Reasoning about Your Own Future Mistakes*

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Abstract

We propose a model of reasoning in dynamic games in which a player, at each information set, holds a conditional belief about his own future choices and the opponents' future choices. These conditional beliefs are assumed to be *cautious*, that is, the player never completely rules out any feasible future choice by himself or the opponents. We impose the following key conditions: (a) a player always believes that he will choose rationally in the future, (b) a player always believes that his opponents will choose rationally in the future, and (c) a player deems his own mistakes infinitely less likely than the opponents' mistakes. Common belief in these conditions leads to the new concept of *perfect quasi-perfect rationalizability.* We show that perfectly quasi-perfectly rationalizable strategies exist in every finite dynamic game. We prove, moreover, that perfect quasi-perfect rationalizability constitutes a refinement of both *perfect rationalizability* (a rationalizability analogue to Selten's (1975) perfect equilibrium) and *quasi-perfect rationalizability* (a rationalizability analogue to van Damme's (1984) quasi-perfect equilibrium).

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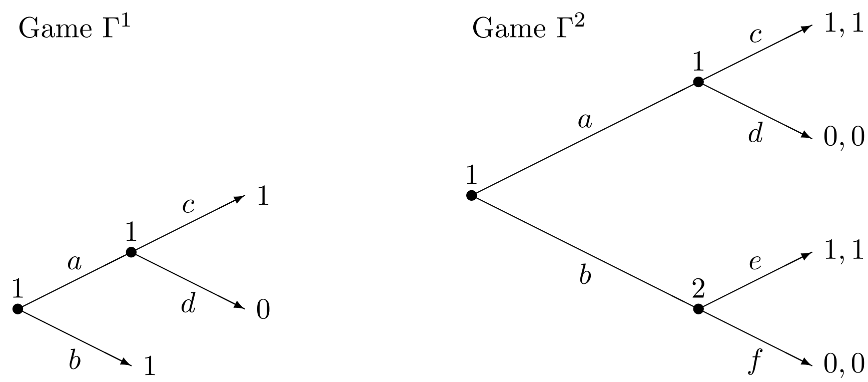

Figure 1: Belief about your own future mistakes

1 Introduction

When reasoning in a game, it is natural to assume that your opponents – and even you yourself – may make mistakes with some small probability. This assumption, to which we often refer as *cautious reasoning,* has first been implemented by Selten (1975) in his concept of *perfect equilibrium.* The main idea is that a player, at each of his information sets in a dynamic game, assigns a – possibly infinitesimal – strictly positive probability to each of the opponents' choices and to each of his own future choices. Since then, Selten's idea of cautious reasoning has inspired many other concepts in game theory, such as *proper equilibrium* (Myerson (1978)), *permissibility* (Brandenburger (1992), Börgers (1994)) and *proper rationalizability* (Schuhmacher (1999), Asheim (2001)) for static games, and *quasi-perfect equilibrium* (van Damme (1984)) and *quasi-perfect rationalizability* (Asheim and Perea (2005)) for dynamic games.

Among these, perfect equilibrium is the only concept in which a player believes that he may make mistakes *himself* in the future. Indeed, in each of the other concepts the cautious reasoning of a player only concerns the strategy choices by his *opponents,* but not the choices by himself. As an illustration, consider the game Γ^1 in the left-hand panel of Figure 1, where player 1 is the only player. If player 1 fears that, by mistake, he may choose d in the future, then his unique best choice is to go for strategy b. This is the only perfect equilibrium strategy for player 1, whereas the other concepts above also allow for strategy (a, c). Intuitively, b seems the only plausible choice for player 1 here. Indeed, why would player 1 risk making a future mistake by choosing a, if by choosing b he achieves the maximum possible utility with no risk of making future mistakes? This argument speaks for perfect equilibrium in this game.

At the same time, perfect equilibrium allows a player to believe that his *own* future mistakes are more likely than the opponents' mistakes. Consider, for instance, the game Γ^2 in the right-hand panel of Figure 1. According to perfect equilibrium, player 1 is free to believe that his own future mistake d is more likely than player 2's mistake f, which allows player 1 to go for strategy b. The other concepts above uniquely select player 1's strategy (a, c), since under these concepts player 1 believes that he will not make mistakes himself. In this particular game, (a, c) seems the only plausible strategy for player 1. Indeed, believing that your own future mistakes are more likely than those of your opponents strikes us as rather counterintuitive.

Summarizing, none of the established concepts above uniquely selects player 1's intuitive choice in both of the games Γ^1 and Γ^2. In fact, we are not aware of any traditional concept in game theory – be it an equilibrium concept or rationalizability concept – that filters player 1's plausible choice in both of the games Γ^1 and Γ^2. This raises the question: Can we develop a new concept that does? This is precisely the question we wish to answer in this paper.

Blume and Meier (2019) have taken a first important step by developing the new concept of *perfect quasi-perfect equilibrium.* Like perfect equilibrium and quasi-perfect equilibrium, they simulate mistakes by trembles – that is, sequences of full-support probability distributions that converge to the strategy profile under consideration. When evaluating the strategy of player i, Blume and Meier (2019) also consider trembles in player i's own strategy, similarly to how perfect equilibrium proceeds. However, the key condition in perfect quasi-perfect equilibrium is that player i deems the trembles in his own strategy much smaller, in fact *infinitely smaller,* than the trembles in the opponents' strategies. By this feature, perfect quasi-perfect equilibrium selects the "right" choice for player 1 in both games Γ^1 and Γ^2 above.

In this paper, we develop a *rationalizability* concept, termed *perfect quasi-perfect rationalizability,* that shares some of the key properties of perfect quasi-perfect equilibrium. More precisely, in perfect quasi-perfect rationalizability a player holds, at each of his information sets, a cautious belief about his opponents' strategies and a cautious belief about his own continuation strategy. Here, by *cautious* we mean that the player never rules out any feasible opponent's strategy, and never rules out any of his own continuation strategies, although he may deem some of these strategies infinitely more likely than other strategies.

We impose the following key conditions on these beliefs: (a) a player always believes that he will himself choose rationally in the future, given his future beliefs about the opponents' strategies and his future beliefs about himself; (b) a player always believes that his opponents will choose rationally in the future; and (c) a player always deems his own future mistakes infinitely less likely than the mistakes by others. In (a) and (b), when we say that a player believes that he, or another player, chooses rationally, we mean that he only assigns infinitesimal probability to all suboptimal strategies. Common belief in the properties (a), (b) and (c) throughout the game yields the concept of perfect quasi-perfect rationalizability. Formally, we define the concept by the recursive elimination of strategies and beliefs in the game (see Section 3).[1] In Theorem

[1] A problem we would like to explore in the near future is characterizing perfect quasi-perfect rationalizability *epistemically.* That is, what are precisely the epistemic conditions on the players' beliefs that lead us to perfect quasi-perfect rationalizability? Our current formulation of this concept is by means of a non-epistemic recursive elimination procedure, although the conditions of "belief in your own future rationality" and "deeming your own mistakes least likely" already have some epistemic flavour.

3.1 we show that this elimination procedure always yields a non-empty output in every finite dynamic game, and hence perfectly quasi-perfectly rationalizable strategies always exist. The existence proof, which can be found in Appendix B, is constructive.

Property (b) shows that, in terms of reasoning, perfect quasi-perfect rationalizability is very similar to *common belief in future rationality* (Perea (2014)), in which players also always believe that their opponents will choose rationally in the future. In that sense, perfect quasi-perfect rationalizability is a true backward induction concept in which the players only reason about the game that lies ahead.

By virtue of the properties (a), (b) and (c) above, especially property (c), the concept of perfect quasi-perfect rationalizability uniquely selects the "right" choice for player 1 in both of the games Γ^1 and Γ^2 above. Indeed, in game Γ^1 perfect quasi-perfect rationalizability uniquely selects strategy b for player 1, as player 1 does not rule out his own future mistake d. The selection here thus coincides with that of perfect equilibrium. In game Γ^2, player 1's reasoning in line with perfect quasi-perfect rationalizability is as follows: Player 1 deems it possible that he himself will make the mistake d and that player 2 will make the mistake f, but by (c) he deems player 2's mistake infinitely more likely than his own mistake. Therefore, player 1 will go for strategy (a, c), which is also the prediction of quasi-perfect equilibrium and quasi-perfect rationalizability.

In this paper we show that this is not a coincidence: In every dynamic game, perfect quasi-perfect rationalizability is a refinement of both *perfect rationalizability* (see Remark 1 in Section 3) and *quasi-perfect rationalizability* (see Theorem 4.1). Here, by perfect rationalizability we mean the rationalizability analogue to perfect equilibrium, just as quasi-perfect rationalizability is the rationalizability analogue to quasi-perfect equilibrium. These two results thus show that, in every dynamic game, perfect quasi-perfect rationalizability inherits the desirable properties of both perfect rationalizability (that you take into account your own future mistakes) and quasi-perfect rationalizability (that you focus primarily on the *opponents'* mistakes).

Compared to Blume and Meier's (2019) perfect quasi-perfect equilibrium concept, there are a few major differences. First, perfect quasi-perfect rationalizability is not an equilibrium concept, and hence does not impose "correct beliefs assumptions" stating that a player must believe that his opponents are correct about his beliefs, or that player i must believe that player j has the same belief about player k as player i has. Perfect quasi-perfect equilibrium, on the other hand, does impose such correct beliefs conditions. Second, beliefs about the opponents' choices and mistakes and beliefs about your own future choices and mistakes are explicitly modelled in perfect quasi-perfect rationalizability, whereas these are only implicitly present – in terms of trembles of the equilibrium strategies – in perfect quasi-perfect equilibrium. Finally, we use *non-standard probability distributions* (Robinson (1973), Hammond (1994) and Halpern (2010)) with infinitesimals to model beliefs about mistakes, which greatly simplifies the presentation and analysis in our case. In contrast, Blume and Meier use the traditional framework of converging sequences of full-support *standard* probability distributions.

This leaves the question why we did not opt for *lexicographic probability systems* (Blume, Brandenburger and Dekel (1991)) to model such cautious beliefs, as is common nowadays in epistemic game theory. The reason is that lexicographic probability systems are not expressive enough for our purposes: In the concept of perfect quasi-perfect rationalizability, player i holds, at each of his information sets, both (a) a cautious belief about his *own* future choices, and (b) a cautious belief about the *opponents'* future choices. To determine which choice is optimal for player i at that information set, we must take the "product" of the belief about his own choices and the belief about the opponents' choices. Taking such products is not well-defined for lexicographic probability distributions, whereas it comes for free when using non-standard probability distributions.

The outline of this paper is as follows. In Section 2 we introduce the necessary notation for dynamic games and provide a brief overview of non-standard analysis, which will be sufficient for understanding the main body of this paper. In Section 3 we introduce the new concept of perfect quasi-perfect rationalizability, show its existence, and observe that it is a refinement of perfect rationalizability. In Section 4 we provide a definition of quasi-perfect rationalizability, which is a slight weakening of the concept proposed in Asheim and Perea (2005), and show that perfect quasi-perfect rationalizability is a refinement of quasi-perfect rationalizability. Appendix A gives a more extensive treatment of non-standard analysis, which is needed for some of the proofs. Appendices B and C contains the proofs for Sections 3 and 4, respectively. Appendix D, finally, explores the relation between our notion of quasi-perfect rationalizability and that defined in Asheim and Perea (2005).

2 Definitions

In this section we introduce the notation for dynamic games, and provide a short overview of non-standard analysis which is needed for our definitions of *perfect quasi-perfect rationalizability* and *quasi-perfect rationalizability.*

2.1 Dynamic Games

In this paper we will focus on finite dynamic games that allow for simultaneous moves and imperfect information. To keep our notation and definitions simple we exclude moves of nature in the definition that follows. However, our definition can easily be generalized to situations that involve moves of nature. Formally, a *finite dynamic game* is a tuple

$$G = (I, X, Z, (X_i)_{i\in I}, (C_i(x))_{i\in I, x\in X_i}, (H_i)_{i\in I}, (u_i)_{i\in I})$$

where

(a) $I = \{1, 2, ..., n\}$ is the finite set of *players*;

(b) X is the finite set of *histories*, consisting of *non-terminal* and *terminal* histories. At every non-terminal history, one or more players must make a choice, whereas at every terminal history the game ends. By $\emptyset$ we denote the history that marks the beginning of the game;

(c) $Z \subseteq X$ is the set of *terminal* histories;

(d) $X_i \subseteq X$ is the set of *non-terminal* histories where player i must make a choice. For a given non-terminal history x, we denote by $I(x) := \{i \in I \mid x \in X_i\}$ the set of *active* players at x. We allow $I(x)$ to contain more than one player, that is, we allow for *simultaneous moves.* At the same time, we require $I(x)$ to be non-empty for every non-terminal history x;

(e) $C_i(x)$ is the finite set of *choices* available to player i at a history $x \in X_i$;

(f) H_i is the collection of *information sets* for player i. Every information set $h \in H_i$ is a subset of histories in X_i such that $\cup_{h \in H_i} h = X_i$ and $h \cap h' = \emptyset$ for every two different $h, h' \in H_i$. That is, H_i is a partition of X_i. Moreover, we assume that $C_i(x) = C_i(y)$ whenever x, y belong to the same information set in H_i, and $C_i(x) \cap C_i(y) = \emptyset$ whenever x and y belong to different information sets in H_i. The interpretation of an information set $h \in H_i$ is that player i at h knows that a history in h has been realized. However, if h contains more than one history, player i does not know which of these histories has been realized. Hence, we allow for *imperfect information*;

(g) $u_i : Z \to \mathbb{R}$ is player i's *utility function*, assigning to every terminal history $z \in Z$ some utility $u_i(z)$.

For practical purposes, we assume that all players are active at the beginning of the game $\emptyset$, that is, $I(\emptyset) = I$. If, in reality, player i does not choose at $\emptyset$, then we define $C_i(\emptyset)$ to be a singleton.

For every non-terminal history x and choice combination $(c_i)_{i \in I(x)}$ in $\times_{i \in I(x)} C_i(x)$, we denote by $x' = (x, (c_i)_{i \in I(x)})$ the (terminal or non-terminal) history that immediately follows this choice combination at x. In this case, we say that x' immediately follows x. We say that a history x *follows* a non-terminal history x' if there is a sequence of histories $x^1, ..., x^K$ such that $x^1 = x'$, $x^K = x$, and x^{k+1} immediately follows x^k for all $k \in \{1, ..., K-1\}$. A history x is said to *weakly follow* x' if either x follows x' or $x = x'$. In the obvious way, we can then also define what it means for x to (*weakly*) *precede* another history x'. Analogously, for two information sets h and h', we say that h (weakly) follows h' if there is some $x \in h$ and $x' \in h'$ such that x (weakly) follows x'. We assume that the dynamic game has *non-overlapping information sets,* that is, for every two information sets h, h' it is never the case that h follows h' and h' follows h.

In view of (f), we can write $C_i(h)$ to denote the (unique) set of choices that player i has available at information set $h \in H_i$. We assume *perfect recall,* that is, for every information set $h \in H_i$ and every two histories $x, y \in H_i$, the sequence of player i choices leading to x is the same as the sequence of player i choices leading to y. In particular, since different information sets in H_i prescribe disjoint sets of available choices, the sequence of player i information sets on the path to x and on the path to y must be the same. That is, player i always remembers, at each of his information sets $h \in H_i$, the choices he made in the past and the information he had in the past.

We view a strategy for player i as a *plan of action* (Rubinstein (1991)), assigning choices only to those histories $h \in H_i$ that are not precluded by previous choices. Formally, consider a collection of information sets $\hat{H}_i \subseteq H_i$, and a mapping $s_i : \hat{H}_i \to \cup_{h \in \hat{H}_i} C_i(h)$ assigning to every information set $h \in \hat{H}_i$ some available choice $s_i(h) \in C_i(h)$. We say that an information set $h \in H$ is *reachable* under s_i if at every information set $h' \in \hat{H}_i$ preceding h, the choice $s_i(h')$ is the unique choice that leads to h. The mapping $s_i : \hat{H}_i \to \cup_{h \in \hat{H}_i} C_i(h)$ is called a *strategy* if $\hat{H}_i$ contains exactly those information sets in H_i that are reachable under s_i.

By S_i we denote the set of strategies for player i. For every information set $h \in H$ and player i, we denote by $S_i(h)$ the set of strategies for player i under which h is reachable. Similarly, for a given strategy s_i we denote by $H_i(s_i)$ the collection of information sets in H_i that are reachable under s_i.

