and $1-\omega < c_j + \frac{\omega}{2} < 2\omega$. Observation 2 then implies that $|I_{1-2\omega}| < |I_\omega|$ and $|I_{1-\omega}| > |I_{2\omega}|$. As a result, (7) and (8) cannot hold. So from now on, we assume that each opponent agent chooses in either $[\frac{\omega}{2}, 1-1.5\omega]$ or $[1.5\omega, 1-\frac{\omega}{2}]$.

Note that if an opponent chooses $c_j \in [\frac{\omega}{2}, 1-1.5\omega]$, then he will attract all clients in $[1-2\omega, \omega]$ and none of the clients in between $[1-\omega, 2\omega]$. Similarly, if an opponent chooses $c_j \in [1.5\omega, 1-\frac{\omega}{2}]$, then he will attract all clients in $[1-\omega, 2\omega]$ and none of the clients in between $[1-2\omega, \omega]$. Hence, (9) can only hold if an equal number of opponents is located in $[\frac{\omega}{2}, 1-1.5\omega]$ as in $[1.5\omega, 1-\frac{\omega}{2}]$. As a result, (9) cannot hold if the number of agents is even. We conclude that if the number of agents is even, then for each agent i, c_i is not optimal for any probability 1 belief on $P_{-i}(1)$ and $P_i = ([\frac{\omega}{2}, 1-1.5\omega], [1.5\omega, 1-\frac{\omega}{2}])$.

Next, consider $\frac{1}{2} < \omega \le \frac{2}{3}$. The choice $c_i \in [\frac{\omega}{2}, 1-\omega]$ is optimal for the belief b_i with $b_i(1,...,1)=1$. Note that $\omega<\frac{2}{3}$ ensures that $\frac{\omega}{2}<1-\omega$. Hence, all clients in between $c_i-\frac{\omega}{2}$ and $c_i+\frac{\omega}{2}$ are only attracted to agent i. Thus, c_i is optimal. Similarly, the choice $c_i\in[\omega,1-\frac{\omega}{2}]$ is optimal for the belief b_i with $b_i=(0,..,0)$. Now suppose by contradiction that $c_i\in(1-\omega,\omega)$ is optimal for some point belief. Choosing c_i should yield at least as much as choosing $1-\omega$, Lemma 3 then implies

$$\int_{1-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx \geq \int_{1-1.5\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx \tag{10}$$

Choosing c_i should also yield at least as much as choosing ω, Lemma 3 then implies

$$\int_{c_i-\frac{\omega}{2}}^{\frac{\omega}{2}} a_{x,i}\,dx \geq \int_{c_i+\frac{\omega}{2}}^{1.5\omega} a_{x,i}\,dx \tag{11}$$

Furthermore, lemma 4 implies

$$\frac{1}{c_i+\omega-1}\int_{1-1.5\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx \geq \frac{1}{\omega-c_i}\int_{c_i-\frac{\omega}{2}}^{\frac{\omega}{2}} a_{x,i}\,dx \tag{12}$$

Similarly, lemma 4 implies

$$\frac{1}{c_i+\omega-1}\int_{1-\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx \leq \frac{1}{\omega-c_i}\int_{c_i+\frac{\omega}{2}}^{1.5\omega} a_{x,i}\,dx \tag{13}$$

Combining (11),(12) and (13) implies that

$$\begin{aligned}\frac{1}{c_i+\omega-1}\int_{1-1.5\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx &\geq \frac{1}{\omega-c_i}\int_{c_i-\frac{\omega}{2}}^{\frac{\omega}{2}} a_{x,i}\,dx\\ \geq\frac{1}{\omega-c_i}\int_{c_i+\frac{\omega}{2}}^{1.5\omega} a_{x,i}\,dx &\geq \int_{1-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx\end{aligned} \tag{14}$$

Then (10) and (14) imply

$$\begin{aligned}\frac{1}{c_i+\omega-1}\int_{1-1.5\omega}^{c_i-\frac{\omega}{2}} a_{x,i}\,dx &= \frac{1}{\omega-c_i}\int_{c_i-\frac{\omega}{2}}^{\frac{\omega}{2}} a_{x,i}\,dx\\ =\frac{1}{\omega-c_i}\int_{c_i+\frac{\omega}{2}}^{1.5\omega} a_{x,i}\,dx &= \int_{1-\frac{\omega}{2}}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx\end{aligned} \tag{15}$$

