Judgment aggregation Winter School

Day two – afternoon session

"From binary to probabilistic and other judgments"

Maastricht, March 2014

Franz Dietrich
Paris School of Economics, CNRS & UEA
www.franzdietrich.net

based on joint with Christian List

Probabilistic judgments

- As before: a given agenda X containing the relevant propositions relative to which judgments are formed.
- So far judgments were binary: each proposition in X was either accepted ('yes') or rejected ('no')
- Now judgments are probabilistic: each proposition in X gets a probability in $[0, 1]$, a *degree* of belief.

How aggregate?

- How aggregate people's probabilistic opinions into collective probabilistic opinions?
- Two popular proposals:
  - take an arithmetic average of people's probabilities ('linear pooling')
  - take a geometric average of people's probabilities.
- Note: such averaging wasn't possible for binary judgments!

Plan for this afternoon

1. **Motivation and overview**
2. **The framework of probabilistic judgment aggregation**
3. **Axioms**
4. **Neutral pooling**
5. **Linear poolingClassical probabilistic opinion pooling results as a special case**
6. **Application: probabilistic preference aggregation**
7. **Unifying binary and probabilistic judgment aggregation**
8. **Towards a unified theory of attitude aggregation**

Part 1

Motivation and overview

Propositions as sets of possible worlds

- We represent propositions as sets of possible worlds ('events').
  –> since you are used to probabilities being defined on such events (rather than, say, on sentences, i.e., syntactic propositions).

Classical agendas: *algebras*

- Classic assumption: the agenda forms a σ-algebra, hence, is closed under taking disjunctions (unions) or conjunctions (intersections) of events.
- The reason is technical: probability measures are defined on algebras.

  –> But technical convenience is ultimately not a justification.

Agendas without interconnections between events

- In practice the group might care about the probability of ‘rain’ and that of ‘heat’ while ignoring that of ‘rain *or* heat’.
- Real-life agendas can be very far away from algebras: they could even contain only logically independent events such as ‘rain’ and ‘heat’ (and their negations such as ‘no rain’ and ‘no heat’), without containing disjunctions or conjunctions of these events.
- Indeed, in practice the events considered are often at most probabilistically dependent (i.e., correlated), not logically dependent.

Why are real-life agendas often not algebras?

Two possible reasons:

- *Either* the probability of ‘artificial’ composite events (like ‘rain-or-not-[heat-and-hunger]’) is simply uninteresting;
- *Or* the individuals are unable to come up with subjective probabilities of such events, so that these events must be excluded from consideration.

Algebra agendas in binary JA

- In binary JA theory, no one would seriously propose to assume an algebra agenda:
  - they are too special/unrealistic,
  - they trivialise JA, as they are so highly interconnected that they are immediately subject to impossibility results.

The key finding

- Probabilistic judgments lead to aggregation possibilities where binary judgments led to aggregation impossibilities.
- The independence axiom no longer impossibility: it leads to *linear* aggregation.

Linear pooling

- 'Linear': the collective probability of any agenda event A is a *linear* function of people's probabilities:

$$P(A) = w_1 P_1(A) + \cdots + w_n P_n(A)$$

for fixed weights $w_1, ..., w_n \geq 0$ of sum one (they might reflect competence levels).

Neutral pooling

- Linear pooling is a special case of 'neutral' or 'systematic' pooling: the collective probability of any agenda event A is *some* (possibly non-linear) function of people's probabilities:

$$P(A) = D(P_1(A), ..., P_n(A))$$

for some fixed function $D : [0, 1]^n \longrightarrow [0, 1]$.

Linear pooling for classic agendas

- Seminal result by Aczél-Wagner (1980) and McConway (1981): For any σ-algebra agenda, pooling is linear $\Leftrightarrow$ pooling is independent and consensus preserving.

$->$ Will be generalized to general agendas.

- Some other linearity characterization: Wagner (1982/1985), Aczél, Ng and Wagner (1984), Genest (1984), Mongin (1995), Chambers (2007), all for σ-algebra agendas.

Neutral pooling for classical agendas

- For a σ-algebra agenda, pooling is neutral $\Leftrightarrow$ pooling is linear !!
- So the additional flexibility which neutral pooling seems to offers over linear pooling is only apparent, as the function D *must* be linear in order for collective probabilities to be well-defined (in particular, additive).