2.2 Non-Standard Numbers

The analysis of non-standard numbers was initiated by Robinson (1973). It has later been incorporated into the analysis of games by Hammond (1994) and Halpern (2010), who review non-standard analysis and connect it to conditional and lexicographic probability systems.

Consider a number $\varepsilon > 0$ with the property that $\varepsilon < a$ for every strictly positive real number $a \in \mathbf{R}$, $a > 0$. The number ε is called an *infinitesimal.* Following Robinson (1973), Hammond (1994) and Halpern (2010), let $\mathbf{R}(\varepsilon)$ be the smallest field that includes all real numbers and the infinitesimal ε. That is, $\mathbf{R}(\varepsilon)$ contains all numbers a that can be written as

$$a = \frac{a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... + a_K\varepsilon^K}{b_0 + b_1\varepsilon + b_2\varepsilon^2 + ... + b_K\varepsilon^K},$$

where $a_k, b_k \in \mathbf{R}$ for all $k \in \{0, ..., K\}$, $b_k \neq 0$ for some $k \in \{0, ..., K\}$, and where either $a_0 \neq 0$ or $b_0 \neq 0$. In other words, $\mathbf{R}(\varepsilon)$ contains all fractions of finite polynomials in ε. Numbers in $\mathbf{R}(\varepsilon)$ are called *non-standard.*

Every finite non-standard number $a \in \mathbf{R}(\varepsilon)$ can uniquely be written as

$$a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... ,$$

where $a_k \in \mathbf{R}$ for every $k \geq 0$. (See Appendix A for a proof). By $st(a) := a_0$ we denote the *standard part* of a, which is the real number that is "closest" to a. By $v(a)$ we denote the smallest index k for which $a_k \neq 0$, and call it the *valuation* of a. For two finite numbers $a, b \in \mathbf{R}(\varepsilon)$, we say that a is of *infinitely smaller size* than b if $v(a) > v(b)$. We use the term "infinitely smaller size" rather than the more familiar "infinitely smaller" because we also apply it to negative numbers. For instance, ε is of infinitely smaller size than -1, although ε is not smaller than -1.

2.3 Non-Standard Probability Distributions

Consider a finite set X. A *non-standard probability distribution* on X is a function $p : X \to \mathbf{R}(\varepsilon)$ such that $p(x) \geq 0$ for all $x \in X$ and $\sum_{x \in X} p(x) = 1$. We say that p is *cautious* on X if

$p(x) > 0$ for all $x \in X$. We say that p *believes* an event $E \subseteq X$ if $\sum_{x \in E} p(x)$ has standard part 1. Consider, for instance, the set $X = \{x, y, z\}$ and the non-standard probability distribution p on X with $p(x) = 1 - \varepsilon - \varepsilon^2$, $p(y) = \varepsilon$ and $p(z) = \varepsilon^2$. Note that p is cautious, since every probability is strictly positive. Moreover, ε is of infinitely smaller size than $1 - \varepsilon - \varepsilon^2$, and ε^2 is of infinitely smaller size than ε. Hence, p can be interpreted as a cautious belief in which you deem event $\{x\}$ infinitely more likely than $\{y\}$, and event $\{y\}$ infinitely more likely than $\{z\}$, while deeming each of these three events possible. Note that p believes the event $\{x\}$.

For a subset $Y \subseteq X$ with $\sum_{x \in Y} p(x) > 0$, the *conditional* probability distribution on Y induced by p is the non-standard probability distribution p_Y on Y given by

$$p_Y(x) := \frac{p(x)}{\sum_{y \in Y} p(y)}$$

for every $x \in Y$.

A more extensive treatment of non-standard analysis, which is needed for some of the proofs, can be found in Appendix A.

3 Perfect Quasi-Perfect Rationalizability

The main ideas behind *perfect quasi-perfect equilibrium* (Blume and Meier (2019)) are that a player (a) is cautious about the opponents' behavior and his own behavior, that is, he assigns a – possibly infinitesimal – strictly positive probability to every opponent's strategy and every continuation strategy by himself, (b) always believes that his opponents and he himself will choose rationally in the future, and (c) believes that he may make mistakes himself in the future, but deems his own future mistakes much less likely – in fact, infinitely less likely – than the mistakes by his opponents. In this section we attempt to incorporate this idea in a rationalizability concept that we call *perfect quasi-perfect rationalizability*. We first give a formal definition of perfect quasi-perfect rationalizability, subsequently prove that perfectly quasi-perfectly rationalizable strategies always exist, and finally illustrate the concept by means of an example.

For every player i, let B_i^{self} be the set of cautious non-standard probability distributions on the set S_i of i's *own* strategies. A member $b_i^{self} \in B_i^{self}$ will be interpreted as a *belief* that player i has about his own future choices in the game. Hence, every belief in B_i^{self} always deems each of his own future choices possible. The assumption that a player holds a belief about his *own* future choices seems natural, but is certainly not standard in game theory. Battigalli, di Tillio and Samet (2013) and Battigalli and de Vito (2019) are some of the few papers that incorporate beliefs about own choices in a game-theoretic setting.

Moreover, let B_i^{opp} be the set of cautious non-standard probability distributions on the set S_{-i} of *opponents'* strategy combinations, where $S_{-i} := \times_{j \neq i} S_j$. A member $b_i^{opp} \in B_i^{opp}$ represents a *belief* of player i about the opponents' strategies. By definition, every belief in B_i^{opp} deems each of the opponents' strategy combinations possible. By B_i we denote the set of belief pairs $b_i = (b_i^{self}, b_i^{opp})$ where $b_i^{self} \in B_i^{self}$ and $b_i^{opp} \in B_i^{opp}$.

Consider an information set $h \in H_i$ and a choice $c_i \in C_i(h)$ available at h. By $S_i(h, c_i)$ we denote the set of strategies $s_i \in S_i(h)$ with $s_i(h) = c_i$. By $S_{-i}(h) := \{s_{-i} \in S_{-i} \mid$ there is some $s_i \in S_i$ such that (s_i, s_{-i}) reaches a history in $h\}$ we denote the set of opponents' strategy combinations that are possible when h is reached. For a given belief pair $b_i = (b_i^{self}, b_i^{opp})$, let $b_i^{self}(h, c_i)$ be the induced conditional belief on $S_i(h, c_i)$, and let $b_i^{opp}(h)$ be the induced conditional belief on $S_{-i}(h)$. By

$$u_i(c_i, b_i, h) := \sum_{s_i \in S_i(h,c_i)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

we denote the expected (non-standard) utility at information set h of making choice c_i under the belief pair b_i. Here, $z(s_i, s_{-i})$ is the outcome reached by the strategy combination (s_i, s_{-i}). Note that defining $u_i(c_i, b_i, h)$ would not be possible by using lexicographic probability systems, as we need to take the product of player i's belief $b_i^{self}(h, c_i)$ over his own future choices and player i's belief $b_i^{opp}(h)$ over the opponents' strategies. If we would use lexicographic probability systems, it is not clear how to define such product.

A choice $c_i \in C_i(h)$ is *locally rational* for the belief pair $b_i = (b_i^{self}, b_i^{opp})$ at information set $h \in H_i$ if

$$u_i(c_i, b_i, h) \geq u_i(c_i', b_i, h) \text{ for all } c_i' \in C_i(h).$$

For every information set h we define

$$S_i^{rat}(b_i, h) := \{s_i \in S_i(h) \mid s_i(h') \text{ locally rational for } b_i \text{ at every } h' \in H_i(s_i) \text{ following } h\}.$$

We say that the belief pair $b_i = (b_i^{self}, b_i^{opp})$ *believes in his own future rationality* if for every information set $h \in H_i$ and every choice $c_i \in C_i(h)$, the induced conditional belief $b_i^{self}(h, c_i)$ believes $S_i^{rat}(b_i, h)$. Note that at information set $h \in H_i$, player i need not believe that his choice at h is optimal. Indeed, it may well be that $c_i(h)$ is suboptimal. The definition above only requires player i to believe at $h \in H_i$ that his own choices *strictly following* h are optimal. This condition is similar to the notion of *optimal planning* in Battigalli and de Vito (2019).

We say that b_i *deems his own mistakes least likely* if for every information set $h \in H_i$, every choice $c_i \in C_i(h)$, every strategy $s_i \in S_i(h, c_i)$ and every strategy combination $s_{-i} \in S_{-i}$ we have that

$$b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i)) \text{ is of infinitely smaller size than } b_i^{opp}(s_{-i}).$$

That is, the infinitesimal mistake part $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i))$ in the belief about i's own strategy is of infinitely smaller size than each of the belief probabilities $b_i^{opp}(s_{-i})$ about the opponent's strategies. Also here it is crucial that we use non-standard probability distributions instead of lexicographic probability systems, as we need to compare the size of two possibly infinitesimal probabilities. Lexicographic probability systems are not expressive enough to do this.

We now define *perfect quasi-perfect rationalizability* by recursively defining sets of strategies $S_i^k(h)$, for every information set h, and sets of belief pairs B_i^k, as follows. Remember that $\emptyset$ is the information set that marks the beginning of the game.

Definition 3.1 (Perfect quasi-perfect rationalizability) *(Initial step) Set $S_i^0(h) := S_i(h)$ and*

$$B_i^0 \ : \ = \{b_i \in B_i \mid b_i \text{ believes in his own future rationality and deems own mistakes least likely}\}$$

for all players i and all information sets h.

(Inductive step) Let $k \geq 1$, and suppose that $S_i^{k-1}(h)$ and B_i^{k-1} have been defined for all players i and all information sets h. Then define, for every player i and every information set h,

$$S_i^k(h) := \{s_i \in S_i^{k-1}(h) \mid \text{there is some } b_i \in B_i^{k-1} \text{ such that } s_i(h') \text{ is locally rational for } b_i \text{ at every } h' \in H_i(s_i) \text{ weakly following } h\}$$

and

$$B_i^k := \{b_i = (b_i^{self}, b_i^{opp}) \in B_i^{k-1} \mid b_i^{opp}(h) \text{ believes } S_{-i}^k(h) \text{ for all } h \in H_i\}.$$

A strategy $s_i \in S_i$ is called perfectly quasi-perfectly rationalizable if $s_i \in S_i^k(\emptyset)$ for all $k \geq 0$.

By construction, if a belief in B_i^k assigns, at information set $h \in H_i$, a non-infinitesimal probability to an opponent's strategy s_j, then s_j must be in $S_j^k(h)$, and hence there must be a belief in B_j^{k-1} for which s_j is optimal from h onwards. In other words, a belief in B_i^k believes, at every information set, that each opponent will choose rationally now and in the future. This resembles the idea of *belief in the opponents' future rationality* as formalized in Perea (2014). Similar ideas can be found in Penta (2015) and Baltag, Smets and Zvesper (2009).

One important question, of course, is whether in every finite dynamic game we can find for every player at least one strategy that is perfectly quasi-perfectly rationalizable. The answer to this question is "yes", as will be shown by the following theorem.

Theorem 3.1 (Existence) *For every finite dynamic game, and every player i, there is at least one strategy for player i that is perfectly quasi-perfectly rationalizable.*

The proof, which is constructive, can be found in Appendix B.

If in the definition of perfect quasi-perfect rationalizability we drop the condition of "own mistakes being deemed least likely" in B_i^0, then we obtain a rationalizability analogue to Selten's (1975) perfect equilibrium which we call *perfect rationalizability.*

Definition 3.2 (Perfect rationalizability) *(Initial step) Set* $S_i^0(h) := S_i(h)$ *and*

$$B_i^0 := \{b_i \in B_i \mid b_i \text{ believes in his own future rationality}\}$$

for all players i *and all information sets* h.

(Inductive step) Let $k \geq 1$, *and suppose that* $S_i^{k-1}(h)$ *and* B_i^{k-1} *have been defined for all players* i *and all information sets* h. *Then define, for every player* i *and every information set* h,

$$S_i^k(h) := \{s_i \in S_i^{k-1}(h) \mid \text{there is some } b_i \in B_i^{k-1} \text{ such that } s_i(h') \text{ is locally rational for } b_i \text{ at every } h' \in H_i(s_i) \text{ weakly following } h\}$$

and

$$B_i^k := \{b_i = (b_i^{self}, b_i^{opp}) \in B_i^{k-1} \mid b_i^{opp}(h) \text{ believes } S_{-i}^k(h) \text{ for all } h \in H_i\}.$$

A strategy $s_i \in S_i$ *is called perfectly rationalizable if* $s_i \in S_i^k(\emptyset)$ *for all* $k \geq 0$.

The following observation is an immediate consequence of the definitions.

Remark 1 *Every strategy that is perfectly quasi-perfectly rationalizable, is also perfectly rationalizable.*

This remark thus states that perfect quasi-perfect rationalizability inherits all the desirable properties from perfect equilibrium, except for the "correct beliefs" conditions that separate it from perfect rationalizability.

The other direction in the remark above is not true, as can be seen from the game Γ^2 in Figure 1. In that game, the strategy b is perfectly rationalizable but not perfectly quasi-perfectly rationalizable. Indeed, if player 1 believes that his own mistake d is infinitely less likely than the opponent's mistake f, then he must go for (a, c).

We will now illustrate the perfect quasi-perfect rationalizability procedure by means of an example.

Example 1: Illustration of perfect quasi-perfect rationalizability procedure.

Consider the game in Figure 2. This game starts at $\emptyset$, where player 1 can choose between a and b. If he chooses a the game reaches information set h_1, where players 1 and 2 can simultaneously choose between c and d, and between e and f, respectively. If he chooses b instead, the game reaches h_2 where players 1 and 2 can simultaneously choose between g and h, and between i and j, respectively.

Remember that b_1^{self} denotes player 1's cautious belief about his own strategy choice, whereas his cautious belief about player 2's strategy choice is denoted by b_1^{opp}. Then, $b_1 = (b_1^{self}, b_1^{opp})$ constitutes player 1's belief about his own behavior and player 2's behavior. Let $b_1^{self}(\emptyset, a)$ be

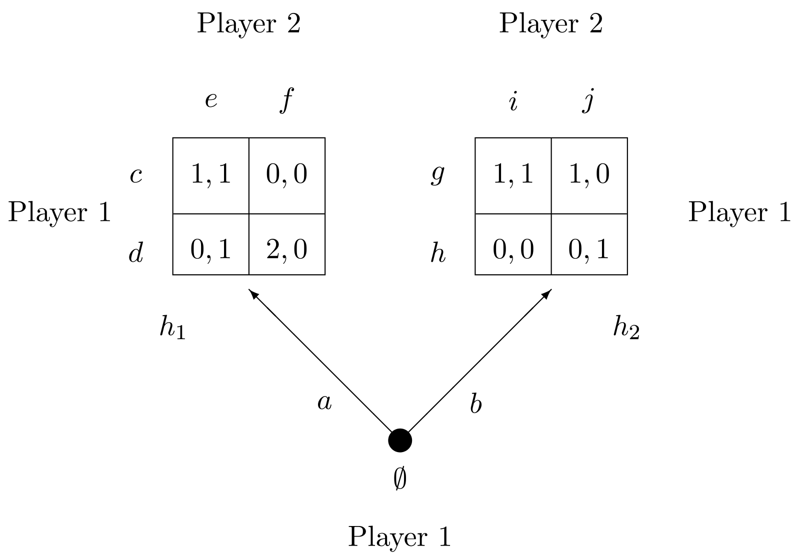

Figure 2: Illustration of perfect quasi-perfect rationalizability

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player 1's conditional belief about his own future choice at h_1, and $b_1^{self}(\emptyset, b)$ his conditional belief about his own future choice at h_2. As an abbreviation, we denote by $b_1^{opp}(e) := b_1^{opp}(e,i) + b_1^{opp}(e,j)$ the probability that player 1 assigns to player 2 choosing e at h_1, and similarly for $b_1^{opp}(f)$. Then, by definition, player 1's expected utility of choosing a at $\emptyset$ is

$$\begin{aligned} u_1(a, b_1, \emptyset) &= b_1^{self}(\emptyset, a)(a, c) \cdot b_1^{opp}(e) \cdot 1 + b_1^{self}(\emptyset, a)(a, d) \cdot b_1^{opp}(f) \cdot 2 \\ &= 1 - b_1^{self}(\emptyset, a)(a, d) \cdot (1 - 3b_1^{opp}(f)) - b_1^{opp}(f). \end{aligned} \tag{3.1}$$

Here, we have used the fact that $b_1^{self}(\emptyset, a)(a, c) = 1 - b_1^{self}(\emptyset, a)(a, d)$, and that $b_1^{opp}(e) = 1 - b_1^{opp}(f)$. Similarly, player 1's expected utility of choosing b at $\emptyset$ is

$$u_1(b, b_1, \emptyset) = b_1^{self}(\emptyset, b)(b, g) = 1 - b_1^{self}(\emptyset, b)(b, h). \tag{3.2}$$

Initial step. Note that (b, g) is player 1's only optimal strategy at h_2, and hence (b, h) is a mistake. As in B_1^0 player 1 must believe in his own future rationality, we have that $b_1^{self}(\emptyset, b)(b, h) \ll 1$ for all $b_1 \in B_1^0$, where $\ll$ is an abbreviation for being "of infinitely smaller size".