The proof of Lemma 4 and the first equality in (15) imply that

$$|I_x|=|I_{x'}|\ \forall x,x'\in[1-1.5\omega,\frac{\omega}{2}]. \tag{16}$$

Similarly, the proof of Lemma 4 and the last equality in (15) implies that

$$|I_x|=|I_{x'}|\ \forall x,x'\in[1-\frac{\omega}{2},1.5\omega]. \tag{17}$$

Furthermore, (6) implies that

$$|I_x| = |I_{1-x}| \ \forall x \in [1-1.5\omega, \frac{\omega}{2}] \text{ and } |I_x| = |I_{1-x}| \ \forall x \in [1-\frac{\omega}{2}, 1.5\omega]. \tag{18}$$

By contradiction, assume that $c_j \in (1-\omega, \omega)$ for some opponent j. Then this opponent attracts clients in between $c_j - \frac{\omega}{2}$ and $c_j + \frac{\omega}{2}$, where $1 - 1.5\omega < c_j - \frac{\omega}{2} < \frac{\omega}{2}$ and $1 - \frac{\omega}{2} < c_j + \frac{\omega}{2} < 1.5\omega$. Observation 1 then implies that $|I_{1-1.5\omega}| < |I_{\frac{\omega}{2}}|$ and $|I_{1-\frac{\omega}{2}}| > |I_{1.5\omega}|$. As a result, (16) and (17) cannot hold. So from now on, we assume that each opponent agent chooses in either $[\frac{\omega}{2}, 1-\omega]$ or $[\omega, 1-\frac{\omega}{2}]$.

Note that if an opponent chooses $c_j \in [\frac{\omega}{2}, 1-\omega]$, then he will attract all clients in $[1-1.5\omega, \frac{\omega}{2}]$ and none of the clients in between $[1-\frac{\omega}{2}, 1.5\omega]$. Similarly, if an opponent chooses $c_j \in [\omega, 1-\frac{\omega}{2}]$, then he will attract all clients in $[1-\frac{\omega}{2}, 1.5\omega]$ and none of the clients in between $[1-1.5\omega, \frac{\omega}{2}]$. Hence, (18) can only hold if an equal number of opponents is located in $[\frac{\omega}{2}, 1-\omega]$ as in $[\omega, 1-\frac{\omega}{2}]$. As a result, (18) cannot hold if the number of agents is even. We conclude that if the number of agents is even, then for each agent i, c_i is not optimal for any probability 1 belief and $P_i(1) = ([\frac{\omega}{2}, 1-\omega], [\omega, 1-\frac{\omega}{2}])$.

Consider an agent i in the next round. Then the choice $c_i = \frac{\omega}{2}$ is optimal for the belief b_i with $b_i(1-\frac{\omega}{2}, ..., 1-\frac{\omega}{2})$. Similarly, the choice $1-\frac{\omega}{2}$ is the optimal choice for agent i for the belief b_i with $b_i(\frac{\omega}{2}, ..., \frac{\omega}{2}) = 1$. By contradiction, assume that $c_i \in \{(\frac{\omega}{2}, 1-\omega], [\omega, 1-\frac{\omega}{2})\}$ is optimal, it must be that c_i yields at least as much as $\frac{\omega}{2}$, so Lemma 3 implies

$$\int_{\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\, dx \geq \int_{0}^{c_i-\frac{\omega}{2}} a_{x,i}\, dx. \tag{19}$$

Similarly, c_i should yield at least as much as $1-\frac{\omega}{2}$, which implies that

$$\int_{c_i-\frac{\omega}{2}}^{1-\omega} a_{x,i}\, dx \geq \int_{c_i+\frac{\omega}{2}}^{1} a_{x,i}\, dx \tag{20}$$

Furthermore, lemma 5 implies

$$\frac{1}{c_i-\frac{\omega}{2}} \cdot \int_{0}^{c_i-\frac{\omega}{2}} a_{x,i}\, dx, \geq \frac{1}{1-c_i-\frac{\omega}{2}} \cdot \int_{c_i-\frac{\omega}{2}}^{1-\omega} a_{x,i}\, dx \tag{21}$$

Similarly, lemma 5 implies

$$\frac{1}{c_i-\frac{\omega}{2}} \cdot \int_{\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\, dx \leq \frac{1}{1-c_i-\frac{\omega}{2}} \cdot \int_{c_i+\frac{\omega}{2}}^{1} a_{x,i}\, dx \tag{22}$$

