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Author(s): Abraham Robinson

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FUNCTION THEORY ON SOME NONARCHIMEDEAN FIELDS

ABRAHAM ROBINSON, Yale University

1. Introduction. Archimedes' axiom states that for any two positive numbers a and b, a smaller than b, the continued addition of a to itself ultimately yields numbers which are greater than b. More formally, if F is an ordered abelian group or, more particularly, an ordered field, then Archimedes' axiom is as follows.

1.1. *If* $0 < a < b$, *where* a *and* b *are elements of* F *then there exists a natural number* n *such that*

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} > b.$$

Throughout the history of mathematics, Archimedes' axiom has been associated with the foundations of the Differential and Integral Calculus. Already in Greek science the method which, much later, was dubbed the method of exhaustion and which, to a large extent, anticipated the ε, δ method in the calculation of areas and volumes, depended on the validity of Archimedes' axiom, which was formulated explicitly for this purpose. On the other hand, when a method of infinitely small and infinitely large numbers is used, as in Nonstandard Analysis, then it is just the nonarchimedean nature of the system which is essential for its success or, more precisely, the superposition of a nonarchimedean field on the archimedean field of real numbers.

Although Nonstandard Analysis (see [**4**] or [**6**]) may perhaps be regarded as the most successful effort in this direction, many other systems have been introduced for the same purpose. Thus, not long ago, D. Laugwitz [**2**] considered a theory of functions on the field L of generalized power series with real coefficients and **real exponents**. The same field was investigated many years earlier by T. Levi-Civita [**3**], also because of its nonarchimedean character, and by A. Ostrowski [**5**], in connection with the theory of valuations.

Laugwitz raised the question whether the functions considered by him satisfy the intermediate value theorem and the mean value theorem of the Differential Calculus. We shall show in the present paper that although these theorems are not valid here in full generality, they are true under rather wide conditions. In order to obtain these results, we shall embed L in the residue class field ${}^{\rho}R$ of a certain subring of a nonstandard model of Analysis, ${}^{*}R$. It appears that ${}^{\rho}R$ has many interesting properties which make it a suitable subject for investigation quite apart from the particular problem just mentioned. In particular, the behavior of a function on ${}^{\rho}R$ is closely connected with the theory of asymptotic expansions, although we shall not pursue this topic in the present paper.

2. Ordered fields and fields with valuation. An **ordered field** F is a commutative field in which an ordering relation $x < y$ (or, equivalently, $y > x$) is defined and satisfies the following conditions.

2.1. *The ordering is transitive,* $x < y$ *and* $y < z$ *implies* $x < z$, *and irreflexive,* $x < y$ *implies* $x \neq y$.

2.2. *The ordering is total, if* $x \neq y$ *then either* $x < y$ *or* $y < x$ *(but not both, by* 2.1).

2.3. *The ordering is related to addition by the requirement that* $x < y$ *implies* $x + z < y + z$; *and to multiplication by the requirement that* $x < y$ *and* $0 < z$ *implies* $xz < yz$.

An ordered field can be characterized also by means of the set of its **positive elements** $P = \{x \mid x > 0\}$. Thus, suppose that a subset P of a field F possesses the following properties.

2.4. $0 \notin P$; for all $x \neq 0$, $x \in P$ or $-x \in P$.

2.5. *If* x, $y \in P$, *then* $x + y \in P$ *and* $xy \in P$.

Then the relation *defined* by

$$x < y \text{ if and only if } y - x \in P$$

satisfies the conditions 2.1–2.3 and P is just the set of positive elements of the field according to this relation.

We shall suppose that the reader is familiar with the elementary properties of ordered fields, e.g., that an ordered field is of characteristic 0 and that $x^2 > 0$ for all $x \neq 0$. As usual, we write $x \leqq y$ or $y \geqq x$ if either $x < y$ or $x = y$.

The rational numbers form an ordered field Q whose positive elements are the fractions (ratios) of natural numbers different from zero, and the real numbers form an ordered field R whose positive elements are just the squares other than zero. In both cases the ordering is unique. Moreover, both Q and R are archimedean, i.e., they satisfy Archimedes' Axiom 1.1.

Perhaps the simplest example of a non-archimedean field is as follows. Let $R(t)$ be a simple transcendental extension of the field of real numbers R. Thus $R(t)$ may be identified with the field of rational functions of the indeterminate t with coefficients in R, each element of $R(t)$ may be written in the form

$$f = \frac{p(t)}{q(t)} = \frac{a_0 + a_1 t + \cdots + a_n t^n}{b_0 + b_1 t + \cdots + b_m t^m}, \tag{2.6}$$

where $q(t) \neq 0$, at least one of the b_j is different from 0. We may then suppose the first $b_j \neq 0$ is actually equal to 1, for if this is not the case from the outset, we may achieve it by multiplying the numerator and denominator on the right hand side of (2.6) by b_j^{-1}. Thus, if $f \neq 0$, we may write

$$f = \frac{a_k t^k + \cdots + a_n t^n}{t^j + b_{j+1} t^{j+1} + \cdots + b_m t^m}, \qquad a_k \neq 0, \qquad 0 \leqq k \leqq n,\ 0 \leqq j \leqq m. \tag{2.7}$$

We now determine an ordering in $R(t)$ by defining that $f \neq 0$ is **positive** if and only if $a_k > 0$. To make sure that this is a good definition one first has to check that it is independent of the particular representation (2.7) chosen for the given f. Next one verifies that the set of positive elements of $R(t)$ defined in this way satisfies the conditions of 2.4 and 2.5. We suppose that these rather simple tasks have been carried out so that $R(t)$ becomes indeed an ordered field with the above definition. Moreover, this ordered field is nonarchimedean. For, by our definitions, $0 < t$, $t < 1$ (since $1 - t$ is positive) and, for any positive integer n,

$$\frac{t + t + \cdots + t}{n \text{ times}} < 1$$

(since $1 - nt$ is positive). This shows that 1.1 is not satisfied.