Step 1. Clearly, $S_1^1(h_2) = \{(b, g)\}$. By construction, every $b_2 \in B_2^1$ must believe $S_1^1(h_2) = \{(b, g)\}$, and therefore $b_2^{opp}(h_2)(b, h) \ll 1$ for all $b_2 \in B_2^1$.

Similarly, note that e is player 2's only optimal choice at h_1, and hence f is a mistake. We thus conclude that $S_2^1(h_1) = \{(e,i),(e,j)\}$. By definition, every $b_1 \in B_1^1$ must believe $S_2^1(h_1)$ at $\emptyset$, and hence $b_1^{opp}(f) \ll 1$ for all $b_1 \in B_1^1$.

Remember that $b_1^{self}(\emptyset,b)(b,h) \ll 1$ for all $b_1 \in B_1^0$. Therefore, $st(b_1^{self}(\emptyset,b)(b,h)) = 0$. As every $b_1 \in B_1^0$ deems his own mistakes least likely, we see that

$$b_1^{self}(\emptyset,b)(b,h) = b_1^{self}(\emptyset,b)(b,h) - st(b_1^{self}(\emptyset,b)(b,h)) \ll b_1^{opp}(f)$$

for all $b_1 \in B_1^0$, and hence in particular for all $b_1 \in B_1^1$. Together with the insight above that $b_1^{opp}(f) \ll 1$ for all $b_1 \in B_1^1$, we conclude that

$$b_1^{self}(\emptyset,b)(b,h) \ll b_1^{opp}(f) \ll 1 \text{ for all } b_1 \in B_1^1. \tag{3.3}$$

Step 2. We have seen that $b_2^{opp}(h_2)(b,h) \ll 1$ for all $b_2 \in B_2^1$. Hence, i is player 2's unique optimal choice at h_2 for every belief $b_2 \in B_2^1$. As e is player 2's unique optimal choice at h_1 for any belief, we conclude that $S_2^2(\emptyset) = \{(e,i)\}$.

We now turn to player 1's beliefs. By combining (3.1), (3.2) and (3.3), it holds for every $b_1 \in B_1^1$ that

$$\begin{aligned} u_1(a,b_1,\emptyset) &= 1 - b_1^{self}(\emptyset,a)(a,d)\cdot(1-3b_1^{opp}(f)) - b_1^{opp}(f) \\ &< 1 - b_1^{opp}(f) < 1 - b_1^{self}(\emptyset,b)(b,h) = u_1(b,b_1,\emptyset). \end{aligned}$$

Hence, b is the only optimal choice for player 1 at $\emptyset$ for every belief $b_1 \in B_1^1$. As g is the only optimal choice for player 1 after b for every $b_1 \in B_1^1$, we conclude that $S_1^2(\emptyset) = \{(b,g)\}$.

We thus see that $S_1^2(\emptyset) = \{(b,g)\}$ and $S_2^2(\emptyset) = \{(e,i)\}$. By Theorem 3.1, there is at least one strategy for player 1 and 2 that survives the procedure, and hence (b,g) and (e,i) must be the *only* strategies for player 1 and 2 that survive the procedure. Therefore, (b,g) and (e,i) are the unique perfectly quasi-perfectly rationalizable strategies for players 1 and 2 in this game.

It turns out that that these are also the only quasi-perfectly rationalizable strategies in this game. However, strategy (a,c) for player 1 is *perfectly rationalizable,* whereas it is not perfectly quasi-perfectly rationalizable. The reason is that according to perfect rationalizability, player 1 is free to believe that his own mistakes are more likely than player 2's mistakes. In particular, player 1 is free to believe that his mistake h after choosing b is much more likely than his own mistake d, and player 2's mistake f, after choosing a. In that case, it would be optimal for player 1 to choose a at $\emptyset$ and c at h_1. □

4 Relation to Quasi-Perfect Rationalizability

In this section we propose the *quasi-perfect rationalizability procedure* as a non-equilibrium counterpart to van Damme's (1984) *quasi-perfect equilibrium.* Like the perfect quasi-perfect rationalizability procedure, it proceeds by iteratively eliminating strategies and beliefs from the game.

The main idea that distinguishes the quasi-perfect rationalizability procedure and quasi-perfect equilibrium from perfect quasi-perfect rationalizability, is that a player believes, at each of his information sets, that his opponents will always make mistakes with some positive infinitesimal probability, but that he will not make mistakes himself. Similarly to perfect quasi-perfect rationalizability, both the quasi-perfect rationalizability procedure and quasi-perfect equilibrium are still based on the assumption that players deem all opponents' strategies possible, and that a player, at each of his information sets, believes in the opponents' future rationality.

We will show that the perfect quasi-perfect rationalizability procedure is always a *refinement* of the quasi-perfect rationalizability procedure. That is, every strategy that is perfectly quasi-perfectly rationalizable is also quasi-perfectly rationalizable. The other direction is not true, as can be seen from the game Γ^1 in Figure 1. Indeed, in that game the strategy (a, c) is quasi-perfectly rationalizable but not perfectly quasi-perfectly rationalizable, since (a, c) induces the risk of making the mistake d in the future. In behavioral terms, this is precisely the key difference between the two concepts: Perfect quasi-perfect rationalizability may additionally rule out strategies that are inferior because of the risk of *own* future mistakes.

Asheim and Perea (2005), from now on AP, provided a definition of quasi-perfect rationalizability which differs both methodologically and behaviorally from ours. Instead of using a procedure that recursively eliminates strategies and beliefs from the game, AP use *belief hierarchies* as a primitive notion to define quasi-perfect rationalizability. That is, AP do not only consider first-order beliefs about the opponents' strategies, as we do in the procedure, but also explore the players' *second-order* beliefs about the opponents' beliefs about the strategies of others, and *higher-order beliefs* as well. AP encode such belief hierarchies by means of epistemic models with types and lexicographic beliefs, and impose epistemic conditions on such belief hierarchies which give rise to their definition of quasi-perfect rationalizability.

In Appendix D we explore, in detail, the formal relation between our quasi-perfect rationalizability procedure on the one hand, and the notion of quasi-perfect rationalizability as defined in Asheim and Perea (2005) on the other hand. We refer to the latter concept as AP-quasi-perfect rationalizability from now on. We show that every strategy that is AP-quasi-perfectly rationalizable survives our quasi-perfect rationalizability procedure, but not *vice versa.* Hence, the concept of Asheim and Perea is stronger than ours. Intuitively, the key difference is the following: According to the quasi-perfect rationalizability procedure, if a player i at information set h deems an opponent's strategy s_j most plausible, then there is a belief b_j that survives all rounds and for which the strategy s_j is optimal from h onwards. The concept of AP-quasi-perfect rationalizability requires more: If player i, at information set h, deems an opponent's belief b_j possible, and at (a *possibly different*) information set h' deems the opponent's strategy s_j most plausible given the opponent's belief b_j, then the strategy s_j must be optimal for b_j from h' onwards. Here, when we say that player i deems strategy s_j most plausible at h, we mean that there is no other strategy $s'_j \in S_j(h)$ that b_i deems infinitely more likely than s_j. Hence, according to AP-quasi-perfect rationalizability, the opponent's belief b_j that player i assigns to his opponent at information set h is not only used to justify his behavior from h

onwards, but also to form his belief about j's behavior at information sets that do not follow h. In that sense, AP-quasi-perfect rationalizability imposes restrictions that go beyond belief in the opponents' future rationality. In contrast, our quasi-perfect rationalizability procedure only imposes rationality restrictions that are in line with belief in the opponents' future rationality.

In order to formally introduce our quasi-perfect rationalizability procedure, we need the following additional notation and definitions. As before, let B_i^{opp} be the set of cautious non-standard probability distributions on the set of opponents' strategy combinations S_{-i}. For a belief $b_i^{opp} \in B_i^{opp}$ and information set $h \in H_i$, let $b_i^{opp}(h)$ be the induced conditional probability distribution on $S_{-i}(h)$. Consider an information set $h \in H_i$ and a strategy $s_i \in S_i(h)$. By

$$u_i(s_i, b_i^{opp}(h)) := \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

we denote the *expected utility* at information set $h \in H_i$ of choosing strategy s_i under the conditional belief $b_i^{opp}(h)$. We say that the strategy $s_i \in S_i(h)$ is *globally rational* for b_i^{opp} at h if

$$u_i(s_i, b_i^{opp}(h)) \geq u_i(s_i', b_i^{opp}(h)) \text{ for all } s_i' \in S_i(h).$$

Strategy s_i is *globally rational* for b_i^{opp} if s_i is globally rational for b_i^{opp} at every $h \in H_i(s_i)$.

In our *quasi-perfect rationalizability procedure*, we recursively define sets of strategies $S_i^k(h)$, for all information sets h, and sets of non-standard beliefs $B_i^{opp,k}$, as follows.

Definition 4.1 (Quasi-perfect rationalizability procedure) *(Initial step) Set $S_i^0(h) := S_i(h)$ and $B_i^{opp,0} := B_i^{opp}$ for all players i and all information sets h.*

(Inductive step) Let $k \geq 1$, and suppose that $S_i^{k-1}(h)$ and $B_i^{opp,k-1}$ have been defined for all players i and all information sets h. Then define, for every player i and every information set h,

$$S_i^k(h) := \{s_i \in S_i^{k-1}(h) \mid \text{there is some } b_i^{opp} \in B_i^{opp,k-1} \text{ such that } s_i \text{ is globally rational for } b_i^{opp} \text{ at every } h' \in H_i(s_i) \text{ weakly following } h\}$$

and

$$B_i^{opp,k} := \{b_i \in B_i^{opp,k-1} \mid b_i^{opp}(h) \text{ believes } S_{-i}^k(h) \text{ for all } h \in H_i\}.$$

A strategy $s_i \in S_i$ is quasi-perfectly rationalizable if $s_i \in S_i^k(\emptyset)$ for all $k \geq 0$.

Similar to the perfect quasi-perfect rationalizability procedure, also the quasi-perfect rationalizability embodies the idea of *belief in the opponents' future rationality* as proposed by Perea (2014). Indeed, if a belief in $B_i^{opp,k}$ assigns, at information set $h \in H_i$, a non-infinitesimal probability to an opponent's strategy s_j, then s_j must be in $S_j^k(h)$, and hence there must be some belief in $B_j^{opp,k-1}$ for which s_j is optimal from h onwards. The crucial difference between quasi-perfect rationalizability and perfect quasi-perfect rationalizability is that in the latter concept, a player also deems possible future mistakes by himself, whereas in the first concept a player only takes into account mistakes by his opponents. However – and that is crucial – under the latter concept the player deems his own mistakes infinitely less likely than the opponents' mistakes. Due to this last property, it can be shown that every perfectly quasi-perfectly rationalizable strategy is also quasi-perfectly rationalizable.

Theorem 4.1 (Relation with quasi-perfect rationalizability) *Every perfectly quasi-perfectly rationalizable strategy is quasi-perfectly rationalizable.*

The other direction is not true. Indeed, in game Γ^1 from Figure 1 the strategy (a, c) is quasi-perfectly rationalizable but not perfectly quasi-perfectly rationalizable. The proof of Theorem 4.1 can be found in Appendix C. What makes the proof challenging is that perfect quasi-perfect rationalizability and quasi-perfect rationalizability are defined in fundamentally different ways: In perfect quasi-perfect rationalizability a player holds beliefs about his own future choices, and optimality of a strategy is defined *locally,* on a choice-by-choice basis. That is, optimality requires that at every information set the prescribed choice is locally optimal, given the player's belief about his own future choices and given his belief about the opponents' strategies. In contrast, quasi-perfect rationalizability does not involve beliefs about the player's own future choices, and optimality is defined *globally.* That is, optimality requires that at every information set the player's strategy is optimal, given his belief about the opponents' strategies. A key step in the proof is to show that a sequence of locally optimal *choices* in the perfect quasi-perfect rationalizability concept always yields a globally optimal *strategy* in the quasi-perfect rationalizability concept. See Corollary 7.1 in Appendix C.

In Appendix D we show that Theorem 4.1 is no longer true if we replace our quasi-perfect rationalizability concept by AP-quasi-perfect rationalizability, as defined in Asheim and Perea (2005). Indeed, we provide a counterexample where some strategy is perfectly quasi-perfectly rationalizable, but not AP-quasi-perfectly rationalizable. Apparently, the extra conditions that AP-quasi-perfect rationalizability imposes relative to quasi-perfect rationalizability, as discussed as the beginning of this section, are not shared by perfect quasi-perfect rationalizability. At the same time, AP-quasi-perfect rationalizability is not a refinement of perfect quasi-perfect rationalizability, as can be seen from the game Γ^1 in Figure 1. In that game, strategy (a, c) is AP-quasi-perfectly rationalizable, but not perfectly quasi-perfectly rationalizable.

It is easily seen that the quasi-perfect rationalizability procedure is a refinement of the backward dominance procedure in Perea (2014). Indeed, every strategy that survives the quasi-perfect rationalizability procedure also survives the backward dominance procedure, but not *vice versa.* The key difference between the two procedures is that the backward dominance procedure does not impose cautious reasoning, as a player, at each of his information sets, is free to assign probability 0 to certain opponents' strategies. Perea (2014) has shown that in every game with perfect information without relevant ties, the only strategies that survive the backward dominance procedure are the backward induction strategies. In the light of Theorem

4.1 it thus follows that in every such game, the only (perfectly) quasi-perfectly rationalizable strategies are the backward induction strategies.

Since we have seen in the previous section that perfect quasi-perfect rationalizability is a refinement of perfect rationalizability, it follows from Theorem 4.1 that perfect quasi-perfect rationalizability refines both perfect rationalizability and quasi-perfect rationalizability. Hence, it inherits all the desirable properties that perfect rationalizability and quasi-perfect rationalizability display. Yet, it adds the requirement that a player deems his own future mistakes infinitely less likely than his opponents' mistakes, without completely discarding his own future mistakes as quasi-perfect rationalizability does.

Corollary 4.1 (Relation with perfect and quasi-perfect rationalizability) *Every perfectly quasi-perfectly rationalizable strategy is both perfectly rationalizable and quasi-perfectly rationalizable.*

In each of the examples we have seen so far, the other direction of the corollary was also true. Indeed, in each of those examples every strategy that was both perfectly rationalizable and quasi-perfectly rationalizable was also perfectly quasi-perfectly rationalizable. It is easily seen that this direction is always true in every game with perfect information and without relevant ties, as in such games the three concepts above all uniquely select the backward induction strategies for the players. Whether this direction is still true for games with perfect information and relevant ties is still an open question to us.

However, as the following example will show, the opposite direction of the corollary is not generally true for games with imperfect information. There are games where a strategy is both perfectly rationalizable and quasi-perfectly rationalizable, but not perfectly quasi-perfectly rationalizable.

Example 2: Combining perfect and quasi-perfect rationalizability does not lead to perfect quasi-perfect rationalizability.

Consider the game in Figure 3. Then, strategy g for player 2 is both perfectly rationalizable and quasi-perfectly rationalizable, but not perfectly quasi-perfectly rationalizable. To see that g is perfectly rationalizable, note first that strategy b is perfectly rationalizable for player 1 if he believes that his own mistake (c, e) is much more likely than player 2's mistake i. Hence, under perfect rationalizability, player 2 may assign at his information set a high probability to player 1 choosing b, which makes g optimal for player 2.

To see that g is quasi-perfectly rationalizable, note that under quasi-perfect rationalizability player 1 believes that he will not make mistakes himself, and hence strategy (a, c, f) will be among his optimal strategies. Therefore, player 2 may assign at his information set a high probability to player 1 choosing (a, c, f), which makes g optimal for player 2.