Combining equation (20),(21) and (22) implies that

$$\begin{aligned}\frac{1}{c_i-\frac{\omega}{2}}\cdot\int_0^{c_i-\frac{\omega}{2}} a_{x,i}\,dx \geq \frac{1}{1-c_i-\frac{\omega}{2}}\cdot\int_{c_i-\frac{\omega}{2}}^{1-\omega} a_{x,i}\,dx \\ \geq \frac{1}{1-c_i-\frac{\omega}{2}}\cdot\int_{c_i+\frac{\omega}{2}}^{1} a_{x,i}\,dx \geq \frac{1}{c_i-\frac{\omega}{2}}\cdot\int_{\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx\end{aligned} \tag{23}$$

Then (19) and (23) imply that

$$\begin{aligned}\frac{1}{c_i-\frac{\omega}{2}}\cdot\int_0^{c_i-\frac{\omega}{2}} a_{x,i}\,dx = \frac{1}{1-c_i-\frac{\omega}{2}}\cdot\int_{c_i-\frac{\omega}{2}}^{1-\omega} a_{x,i}\,dx \\ = \frac{1}{1-c_i-\frac{\omega}{2}}\cdot\int_{c_i+\frac{\omega}{2}}^{1} a_{x,i}\,dx = \frac{1}{c_i-\frac{\omega}{2}}\cdot\int_{\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx\end{aligned} \tag{24}$$

The proof of Lemma 5 and the first equality in (24) imply that

$$|I_x| = |I_{x'}| \;\forall x, x' \in [0, 1-\omega]. \tag{25}$$

Similarly, the proof of Lemma 5 and the last equality in (24) imply that

$$|I_x| = |I_{x'}| \;\forall x, x' \in [\omega, 1]. \tag{26}$$

Furthermore, equation 24 implies that

$$|I_x| = |I_{1-x}| \;\forall x \in [0, 1-\omega] \text{ and } |I_x| = |I_{1-x}| \;\forall x \in [\omega, 1]. \tag{27}$$

For (25) to be true, it must be that the point belief of agent i about the choices of the opponents cannot consist of choices in $\{(\frac{\omega}{2}, 1-\omega], [\omega, 1-\frac{\omega}{2})\}$. By contradiction, suppose that one opponent chooses $c_j \in (\frac{\omega}{2}, \frac{1}{2}]$. Hence, $0 < c_j - \frac{\omega}{2} < 1-\omega$. Observation 2 then implies that $|I_0| < |I_{1-\omega}|$, so (25) cannot be true. Similarly, For (26) to be true, it must be that the point belief of agent i about the choices of the opponents cannot consist of choices in $[\omega, 1-\frac{\omega}{2})$. Hence, the point belief of agent i about the choices of the opponents can only consist of the choices $\frac{\omega}{2}$ and $1-\frac{\omega}{2}$.

Finally, (27) can only hold if half of the opponents is positioned at $\frac{\omega}{2}$ and the other half is positioned at $1-\frac{\omega}{2}$. If the number of agents is even, then (27) cannot hold, and we can conclude that $c_i \in \{(\frac{\omega}{2}, 1-\omega], [\omega, 1-\frac{\omega}{2})\}$ is not optimal for any point belief in $P_{-i}(1)$. As a result, the point rationalizable choices of each agent i are $P_i = \{\frac{\omega}{2}, 1-\frac{\omega}{2}\}$.

Lastly, consider $\frac{2}{3} < \omega \leq 1$. Then again, the choice $c_i = \frac{\omega}{2}$ is optimal for the belief b_i with $b_i(1-\frac{\omega}{2}, ..., 1-\frac{\omega}{2})$. Similarly, the choice $1-\frac{\omega}{2}$ is the optimal choice for agent i for the belief b_i with $b_i(\frac{\omega}{2}, ..., \frac{\omega}{2}) = 1$. By contradiction, assume that $c_i \in (\frac{\omega}{2}, 1-\frac{\omega}{2})$ is optimal, it must be that c_i yields at least as much as $\frac{\omega}{2}$, so Lemma 3 implies

$$\int_{\omega}^{c_i+\frac{\omega}{2}} a_{x,i}\,dx \geq \int_0^{c_i-\frac{\omega}{2}} a_{x,i}\,dx. \tag{28}$$

Similarly, c_i should yield at least as much as $1 - \frac{\omega}{2}$, which implies that

$$\int_{c_i - \frac{\omega}{2}}^{1-\omega} a_{x,i}\, dx \geq \int_{c_i + \frac{\omega}{2}}^{1} a_{x,i}\, dx \tag{29}$$