In any ordered field, the absolute value of a number a is defined to be $|a| = a$ if $a \geqq 0$, otherwise $|a| = -a$. Then $|ab| = |a|\,|b|$ and $|a + b| \leqq |a| + |b|$ (triangle inequality).

Let F be a nonarchimedean ordered field. Then F is of characteristic 0 and, hence, contains the field of rational numbers Q. An element $a \in F$ is said to be **infinite** if $|a| > q$ for all $q \in Q$. Also, $a \in F$ is said to be **infinitely small** or **infinitesimal** if $|a| < q$ for all positive $q \in Q$. $a \in F$ is **finite** if it is not infinite. This will be the case if and only if $|a| < q$ for some $q \in Q$.

The finite elements of F constitute a subring F_0 of F. The infinitesimal elements of F constitute a proper ideal F_1 within F_0. F_1 is maximal in F_0 as can be seen by the following argument. Suppose that $F_1 \subset J \subset F_0$ where J is an ideal in F_0, such that $J - F_1 \neq \varnothing$. Let $a \in J - F_1$ then a is not infinitesimal. We conclude without difficulty that a^{-1} is finite, so $a^{-1} \in F_0$, $aa^{-1} = 1 \in J$. But then $J = F_0$, F_1 is maximal in F_0.

It follows that $F' = F_0/F_1$ is a field. F' is called the **residue class field of the ordering**. The canonical mapping $F_0 \xrightarrow{\psi} F'$ induces an ordering in F' according to the rule that, for any $a \in F'$, $a \neq 0$, a is to be positive in F' if and only if one (and hence, all) of the elements of $\psi^{-1}a$ is (are) positive. It is not difficult to show that F' is archimedean according to this ordering and (hence) that it is isomorphic and order-isomorphic to a subfield of R.

The cosets of F_1 as an additive subgroup of F are called **monads.** If a is any element of F then we denote the monad containing it by $\mu(a)$. In particular, $\mu(0) = F_1$. The monads which are subsets of F_0 may be identified with the elements of F'.

As a tool in our investigation of nonarchimedean fields we shall require also the notion of a **field with valuation**, more particularly, the notion of a **field with non-**

archimedean valuation in the real numbers. This concept is given by a field F together with a mapping $v(x)$ from $F - \{0\}$ into the real numbers R such that the following conditions are satisfied:

2.8. *For all* $x \neq 0$, $y \neq 0$ *in* F, $v(xy) = v(x) + v(y)$.
2.9. *For all* x, y *in* F *such that* $x \neq 0$, $y \neq 0$, $x + y \neq 0$,

$$v(x + y) \geqq \min(v(x), v(y)).$$

If we add to R an **element** ∞ (usually called "a symbol") with the rules $x + \infty = \infty + x = \infty + \infty = \infty$ and the stipulation that $\infty > x$ for all real x, then the auxiliary definition $v(0) = \infty$ ensures that the equations of 2.8 and 2.9 are satisfied without any restriction on x and y.

The set $O_F = \{x \in F \mid v(x) \geqq 0\}$ is a subring of F, the **valuation ring**, and the set $J_F = \{x \in F \mid v(x) > 0\}$ constitutes a maximal ideal in O_F, the **valuation ideal**. The field $\bar{F} = O_F / J_F$ is called the residue class field of the given valuation.

Let c be an arbitrary but fixed constant greater than 1. Then the definition of distance

$$d(x, y) = c^{-v(x-y)},$$

where $c^{-\infty}$ is interpreted as 0, turns F into a metric space. If every Cauchy sequence in that space has a limit then F is said to be **complete** for the given valuation.

See [1], [7] or [8] for basic facts in valuation theory. From now on such facts will be taken for granted.

3. The field L. The field $R(t)$ is inadequate for the development of the calculus because we cannot extend to it even some of the most common functions defined in the field of real numbers, e.g., the function $y = \sqrt{x}$. Passing to the field of formal Laurent series $\sum_{k=-n}^{\infty} a_k t^k$, $a_k \in R$, does not remedy the situation. Following Laugwitz, we therefore consider the field of generalized power series L, which is defined as follows:

The elements of L are the formal expressions

$$\sum_{k=0}^{\infty} a_k t^{\nu_k} \qquad a_k, \nu_k \in R, \qquad \nu_k \uparrow \infty, \tag{3.1}$$

(where the last symbol implies $\nu_0 < \nu_1 < \nu_2 < \cdots$). Two expressions (3.1) are, by definition regarded as equal if for any term a_ν which occurs in one but not in the other, $a = 0$. We shall also write $a_0 t^{\nu_0} + a_1 t^{\nu_1} + \cdots + a_k t^{\nu_k}$ for an expression for which $a_{k+1} = a_{k+2} = \cdots = 0$.