Under perfect quasi-perfect rationalizability, however, player 1 believes that he will make the mistake (c, e) with positive probability, but he deems the probability of player 2 making the

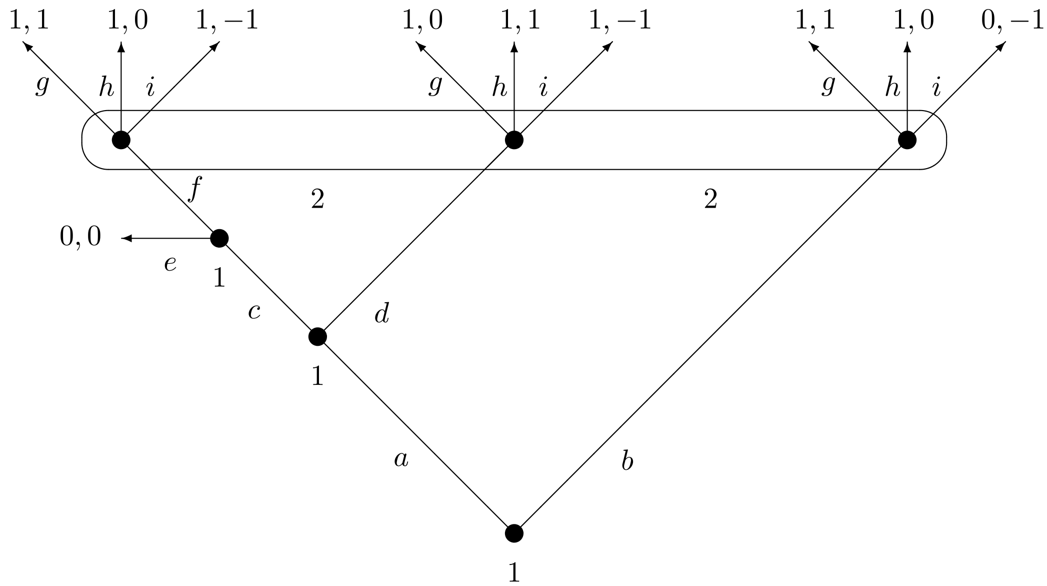

Figure 3: Combining perfect and quasi-perfect rationalizability does not lead to perfect quasi-perfect rationalizability

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mistake i much higher. As such, the only perfectly quasi-perfectly rationalizable strategy for player 1 is (a, d). Hence, player 2 must at his information set assign a high probability to player 1 choosing (a, d), which implies that player 2 must choose h. That is, h is the only perfectly quasi-perfectly rationalizable strategy for player 2. In particular, g is not perfectly quasi-perfectly rationalizable. □

5 Appendix A: Non-Standard Analysis

5.1 Non-Standard Numbers

Recall that the field of non-standard numbers $\mathbf{R}(\varepsilon)$ contains all numbers a that can be written as

$$a = \frac{a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... + a_K\varepsilon^K}{b_0 + b_1\varepsilon + b_2\varepsilon^2 + ... + b_K\varepsilon^K}, \tag{5.1}$$

where $a_k, b_k \in \mathbf{R}$ for all $k \in \{0, ..., K\}$, $b_k \neq 0$ for some $k \in \{0, ..., K\}$, and where either $a_0 \neq 0$ or $b_0 \neq 0$. We call the non-standard number $a \in \mathbf{R}(\varepsilon)$ *finite* if there is some number $b \in \mathbf{R}$ such that $|a| < b$. It is easily seen that a is finite, if and only if, $b_0 \neq 0$. We now show that every finite non-standard number can be written as a (possibly infinite) polynomial in ε. Since $b_0 \neq 0$, we can write the denominator in (5.1) as

$$b_0(1 + \frac{b_1}{b_0}\varepsilon + \frac{b_2}{b_0}\varepsilon^2 + ... + \frac{b_K}{b_0}\varepsilon^K).$$

Moreover, by the property of ε we know that $|\frac{b_k}{b_0}\varepsilon^k| < (\frac{1}{2})^k$ for every $k \in \{1, ..., K\}$, and hence

$$\left|\frac{b_1}{b_0}\varepsilon + \frac{b_2}{b_0}\varepsilon^2 + ... + \frac{b_K}{b_0}\varepsilon^K\right| \leq \sum_{k=1}^{K}\left|\frac{b_k}{b_0}\varepsilon^k\right| < \sum_{k=1}^{K}(\frac{1}{2})^k < 1.$$

But then, by the formula for geometric series it immediately follows that

$$(1 + \frac{b_1}{b_0}\varepsilon + \frac{b_2}{b_0}\varepsilon^2 + ... + \frac{b_K}{b_0}\varepsilon^K)^{-1} = 1 + \sum_{m=1}^{\infty}(-1)^m\left(\frac{b_1}{b_0}\varepsilon + \frac{b_2}{b_0}\varepsilon^2 + ... + \frac{b_K}{b_0}\varepsilon^K\right)^m.$$

Combining this with (5.1) then yields

$$a = \frac{1}{b_0}\left(a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... + a_K\varepsilon^K\right)\left(1 + \sum_{m=1}^{\infty}(-1)^m\left(\frac{b_1}{b_0}\varepsilon + \frac{b_2}{b_0}\varepsilon^2 + ... + \frac{b_K}{b_0}\varepsilon^K\right)^m\right)$$

which is a power series in ε. We thus conclude that every *finite* number $a \in \mathbf{R}(\varepsilon)$ can be written as

$$a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... , \tag{5.2}$$

where $a_k \in \mathbf{R}$ for every $k \geq 0$. We call this a *power series representation* of the number a. Below, we will show that this power series representation is unique.

5.2 Properties of Non-Standard Numbers

In this subsection we will investigate some important properties of finite non-standard numbers. First, we show that the sign of a non-standard number is fully determined by the sign of the leading coefficient in the power series representation (5.2). This property thus illustrates the lexicographic nature of the power series representation of non-standard numbers, as the leading coefficient a_k turns out to be " infinitely more important" than the collection of all the coefficients that follow.

Lemma 5.1 (Leading coefficient determines sign) *Consider a finite number $a \in \mathbf{R}(\varepsilon)$ where $a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + ...$. Then, $a > 0$ if and only if there is some $k \geq 0$ with $a_k > 0$ and $a_m = 0$ for all $m < k$.*

Proof. For the "if" part, suppose that $a_k > 0$ and $a_m = 0$ for all $m < k$. Hence,

$$a = a_k\varepsilon^k + a_{k+1}\varepsilon^{k+1} + ...$$

where $a_k > 0$. By the property of ε we know that

$$|a_m|\,\varepsilon^m < a_k \left(\frac{1}{2}\right)^m \varepsilon^k$$

for every $m \geq k+1$. Hence,

$$\left|\sum_{m=k+1}^{\infty} a_m\varepsilon^m\right| \leq \sum_{m=k+1}^{\infty} |a_m|\,\varepsilon^m < \sum_{m=k+1}^{\infty} a_k \left(\frac{1}{2}\right)^m \varepsilon^k \leq a_k\varepsilon^k,$$

which immediately implies that $a > 0$.

For the "only if" part, assume that $a > 0$. If $a_k = 0$ for all $k \geq 0$, then $a = 0$, which would be a contradiction. Hence, there must be some $k \geq 0$ with $a_k \neq 0$ and $a_m = 0$ for all $m < k$. If $a_k < 0$, then it follows by the "if" part above that $a < 0$, which would be a contradiction. Hence, we conclude that $a_k > 0$. ■

The lemma above really is the key result in this section, as all other properties follow rather directly from this lemma. A first consequence of Lemma 5.1 is that a non-standard number is 0 precisely when all coefficients in the power series representation are equal to 0.

Lemma 5.2 (Zero has unique representation) *Consider a finite number $a \in \mathbf{R}(\varepsilon)$ where $a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + ...$. Then, $a = 0$ if and only if $a_k = 0$ for all $k \geq 0$.*

Proof. The "if" direction is trivial. For the "only if" direction, assume that $a = 0$. Contrary to what we want to show, assume that there is some $k \geq 0$ with $a_k \neq 0$ and $a_m = 0$ for all $m < k$. If $a_k > 0$ then it follows from Lemma 5.1 that $a > 0$, which would be a contradiction. If $a_k < 0$ then it follows by Lemma 5.1 that $a < 0$, which would also be a contradiction. Hence, we conclude that $a_k = 0$ for all $k \geq 0$. ■

This lemma implies that for every finite non-standard number a, the power series representation is unique. Indeed, suppose that

$$a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + ... = b_0 + b_1\varepsilon + b_2\varepsilon^2 + ... \,.$$

Then,

$$(a_0 - b_0) + (a_1 - b_1)\varepsilon + (a_2 - b_2)\varepsilon^2 + ... = 0,$$

which implies by Lemma 5.2 that $a_k = b_k$ for all $k \geq 0$. We can therefore refer to (5.2) as *the* power series representation of the non-standard number a.

If $a \neq 0$, we denote by $v(a)$ the smallest index k for which $a_k \neq 0$, and call it the *valuation* of the number a. We set $v(0) := \infty$. For two finite non-standard numbers a and b we say that a is of *infinitely smaller size* than b if $v(a) > v(b)$. We use the term "infinitely smaller size" rather than the more familiar "infinitely smaller" because we also apply it to negative numbers. For instance, ε is of infinitely smaller size than -1, although ε is not smaller than -1. Note that, by definition, 0 is of infinitely smaller size than any finite non-zero non-standard number.

For a finite non-standard number a with power series representation (5.2), and a given $k \geq 0$, we call

$$trunc^k(a) = a_0 + a_1\varepsilon + ...a_k\varepsilon^k$$

the k-th order truncation of a. The 0-th order truncation $a_0 \in \mathbf{R}$ is also called the *standard part* of a, and is denoted by $st(a)$. Hence, $st(a)$ is the unique real number that is closest to a.

A consequence of Lemma 5.2 and Lemma 5.1 is that the k-th order truncation of a will either be zero, or have the same sign as a. This property will be important for our proofs.

Lemma 5.3 (Truncation has the same sign) *Consider a finite number $a \in \mathbf{R}(\varepsilon)$ with $a \geq 0$. Then, $trunc^k(a) \geq 0$ for every $k \geq 0$.*

Proof. Suppose first that $a = 0$. Then, by Lemma 5.2, $a_k = 0$ for all $k \geq 0$, and hence $trunc^k(a) = 0$ for all $k \geq 0$.

Assume next that $a > 0$. Then, by Lemma 5.1, there is some r with $a_r > 0$ and $a_m = 0$ for all $m < r$. If $k < r$, then $trunc^k(a) = 0$. If $k \geq r$, then $trunc^k(a) > 0$ by Lemma 5.1. ■

In the following subsection we will use the properties above to investigate non-standard probability distributions.

5.3 Non-Standard Probability Distributions

Consider a finite set X. A *non-standard probability distribution* on X is a function $p : X \to \mathbf{R}(\varepsilon)$ such that $p(x) \geq 0$ for all $x \in X$ and $\sum_{x\in X} p(x) = 1$. By $\Delta^{ns}(X)$ we denote the set of non-standard probability distributions on X. Such non-standard probability distributions will often be interpreted as *beliefs.* We therefore use the terms "non-standard probability distribution" and "belief" interchangeably in this paper. For two elements x and y in X, we say that p deems x *infinitely more likely than* y if $p(y)$ is of infinitely smaller size than $p(x)$.

Consider a non-standard probability distribution p on X. For a subset $Y \subseteq X$ with $\sum_{x\in Y} p(x) > 0$, the *conditional* probability distribution on Y induced by p is the non-standard probability distribution p_Y on Y given by

$$p_Y(x) := \frac{p(x)}{\sum_{y\in Y} p(y)}$$

for every $x \in Y$.

We say that p is *cautious* on X if $p(x) > 0$ for all $x \in X$, such that conditional probability distributions can be formed for every subset $Y \subseteq X$. We call p a *standard* probability distribution on X if $p(x) \in \mathbf{R}$ for all $x \in X$, and the set of standard probability distributions on X is denoted by $\Delta(X)$. A *standard zero-sum distribution* on X is a function $f : X \to \mathbf{R}$ with $\sum_{x \in X} f(x) = 0$.

Consider a non-standard probability distribution p on X. From above we know that every probability $p(x)$ has a unique power series representation

$$p(x) = p_0(x) + p_1(x)\varepsilon + p_2(x)\varepsilon^2 + \dots ,$$

where $p_k(x) \in \mathbf{R}$ for every $k \geq 0$. As $\sum_{x \in X} p(x) = 1$, it follows that

$$(\sum_{x \in X} p_0(x) - 1) + \varepsilon \left(\sum_{x \in X} p_1(x) \right) + \varepsilon^2 \left(\sum_{x \in X} p_2(x) \right) + \dots = 0.$$

By Lemma 5.2 we thus conclude that

$$\sum_{x \in X} p_0(x) = 1 \text{ and } \sum_{x \in X} p_k(x) = 0 \text{ for all } k \geq 1.$$

Moreover, since $p(x) \geq 0$ for every $x \in X$, it follows by Lemma 5.3 that $p_0(x) = trunc^0(p(x)) \geq 0$ for every $x \in X$. Hence, we conclude that $p_0 := (p_0(x))_{x \in X}$ is a standard probability distribution in $\Delta(X)$, and that $p_k := (p_k(x))_{x \in X}$ is a standard zero-sum distribution on X.

As such, every non-standard probability distribution p on X can uniquely be written as

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots \tag{5.3}$$

where $p_0 \in \Delta(X)$ is a standard probability distribution on X, and p_k is a standard zero-sum distribution on X for every $k \geq 1$. We call this the (unique) *power series representation* of the non-standard probability distribution p. This representation will be important for our game-theoretic analysis later on.

By $st(p) := p_0$ we denote the standard part of the non-standard probability distribution p. For a subset $Y \subseteq X$ we say that p *believes* Y if $st(\sum_{x \in Y} p(x)) = 1$. By the representation (5.3), this thus means that $\sum_{x \in Y} p_0(x) = 1$.

Consider an element $x \in X$. As $p(x) \geq 0$, it follows from (5.3) and Lemma 5.3 that $trunc^k(p(x)) = p_0(x) + \varepsilon p_1(x) + \dots + \varepsilon^k p_k(x) \geq 0$ for every $k \geq 0$. Hence, $p_0 + \varepsilon p_1 + \dots + \varepsilon^k p_k$ is a non-standard probability distribution on X as well.

Suppose that p is cautious on X, and let k be the minimal index such that the truncated non-standard probability distribution $p_0 + \varepsilon p_1 + \dots + \varepsilon^k p_k$ is cautious on X. Then, $p_0 + \varepsilon p_1 + \dots + \varepsilon^k p_k$ is called the *minimal cautious truncation* of p on X.

6 Appendix B: Proof of Theorem 3.1

Remember that B_i^k denotes the set of belief pairs (b_i^{self}, b_i^{opp}) for player i that survives round k of the procedure. Similarly, $S_i^k(h)$ is the set of strategies for player i that survive round k of the procedure at information set h. By construction, $B_i^k \subseteq B_i^{k-1}$ and $S_i^k(h) \subseteq S_i^{k-1}(h)$ for all $k \geq 1$. Since the collection of information sets is finite, and the set of strategies $S_i(h)$ is finite for every player i and every information set h, the procedure must terminate within finitely many steps. To prove the existence of perfectly quasi-perfectly rationalizable strategies, it is therefore sufficient to show that B_i^k and $S_i^k(h)$ are always non-empty for every player i, every information set h and every $k \geq 0$. We prove so by induction on k.

For $k = 0$ we have that $S_i^0(h) = S_i(h)$, and hence $S_i^0(h)$ is non-empty. To prove that B_i^0 is non-empty, we must show that there is a belief pair (b_i^{self}, b_i^{opp}) for player i that believes in his own future rationality and deems his own mistakes least likely.

Take an arbitrary cautious non-standard probability distribution b_i^{opp} on the set S_{-i} of opponents' strategy combinations. For every opponents' strategy combination, let $v(b_i^{opp}(s_{-i}))$ be the valuation of the probability $b_i^{opp}(s_{-i})$, as defined in Section 5.2. That is, if $b_i^{opp}(s_{-i}) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + ...$, then $v(b_i^{opp}(s_{-i}))$ is the smallest number m such that $a_m \neq 0$. Let $v := \max\{v(b_i^{opp}(s_{-i})) \mid s_{-i} \in S_{-i}\}$ be the maximal valuation of $b_i^{opp}(s_{-i})$ across all opponents' strategy combinations $s_{-i} \in S_{-i}$.