–> But for general agendas neutrality and linearity will come apart: there may be neutral, non-linear pooling operators.

Terminological comparison

binary JA jargon	**probabilistic JA jargon**
"judgments"	"opinions"
"aggregating"	"pooling"
"systematic"	"neutral"

Part 2

Framework

The individuals

A group of $n \geq 2$ individuals, labelled $i = 1, ..., n$, who have to assign collective probabilities to some relevant events.

The agenda

- We use a **semantic** agenda, i.e., the objects of opinions are sets of worlds (rather than sentences, say)
  - Not a restriction!
  - But reflects a common convention in probability theory.
- Ω : a non-empty set of possible *worlds* (or *states*).
- Each subset A of Ω is an *event* (its complement is $A^c = \Omega \backslash A$).
- Those events which are relevant – i.e., on which collective beliefs are to be formed – make up the agenda. Formally, an *agenda* is a non-empty set X of events which is closed under taking complements, i.e., $A \in X \Rightarrow A^c \in X$.
- Examples: $X = \{$rain, no rain$\}$, or $X = \{$rain, no rain, snow, no snow$\}$.

Classical agendas

- In classical opinion pooling, the agenda X is a σ-algebra, i.e., is closed not just under complement, but also under taking countable unions of events (and hence also under taking countable intersections of events).

Example

- Each world in Ω is a triple (j, k, l) where
  - j is 1 if CO_2 emissions are above some critical threshold, and 0 otherwise
  - k is 1 if CO_2 emissions above that threshold would cause Arctic summers are ice-free, and 0 otherwise
  - l is 1 if Arctic summers are ice-free, and 0 otherwise.
- So, $\Omega = \{0, 1\}^3 \backslash \{(1, 1, 0)\}$. The triple $(1, 1, 0)$ is excluded because it is inconsistent.

Example (cont.)

- Imagine an expert committee needs opinions on the agenda X consisting of the events
  - $A := \{(j,k,l) \in \Omega : j = 1\}$ (high emissions)
  - $A \rightarrow B := \{(j,k,l) \in \Omega : k = 1\}$ (high emissions cause global warming)
  - $B := \{(j,k,l) : l = 1\}$ (global warming)
  - and of course the negations A^c, $(A \rightarrow B)^c$, B^c.
- N.B.: X contains non-trivial interconnections: A, $A \rightarrow B$ and B^c can't be true simultaneously.

Probability functions

- If the agenda were a σ-algebra, we could represent an agent's (probabilistic) opinions by a probability function.
- A *probability function* on a σ-algebra $\mathbf{\Sigma}$ of events is a function $P : \mathbf{\Sigma} \longrightarrow [0, 1]$ such that $P(\Omega) = 1$ and P is σ-additive (i.e., $P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$ for every sequence of pairwise disjoint events $A_1, A_2, ... \in \mathbf{\Sigma}$).

Opinions

- For our general agenda X, we capture opinions by a so-called 'opinion function'.
- An *opinion function* is a probability assignment $P : X \to [0,1]$ which is extendible to a probability function on the σ-algebra generated by X. This σ-algebra is denoted $\sigma(X)$ and is the smallest σ-algebra which includes X.
- $\sigma(X)$ can be constructed by closing X under (countable) unions and complements, i.e., adding to X any combinations of agenda events formed using (countable) unions and complements. So, whenever X contains A, B, then $\sigma(X)$ also contains $A \cup B$, $(A \cup B)^c$, $(A \cup B)^c \cup B$, ...
- Often, $\sigma(X)$ is simply the set 2^Ω of *all* events.

Example

- In our CO_2 emissions example, an opinion function can *not* assign probability one to each of the events A, $A \longrightarrow B$ and B^c: this would be incoherent since $A \cap (A \longrightarrow B) \cap B^c = \varnothing$.

Notation

- $\mathcal{P}_X$ is the set of all opinion functions for agenda X.
- Note: If X happens to be a σ-algebra, $\mathcal{P}_X$ is simply the set of all probability functions on X.