The **sum** of two expressions $\sum a_k t^{\nu_k}$ and $\sum b_k t^{\mu_k}$ as in (3.1) is the expression $\sum c_k t^{\lambda_k}$ which is defined as follows. The sequence $\{\lambda_k\}$ is the set theoretical union of the sequences $\{\nu_k\}$ and $\{\mu_k\}$ arranged in increasing order. If a particular λ_m occurs both in $\{\nu_k\}$ and in $\{\mu_k\}$, e.g., $\lambda_m = \nu_p = \mu_q$ then $c_m = a_p + b_q$; if $\lambda_m = \nu_p$ but λ_m does not occur in $\{\mu_k\}$ then $c_m = a_p$; and if $\lambda_m = \mu_q$ but λ_m does not occur in $\{\nu_k\}$ then

$c_m = b_q$. Thus, briefly, the sum $\sum c_k t^{\lambda_k}$ is obtained by the formal addition of the terms of $\sum a_k t^{\nu_k}$ and $\sum b_k t^{\mu_k}$. Similarly, the **product** $\sum c_k t^{\lambda_k}$ of $\sum a_k t^{\nu_k}$ and $\sum b_k t^{\mu_k}$ as in 3.1 is obtained by formal multiplication. Thus, the sequence $\{\lambda_k\}$ consists of the sums $\nu_p + \mu_q$ arranged in increasing order and $c_k = \sum a_p b_q$ where p and q range over the natural numbers such that $\nu_p + \mu_q = \lambda_k$. It is not difficult to see that all these sums are finite and that the resulting expression satisfies the conditions of (3.1). Moreover, our definitions of sums and products are compatible with the relation of equality introduced earlier, and they turn L into a ring whose zero and unit elements may be written as $0t^0 + 0t^1 + 0t^2 + \cdots$, or 0, and as $1t^0 + 0t^1 + 0t^2 + \cdots$, or 1.

Now let $\alpha = 1 + \sum_{k=1}^{\infty} a_k t^{\nu_k}$, $0 < \nu_1 < \nu_2 < \cdots \to \infty$, i.e., α is an element of L as in 3.1 with $\nu_0 = 0$, $a_0 = 1$. We wish to show that α possesses a multiplicative inverse in L. For this purpose we define β as the formal expansion in powers of t of the expression

$$1 - \left(\sum_{k=1}^{\infty} a_k t^{\nu_k}\right) + \left(\sum_{k=1}^{\infty} a_k t^{\nu_k}\right)^2 - \left(\sum_{k=1}^{\infty} a_k t^{\nu_k}\right)^3 + \cdots .$$

Again it is not difficult to see that this expansion can be worked out and that it is of the form $\beta = 1 + \sum_{k=1}^{\infty} b_k t^{\mu_k}$ where $0 < \mu_1 < \mu_2 < \cdots \to \infty$, so that β belongs to L.

We now claim that $\alpha\beta = 1$. To see this, consider the identity

$$(1 + \gamma)(1 - \gamma + \gamma^2 - \gamma^3 + \cdots + \gamma^{2m}) = 1 + \gamma^{2m+1} \tag{3.2}$$

which holds in L for arbitrary natural m. We may substitute $\sum_{k=1}^{\infty} a_k t^{\nu_k}$ for γ and expand on both sides of (3.2). This yields an equation

$$\alpha\beta' = \gamma', \tag{3.3}$$

where β' is the expansion of

$$1 - \left(\sum_{k=1}^{\infty} a_k t^{\nu_k}\right) + \left(\sum_{k=1}^{\infty} a_k t^{\nu_k}\right)^2 - \cdots + \left(\sum_{k=1}^{\infty} a_k t^{\nu_k}\right)^{2m}$$

and γ' is the expansion of $1 + (\sum_{k=1}^{\infty} a_k t^{\nu_k})^{2m+1}$. But then β' differs from β only in powers of t whose exponent is at least $(2m + 1)\nu_1$ and γ' differs from 1 only in powers of t whose exponent also is at least $(2m + 1)\nu_1$. Since m is an arbitrary natural number, we conclude that $\alpha\beta = 1$, $\beta = \alpha^{-1}$.

Now let $\alpha \in L$ be different from zero, otherwise arbitrary. Then $\alpha = \sum_{k=0}^{\infty} a_k t^{\nu_k}$, where we may assume that $a_0 \neq 0$. Putting $\alpha = a_0 t^{\nu_0} \alpha'$ where

$$\alpha' = 1 + \sum_{k=1}^{\infty} (a_k / a_0) t^{\nu_k - \nu_0},$$

we then obtain $a_0^{-1} t^{-\nu_0} \alpha'^{-1}$ as the multiplicative inverse of α.

Thus, L is a field. We introduce an ordering of L by defining that an element

$\alpha \in L$, $\alpha \neq 0$ is positive if and only if the nonvanishing coefficient a_k with lowest subscript m in the expression $\alpha = \sum_{k=0}^{\infty} a_k t^{\nu_k}$ is positive. Also, L obtains a valuation by defining $v(\alpha) = \nu_m$ (so that $a_m \neq 0$, $a_k = 0$ for $k < m$), for $\alpha \neq 0$, together with $v(0) = \infty$ in accordance with our general convention.