We now define the cautious non-standard probability distribution b_i^{self} on the set S_i of i's own strategies by a backward induction construction, as follows. For every $m \geq 0$, let H_i^m be the collection of information sets in H_i that are followed by at most m consecutive information sets in H_i.

We start by considering all information sets in H_i^0, that is, player i information sets that are not followed by any other player i information set. Consider an information set $h \in H_i^0$, and let $c_i^*(h)$ be an optimal choice for player i at h given the conditional belief $b_i^{opp}(h)$. Let σ_{ih} be the cautious non-standard probability distribution on the set of available choices $C_i(h)$ given by

$$\sigma_{ih}(c_i) := \begin{cases} 1 - (|C_i(h)| - 1) \cdot \varepsilon^{v+1}, & \text{if } c_i = c_i^*(h) \\ \varepsilon^{v+1}, & \text{if } c_i \neq c_i^*(h) \end{cases} . \tag{6.1}$$

Now let $m \geq 1$ and consider some $h \in H_i^m$. Suppose that the choice $c_i^*(h')$ and the cautious non-standard probability distribution $\sigma_{ih'}$ have been defined at all $h' \in H_i^l$ where $l \leq m - 1$. In particular, $c_i^*(h')$ and $\sigma_{ih'}$ have been defined for all information sets $h' \in H_i$ following h. For every choice $c_i \in C_i(h)$, let $u_i(c_i, ((\sigma_{ih'})_{h' \in H_i : h' \succ h}, b_i^{opp}), h)$ be the expected utility of making choice c_i at h, given the conditional belief $b_i^{opp}(h)$ about the opponents' strategy combinations, and given the non-standard probability distributions $\sigma_{ih'}$ on i's own choices at h' for every $h' \in H_i$ that follows h. Above, we have used the expression "$h' \succ h$" as a shortcut for "h' follows h". Let $c_i^*(h)$ be an optimal choice for player i at h, that is,

$$u_i(c_i^*(h), ((\sigma_{ih'})_{h' \in H_i : h' \succ h}, b_i^{opp}), h) \geq u_i(c_i, ((\sigma_{ih'})_{h' \in H_i : h' \succ h}, b_i^{opp}), h) \text{ for all } c_i \in C_i(h). \tag{6.2}$$

Moreover, let σ_{ih} be the cautious non-standard probability distribution on the set of available choices $C_i(h)$ given by

$$\sigma_{ih}(c_i) := \begin{cases} 1 - (|C_i(h)| - 1) \cdot \varepsilon^{v+1}, & \text{if } c_i = c_i^*(h) \\ \varepsilon^{v+1}, & \text{if } c_i \neq c_i^*(h) \end{cases}. \tag{6.3}$$

By induction on m we have thus defined, for every information set $h \in H_i$, the cautious non-standard probability distribution σ_{ih} on $C_i(h)$.

Let b_i^{self} be the cautious non-standard probability distribution on i's own strategies given by

$$b_i^{self}(s_i) := \prod_{h \in H_i(s_i)} \sigma_{ih}(s_i(h)) \text{ for every } s_i \in S_i. \tag{6.4}$$

We will now show that $b_i = (b_i^{self}, b_i^{opp})$ believes in his own future rationality and deems his own mistakes least likely.

To prove that b_i believes in his own future rationality, we must show that $b_i^{self}(h, c_i)$ believes $S_i^{rat}(b_i, h)$ for every $h \in H_i$ and $c_i \in C_i(h)$. Take some $h \in H_i$ and $c_i \in C_i(h)$. By (6.4) we conclude that

$$u_i(c_i, b_i, h) = u_i(c_i, ((\sigma_{ih'})_{h' \in H_i : h' \succ h}, b_i^{opp}), h). \tag{6.5}$$

By (6.5) and (6.2) it then follows that

$$u_i(c_i^*(h), b_i, h) \geq u_i(c_i, b_i, h) \text{ for all } c_i \in C_i(h). \tag{6.6}$$

Let $s_i^*(h, c_i)$ be the unique strategy in $S_i(h, c_i)$ such that $s_i^*(h, c_i)$ prescribes the optimal choice $c_i^*(h')$ at every $h' \in H_i(s_i^*(h, c_i))$ not weakly preceding h. Hence, in particular, $s_i^*(h, c_i)$ prescribes the optimal choice $c_i^*(h')$ at every $h' \in H_i(s_i^*(h, c_i))$ following h. Then, by (6.6) applied to every $h' \in H_i(s_i^*(h, c_i))$ following h, we conclude that $s_i^*(h, c_i) \in S_i^{rat}(b_i, h)$.

Moreover, by (6.4) and (6.3) we conclude that the standard part of the conditional non-standard probability distribution $b_i^{self}(h, c_i)$ assigns probability 1 to $s_i^*(h, c_i) \in S_i^{rat}(b_i, h)$. This implies that $b_i^{self}(h, c_i)$ believes $S_i^{rat}(b_i, h)$. As this is true for every $h \in H_i$ and $c_i \in C_i(h)$, we conclude that b_i believes in his own future rationality.

We next prove that $b_i = (b_i^{self}, b_i^{opp})$ deems his own mistakes least likely. Consider an information set $h \in H_i$ and a choice $c_i \in C_i(h)$. From (6.4) and (6.3) we see that for every strategy $s_i \in S_i(h, c_i)$, the infinitesimal mistake part $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i))$ has a valuation which is at least $v+1$. On the other hand, the non-standard probability $b_i^{opp}(s_{-i})$ has a valuation of at most v for every opponents' strategy combination s_{-i}, by definition of v. We thus conclude that $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i))$ is of infinitely smaller size than $b_i^{opp}(s_{-i})$. As this holds for every information set $h \in H_i$, choice $c_i \in C_i(h)$, strategy $s_i \in S_i(h, c_i)$ and opponents' strategy combination $s_{-i} \in S_{-i}$, we know that $b_i = (b_i^{self}, b_i^{opp})$ deems his own mistakes least likely.

Overall, we have thus constructed a belief $b_i = (b_i^{self}, b_i^{opp})$ that believes in his own future rationality and deems his own mistakes least likely. Hence, by definition, $b_i \in B_i^0$, which implies that B_i^0 is non-empty.

Now, let $k \geq 1$ and assume that $S_i^{k-1}(h)$ and B_i^{k-1} are non-empty for every player i and every information set h. Consider some player i and information set h. We will show that $S_i^k(h)$ and B_i^k are non-empty.

To show that $S_i^k(h)$ is non-empty, take some $b_i = (b_i^{self}, b_i^{opp})$ in B_i^{k-1}. This is possible since we assume that B_i^{k-1} is non-empty. At every information set $h' \in H_i$, let $c_i[h']$ be a locally rationally choice for $b_i = (b_i^{self}, b_i^{opp})$ at h'. Let s_i be a strategy in $S_i(h)$ such that $s_i(h') = c_i[h']$ for every $h' \in H_i(s_i)$ weakly following h. Then, by construction, $s_i \in S_i^k(h)$, and hence $S_i^k(h)$ is non-empty.

We will next construct a belief $b_i = (b_i^{self}, b_i^{opp})$ in B_i^k. We start by defining b_i^{opp}. That is, we must find a belief b_i^{opp} such that $b_i^{opp}(h)$ believes $S_{-i}^k(h)$ for all $h \in H_i$. For every opponent $j \neq i$, let b_j be an arbitrary belief pair in B_j^{k-1}. This is possible since we assume that B_j^{k-1} is non-empty. For every information set $h \in H_i$ let $s_{-i}[h] = (s_j[h])_{j \neq i}$ be an opponents' strategy combination in $S_{-i}(h)$ with the following property: For every opponent $j \neq i$ and every information set $h' \in H_j(s_j)$ that does not precede h, the choice $(s_j[h])(h')$ is locally rational for b_j at h'. Clearly, such a strategy $s_j[h]$ can always be found. Then, by construction, $s_j[h] \in S_j^k(h')$ for every h' that does not precede h and such that $s_j[h] \in S_j(h')$. Hence,

$$\begin{array}{c} s_{-i}[h] \in S_{-i}^k(h') \text{ for every } h' \text{ that does not precede } h \\ \text{and such that } s_{-i}[h] \in S_{-i}(h'). \end{array} \tag{6.7}$$

Since $s_{-i}[h] \in S_{-i}(h)$, it follows in particular that $s_{-i}[h] \in S_{-i}^k(h)$.

Let $h_i^0, h_i^1, ..., h_i^M$ be a numbering of the information sets of player i which respects their precedence ordering. That is, if h_i^l precedes h_i^m then $l < m$. Hence, it must be that $h_i^0 = \emptyset$. Let b_i^{opp} be the cautious non-standard belief about the opponents' strategy combinations given by

$$b_i^{opp}(s_{-i}) = \begin{cases} 1 - a, & \text{if } s_{-i} = s_{-i}[h_i^0] \\ \varepsilon^m, & \text{if } s_{-i} \neq s_{-i}[h_i^0], \text{ and } m \in \{1, ..., M\} \text{ is minimal with } s_{-i} = s_{-i}[h^m], \\ \varepsilon^{M+1}, & \text{otherwise,} \end{cases}$$

where a is chosen such that $\sum_{s_{-i} \in S_{-i}} b_i^{opp}(s_{-i}) = 1$. Hence, $st(a) = 0$.

We will now show that $b_i^{opp}(h)$ believes $S_{-i}^k(h)$ for all $h \in H_i$. Take some arbitrary $h \in H_i$, and let $h = h_i^m$. Let $l \in \{0, 1, ..., M\}$ be the smallest number such that $s_{-i}[h_i^l] \in S_{-i}(h_i^m)$. Then, by construction, the standard part of the conditional belief $b_i^{opp}(h_i^m)$ assigns probability 1 to $s_{-i}[h_i^l]$. That is, $b_i^{opp}(h_i^m)$ believes $\{s_{-i}[h_i^l]\}$. Since $s_{-i}[h_i^m] \in S_{-i}(h_i^m)$, we know that $l \leq m$, and hence h_i^m does not precede h_i^l. We thus conclude that $s_{-i}[h_i^l] \in S_{-i}(h_i^m)$ and that h_i^m does not precede h_i^l. But then, by (6.7), $s_{-i}[h_i^l] \in S_{-i}^k(h_i^m)$. As the conditional belief $b_i^{opp}(h_i^m)$ believes

$\{s_{-i}[h_i^l]\}$, we conclude that $b_i^{opp}(h_i^m)$ believes $S_{-i}^k(h_i^m)$. This holds for every m, and hence $b_i^{opp}(h)$ believes $S_{-i}^k(h)$ for all $h \in H_i$.

In this way, we can construct a cautious non-standard belief b_i^{opp} on the opponents' strategy combinations such that $b_i^{opp}(h)$ believes $S_{-i}^k(h)$ for all $h \in H_i$. With b_i^{opp} at hand, we can then define the belief b_i^{self} in the same way as above, guaranteeing that $b_i = (b_i^{self}, b_i^{opp})$ believes in his own future rationality and deems his own mistakes least likely. Hence, $b_i \in B_i^0$. Since, moreover, $b_i^{opp}(h)$ believes $S_{-i}^k(h)$ for all $h \in H_i$, we conclude that $b_i \in B_i^k$. We have thus shown that B_i^k is non-empty.

By induction on k, it follows that $S_i^k(h)$ and B_i^k are always non-empty for every player i, every information set h and every $k \geq 0$. In particular, $S_i^k(\emptyset)$ is always non-empty for all $k \geq 0$. Since the procedure terminates within finitely many steps, it follows that for every player i there is at least one perfectly quasi-perfectly rationalizable strategy. ■

7 Appendix C: Proof of Theorem 4.1

To prove Theorem 4.1, we proceed by three preparatory steps.

For the first step, consider a belief pair (b_i^{self}, b_i^{opp}) in B_i. For every information set $h \in H_i$ and choice $c_i \in C_i(h)$, let $st(b_i^{self}(h, c_i))$ be the standard part of the conditional belief $b_i^{self}(h, c_i)$ on $S_i(h, c_i)$. Moreover, let $tr(b_i^{opp})$ be the minimal cautious truncation of the cautious belief b_i^{opp} on S_{-i}, as defined in Section 5.3. For every $h \in H_i$, this truncated belief $tr(b_i^{opp})$ induces a conditional cautious belief $tr(b_i^{opp})(h)$ on $S_{-i}(h)$. We say that a choice $c_i^* \in C_i(h)$ is *locally rational* for $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$ at h if

$$\begin{aligned} & \sum_{s_i \in S_i(h, c_i^*)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i^*))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\ \geq & \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \end{aligned}$$

for every $c_i \in C_i(h)$.

Lemma 7.1 (Truncation preserves local rationality) *Let (b_i^{self}, b_i^{opp}) be a belief pair in B_i that deems own mistakes least likely, $h \in H_i$ an information set for player i, and $c_i^* \in C_i(h)$ a choice for player i at h. If c_i^* is locally rational for (b_i^{self}, b_i^{opp}) at h, then c_i^* is also locally rational at h for the truncated beliefs $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$.*

Proof. Let the power series representation of the belief b_i^{opp} on S_{-i}, as defined in Section 5.3, be given by

$$b_i^{opp} = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots ,$$

and let the minimal cautious truncation on S_{-i} be

$$tr(b_i^{opp}) = p_0 + \varepsilon p_1 + ... + \varepsilon^K p_K.$$

By definition of the minimal cautious truncation, there must be some $s_{-i}^* \in S_{-i}$ such that

$$b_i^{opp}(s_{-i}^*) = \varepsilon^K p_K(s_{-i}^*) + \varepsilon^{K+1} p_{K+1}(s_{-i}^*) + ... \tag{7.1}$$

with $p_K(s_{-i}^*) > 0$.

Then, the conditional beliefs at h induced by b_i^{opp} and $tr(b_i^{opp})$ are given by

$$b_i^{opp}(h)(s_{-i}) = \frac{1}{a}(p_0(s_{-i}) + \varepsilon p_1(s_{-i}) + \varepsilon^2 p_2(s_{-i}) + ...) \tag{7.2}$$

for every $s_{-i} \in S_{-i}(h)$, where $a := \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(s_{-i})$, and

$$tr(b_i^{opp})(h)(s_{-i}) = \frac{1}{b}(p_0(s_{-i}) + \varepsilon p_1(s_{-i}) + ... + \varepsilon^K p_K(s_{-i})) \tag{7.3}$$

for every $s_{-i} \in S_{-i}(h)$, where $b := \sum_{s_{-i} \in S_{-i}(h)} tr(b_i^{opp})(s_{-i})$.

For every choice $c_i \in C_i(h)$, let the power series representation of the conditional belief $b_i^{self}(h, c_i)$ on $S_i(h, c_i)$ be given by

$$b_i^{self}(h, c_i) = q_0^{c_i} + \varepsilon q_1^{c_i} + \varepsilon^2 q_2^{c_i} + ... ,$$

which implies that

$$b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i))(s_i) = \varepsilon q_1^{c_i}(s_i) + \varepsilon^2 q_2^{c_i}(s_i) + ... \tag{7.4}$$

for every $s_i \in S_i(h, c_i)$.