Pooling functions ("aggregation rules")

- Given the agenda X, a combination of opinion functions across the n individuals, $(P_1, ..., P_n)$, is called a *profile*.
- An *opinion pooling function* – for short *pooling function* – is a function $F : \mathcal{P}_X^n \to \mathcal{P}_X$, which assigns to each profile $(P_1, ..., P_n)$ of individual opinion functions a collective opinion function $P = F(P_1, ..., P_n)$, often denoted $P_{P_1,...,P_n}$ for short.
- For instance, $P_{P_1,...,P_n}$ could be the arithmetic average $\frac{1}{n}P_1 + ... + \frac{1}{n}P_n$.

Two implicit requirements

Our definition of pooling function implicitly assumes

- a **universal domain** (since *every* profile of opinion functions is permissible)
- **collective rationality** (since $P_{P_1,\ldots,P_n}$ is an opinion function rather than any function from X to $[0,1]$).

Linear pooling

- We call the pooling function *linear* if there exist 'weights' $w_1, ..., w_n \geq 0$ with sum 1 such that, for every profile $(P_1, ..., P_n) \in \mathcal{P}_X^n$,

$$P_{P_1,...,P_n}(A) = \sum_{i=1}^{n} w_i P_i(A) \text{ for all } A \in X,$$

or in short, $P_{P_1,...,P_n} = \sum_{i=1}^{n} w_i P_i$.

- In the extreme case that $w_i = 1$ for some 'expert' i, we obtain an expert rule given by $P_{P_1,...,P_n} = P_i$.

Neutral pooling

- More generally, we call the pooling function *neutral* if there exists a (possibly non-linear) function $D : [0,1]^n \to [0,1]$ such that, for every profile $(P_1, ..., P_n) \in \mathcal{P}_X^n$,

$$P_{P_1,...,P_n}(A) = D(P_1(A), ..., P_n(A)) \text{ for all } A \in X. \quad (1)$$

- We call D the *local pooling criterion*.
- Linearity is the special case in which $D(x) = \sum_{i=1}^{n} w_i x_i$ for all $x \in [0,1]^n$.

Neutral pooling (cont.)

- While every combination of weights $w_1, ..., w_n \geq 0$ with sum 1 defines a proper linear pooling rule, it's far from clear whether a *given* non-linear function $D : [0,1]^n \longrightarrow [0,1]$ defines a proper pooling function, since (1) might not define an opinion function.
- It's not even obvious whether there exist any neutral but non-linear pooling functions. For algebra agendas, the answer is: NO!

Terminology

- An event A is *contingent* if it is neither $\varnothing$ (impossible) nor Ω (necessary).
- A set S of events is *consistent* if $\cap_{A \in S} A \neq \varnothing$.
- S *entails* another event B if $\cap_{A \in S} A \subseteq B$.

Two classes of applications

The group could be interested in

(a) EITHER the probabilities of certain *propositions (statements) of natural language* such as 'it will rain' or 'law X will be rejected by the constitutional court';

(b) OR the distribution of some real-valued (or vector-valued) random variable, such as rainfall or the number of insurance claims in 2013.

- Non-algebra agendas are more relevant to (a) than (b), since probabilities over an entire algebra are:
  – easier to come up with in (b)
  – more likely to be of interest in (b).

Part 3

Axioms

The axiom of independence

Independence: The collective belief on any event in the agenda is a function of the individuals' beliefs on *this* event only; i.e., for each event $A \in X$, there exists a function $D_A : [0,1]^n \longrightarrow [0,1]$ (the *local pooling criterion* for A) such that, for all $P_1, ..., P_n \in \mathcal{P}_X$,

$$P_{P_1,...,P_n}(A) = D_A(P_1(A), ..., P_n(A)).$$

Is the independence axiom plausible?

- **Normative defense:** local conception of democracy
- **Normative objection:** implausible were applied to ‘artificial’ events/propositions like ‘it rains or it is sunny’
  - So hard to defend if the agenda is a σ-algebra.
- **Pragmatic defence**: preventing agenda manipulation

Three types of beliefs

- Explicitly revealed beliefs
- Implicitly revealed beliefs
- Unrevealed beliefs

Explicitly revealed beliefs

- Individual i's *explicitly revealed* beliefs are the beliefs about events in the agenda X. Such beliefs are directly expressed by the submitted opinion function P_i.