In this valuation, the valuation ring O_L consists of all elements of L which can be written as $\sum a_k t^{\nu_k}$ with $\nu_0 \geqq 0$, and this is also the ring of **finite** elements of L in the ordering of L; and the valuation ideal J_L consists of all $\sum a_k t^{\nu_k}$ with $\nu_0 > 0$ and coincides with the set of **infinitesimal** elements of L. Thus, the residue class field of L with respect to its valuation coincides with the residue class field of L with respect to its ordering and is, in fact, the field of real numbers R. Also, since $J_L \neq \{0\}$, L is nonarchimedean.

There is a natural (and obvious) embedding (injection) of R into L: $a \to a = at^0 + 0t^1 + 0t^2 + \cdots$ and this extends, equally obviously, to an embedding of $R[t]$ into L:

$$a_0 + a_1 t + \cdots + a_n t^n \to a_0 t^0 + a_1 t^1 + a_2 t^2 + \cdots + a_n t^n + 0t^{n+1} + \cdots$$

and hence, to an embedding of $R(t)$ into L. The embedding is order preserving for the ordering of $R(t)$ defined in section 2 above.

It is shown in [**5**] that L is complete. It is also shown there that the field L' which is obtained by taking complex coefficients in place of the real coefficients in L, is algebraically closed. Since $L' = L(\sqrt{-1})$ it follows (compare [**7**]) that L is real-closed, i.e., that every positive element of L possesses a square root in L and that every polynomial of odd degree in $L[x]$ possesses a root in L. It follows in particular that a positive element of L possesses roots of all orders $n = 2, 3, 4, \cdots$. The same result is established by elementary means in [**2**] and will be used later in this paper.

Now let $f(x)$ be a real-valued infinitely differentiable function of a real variable which is defined in an interval $a < x < b$, $a, b \in R$. On passing from R to L, we find that the interval $a < x < b$ in L consists of points $x = \xi + \sum_{k=1}^{\infty} a_k t^{\nu_k}$, $0 < \nu_1 < \cdots \to \infty$, of three kinds,

(i) $$a < \xi < b,$$

(ii) $$\xi = a, \sum_{k=1}^{\infty} a_k t^{\nu_k} > 0, \text{ and}$$

(iii) $$\xi = b, \sum_{k=1}^{\infty} a_k t^{\nu_k} < 0.$$

In all these cases ξ is the unique real number which is infinitely close to x, i.e., such that $x - \xi$ is infinitely small and (by analogy with the terminology in Nonstandard Analysis) we call ξ the **standard part** of x, $\xi = {}^0x$.

Laugwitz extends the function $f(x)$ to values of x in L with standard part ξ, $a < \xi < b$ by using the formal Taylor expansion of $f(x)$,

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)h^n.$$

Thus, he defines for $x = \xi + \sum_{k=1}^{\infty} a_k t^{v_k}$,

$$(3.4) \qquad {}^L f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\xi) \left(\sum_{k=1}^{\infty} a_k t^{v_k} \right)^n ,$$

where it is understood that ${}^L f(x)$ is the element of L which is obtained by expanding the right hand side of (3.4) and rearranging it in powers of t. Once again, the condition $v_0 > 0$ shows that this can be done.

We shall show in the following sections that the definition proposed by Laugwitz is obtained in a natural way by relating L to a nonstandard model of Analysis.

4. The field ${}^\rho R$. Let ${}^*R$ be a nonstandard model of Analysis (cf. [**4**] and [**6**]). We shall suppose that ${}^*R$ is **sequentially comprehensive**. That is to say, if $a_0, a_1, a_2, \cdots, a_n, \cdots, n \in N$, is a sequence of entities of ${}^*R$ (of the same type, if type restrictions are adopted), e.g., a sequence of numbers of ${}^*R$, then there exists an internal sequence $\{s_n\}$ in ${}^*R$ (where n now ranges over ${}^*N$) such that $s_n = a_n$ for all finite n.

There exist sequentially comprehensive ${}^*R$. More particularly, all ${}^*R$ which are ultrapowers are sequentially comprehensive. Thus, suppose ${}^*R = R^I/D$ where D is a free ultrafilter on the index set I. Every internal entity of ${}^*R$ is represented by (is an equivalence class of) functions $f(v)$ on I. Let $f_n(v)$ represent a_n, $n = 0, 1, 2, \cdots$, and for each $v \in I$, consider $s(v) = \{f_n(v)\}$. Then $s(v)$, v ranging over I represents an internal sequence $\{s_n\}$ in ${}^*R$. We claim that for each finite k, the value of that sequence is just a_k. Now, in order to obtain the value of $\{s_n\}$ for $n = k$, we have to substitute the function $f(v) \equiv k$ for each n in $f_n(v)$. This yields precisely $f_k(v)$, i.e., a_k.