As (b_i^{self}, b_i^{opp}) deems own mistakes least likely, we must have that $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i))(s_i)$ is of infinitely smaller size than $b_i^{opp}(s_{-i}^*)$ for every $c_i \in C_i(h)$ and every $s_i \in S_i(h, c_i)$. By (7.1) and (7.4) it thus follows that

$$b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i))(s_i) = \varepsilon^{K+1} q_{K+1}^{c_i}(s_i) + \varepsilon^{K+2} q_{K+2}^{c_i}(s_i) + ... ,$$

and hence

$$b_i^{self}(h, c_i) = q_0^{c_i} + \varepsilon^{K+1} q_{K+1}^{c_i} + \varepsilon^{K+2} q_{K+2}^{c_i} + ... \tag{7.5}$$

Let $b_i = (b_i^{self}, b_i^{opp})$. By (7.2) and (7.5) it follows, for every choice $c_i \in C_i(h)$, that

$$\begin{aligned}
u_i(c_i, b_i, h) &= \sum_{s_i \in S_i(h,c_i)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} \left(q_0^{c_i}(s_i) + \varepsilon^{K+1} q_{K+1}^{c_i}(s_i) + \varepsilon^{K+2} q_{K+2}^{c_i}(s_i) + \ldots\right) \cdot \\
&\quad \cdot \frac{1}{a} \left(p_0(s_{-i}) + \varepsilon p_1(s_{-i}) + \varepsilon^2 p_2(s_{-i}) + \ldots\right) \cdot u_i(z(s_i, s_{-i})) \\
&= \frac{1}{a} \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} (q_0^{c_i}(s_i) p_0(s_{-i}) + \varepsilon q_0^{c_i}(s_i) p_1(s_{-i}) + \ldots \\
&\quad + \varepsilon^K q_0^{c_i}(s_i) p_K(s_{-i}) + d^{c_i}(s_i, s_{-i})) \cdot u_i(z(s_i, s_{-i})),
\end{aligned}$$

where $v(d^{c_i}(s_i, s_{-i})) \geq K+1$. Remember that $v(d^{c_i}(s_i, s_{-i}))$ denotes the valuation of $d^{c_i}(s_i, s_{-i})$, which is the index of the leading coeficient in the power series representation of $d^{c_i}(s_i, s_{-i})$. Moreover, recall that the choice c_i^* is locally rational for $b_i = (b_i^{self}, b_i^{opp})$ at h. Then, for every choice $c_i \in C_i(h)$,

$$\begin{aligned}
u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h) &= \frac{1}{a} \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} ((q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_0(s_{-i}) + \\
&\quad + \varepsilon (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_1(s_{-i}) + \ldots \\
&\quad + \varepsilon^K (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_K(s_{-i}) + \hat{d}^{c_i}(s_i, s_{-i})) \cdot u_i(z(s_i, s_{-i})) \quad (7.6)
\end{aligned}$$

where $v(\hat{d}^{c_i}(s_i, s_{-i})) \geq K+1$.

Since c_i^* is locally rational for $b_i = (b_i^{self}, b_i^{opp})$ at h, we have that $u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h) \geq 0$ for all $c_i \in C_i(h)$. This implies that $\frac{a}{b} \cdot (u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h)) \geq 0$. By Lemma 5.3 we thus know that $trunc^K(\frac{a}{b} \cdot (u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h))) \geq 0$.

By (7.6) we thus conclude that

$$trunc^K(\frac{a}{b} \cdot (u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h))) = \frac{1}{b}(\sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} ((q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_0(s_{-i}) +$$

$$+ \varepsilon (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_1(s_{-i}) + \ldots + \varepsilon^K (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_K(s_{-i})) \cdot u_i(z(s_i, s_{-i}))) \geq 0$$

for every $c_i \in C_i(h)$. Hence,

$$\begin{aligned} & \frac{1}{b} \cdot \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} (q_0^{c_i^*}(s_i) p_0(s_{-i}) + \varepsilon q_0^{c_i^*}(s_i) p_1(s_{-i}) + ... + \varepsilon^K q_0^{c_i^*}(s_i) p_K(s_{-i})) \cdot u_i(z(s_i, s_{-i})) \\ \geq \ & \frac{1}{b} \cdot \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} (q_0^{c_i}(s_i) p_0(s_{-i}) + \varepsilon q_0^{c_i}(s_i) p_1(s_{-i}) + ... + \varepsilon^K q_0^{c_i}(s_i) p_K(s_{-i})) \cdot u_i(z(s_i, s_{-i})) \end{aligned}$$

for all $c_i \in C_i(h)$. As $st(b_i^{self}(h, c_i)) = q_0^{c_i}$ for all $c_i \in C_i(h)$ and $tr(b_i^{opp})(h)$ is given by (7.3), the above inequality is equivalent to

$$\begin{aligned} & \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i^*))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\ \geq \ & \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \end{aligned}$$

for all $c_i \in C_i(h)$. This, in turn, means that c_i^* is locally rational at h for $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$, which was to show. ■

As a second step, we prove that if a player believes in his own future rationality, then *local* rationality of a strategy at all information sets weakly following information set h^* implies *global* rationality of this strategy at h^*. To define this lemma formally, we need some additional notation and definitions. Let $\hat{b}_i^{self} = (\hat{b}_i^{self}(h, c_i))_{h \in H_i, c_i \in C_i(h)}$, where $\hat{b}_i^{self}(h, c_i)$ is a standard probability distribution on $S_i(h, c_i)$ for every $h \in H_i$ and every $c_i \in C_i(h)$. Moreover, let $b_i^{opp} \in B_i^{opp}$. For a given information set $h \in H_i$ and choice $c_i \in C_i(h)$, we define the expected utility

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) := \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} \hat{b}_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})).$$

We call c_i locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at h if

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) \geq u_i(c_i', (\hat{b}_i^{self}, b_i^{opp}), h) \text{ for all } c_i' \in C_i(h).$$

We say that $(\hat{b}_i^{self}, b_i^{opp})$ *believes in his own future rationality* if for every $h \in H_i$ and $c_i \in C_i(h)$, the standard probability distribution $\hat{b}_i^{self}(h, c_i)$ only assigns positive probability to strategies $s_i \in S_i(h, c_i)$ where $s_i(h')$ is locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at every $h' \in H_i(s_i)$ following h.

Lemma 7.2 (When local rationality implies global rationality) *Let $\hat{b}_i^{self} = (\hat{b}_i^{self}(h, c_i))_{h \in H_i, c_i \in C_i(h)}$ where $\hat{b}_i^{self}(h, c_i)$ is a standard probability distribution on $S_i(h, c_i)$ for every $h \in H_i$ and every $c_i \in C_i(h)$. Let $b_i^{opp} \in B_i^{opp}$ and assume that $(\hat{b}_i^{self}, b_i^{opp})$ believes in his own future rationality. Let $s_i^* \in S_i$ and $h^* \in H_i(s_i^*)$ such that $s_i^*(h)$ is locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at every $h \in H_i(s_i^*)$ weakly following h^*. Then, s_i^* is globally rational for b_i^{opp} at h^*.*

Proof. We first introduce some additional notation. For every information set $h \in H_i$ and every choice $c_i \in C_i(h)$, let

$$u_i^{\max}(b_i^{opp}, h) := \max_{s_i \in S_i(h)} u_i(s_i, b_i^{opp}(h))$$

and

$$u_i^{\max}(b_i^{opp}, h, c_i) := \max_{s_i \in S_i(h, c_i)} u_i(s_i, b_i^{opp}(h)).$$

Then, we have that

$$u_i^{\max}(b_i^{opp}, h) = \max_{c_i \in C_i(h)} u_i^{\max}(b_i^{opp}, h, c_i) \tag{7.7}$$

for every $h \in H_i$. Moreover, strategy s_i is globally rational for b_i^{opp} at $h \in H_i(s_i)$ if

$$u_i(s_i, b_i^{opp}(h)) = u_i^{\max}(b_i^{opp}, h).$$

For every information set $h \in H_i$ and every choice $c_i \in C_i(h)$ we also define

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) := \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} \hat{b}_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})).$$

Hence, strategy s_i is locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at $h \in H_i(s_i)$ if

$$u_i(s_i(h), (\hat{b}_i^{self}, b_i^{opp}), h) \geq u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) \text{ for all } c_i \in C_i(h).$$

We prove the statement in the lemma by induction on the number of player i information sets that follow h^*. If h^* is not followed by any player i information set, then the statement holds because local rationality for $(\hat{b}_i^{self}, b_i^{opp})$ at h^* coincides with global rational for b_i^{opp} at h.

Suppose now that h^* is followed by $k \geq 1$ consecutive player i information sets, and that the statement holds for every player i information set that follows h^*. Consider some choice $c_i \in C_i(h^*)$. By $H_i^+(h^*, c_i)$ we denote the collection of information sets $h \in H_i$ such that h weakly follows h^* and c_i, and there is no $h' \in H_i$ preceding h that also weakly follows h^* and c_i. Let $S_{-i}^{not}(h^*, c_i)$ be the collection of those opponents' strategy combinations $s_{-i} \in S_{-i}(h^*)$ that after h^* and c_i do not lead to any player i information set.

Then, for every $c_i \in C_i(h^*)$ we have that

$$\begin{aligned}
u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{s_i \in S_i(h^*, c_i)} \sum_{s_{-i} \in S_{-i}(h^*)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \\
&\quad \cdot [\sum_{h \in H_i^+(h^*, c_i)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) + \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))] \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \\
&\quad \cdot [\sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \sum_{s_{-i} \in S_{-i}(h)} \frac{b_i^{opp}(h^*)(s_{-i})}{b_i^{opp}(h^*)(S_{-i}(h))} \cdot u_i(z(s_i, s_{-i})) \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))] \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \\
&\quad [\sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))] \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot [\sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i(s_i, b_i^{opp}(h)) \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))]. \qquad (7.8)
\end{aligned}$$

Here, the fourth equality follows from the rules of conditional probabilities.

As $(\hat{b}_i^{self}, b_i^{opp})$ believes in his own future rationality, $\hat{b}_i^{self}(h^*, c_i)$ only assigns positive probability to $s_i \in S_i(h^*, c_i)$ where s_i is locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at every $h \in H_i^+(h^*, c_i)$, and every $h' \in H_i(s_i)$ that follows h. By the induction assumption, we know that every such s_i is globally rational at every $h \in H_i^+(h^*, c_i)$. Hence, $\hat{b}_i^{self}(h^*, c_i)$ only assigns positive probability to $s_i \in S_i(h^*, c_i)$ where

$$u_i(s_i, b_i^{opp}(h)) = u_i^{\max}(b_i^{opp}, h)$$

for every $h \in H_i^+(h^*, c_i)$. Together with (7.8) we conclude that

$$\begin{aligned}
u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot [\sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i^{\max}(b_i^{opp}, h) \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))] \\
&= \sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i^{\max}(b_i^{opp}, h) + \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \qquad (7.9) \\
&= u_i^{\max}(b_i^{opp}, h^*, c_i).
\end{aligned}$$

Here, the last equality follows from the fact that the terminal node $z(s_i, s_{-i})$ does not depend on the specific $s_i \in S_i(h^*, c_i)$ if $s_{-i} \in S_{-i}^{not}(h^*, c_i)$. Hence, we see that

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) = u_i^{\max}(b_i^{opp}, h^*, c_i) \text{ for all } c_i \in C_i(h^*). \qquad (7.10)$$

As $s_i^*(h^*)$ is locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at h^*, we know that

$$\begin{aligned}
u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \max_{c_i \in C_i(h^*)} u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) \\
&= \max_{c_i \in C_i(h^*)} u_i^{\max}(b_i^{opp}, h^*, c_i) = u_i^{\max}(b_i^{opp}, h^*), \qquad (7.11)
\end{aligned}$$

where the second equality follows from (7.10) and the last equality from (7.7).

On the other hand, we know by (7.9) that

$$\begin{aligned}
u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{h \in H_i^+(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i^{\max}(b_i^{opp}, h) + \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})). \qquad (7.12)
\end{aligned}$$

As $s_i^*(h)$ is assumed to be locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at every $h \in H_i(s_i^*)$ weakly following h^*, we know that, for every $h \in H_i^+(h^*, s_i^*(h^*))$, the choice $s_i^*(h')$ is locally rational for $(\hat{b}_i^{self}, b_i^{opp})$ at every $h' \in H_i(s_i^*)$ weakly following h. Hence, by the induction assumption, s_i^* is globally rational for b_i^{opp} at every $h \in H_i^+(h^*, s_i^*(h^*))$, which means that

$$u_i(s_i^*, b_i^{opp}(h)) = u_i^{\max}(b_i^{opp}, h) \text{ for every } h \in H_i^+(h^*, s_i^*(h^*)). \qquad (7.13)$$

By combining (7.12) and (7.13) we obtain that

$$\begin{aligned} u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{h \in H_i^+(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i(s_i^*, b_i^{opp}(h)) + \\ &\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\ &= u_i(s_i^*, b_i^{opp}(h^*)). \end{aligned} \tag{7.14}$$

From (7.14) and (7.11) we can thus conclude that

$$u_i(s_i^*, b_i^{opp}(h^*)) = u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) = u_i^{\max}(b_i^{opp}, h^*).$$

This means that s_i^* is globally rational for b_i^{opp} at h^*, which was to show. By induction, the proof is thus complete. ■

As a third step, we are able to derive the following important result by combining Lemma 7.1 and Lemma 7.2. This step will be crucial for proving our main theorem below.

Corollary 7.1 (From local to global rationality) *Let (b_i^{self}, b_i^{opp}) be a belief pair in B_i that deems own mistakes least likely and believes in his own future rationality. Let $s_i^* \in S_i$ and $h^* \in H_i(s_i^*)$ such that s_i^* is locally rational for (b_i^{self}, b_i^{opp}) at every $h \in H_i(s_i^*)$ that weakly follows h^*. Then, s_i^* is globally rational at h^* for the minimal cautious truncation $tr(b_i^{opp})$ of b_i^{opp}.*

Proof. Suppose that s_i^* is locally rational for (b_i^{self}, b_i^{opp}) at every $h \in H_i(s_i^*)$ that weakly follows h^*. That is, $s_i^*(h)$ is locally rational for (b_i^{self}, b_i^{opp}) at every $h \in H_i(s_i^*)$ that weakly follows h^*. Since (b_i^{self}, b_i^{opp}) deems own mistakes least likely, it follows from Lemma 7.1 that $s_i^*(h)$ is also locally rational for the truncated belief pair $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$ at every $h \in H_i(s_i^*)$ that weakly follows h^*.

Define $\hat{b}_i^{self} := (st(b_i^{self}(h, c_i)))_{h \in H_i, c_i \in C_i(h)}$. We will show that $(\hat{b}_i^{self}, tr(b_i^{opp}))$ believes in his own future rationality. As the original belief pair (b_i^{self}, b_i^{opp}) believes in his own future rationality, we know that for every $h \in H_i$ and $c_i \in C_i(h)$, the standard part of $b_i^{self}(h, c_i)$ only assigns positive probability to strategies $s_i \in S_i(h, c_i)$ where $s_i(h')$ is locally rational for (b_i^{self}, b_i^{opp}) at every $h' \in H_i(s_i)$ following h. By Lemma 7.1 we know that every such $s_i(h')$ is also locally rational for $(\hat{b}_i^{self}, tr(b_i^{opp}))$ at h'. Hence, we conclude that for every $h \in H_i$ and $c_i \in C_i(h)$, the belief $\hat{b}_i^{self}(h, c_i) = st((b_i^{self}(h, c_i))$ only assigns positive probability to strategies $s_i \in S_i(h, c_i)$ where $s_i(h')$ is locally rational for $(\hat{b}_i^{self}, tr(b_i^{opp}))$ at every $h' \in H_i(s_i)$ following h. Therefore, $(\hat{b}_i^{self}, tr(b_i^{opp}))$ believes in his own future rationality.

As such, we conclude that $s_i^*(h)$ is locally rational for the truncated belief pair $(\hat{b}_i^{self}, tr(b_i^{opp}))$ at every $h \in H_i(s_i^*)$ that weakly follows h^*, and that $(\hat{b}_i^{self}, tr(b_i^{opp}))$ believes in his own future rationality. By Lemma 7.2 it follows that s_i^* is globally rational for $tr(b_i^{opp})$ at h^*, which was to show. ■

We are now fully equipped to prove Theorem 4.1.

Proof of Theorem 4.1. For every $k \geq 0$, every player i and every $h \in H_i$, let $S_{i,qp}^k(h)$ and $B_{i,qp}^{opp,k}$ be the sets of strategies and beliefs that survive round k of the quasi-perfect rationalizability procedure. Similarly, let $S_{i,pqp}^k(h)$ and $B_{i,pqp}^k$ be the sets of strategies and belief pairs that survive round k of the perfect quasi-perfect rationalizability procedure. As before, for every $b_i^{opp} \in B_i^{opp}$ we denote by $tr(b_i^{opp})$ the minimal cautious truncation of b_i^{opp} on S_{-i}. We prove the following claim.

Claim. For every $k \geq 0$, every player i and every $h^* \in H_i$, (a) $S_{i,pqp}^k(h^*) \subseteq S_{i,qp}^k(h^*)$, and (b) for every $(b_i^{self}, b_i^{opp}) \in B_{i,pqp}^k$ it holds that $tr(b_i^{opp}) \in B_{i,qp}^{opp,k}$.

Proof of claim. We prove so by induction on k. For $k = 0$ the statement is trivial since $S_{i,pqp}^0(h^*) = S_{i,qp}^0(h^*) = S_i(h^*)$ and $B_{i,qp}^{opp,0}$ is the set of all cautious non-standard probability distributions on S_{-i}.