Implicitly revealed beliefs

- Individual i's *implicitly revealed* beliefs are given by any probabilities of events in $\sigma(X)\backslash X$ which follow from the explicitly revealed beliefs, i.e., which hold under any probability function extending the submitted opinion function P_i to the σ-algebra $\sigma(X)$.
- E.g., if P_i assigns probability one to the 'rain' event A in the agenda X, then agent i explicitly reveals certainty of A, and implicitly reveals certainty of events $B \supseteq A$ not contained in the agenda.

Unrevealed beliefs

- Individual i's *unrevealed* beliefs are any probabilistic beliefs about events in $\sigma(X)\backslash X$ which are privately held without being deducible from the submitted opinion function P_i. Such beliefs are subjectively held without following from the submitted opinion function P_i.
- E.g., the individual could be certain of the 'snow' event B outside the agenda, yet the submitted opinion function P_i might be compatible with 'snow' having probability 0.8.

A unanimity axiom for explicitly revealed beliefs

Consensus preservation. For all $A \in X$ and all $P_1, ..., P_n \in \mathcal{P}_X$, if $P_i(A) = 1$ for all individuals i then $P_{P_1,...,P_n}(A) = 1$.

=> Intuitively appealing (but not totally obvious)

A unanimity axiom for implicitly revealed beliefs

Implicit consensus preservation. For all $A \in \sigma(X)$ and all $P_1, ..., P_n \in \mathcal{P}_X$, if each P_i implies certainty of A (i.e., $\overline{P}_i(A) = 1$ for every extension $\overline{P}_i$ of P_i to a probability function on $\sigma(X)$), then so does $P_{P_1,...,P_n}$.

Unanimity axiom for unrevealed beliefs

- Let's require collective opinions to be compatible with any unanimously held certainty of an event – including any unrevealed certainty, which does not follow from the submitted opinion functions but is merely possible based on these opinion functions:

Unrevealed consensus preservation. For all $A \in \sigma(X)$ and all $P_1, ..., P_n \in \mathcal{P}_X$, if each P_i is consistent with certainty of A (i.e., $\overline{P}_i(A) = 1$ for *some* extension $\overline{P}_i$ of P_i to a probability function on $\sigma(X)$), then so is $P_{P_1,...,P_n}$.

The rationale of the last axiom

- The axiom avoids overruling any unanimously held (revealed or unrevealed) certainty of an event.
- Why does it so? Suppose $P_1, ..., P_n$ are each compatible with certainty of event A; so it is *possible* that all individuals are certain of A.
- To avoid overruling a unanimous certainty – should it exist – the collective opinion function needs indeed to be compatible with certainty of A.

Towards another unanimity axiom

- Idea: respecting unanimous *conditional* beliefs.
- E.g., if everyone is certain that there will be a famine given that there will be a civil war, then so should the group.
- Problem: for events $A, B \in X$, the belief on A given B may be unrevealed, even though A and B belong to the agenda. Indeed, for an opinion function P_i (with $P_i(B) \neq 0$), $P_i(A|B) = P_i(A \cap B)/P(B)$ is undefined if $A \cap B \notin X$.

Towards another unanimity axiom (cont.)

- Let's therefore require that if each individual *could* be certain of A given B (given his opinion function), then also the collective opinion function should be compatible with certainty of A given B.

Towards another unanimity axiom (cont.)

- In fact, let's require something subtly stronger: if each individual could be *simultaneously* certain of A given B, and of A' given B', and of A'' given B'' etc. (for events $A, B, A', B', A'', B'', \ldots \in X$), then also the collective opinion function should be simultaneously compatible with all of these conditional certainties.

The axiom

Conditional consensus preservation. For all $P_1, ..., P_n \in \mathcal{P}_X$, and all finite sets S of pairs (A, B) of events in X, if every opinion function P_i is consistent with certainty of A given B for all (A, B) in S (i.e., some extension $\overline{P}_i$ of P_i to a probability function on $\sigma(X)$ satisfies $\overline{P}_i(A|B) = 1$ for all pairs $(A, B) \in S$ such that $P_i(B) \neq 0$), then so is the collective opinion function $P_{P_1,...,P_n}$.