Furthermore, lemma 4 implies

$$\frac{1}{c_i - \frac{\omega}{2}} \cdot \int_0^{c_i - \frac{\omega}{2}} a_{x,i}\, dx, \geq \frac{1}{1 - c_i - \frac{\omega}{2}} \cdot \int_{c_i - \frac{\omega}{2}}^{1-\omega} a_{x,i}\, dx \tag{30}$$

Similarly, lemma 4 implies

$$\frac{1}{c_i - \frac{\omega}{2}} \cdot \int_{\omega}^{c_i + \frac{\omega}{2}} a_{x,i}\, dx \leq \frac{1}{1 - c_i - \frac{\omega}{2}} \cdot \int_{c_i + \frac{\omega}{2}}^{1} a_{x,i}\, dx \tag{31}$$

Combining equation (29),(30) and (31) implies that

$$\begin{aligned} &\frac{1}{c_i - \frac{\omega}{2}} \cdot \int_0^{c_i - \frac{\omega}{2}} a_{x,i}\, dx \geq \frac{1}{1 - c_i - \frac{\omega}{2}} \cdot \int_{c_i - \frac{\omega}{2}}^{1-\omega} a_{x,i}\, dx \\ \geq &\frac{1}{1 - c_i - \frac{\omega}{2}} \cdot \int_{c_i + \frac{\omega}{2}}^{1} a_{x,i}\, dx \geq \frac{1}{c_i - \frac{\omega}{2}} \cdot \int_{\omega}^{c_i + \frac{\omega}{2}} a_{x,i}\, dx \end{aligned} \tag{32}$$

Then (28) and (32) imply that

$$\begin{aligned} &\frac{1}{c_i - \frac{\omega}{2}} \cdot \int_0^{c_i - \frac{\omega}{2}} a_{x,i}\, dx = \frac{1}{1 - c_i - \frac{\omega}{2}} \cdot \int_{c_i - \frac{\omega}{2}}^{1-\omega} a_{x,i}\, dx \\ = &\frac{1}{1 - c_i - \frac{\omega}{2}} \cdot \int_{c_i + \frac{\omega}{2}}^{1} a_{x,i}\, dx = \frac{1}{c_i - \frac{\omega}{2}} \cdot \int_{\omega}^{c_i + \frac{\omega}{2}} a_{x,i}\, dx \end{aligned} \tag{33}$$

The proof of Lemma 4 and the first equality in (33) imply that

$$|I_x| = |I_{x'}| \ \forall x, x' \in [0, 1-\omega]. \tag{34}$$

Similarly, the proof of Lemma 4 and the last equality in (33) imply that

$$|I_x| = |I_{x'}| \ \forall x, x' \in [\omega, 1]. \tag{35}$$

Furthermore, equation 33 implies that

$$|I_x| = |I_{1-x}| \ \forall x \in [0, 1-\omega] \text{ and } |I_x| = |I_{1-x}| \ \forall x \in [\omega, 1]. \tag{36}$$

For (34) to be true, it must be that the point belief of agent i about the choices of the opponents cannot consist of choices in $(\frac{\omega}{2}, \frac{1}{2}]$. By contradiction, suppose that one opponent chooses $c_j \in (\frac{\omega}{2}, \frac{1}{2}]$. Hence, $0 < c_j - \frac{\omega}{2} < 1 - \omega < \frac{\omega}{2}$. Observation 1 implies that $|I_0| < |I_{\frac{\omega}{2}}|$, so (34) cannot be true. Similarly, For (35) to be true, it must be that the point belief of agent i about the choices of the opponents cannot consist of choices in $[\frac{1}{2}, 1-\frac{\omega}{2})$. Hence, the point belief of agent i about the choices of the opponents can only consist of the choices $\frac{\omega}{2}$ and $1-\frac{\omega}{2}$.

Finally, (36) can only hold if half of the opponents is positioned at $\frac{\omega}{2}$ and the other half is positioned at $1 - \frac{\omega}{2}$. If the number of agents is even, then (36) cannot hold, and we can conclude that $c_i \in (\frac{\omega}{2}, 1 - \frac{\omega}{2})$ is not optimal for any point belief. As a result, when the total number of agents in the game is even, then the point rationalizable choices of each agent i are $P_i = P_i(1) = \{\frac{\omega}{2}, 1 - \frac{\omega}{2}\}$. $\square$

8 References

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