Supposing, from now on, that ${}^*R$ is sequentially comprehensive, we wish to show that the set of infinite natural numbers, ${}^*N - N$, cannot be coinitial with ω^*. In other words:

4.1. THEOREM. *Let $a_0 > a_1 > a_2 > \cdots > a_n > \cdots, n \in N$ be a strictly decreasing sequence of infinite natural numbers, internal or external. Then there exists an infinite natural number a, such that $a_n > a$ for all $n \in N$.*

Proof. Since ${}^*R$ is sequentially comprehensive, we may suppose that, for all $n \in N$, $a_n = s_n$ where $\{s_n\}$ is an internal sequence of numbers of ${}^*R$. Consider the internal sequence

$$t_n = \frac{n}{\min(s_0, s_1, \cdots, s_n)}, \qquad n \in {}^*N.$$

Then $0 \leqq t_n < 1$ for all finite n but $t_n > 1$ for large infinite n. Hence there exists a smallest m, which must then be infinite, such that $0 \leqq t_m < 1$ does not hold.

Thus, for $k = m - 1$,

$$0 \leqq \frac{k}{\min(s_0, s_1, \cdots, s_k)} < 1.$$

This shows that $k < a_0$, $k < a_1, \cdots, k < a_n, \cdots$ for all finite n and proves the theorem.

Now let ρ be an arbitrary but fixed positive infinitesimal number in $^*R$. We define subsets M_0 and M_1 of $^*R$ by

$$M_0 = \{x \in {}^*R \mid |x| \leqq \rho^{-n} \text{ for some finite positive integer } n\},$$

$$M_1 = \{x \in {}^*R \mid |x| \leqq \rho^{n} \text{ for all finite positive integers } n\}.$$

Evidently, $M_1 \subset M_0$ and $M_0 \supset R$. Both M_0 and M_1 are rings under the operations of $^*R$. For if $|x| \leqq \rho^{-n}$, $|y| \leqq \rho^{-m}$, with $n \leqq m$ say, then

$$|x + y| \leqq |x| + |y| \leqq 2\rho^{-m} \leqq \rho^{-(m+1)}$$

and $|xy| \leqq \rho^{-(n+m)}$, so M_0 is a ring. And if $|x| \leqq \rho^n$, $|y| \leqq \rho^n$ then $|x \pm y| \leqq 2\rho^n \leqq \rho^{n-1}$, $|xy| \leqq \rho^{2n}$. Since, in the definition of M_1, n is arbitrary, this shows that M_1 also is a ring.

Moreover, M_1 is an ideal in M_0, for if $x \in M_1$ and $y \in M_0$ then $|y| \leqq \rho^{-n}$ for some natural number n, and since $|x| \leqq \rho^{m+n}$ for all natural n, it follows that $|xy| \leqq \rho^m$ for all natural m, $xy \in M_1$. M_1 is a proper ideal since it does not contain 1. Finally, M_1 is a *maximal* ideal in M_0. For let $J \supset M_1$ be another ideal in M_0 such that $J - M_1$ is not empty, and let $x \in J - M_1$. Then $|x| > \rho^m$ for some finite natural number m and so $|x^{-1}| < \rho^{-m}$, $x^{-1} \in M_0$. Hence $1 = xx^{-1} \in J$, $J = M_0$, showing that M_1 is maximal.

We conclude that the quotient ring ${}^\rho R = M_0/M_1$ is a field. Moreover, the canonical map

$$\psi: M_0 \to {}^\rho R \tag{4.2}$$

induces an ordering in ${}^\rho R$. For let $x \in M_0 - M_1$, $x > 0$, and let $x + y$, $y \in M_1$ be any other element of the coset of x with respect to M_1. Then $|x| > \rho^m$ for some finite natural number m and $|y| \leqq \rho^n$ for all finite natural numbers n. Hence $|y| < |x|$, and so $x + y \geqq x - |y| = |x| - |y| > 0$, all elements of the coset of x are positive. Accordingly, we may define an ordering in ${}^\rho R$ by defining that an element $\alpha \in {}^\rho R$, $\alpha \neq 0$, is positive if and only if the elements of $\psi^{-1}\alpha$ are positive. Then the sum and product of positive elements of ${}^\rho R$ are positive but $0 \in {}^\rho R$ is not positive. This shows that our definition turns ${}^\rho R$ into an ordered field. We also observe that for any $\alpha \in {}^\rho R$, $\psi^{-1}\alpha$ is an interval in M_0 and $^*R$. Finally, since M_1 contains only the single standard number 0, ψR provides an embedding of R (as a subfield of $^*R$) in ${}^\rho R$.

Next, we define a valuation in ${}^\rho R$, as follows. For any $\alpha \in {}^\rho R$, $\alpha \neq 0$, let x and $x + y$ be elements of $\psi^{-1}\alpha$, $y \in M_1$ and consider $\log_\rho |x|$ and $\log_\rho |x + y|$. Since $|x|$

and $|x+y|$ are greater than some positive, and smaller than some negative power of ρ, $\log_\rho|x|$ and $\log_\rho|x+y|$ are finite and possess standard parts. We claim that

$$^0(\log_\rho|x|) = {}^0(\log_\rho|x+y|),$$

i.e., that

$$\log_\rho|x+y| - \log_\rho|x| = \log_\rho|1 + y/x|$$

is infinitesimal. But $\log_\rho|1+(y/x)| = \ln|1+(y/x)|/\ln\rho$. Since y/x is infinitesimal and $\ln|w|$ is a standard function which is continuous at $w = 1$, $\ln|1+(y/x)|$ is infinitesimal. Hence $\log_\rho|1+(y/x)|$ also is infinitesimal, as asserted.