Relationship between our unanimity axioms

Proposition 1

(a) *Each extended consensus preservation condition (i.e., implicit, unrevealed and conditional consensus preservation) implies consensus preservation, and is equivalent to it if the agenda* X *is classical, i.e., a* σ*-algebra.*

(b) *Unrevealed consensus preservation implies conditional consensus preservation.*

N.B.: our three new unanimity axioms all strengthen simple consensus preservation

Comparison with binary JA

- Recall that Arrow's Theorem in binary JA was obtained in two steps.
- Roughly (recall that neutrality = systematicity):

(1) "independence implies neutrality"

(2) "neutrality implies dictatorship"

- For probilistic opinion pooling, step (1) happens again, but instead of step (2) we get:

(2*) "neutrality implies linearity".

We will focus

first on step (1) (see part 4), and

then on step (2*) (part 5)

Part 4

Neutral pooling

Preview of the findings

- We show: for many agendas, the neutral pooling functions are the only independent pooling functions which respect consensus in an appropriate sense, namely in the sense of either consensus preservation, or conditional consensus preservation, or unrevealed consensus preservation. (The remaining unanimity axiom – implicit consensus preservation – won't be put to work.)
- The stronger the consensus principle invoked, the wider the class of agendas for which neutrality follows.

Nested agendas

- An agenda X is *nested* if it takes the very special form $X = \{A, A^c : A \in X_+\}$ for some set of events X_+ which is linearly ordered by set-inclusion.
- Examples of nested agendas:
  - $X = \{A, A^c\}$ (take $X_+ := \{A\}$).
  - $X = \{(-\infty, t], (t, \infty) : t \in \mathbb{R}\}$, where the set of worlds is $\Omega = \mathbb{R}$ (take $X^+ := \{(-\infty, t] : t \in \mathbb{R}\}$).
- Examples of non-nested agendas:
  - Our CO2-emissions agenda.

First characterization of neutral pooling

Theorem 1

(a) *For any non-nested agenda* X*, all independent and unrevealed consensus preserving pooling functions* $F : \mathcal{P}_X^n \to \mathcal{P}_X$ *are neutral.*

(b) *For any nested agenda* X *(*$\neq \{\varnothing, \Omega\}$*), some independent and unrevealed consensus preserving pooling function* $F : \mathcal{P}_X^n \to \mathcal{P}_X$ *is not neutral.*

Part (b) shows that the agenda condition used in part (a) is tight: as soon as the agenda becomes nested, non-neutral 'solutions' emerge.

Second characterization of neutral pooling

- The previous theorem stays true if we weaken unrevealed consensus preservation to conditional consensus preservation (but part (a) becomes logically stronger, and part (b) logically weaker):

Theorem 2

(a) *For any non-nested agenda X, all independent and conditional consensus preserving pooling functions $F : \mathcal{P}_X^n \to \mathcal{P}_X$ are neutral.*

(b) *For any nested agenda X ($\neq \{\varnothing, \Omega\}$), some independent and conditional consensus preserving pooling function $F : \mathcal{P}_X^n \to \mathcal{P}_X$ is not neutral.*

Pathconnected agendas

- The situation changes once we further relax the consensus axiom, namely to the familiar axiom of consensus preservation.
- The class of agendas for which neutrality follows shrinks considerably, namely to the class of *pathconnected* agendas, familiar from binary judgment aggregation theory (e.g., Nehring and Puppe 2010, Dietrich and List 2007, Dokow and Holzman 2010).
  –> definition on request!
- Our climate committee's agenda isn't pathconnected.
- The preference agenda is pathconnected.

Third characterization of neutral pooling

Theorem 3

(a) *For any pathconnected agenda X, all independent and consensus preserving pooling functions $F : \mathcal{P}_X^n \longrightarrow \mathcal{P}_X$ are neutral.*

(b) *For any non-pathconnected (finite) agenda X, some independent and consensus preserving pooling function $F : \mathcal{P}_X^n \longrightarrow \mathcal{P}_X$ is not neutral.*

Part 5

Linear pooling

Preview of the findings

- As just seen, many agendas force all independent pooling functions which respect consensus (in one of three senses) to be neutral.
- We now show: not just neutrality but even linearity follows if in each of Theorems 1-3 we suitably restrict the class of agendas considered in part (a).