Let $k \geq 1$, and suppose that (a) and (b) are true for $k-1$. To show (a) for k, take some strategy $s_i^* \in S_{i,pqp}^k(h^*)$. Then, by definition, $s_i^* \in S_{i,pqp}^{k-1}(h^*)$, and there is some $(b_i^{self}, b_i^{opp}) \in B_{i,pqp}^{k-1}$ such that $s_i^*(h)$ is locally rational for (b_i^{self}, b_i^{opp}) at every $h \in H_i(s_i^*)$ weakly following h^*. Since, by the induction assumption on (a), $S_{i,pqp}^{k-1}(h^*) \subseteq S_{i,qp}^{k-1}(h^*)$, we know that $s_i^* \in S_{i,qp}^{k-1}(h^*)$. Moreover, as $(b_i^{self}, b_i^{opp}) \in B_{i,pqp}^{k-1} \subseteq B_{i,pqp}^0$ we know that (b_i^{self}, b_i^{opp}) deems own mistakes least likely and believes in his own future rationality. Since s_i^* is locally rational for (b_i^{self}, b_i^{opp}) at every $h \in H_i(s_i^*)$ weakly following h^*, we thus conclude by Corollary 7.1 that s_i^* is globally rational for $tr(b_i^{opp})$ at every $h \in H_i(s_i^*)$ weakly following h^*. Moreover, as $(b_i^{self}, b_i^{opp}) \in B_{i,pqp}^{k-1}$, we know by the induction assumption on (b) that $tr(b_i^{opp}) \in B_{i,qp}^{opp,k-1}$.

Summarizing, we see that $s_i^* \in S_{i,qp}^{k-1}(h^*)$, and that s_i^* is globally rational for $tr(b_i^{opp}) \in B_{i,qp}^{opp,k-1}$ at every $h \in H_i(s_i^*)$ weakly following h^*. Hence, by definition, $s_i^* \in S_{i,qp}^k(h^*)$. We thus conclude that $S_{i,pqp}^k(h^*) \subseteq S_{i,qp}^k(h^*)$.

To show (b), take some $(b_i^{self}, b_i^{opp}) \in B_{i,pqp}^k$. Then, by definition, $(b_i^{self}, b_i^{opp}) \in B_{i,pqp}^{k-1}$ and $b_i^{opp}(h)$ believes $S_{-i,pqp}^{k-1}(h)$ for every $h \in H_i$. By the induction assumption on (b) we already know that $tr(b_i^{opp}) \in B_{i,qp}^{opp,k-1}$. Since $b_i^{opp}(h)$ believes $S_{-i,pqp}^{k-1}(h)$, the standard part of $b_i^{opp}(h)$ only assigns positive probability to opponents' strategy combinations $s_{-i} \in S_{-i,pqp}^{k-1}(h)$. Note that the standard part of $b_i^{opp}(h)$ is the same as the standard part of $tr(b_i^{opp})(h)$. Hence, the standard part of $tr(b_i^{opp})(h)$ only assigns positive probability to $s_{-i} \in S_{-i,pqp}^{k-1}(h)$. By the induction assumption on (a) we know that $S^{k-1}_{-i,pqp}(h) \subseteq S^{k-1}_{-i,qp}(h)$, and therefore the standard part of $tr(b_i^{opp})(h)$ only assigns positive probability to $s_{-i} \in S^{k-1}_{-i,qp}(h)$. In other words, $tr(b_i^{opp})(h)$ believes $S^{k-1}_{-i,qp}(h)$.

Summarizing, we see that $tr(b_i^{opp}) \in B^{opp,k-1}_{i,qp}$ and that $tr(b_i^{opp})(h)$ believes $S^{k-1}_{-i,qp}(h)$ for every $h \in H_i$. Hence, by definition, $tr(b_i^{opp}) \in B^{opp,k}_{i,qp}$, as was to show.

By induction on k, (a) and (b) are true for every $k \geq 0$, which completes the proof of the claim.

To prove the theorem, consider some perfectly quasi-perfectly rationalizable strategy s_i^* for player i. Then, by definition, $s_i^* \in S^k_{i,pqp}(\emptyset)$ for all $k \geq 0$. Hence, by part (a) of the claim, $s_i^* \in S^k_{i,qp}(\emptyset)$ for all $k \geq 0$, which means that s_i^* is quasi-perfectly rationalizable. This completes the proof. ■

8 Appendix D: Relation with Asheim and Perea (2005)

In this section we will compare our definition of quasi-perfect rationalizability to the one given by Asheim and Perea (2005). To that purpose, we first review the definition of quasi-perfect rationalizability as given in Asheim and Perea (2005), which we will call AP-quasi-perfect rationalizability from now on. We then show that in all games, every AP-quasi-perfectly rationalizable strategy is also quasi-perfectly rationalizable in our sense. Subsequently, we show by means of a counterexample that there are quasi-perfectly rationalizable strategies which are not AP-quasi-perfectly rationalizable. Hence, AP-quasi-perfect rationalizability is a strict refinement of our notion of quasi-perfect rationalizability. The same example also demonstrates that a even a perfectly quasi-perfectly rationalizable strategy need not be AP-quasi-perfectly rationalizable.

8.1 Quasi-Perfect Rationalizability in Asheim and Perea (2005)

We have defined quasi-perfect rationalizability by means of a procedure, that recursively eliminates strategies and beliefs from the game. Asheim and Perea (2005) (AP from now on) take a different approach, since they define quasi-perfect rationalizability by looking at *belief hierarchies* encoded by types within an epistemic model. Also, they use *lexicographic beliefs* (Blume, Brandenburger and Dekel (1991)) rather than non-standard beliefs to model cautious reasoning. That is, they take as a primitive not only beliefs about the opponents' strategies, as we do, but also beliefs about the opponents' beliefs about the other players' strategies (second-order beliefs), and higher-order beliefs. AP then define quasi-perfectly rationalizability by imposing epistemic conditions on such belief hierarchies. Since we have used non-standard beliefs, rather than lexicographic beliefs, to define (perfect) quasi-perfect rationalizability, we will reproduce AP's definition by using non-standard beliefs instead of lexicographic beliefs.

Definition 8.1 (Epistemic model with non-standard beliefs) *For a given dynamic game G, a finite epistemic model with non-standard beliefs is a tuple $M = (T_i, \beta_i)_{i \in I}$ such that, for* *every player i,*

(a) T_i is a finite set of types, and

(b) β_i is a function that assigns to every type $t_i \in T_i$ a non-standard belief $\beta_i(t_i)$ on $S_{-i} \times T_{-i}$.

An epistemic model is used to *encode* non-standard belief hierarchies for the players, including beliefs about the opponents' strategies, beliefs about the opponents' beliefs about their opponents' strategies, and so on. The concept of AP-quasi-perfect rationalizability restricts to types that express common full belief in "caution" and the "event that types induce sequentially rational behavioral strategies". We will now formally define these events.

Definition 8.2 (Caution) *Consider a finite epistemic model $M = (T_i, \beta_i)_{i \in I}$ for a dynamic game G. A type $t_i \in T_i$ is cautious if for every opponents' type combination $t_{-i} \in T_{-i}$ with $\beta_i(t_i)(t_{-i}) > 0$, it holds that $\beta_i(t_i)(s_{-i}, t_{-i}) > 0$ for every $s_{-i} \in S_{-i}$.*

Here, $\beta_i(t_i)(t_{-i})$ is an abbreviation for the marginal probability $\beta_i(t_i)(S_{-i} \times \{t_{-i}\})$. We will use such abbreviations for marginals more often in the remainder of this section. Hence, caution states that if t_i seems possible a type combination t_{-i} for his opponents, then he must deem possible every strategy combination for that type combination. In particular, t_i holds a cautious non-standard belief on the set S_{-i} of opponents' strategy combinations. Consider a cautious type t_i and an information set $h \in H_i$. By $\beta_i(t_i, h)$ we denote the induced (cautious) conditional belief on $S_{-i}(h) \times T_{-i}$. For every strategy $s_i \in S_i(h)$ we denote by

$$u_i(s_i, t_i, h) := \sum_{(s_{-i}, t_{-i}) \in S_{-i}(h) \times T_{-i}} \beta_i(t_i, h)(s_{-i}, t_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

the expected (non-standard) utility at h of choosing strategy s_i under the conditional belief $\beta_i(t_i, h)$. We say that a strategy $s_i \in S_i(h)$ is *globally rational* for the cautious type t_i at $h \in H_i$ if

$$u_i(s_i, t_i, h) \geq u_i(s_i', t_i, h) \text{ for all } s_i' \in S_i(h).$$

To define what it means for a type to "induce a sequentially rational behavioral strategy" we need some additional terminology. A *behavioral strategy* for player i is a tuple $\sigma_i = (\sigma_i(h))_{h \in H_i}$ such that $\sigma_i(h)$ is a (standard) probability distribution on the set of choices $C_i(h)$ available for player i at h. For a behavioral strategy σ_i and a strategy $s_i \in S_i$, let

$$\sigma_i(s_i) := \prod\nolimits_{h \in H_i(s_i)} \sigma_i(s_i(h)) \tag{8.1}$$

be the induced probability that σ_i assigns to the strategy s_i. For a given behavioral strategy σ_i and information set $h \in H_i$, let $\sigma_i|_h$ be the behavioral strategy that (i) at every $h' \in H_i$ preceding h assigns probability 1 to the unique choice for player i at h' leading to h, and (ii) coincides with

σ_i at all other information sets. We say that a behavioral strategy σ_i is *sequentially rational* for a cautious type t_i if at every information set $h \in H_i$, we have that $\sigma_i|_h(s_i) > 0$ only if s_i is globally rational for t_i at h.

Recall that, for an information set $h \in H_i$ and a choice $c_i \in C_i(h)$, we denote by $S_i(h, c_i)$ the set of strategies $s_i \in S_i(h)$ with $s_i(h) = c_i$. For a cautious type t_i and an opponent's type t_j with $\beta_i(t_i)(t_j) > 0$, let $\sigma_j^{t_i|t_j}$ be the induced behavioral strategy for player j given by

$$\sigma_j^{t_i|t_j}(h)(c_j) := st\left(\frac{\beta_i(t_i)(S_j(h, c_j) \times \{t_j\}}{\beta_i(t_i)(S_j(h) \times \{t_j\}}\right) \tag{8.2}$$

for every information set $h \in H_j$ and every choice $c_j \in C_j(h)$.

Definition 8.3 (Inducing sequentially rational behavioral strategies) *Consider a finite epistemic model $M = (T_i, \beta_i)_{i \in I}$ for a dynamic game G. A cautious type $t_i \in T_i$ induces sequentially rational behavioral strategies if for every opponent $j \neq i$ and every type $t_j \in T_j$ with $\beta_i(t_i)(t_j) > 0$, the induced behavioral strategy $\sigma_j^{t_i|t_j}$ is sequentially rational for t_j.*

We are now ready to define AP-quasi-perfectly rationalizable types as those types that are cautious, induce sequentially rational behavioral strategies, and express common full belief in these two events. Formally, a type t_i expresses *1-fold* full belief in caution and the event that types induce sequentially rational behavioral strategies if $\beta_i(t_i)$ only assigns positive (non-standard) probability to opponents' types that are cautious and induce sequentially rational behavioral strategies. For every $k \geq 2$, type t_i expresses *k-fold* full belief in caution and the event that types induce sequentially rational behavioral strategies if $\beta_i(t_i)$ only assigns positive (non-standard) probability to opponents' types that express $(k-1)$-fold full belief in caution and the event that types induce sequentially rational behavioral strategies. A type t_i expresses *common* full belief in caution and the event that types induce sequentially rational behavioral strategies if t_i expresses k-fold full belief in caution and the event that types induce sequentially rational behavioral strategies, for every $k \geq 1$.

Definition 8.4 (AP-quasi-perfect rationalizability) *Consider a finite epistemic model $M = (T_i, \beta_i)_{i \in I}$ for a dynamic game G. A type $t_i \in T_i$ is AP-quasi-perfectly rationalizable if it is cautious, induces sequentially rational behavioral strategies, and expresses common full belief in caution and the event that types induce sequentially rational behavioral strategies. A strategy $s_i \in S_i$ is AP-quasi-perfectly rationalizable if there is a finite epistemic model $M = (T_i, \beta_i)_{i \in I}$ and an AP-quasi-perfectly rationalizable type $t_i \in T_i$, such that s_i is globally rational for t_i at every $h \in H_i(s_i)$.*

In the following subsection we will show that every AP-quasi-perfectly rationalizable strategy is quasi-perfectly rationalizable in our sense, but not *vice versa.*

8.2 Relation Between the Two Quasi-Perfect Rationalizability Concepts

We first show that, in all dynamic games, every strategy that is AP-quasi-perfectly rationalizable is also quasi-perfectly rationalizable in our sense.

Theorem 8.1 (Relation with AP-quasi-perfect rationalizability) *Consider a dynamic game G. Then, every strategy that is AP-quasi-perfectly rationalizable is also quasi-perfectly rationalizable.*

Proof. Let $S_i^k(h)$ and $B_i^{opp,k}$ be the sets of strategies and beliefs that survive round k of our quasi-perfect rationalizability procedure. Consider a finite epistemic model $M = (T_i, \beta_i)_{i \in I}$ for G, as in AP. We prove, by induction on k, that for every player i, every AP-quasi-perfectly rationalizable type $t_i \in T_i$, and every information set $h \in H_i$, we have that (a) every strategy $s_i \in S_i(h)$ that is globally rational for t_i at every $h' \in H_i(s_i)$ weakly following h is in $S_i^k(h)$, and (b) the marginal of $\beta_i(t_i)$ on S_{-i} is in $B_i^{opp,k}$.

For $k = 0$ this statement is true because $S_i^0(h) = S_i(h)$, the type t_i is cautious, and $B_i^{opp,0} = B_i^{opp}$ contains all cautious beliefs on S_{-i}.

Now let $k \geq 1$ and suppose that (a) and (b) are true for $k-1$ and all players i. Consider a player i, an AP-quasi-perfectly rationalizable type $t_i \in T_i$, and an information set $h \in H_i$. To show (a), take some strategy $s_i \in S_i(h)$ that is globally rational for t_i at every $h' \in H_i(s_i)$ weakly following h. By the induction assumption on (a) we know that $s_i \in S_i^{k-1}(h)$. Let $b_i^{opp}(t_i)$ be the marginal of $\beta_i(t_i)$ on S_{-i}. By the induction assumption on (b) we know that $b_i^{opp}(t_i) \in B_i^{opp,k-1}$. Hence, $s_i \in S_i^{k-1}(h)$ is globally rational for $b_i^{opp}(t_i) \in B_i^{opp,k-1}$ at every $h' \in H_i(s_i)$ weakly following h. This implies that $s_i \in S_i^k(h)$, which completes the induction step for (a).

To show (b), let $b_i^{opp}(t_i)$ be the marginal of $\beta_i(t_i)$ on S_{-i}. By the induction assumption on (b) we know that $b_i^{opp}(t_i) \in B_i^{opp,k-1}$. To show that $b_i^{opp}(t_i) \in B_i^{opp,k}$, we must show that $b_i^{opp}(t_i)(h)$ believes $S_{-i}^k(h)$ for all $h \in H_i$. That is, we must show that $st(b_i^{opp}(t_i)(h)(s_{-i})) > 0$ only if $s_{-i} \in S_{-i}^k(h)$.

Consider some information set $h \in H_i$ and some opponents' strategy combination s_{-i} such that $st(b_i^{opp}(t_i)(h)(s_{-i})) > 0$. We will show that $s_{-i} \in S_{-i}^k(h)$. As, by the induction assumption on (b), $b_i^{opp}(t_i) \in B_i^{opp,k-1}$, it follows that $b_i^{opp}(t_i)(h)$ believes $S_{-i}^{k-1}(h)$, and hence $s_{-i} \in S_{-i}^{k-1}(h)$. Let $s_{-i} = (s_j)_{j \neq i}$. To show that $s_{-i} \in S_{-i}^k(h)$, we will show that for every opponent $j \neq i$ there is some $b_j^{opp} \in B_j^{opp,k-1}$ such that s_j is globally rational for b_j^{opp} at every $h' \in H_j(s_j)$ weakly following h.