Accordingly, we obtain a unique definition of a function $v(\alpha)$ for $\alpha \in {}^\rho R$, $\alpha \neq 0$, by putting $v(\alpha) = {}^0(\log_\rho|x|)$ for any $x \in \psi^{-1}\alpha$. We claim that this defines a valuation of the field ${}^\rho R$.

Let α, $\beta \in {}^\rho R$, $\alpha \neq 0$, $\beta \neq 0$ and let $x \in \psi^{-1}\alpha$, $y \in \psi^{-1}\beta$. Then

$$^0(\log_\rho|xy|) = {}^0(\log_\rho|x|) + {}^0(\log_\rho|y|)$$

and so $v(\alpha\beta) = v(\alpha) + v(\beta)$, as required. Next, suppose $\alpha + \beta \neq 0$, then we have to show that $v(\alpha+\beta) \geqq \min(v(\alpha), v(\beta))$ or, equivalently, that

$$^0(\log_\rho|x+y|) \geqq \min({}^0(\log_\rho|x|), {}^0(\log_\rho|y|)). \tag{4.3}$$

We may suppose without essential loss of generality that $\log_\rho|x| \geqq \log_\rho|y|$. Then (4.3) will hold precisely if there is an infinitesimal η such that

$$\log_\rho|x+y| \geqq \log_\rho|y| - \eta,$$

i.e., such that

$$\log_\rho\left|1 + \frac{x}{y}\right| \geqq -\eta.$$

Putting $x/y = w$, we have to show $\log_\rho|1+w| \geqq -\eta$ for $\log_\rho|w| \geqq 0$, (where we may rule out $w = -1$ because of $\alpha + \beta \neq 0$). Put $\sigma = \log_\rho|w|$, $|w| = \rho^\sigma$, where $\sigma \geqq 0$, then

$$|1+w| \leqq 1 + |w| = 1 + \rho^\sigma \leqq 2\rho^\sigma = \rho^{\sigma + \log_\rho 2}$$
$$\log_\rho|1+w| \geqq \sigma + \log_\rho 2 \geqq \log_\rho 2.$$

But $\ln_\rho 2$ is (negative) infinitesimal, and so (4.3) is proved. We supplement the definition of $v(x)$ as usual by putting $v(0) = \infty$.

The valuation ring of the valuation just defined will be denoted by O_ρ. It is not difficult to see that O_ρ includes the ψ-images of all finite elements of $^*R$. However, O_ρ includes other elements as well. For example, let $\lambda = \psi\ln\rho$. Then $v(\lambda) = {}^0(\log_\rho|\ln\rho|)$ $= {}^0(\ln|\ln\rho|/\ln\rho)$. But the expression in the parentheses on the right hand side is infinitesimal, for $\ln \rho$ is (negative) infinite and

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0.$$

Hence $v(\lambda) = 0$.

We shall now show that the field ${}^{\rho}R$ is *complete* for the valuation defined above. Defining the distance between two elements of ${}^{\rho}R$, α and β, by $d(\alpha, \beta) = c^{-v(\alpha-\beta)}$ (see the end of section 2 above) let $\{\alpha_n\}$ be a Cauchy sequence in this metric.

$$\lim_{\substack{n \to \infty \\ m \to \infty}} d(\alpha_n, \alpha_m) = 0. \tag{4.4}$$

Then we have to show that $\{\alpha_n\}$ converges to a limit α in ${}^{\rho}R$.

Choose elements $x_n \in \psi^{-1}\alpha_n$, $n = 0, 1, 2, \cdots, n \in N$. Since $^*R$ is sequentially comprehensive there exists an internal sequence $\{s_n\}$ of elements of $^*R$ such that $s_n = x_n$ for all finite n. We shall write x_n in place of s_n also for infinite n. By (4.4)

$$\lim_{\substack{n \to \infty \\ m \to \infty}} v(\alpha_n - \alpha_m) = \infty.$$

Equivalently, given any finite natural number k, there exists a finite natural $j = j_k$ such that

$$\log_\rho \left| x_n - x_m \right| > k \text{ for } n, m > j_k, \ n, m \in N. \tag{4.5}$$

Now since 4.5 holds for all finite n and m greater than j, it holds for all $n > j$, $m > j$, $n + m$ finite, $j = j_k$. A standard argument of Nonstandard Analysis, which was exemplified in the proof of 4.1, now shows that there exists an *infinite* natural $\omega = \omega_k$ such that (4.5) holds for all $n > j$, $m > j$, and $n + m < 2\omega_k$ and hence, in particular, for all $n > j$, $m > j$ and $n < \omega_k$, $m < \omega_k$. Moreover, by determining $\omega_0, \omega_1, \omega_2, \cdots$ one after the other, we may evidently assume that $\omega_0 > \omega_1 > \omega_2 > \cdots$. Appealing to 4.1, we may then choose an infinite natural number Ω which is smaller than ω_0, ω_1, ω_2 and—obviously, being infinite, larger than $j_0, j_1, j_2, \cdots$. Then,

$$\log_\rho \left| x_n - x_\Omega \right| > k \text{ for } n > j_k, \qquad n \in N, \quad k \in N. \tag{4.6}$$

(4.6) shows in the first place, that $x_\Omega \in M_0$. To see this, choose $n > j_0$ then $\log_\rho \left| x_n - x_\Omega \right| > 0$, so $\left| x_n - x_\Omega \right|$ is finite. Also, $x_n \in M_0$, so $\left| x_n \right| \leqq \rho^{-m}$ for some positive integer m and $\left| x_\Omega \right| \leqq \left| x_\Omega - x_n \right| + \left| x_n \right| \leqq 2\rho^{-m} < \rho^{-(m+1)}$, $x_\Omega \in M_0$.