First characterization of linear pooling

If in Theorem 1 we additionally require (in part (a)) that the agenda is ‘not very small’, then we obtain linearity rather than just neutrality:

Theorem 4

(a) *For any non-nested agenda X with $|X \backslash \{\Omega, \varnothing\}| > 4$, all independent and unrevealed consensus preserving pooling functions $F : \mathcal{P}_X^n \rightarrow \mathcal{P}_X$ are linear.*

(b) *For any other agenda X ($\neq \{\varnothing, \Omega\}$), some independent and unrevealed consensus preserving pooling function $F : \mathcal{P}_X^n \rightarrow \mathcal{P}_X$ is not linear.*

Non-simple agendas

- To turn Theorem 2 into a linearity characterization, we must strengthen the agenda condition of non-nestedness in part (a) to 'non-simplicity' (another familiar type of agenda in JA theory).
- An agenda X is *non-simple* if it has a minimal inconsistent subset Y where $|Y| \geq 3$.[1]
- Our climate committee agenda is non-simple, since the subset $\{A, A \to B, B^c\}$ is minimal inconsistent.
- In the preference agenda, $\{xPy, yPz, zPx\}$ is minimal inconsistent.

[1] and where Y isn't uncountably infinite

Second characterization of linear pooling

Theorem 5

(a) *For any non-simple agenda X, all independent and conditional consensus preserving pooling functions $F : \mathcal{P}_X^n \longrightarrow \mathcal{P}_X$ are linear.*

(b) *For any simple agenda X (finite and not $\{\varnothing, \Omega\}$), some independent and conditional consensus preserving pooling function $F : \mathcal{P}_X^n \longrightarrow \mathcal{P}_X$ is not linear.*

Third characterization of linear pooling

- How must we strengthen Theorem 3's agenda condition (pathconnectedness) so as to get linearity rather than just neutrality?
- We require the agenda to be *partitional*.
  - definition on request

Theorem 6

(a) *For any pathconnected and partitional agenda* X*, all independent and consensus preserving pooling functions* $F : \mathcal{P}_X^n \to \mathcal{P}_X$ *are linear.*

(b) *For any non-pathconnected (finite) agenda* X*, some independent and consensus preserving pooling function* $F : \mathcal{P}_X^n \to \mathcal{P}_X$ *is not linear.*

$=>$ This result is imperfect: in (a), partitionality isn't *necessary*; but it's *non-redundant*.

Part 6

Classical opinion pooling results as a special case

Algebra agendas

How do we generalize the classic linearity characterization for σ-algebra agendas (Aczél and Wagner 1980 and McConway 1981)?

Lemma 1 *For any σ-algebra agenda X ($\neq \{\Omega, \varnothing\}$), the agenda conditions of non-nestedness, non-simplicity, pathconnectedness and partitionality are all equivalent, and they hold if and only if $|X| > 4$ (i.e., X is of the form $\{A, A^c, \Omega, \varnothing\}$).*

Our theorems applied to algebra agendas

By the above lemma, for the special case of a σ-algebra agenda X our six theorems reduce to two classical results:

- Theorems 1-3 all reduce to the result that independent and consensus preserving pooling must be neutral if $|X| > 4$, but not if $|X| = 4$;
- Theorems 4-6 all reduce to the result that independent and consensus preserving pooling must be linear if $|X| > 4$, but not if $|X| = 4$.

(The case $|X| < 4$ is uninteresting as it means that $X = \{\varnothing, \Omega\}$ given that X is a σ-algebra.)

Part 7

Application: probabilistic preference aggregation

The preference agenda

- Let K be a set of alternatives (as in preference aggregation theory).
- Consider the *preference agenda* X_K, consisting of all events of the sort 'x is better than y' for distinct alternatives $x, y \in K$.

Relationship to preference aggregation

- Opinion pooling for this agenda relates to preference aggregation, except that:
  - classical preference aggregation deals with binary, non-graded attitudes
  - preferring x to y differs interpretationally from believing that x is better than y; but we may formally re-interpret probabilities of betterness as fuzzy or vague preferences.