Fix an opponent j. Since $st(b_i^{opp}(t_i)(h)(s_j)) > 0$, the belief $b_i^{opp}(t_i)$ is the marginal of $\beta_i(t_i)$ on S_{-i}, and $b_i^{opp}(t_i)(h)$ is the induced conditional belief on $S_{-i}(h)$, there must be some type $t_j \in T_j$ with $\beta_i(t_i)(t_j) > 0$ such that

$$st\left(\frac{\beta_i(t_i)(s_j, t_j)}{\beta_i(t_i)(S_j(h) \times \{t_j\}}\right) > 0. \tag{8.3}$$

Now, let $b_j^{opp}(t_j)$ be the marginal of $\beta_j(t_j)$ on S_{-j}. We show that $b_j^{opp}(t_j) \in B_j^{opp,k-1}$ and that s_j is globally rational for $b_j^{opp}(t_j)$ at every $h' \in H_j(s_j)$ weakly following h.

As $\beta_i(t_i)(t_j) > 0$ and t_i is AP-quasi-perfectly rationalizable, it must be that t_j is AP-quasi-perfectly rationalizable as well. Hence, by our induction assumption on (b) we conclude that $b_j^{opp}(t_j) \in B_j^{opp,k-1}$.

Consider now some $h' \in H_j(s_j)$ weakly following h. We show that s_j is globally rational for $b_j^{opp}(t_j)$ at h'. Take some arbitrary $h'' \in H_j(s_j)$ weakly following h'. Then, h'' weakly follows h and hence $S_j(h'') \subseteq S_j(h)$. Moreover, $s_j \in S_j(h'', s_j(h''))$. It thus follows by (8.3) that

$$\begin{aligned} st\left(\frac{\beta_i(t_i)(S_j(h'', s_j(h'')) \times \{t_j\}}{\beta_i(t_i)(S_j(h'') \times \{t_j\}}\right) &\geq st\left(\frac{\beta_i(t_i)(S_j(h'', s_j(h'')) \times \{t_j\}}{\beta_i(t_i)(S_j(h) \times \{t_j\}}\right) \\ &\geq st\left(\frac{\beta_i(t_i)(s_j, t_j)}{\beta_i(t_i)(S_j(h) \times \{t_j\}}\right) > 0. \end{aligned}$$

We therefore conclude by (8.2) that

$$\sigma_j^{t_i|t_j}(h'')(s_j(h'')) = st\left(\frac{\beta_i(t_i)(S_j(h'', s_j(h'')) \times \{t_j\}}{\beta_i(t_i)(S_j(h'') \times \{t_j\}}\right) > 0$$

for all $h'' \in H_j(s_j)$ weakly following h'. But then, it follows by (8.1) that there is some $\hat{s}_j \in S_j(h')$ with $\hat{s}_j(h'') = s_j(h'')$ for all $h'' \in H_j(s_j)$ weakly following h' such that

$$\sigma_j^{t_i|t_j}|_{h'}(\hat{s}_j) > 0. \tag{8.4}$$

Since t_i is AP-quasi-perfectly rationalizable, we know in particular that t_i induces sequentially rational behavioral strategies. Hence, the induced behavioral strategy $\sigma_j^{t_i|t_j}$ must be sequentially rational for t_j. Since by (8.4) we have that $\sigma_j^{t_i|t_j}|_{h'}(\hat{s}_j) > 0$, it follows that $\hat{s}_j$ must be globally rational for t_j at h'. Since s_j and $\hat{s}_j$ coincide at all $h'' \in H_j(\hat{s}_j)$ that weakly follow h', it follows that also s_j is globally rational for t_j at h'. But then, we conclude that s_j is globally rational for $b_j^{opp}(t_j)$ at h'. As $h' \in H_j(s_j)$ weakly following h was chosen arbitrarily, it follows that s_j is globally rational for $b_j^{opp}(t_j)$ at every $h' \in H_j(s_j)$ weakly following h. Since we have seen that $b_j^{opp}(t_j) \in B_j^{opp,k-1}$ and $s_j \in S_j^{k-1}(h)$, we conclude that $s_j \in S_j^k(h)$.

We thus see that $st(b_i^{opp}(t_i)(h)(s_j)) > 0$ only if $s_j \in S_j^k$. Since this holds for every $h \in H_i$ and every opponent j, it follows that $b_i^{opp}(t_i)$ believes $S_{-i}^k(h)$ for all $h \in H_i$. As $b_i^{opp}(t_i) \in B_i^{opp,k-1}$, we conclude that $b_i^{opp}(t_i) \in B_i^{opp,k}$, which completes the induction step for (b).

By induction on k, it thus follows that for every player i, every AP-quasi-perfectly rationalizable type $t_i \in T_i$, and every information set $h \in H_i$, (a) every strategy $s_i \in S_i(h)$ that is globally rational for t_i at every $h' \in H_i(s_i)$ weakly following h is in $S_i^k(h)$ for all $k \geq 0$, and (b) the marginal of $\beta_i(t_i)$ on S_{-i} is in $B_i^{opp,k}$ for all $k \geq 0$.

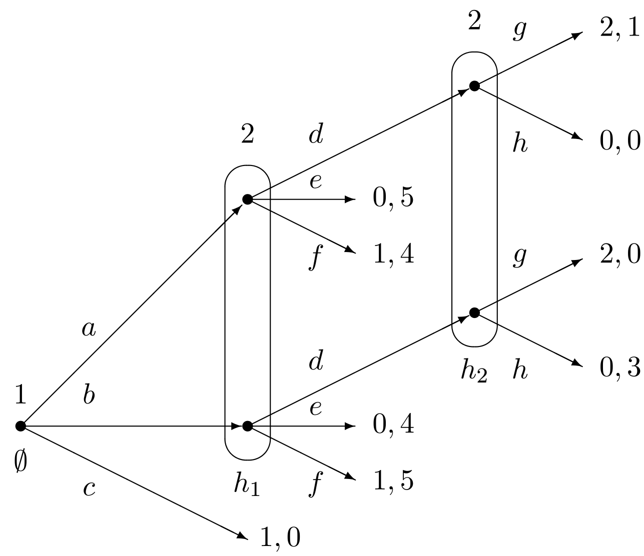

Figure 4: Quasi-perfect rationalizability does not imply AP-quasi-perfect rationalizability

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Now, take a player i, and an AP-quasi-perfectly rationalizable strategy $s_i \in S_i$. Then, there is an epistemic model $M = (T_i, \beta_i)_{i \in I}$ and an AP-quasi-perfectly rationalizable type $t_i \in T_i$ such that s_i is globally rational for t_i at every $h \in H_i(s_i)$. Then, by (a) above, $s_i \in S_i^k(\emptyset)$ for all $k \geq 0$, and hence s_i survives the quasi-perfect rationalizability procedure. This completes the proof. ■

We next prove, by means of a counter-example, that the opposite direction of this theorem is not true. Consider the dynamic game in Figure 4. Note that player 1 is always indifferent between his strategies a and b. We will show that the strategy a is quasi-perfectly rationalizable in our sense, but not AP-quasi-perfectly rationalizable.

Before we give a formal proof, we first provide an informal intuitive argument. According to our quasi-perfect rationalizability procedure, player 1 can rationally choose a because he may deem player 2's strategy f infinitely more likely than (d, g), strategy (d, g) infinitely more likely than (d, h), and (d, h) infinitely more likely than e. Indeed, under such belief player 1 would assign, at the beginning of the game $\emptyset$, only non-infinitesimal probability to player 2's strategy f, which is optimal for player 2 from $\emptyset$ onwards if player 2 assigns a high probability to player 1 choosing b.

Such a belief, however, is not possible under the concept of AP-quasi-perfect rationalizability. In order for player 1 to rationally choose a, he must deem player 2's strategy (d, g) at least as likely as (d, h). Hence, conditional on information set h_2 being reached, player 1 must assign a non-infinitesimal probability to player 2 choosing g. According to AP-quasi-perfect rationalizability, this is only possible if player 1 believes that g is optimal for player at h_2. Hence, player 1 must believe, conditional on h_2 being reached, that player 2 holds a belief b_2 that assigns probability at least 3/4 to player 1 having chosen a. Under such a belief b_2, however, e would be the only optimal strategy for player 2 at h_1. According to AP-quasi-perfect-rationalizability, player 1 must induce a sequentially rational behavioral strategy for player 2. In particular, conditional on player 2's belief b_2, and conditional on the information set h_1, player 1 must only assign non-infinitesimal probability to strategies that are optimal for player 2 under the belief b_2 at h_1. That is, conditional on player 2's belief b_2, and conditional on the information set h_1, player 1 must only assign non-infinitesimal probability to strategy e. In particular, this means that player 1 must deem player 2's strategy e infinitely more likely than (d, g). However, if that is the case player 1's expected utility from choosing a will always be lower than 1, and therefore a cannot be an AP-quasi-perfectly rationalizable strategy.

We will now turn the formal proof. We first show that a survives our quasi-perfect rationalizability procedure. Let $S_i^k(h)$ and $B_i^{opp,k}$ be the sets of strategies and beliefs that survive round k of the quasi-perfect rationalizability procedure, and define $S_i^\infty(h) := \cap_{k\geq 1} S_i^k(h)$ and $B_i^{opp,\infty} := \cap_{k\geq 1} B_i^{opp,k}$. Then, it is easily verified that $S_2^\infty(\emptyset) = \{e, f\}$. Consider player 1's belief

$$b_1^{opp} := (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot f + \varepsilon \cdot (d, g) + \varepsilon^2 \cdot (d, h) + \varepsilon^3 \cdot e$$

which clearly is in B_1^{opp}. As

$$st(b_1^{opp}(\emptyset)) = f \text{ where } f \in S_2^\infty(\emptyset)$$

it follows that $b_1^{opp}(\emptyset)$ believes $S_{-1}^\infty(\emptyset)$. Hence, $b_1^{opp} \in B_1^{opp,\infty}$.

We now verify that strategy a is globally rational for b_1^{opp} at $\emptyset$. By construction of the belief b_1^{opp},

$$\begin{aligned} u_1(a, b_1^{opp}(\emptyset)) = u_1(b, b_1^{opp}(\emptyset)) &= (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot 1 + \varepsilon \cdot 2 + \varepsilon^2 \cdot 0 + \varepsilon^3 \cdot 0 \\ &= 1 + \varepsilon - \varepsilon^2 - \varepsilon^3 > 1, \text{ and } u_1(c, b_1^{opp}(\emptyset)) = 1, \end{aligned}$$

and hence a is indeed globally rational for b_1^{opp} at $\emptyset$. As $b_1^{opp} \in B_1^{opp,\infty}$, it follows that $a \in S_1^\infty(\emptyset)$, and hence a is quasi-perfectly rationalizable.

We next prove that a is not AP-quasi-perfectly rationalizable. Suppose, contrary to what we want to show, that a were AP-quasi-perfectly rationalizable. Then, there is an epistemic model $M = (T_i, \beta_i)_{i\in I}$ for G, as in AP, and a type $t_1 \in T_1$, such that t_1 is AP-quasi-perfectly rationalizable, and a is globally rational for t_1 at $\emptyset$. Let $\beta_1(t_1)$ be the cautious non-standard belief that t_1 holds on $S_2 \times T_2$. For every $s_2 \in S_2$, let $\beta_1(t_1)(s_2)$ be the marginal probability that $\beta_1(t_1)$ assigns to s_2. Then, the expected utility of strategy a at $\emptyset$ for type t_2 is given by

$$u_1(a, t_1, \emptyset) = \beta_1(t_1)(f) \cdot 1 + \beta_1(t_1)(d, g) \cdot 2 + \beta_1(t_1)(d, h) \cdot 0 + \beta_1(t_1)(e) \cdot 0. \tag{8.5}$$

Since $u_1(c, t_1, \emptyset) = 1$ and a is globally rational for t_1 at $\emptyset$, we must have that $u_1(a, t_1, \emptyset) \geq 1$, which is only possible if $\beta_1(t_1)(d, g) \geq \beta_1(t_1)(d, h)$. Let $t_2 \in T_2$ be such that

$$\beta_1(t_1)((d, g), t_2) \geq \beta_1(t_1)((d, g), t_2') \text{ for all } t_2' \in T_2. \tag{8.6}$$

Since $\beta_1(t_1)(d, g) \geq \beta_1(t_1)(d, h)$ we must have, by (8.6), that $\beta_1(t_1)((d, g), t_2)$ is not of infinitely smaller size than $\beta_1(t_1)((d, h), t_2)$. This implies, by (8.2), that $\sigma_2^{t_1|t_2}|_{h_2}(d, g) > 0$. Since t_1 is AP-quasi-perfectly rationalizable, the behavioral strategy $\sigma_2^{t_1|t_2}$ must be sequentially rational for t_2. In particular, $\sigma_2^{t_1|t_2}|_{h_2}(d, g) > 0$ implies that (d, g) must be globally rational for t_2 at h_2. This, in turn, is only possible if $\beta_2(t_2)(a) \geq \frac{3}{4}$. Hence, the only strategy that is globally rational for t_2 at h_1 is e. Since the behavioral strategy $\sigma_2^{t_1|t_2}$ must be sequentially rational for t_2, we must have that $\sigma_2^{t_1|t_2}|_{h_1}(e) = 1$. Hence, in particular, $\beta_1(t_1)(e, t_2)$ must be of infinitely larger size than $\beta_1(t_1)((d, g), t_2)$. But then, it follows by (8.6) that $\beta_1(t_1)(e)$ is of infinitely larger size than $\beta_1(t_1)(d, g)$. However, this insight, together with (8.5), would imply that $u_1(a, t_1, \emptyset) < 1$, and hence a cannot be globally rational for t_1 at $\emptyset$. That is a contradiction. We thus conclude that a cannot be AP-quasi-perfectly rationalizable. Hence, we have found a strategy a that is quasi-perfectly rationalizable, but not AP-quasi-perfectly rationalizable.

In fact, we can show even more in this example. The strategy a is not only quasi-perfectly rationalizable, it is even *perfectly* quasi-perfectly rationalizable. To see this, let $S_i^k(h)$ and B_i^k be the sets of strategies and beliefs that survive round k of the perfect quasi-perfect rationalizability procedure, and define $S_i^\infty(h) := \cap_{k\geq 1} S_i^k(h)$ and $B_i^\infty := \cap_{k\geq 1} B_i^k$. Then, it may be verified that $S_2^\infty(\emptyset) = \{e, f\}$. Consider player 1's belief $b_1 = (b_1^{self}, b_1^{opp})$ where

$$b_1^{self} := \tfrac{1}{2}(1 - \varepsilon) \cdot a + \tfrac{1}{2}(1 - \varepsilon) \cdot b + \varepsilon \cdot c$$

and

$$b_1^{opp} := (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot f + \varepsilon \cdot (d, g) + \varepsilon^2 \cdot (d, h) + \varepsilon^3 \cdot e.$$

Then,

$$\begin{aligned} u_1(a, b_1(\emptyset)) = u_1(b, b_1(\emptyset)) &= (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot 1 + \varepsilon \cdot 2 + \varepsilon^2 \cdot 0 + \varepsilon^3 \cdot 0 \\ &= 1 + \varepsilon - \varepsilon^2 - \varepsilon^3 > 1, \text{ and } u_1(c, b_1^{opp}(\emptyset)) = 1, \end{aligned}$$

which implies that choices a and b are locally rational for b_1 at $\emptyset$. Therefore, $S_1^{rat}(b_1, \emptyset) = \{a, b\}$. As $b_1^{self}(\emptyset)$ believes $\{a, b\}$, we conclude that b_1 believes in his own future rationality. Note that player 1 only makes a choice at $\emptyset$, and therefore he trivially deems his own mistakes least likely under the belief b_1. We thus conclude that $b_1 \in B_1^0$.

Since

$$st(b_1^{opp}(\emptyset)) = f \text{ where } f \in S_2^\infty(\emptyset)$$

it follows that $b_2^{opp}(\emptyset)$ believes $S_{-1}^\infty(\emptyset)$. It therefore follows that $b_1 = (b_1^{self}, b_1^{opp}) \in B_1^\infty$.

Since we have seen above that a is locally rational for b_1 at $\emptyset$, it follows that $a \in S_2^\infty(\emptyset)$, and hence a is perfectly quasi-perfectly rationalizable. We have thus found a strategy a that is perfectly quasi-perfectly rationalizable but not AP-quasi-perfectly rationalizable. This means that Theorem 4.1 is no longer true if quasi-perfect rationalizability is replaced by AP-quasi-perfect rationalizability.

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