Now let $\alpha = \psi x_\Omega$, then we wish to show that $\lim_{n \to \infty} \alpha_n = \alpha$ or, which is equivalent, that

$$\lim_{n \to \infty} v(\alpha_n - \alpha) = \infty. \tag{4.7}$$

But this is an immediate consequence of (4.6), since (4.6) implies

$$^{0}(\log_\rho |x_n - x_\Omega|) > k - 1 \text{ for } n > j_k, \quad n \in N, \quad k \in N$$

and this is the same as

$$v(\alpha_n - \alpha) > k - 1 \text{ for } n > j_k, \quad n \in N, \quad k \in N$$

which is just an explicit expression for the validity of (4.7). Thus, we have shown that $^{\rho}R$ is complete.

Let $\bar{\rho} = \psi\rho$ and consider any infinite series in $^{\rho}R$ of the form

$$(4.8) \qquad a_0\bar{\rho}^{\nu_0} + a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots, \qquad a_n \in R \subset {}^{\rho}R,$$
$$\nu_0 < \nu_1 < \nu_2 < \cdots \to \infty,$$

where the ν_j are standard real. The partial sums of (4.8) are

$$\sigma_k = a_0\bar{\rho}^{\nu_0} + a_1\bar{\rho}^{\nu_1} + \cdots + a_k\bar{\rho}^{\nu_k}, \qquad k = 0, 1, 2, \cdots.$$

The value of any monomial in (4.8) is, for $a_j \neq 0$, $v(a_j\bar{\rho}^{\nu_j}) = v(a_j) + v(\bar{\rho}^{\nu_j}) = 0 + \nu_j = \nu_j$, with $v(a_j\bar{\rho}^{\nu_j}) = \infty$ for $a_j = 0$. Hence $v(\sigma_k) = \nu_j$ where j is the smallest subscript $\leqq k$ for which $a_j \neq 0$, if any, otherwise $v(\sigma_k) = \infty$. Also, for $0 \leqq k < l$

$$\sigma_l - \sigma_k = a_{k+1}\bar{\rho}^{\nu_{k+1}} + \cdots + a_l\bar{\rho}^{\nu_l}$$

and so $v(\sigma_l - \sigma_k) \geqq \nu_{k+1}$. This shows that $\{\sigma_k\}$ is a Cauchy sequence, and the limit of that sequence, σ is just the sum of (4.8). Also, $v(\sigma) = \nu_j$ where j is the lowest subscript for which $a_j \neq 0$ or, if there is no such j, i.e., if all a_j vanish, $v(\sigma) = \infty$ and this is the case if and only if $\sigma = 0$. As usual in the theory of infinite series, we identify (4.8) with its sum in $^{\rho}R$. It is then not difficult to verify that the sum of two numbers of $^{\rho}R$, σ and τ, given by (4.8) and

$$(4.9) \qquad b_0\bar{\rho}^{\mu_0} + b_1,\bar{\rho}^{\mu_1} + b_2\bar{\rho}^{\mu_2} + \cdots, b_n \in R \subset {}^{\rho}R,$$
$$\mu_0 < \mu_1 < \mu_2 < \cdots \to \infty,$$

is represented by an expression

$$c_0\bar{\rho}^{\lambda_0} + c_1\bar{\rho}^{\lambda_1} + c_2\bar{\rho}^{\lambda_2} + \cdots$$

which is obtained from (4.8) and (4.9) just as the sum $\sum c_k t^{\lambda_k}$ was obtained from $\sum a_k t^{\nu_k}$ and $b_k t^{\mu_k}$ as elements of L in section 3 above. The product of (4.8) and (4.9) also is obtained by the procedure described in section 3, with $\bar{\rho}$ for t. It follows that the mapping

$$(4.10) \qquad \Phi: a_0 t^{\nu_0} + a_1 t^{\nu_1} + a_2 t^{\nu_2} + \cdots \to a_0\bar{\rho}^{\nu_0} + a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots,$$
$$a_j \in R, \ \nu_0 < \nu_1 < \nu_2 < \cdots \to \infty,$$