Structure of the preference agenda

Lemma 2 *For the preference agenda X_K, the conditions of non-nestedness, non-simplicity and pathconnectedness are equivalent, and hold if and only if $|K| > 2$; the condition of partitionality is violated (whatever the size of K).*

One of our theorems applied to the preference agenda

Let's apply Theorem 5 to the preference agenda:

Corollary 1 *For the preference agenda* X_K,

(a) *if* $|K| > 2$, *all independent and conditional consensus preserving pooling functions are linear;*

(b) *if* $|K| = 2$, *some independent and conditional consensus preserving pooling function is not linear.*

Part 8

Unifying binary and probabilistic judgment aggregation

Binary judgments as special probabilistic judgments

- A complete and consistent judgment set $J \subseteq X$ is equivalent to the opinion function P given by
$$P(A) = \begin{cases} 1 & \text{if } A \in J \\ 0 & \text{if } A \notin J. \end{cases}$$
- See why P is an opinion function, i.e., extentible to a probability function on $\sigma(X)$?

Judgment aggregation rules as restricted opinion pooling functions

- Judgment aggregation rules (satisfying universal domain and collective rationality) can be seen as restricted opinion pooling functions whose domain and co-domain only allows for $\{0, 1\}$-valued opinion functions.
- Let's call the so-restricted opinion pooling functions *binary opinion pooling functions*.

Linearity and dictatorship

- Notice: A binary opinion pooling function can only be linear if it is dictatorial.
- Why?

  $=>$ So linearity collapes into dictatorship if beliefs are binarized.

A unified theorem

- Using that in the binary case "linear = dictatorial", Arrow's theorem for binary JA suddenly resembles Theorem 6.
- The two theorems can be unified:

Theorem 6* *For any pathconnected and partitional agenda X, all independent and consensus preserving* **probabilistic or binary** *pooling functions are linear.*

Note: partitionality is stronger than necessary in the binary case

Two more unified theorems

Just as Theorem 6, so Theorems 4 and 5 have counterparts for binary JA, and can be merged with their counterparts:

Theorem 4* *For any non-nested agenda* X *with* $|X\backslash\{\Omega,\varnothing\}|$ 4, *all independent and unrevealed consensus preserving* **binary or probabilistic** *pooling functions are linear.*

Theorem 5* *For any non-simple agenda* X, *all independent and conditional consensus preserving* **binary or probabilistic** *pooling functions are linear.*

Part 9

Towards a unified theory of attitude aggregation

A taxonomy of attitudes

	binary	**discrete**	**continuous**
belief attitudes (cognitive)	judgments	credence ratings	subjective probabilities
desire attitudes (emotive)	categorical desires	evaluative ratings	utilities

We focus on belief attitudes

Agenda, attitudes, attitude functions

- X : an agenda (of whatever kind)
- V : a set of values that an attitude on a proposition can take.
  - $V = \{0, 1\} = \{accept, reject\}$ for binary attitudes
  - $V = [0, 1]$ for probabilistic attitudes
  - ...
- An **attitude function** is any function $A : X \to V$

Rationality

- Not every attitude function is rational.
  - A binary attitude function $A : X \to \{0, 1\}$ must be logically coherent, i.e., the set of accepted propositions $\{p \in X : A(p) = 1\}$ must be complete and consistent.
  - A probabilistic attitude function $A : X \to [0, 1]$ must be probabilistically coherent, i.e., an opinion function.
- Let $\mathbf{R}$ be the set of rational attitude functions $A : X \to V$.

Further examples of attitudes

Rational attitude functions might be:

- (Dempster-Schafer theory) lower-probability functions into $V = [0, 1]$
- (Spohnian ranking theory) ranking functions into $V = \{0, 1, 2, ...\} \cup \{\infty\}$
- (T-valued logic) T-valued truth functions into $V = \{0, 1, 2, ..., T-1\}$
- functions into $V = \{0, 1,$'undecided out of conflicting info', 'undecided out of conflicting intuition'$\}$ which are wakly increasing w.r.t. to the partial order $\leq$ on V which ranks 0 bottom, 1 top, and each 'undecidedness' value between 0 and 1.

Aggregation rules

- An **aggregation rule/function** $F : \mathbf{R}^n \longrightarrow \mathbf{R}$ maps any profile $(A_1, ..., A_n)$ of rational individual attitude functions to a rational collective attitude function $F(A_1, ..., A_n)$.

General theorems?

- The result "independence $+$ consensus-preservation implies neutrality" generalizes to several kinds of attitude functions.
- We suspect the result "independence $+$ consensus-preservation implies linearity" generalizes to V-valued opinion functions for a large class of sets of values $V \subseteq [0,1]$ (not just $V = \{0,1\}$ and $V = [0,1]$).