where the ν_j are standard real, is a homomorphism from L into $^{\rho}R$. This homomorphism is an injection since $\Phi\alpha = 0$ implies $a_0 = a_1 = a_2 = \cdots = 0$ (see above) and, hence, $\alpha = 0$. It follows that ΦL is a field which is isomorphic to L and we write $\Phi L = {}^{\rho}L$. Evidently, Φ is analytic (i.e., value preserving, $v(\Phi(\alpha)) = v(\alpha)$). But Φ is also order preserving, as can be shown by verifying that, for any $\alpha \in L$, $\Phi\alpha > 0$ if and only if $\alpha > 0$. Now for $\alpha \neq 0$, $\alpha > 0$ if and only if the first nonvanishing a_j is positive, so we only have to show that an expression as in (4.8), $\bar{\sigma} = a_0\bar{\rho}^{\nu_0} + a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots$ is positive provided (without loss of generality) $a_0 > 0$. Now, we may write $\bar{\sigma} = \psi\sigma$, where $\sigma = a_0\rho^{\nu_0} + \tau$, ${}^0(\log_\rho|\tau|) \geqq \nu_1$. It follows that if ν is an arbitrary standard real number between ν_0 and $\nu_1, \nu_0 < \nu < \nu_1$ then $\log_\rho|\tau| > \nu$, $|\tau| < \rho^\nu$, $a_0\rho^{\nu_0} > |\tau|$ and so

$$\sigma = a_0\rho^{\nu_0} + \tau \geqq a_0\rho^{\nu_0} - |\tau| > 0$$

and hence, $\bar{\sigma} > 0$. Thus Φ is order preserving, as asserted.

5. Functions in ${}^{\rho}R$. Let $f(x)$ be any real-valued function defined for $a < x < b$, $a, b \in R$. On passing to $^*R$, $f(x)$ is extended automatically to a function $^*f(x)$ which is defined for $a < x < b$ in $^*R$. We wish to find a natural extension of the function $f(x)$ as we pass from R to ${}^{\rho}R$.

Such an extension can be obtained, under certain conditions, as follows. Let ξ be any element of ${}^{\rho}R$ between a and b, $a < \xi < b$. Let ψ be the canonical homomorphism from M_0 to ${}^{\rho}R$ as before (see (4.2) above). Then we define

$$ {}^{\rho}f(\xi) = \psi(^*f(x)) \text{ for } x \in \psi^{-1}\xi, \qquad a < x < b \tag{5.1}$$

provided the expression on the right hand side of (5.1) is independent of the particular choice of x subject to the stated conditions ($a < x < b$, $\psi x = \xi$).

Suppose in particular that $f(x)$ satisfies a Lipschitz condition in any closed subinterval of $a < x < b$. Thus, for any $a < a' < b' < b$ there exists a $k = k(a', b')$ such that for any $a' \leqq x_1 < x_2 \leqq b'$,

$$ |f(x_2) - f(x_1)| \leqq k|x_2 - x_1|. \tag{5.2}$$

Passing from R to $^*R$, we see that (5.2) still holds, for standard a', b' and for arbitrary x_1, x_2 in the interval $\langle a', b'\rangle$, if we affix a star to $f(x_2)$ and $f(x_1)$. In particular, it therefore holds for two points x_1, x_2 of $^*R$ which are infinitely close to some standard x_0, $a < x_0 < b$ (where the constant k may depend on x_0).

Now let $\xi \in {}^{\rho}R$ be infinitely close to $x_0 \in R$. Then if x_1, x_2 belong to $\psi^{-1}\xi$, both x_1 and x_2 are infinitely close to x_0 in $^*R$, and (5.2) applies for an appropriate standard k. But then $x_2 - x_1 \in M_1$ and so, by (5.2), $f(x_2) - f(x_1) \in M_1$, $\psi f(x_2) = \psi f(x_1)$. This shows that in this case, (5.1) provides a unique definition for ${}^{\rho}f(\xi)$.

In particular, the Lipschitz condition is satisfied if $f(x)$ has a continuous derivative for $a < x < b$ or, more particularly, if $f(x)$ is infinitely differentiable in that interval. Suppose that this is the case and consider the restriction of ${}^{\rho}f(x)$ to points

$$\xi = a_0 + a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots,\ 0 < \nu_1 < \nu_2 < \cdots, \qquad a < a_0 < b.$$

We may compare ${}^{\rho}f(x)$ for such a point with the function which is obtained by transferring Laugwitz' definition from L to ${}^{\rho}L$, i.e., with the function

$$F(x) = \Phi({}^{L}f(\Phi^{-1}x)).$$

We propose to show that ${}^{\rho}f(x)$ actually *coincides* with $F(x)$ for such argument values,

$$(5.3) \qquad {}^{\rho}f(\xi) = \Phi({}^{L}f(\Phi^{-1}\xi)).$$

In order to verify this identity, we observe that, except for rearrangements (which can be justified without difficulty within ${}^{\rho}R$), the right hand side of (5.3) is simply the formal Taylor expansion in ${}^{\rho}R$ of $f(x)$ about the point a_0. Thus, our claim is that

$$(5.4) \qquad \begin{aligned} {}^{\rho}f(\xi) = f(a_0) &+ \frac{f'(a_0)}{1!}(a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots) + \frac{f''(a_0)}{2!}(a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots)^2 \\ &+ \cdots + \frac{f^{(n)}(a_0)}{n!}(a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots)^n + \cdots, \end{aligned}$$

in other words, that the Taylor series of ${}^{\rho}f$ about a_0 converges at ξ. Put $\eta = a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots$ and choose $h \in \psi^{-1}\eta$, so that $a_0 + h \in \psi^{-1}\xi$. By Taylor's formula with Lagrange's remainder term

$$\begin{aligned} {}^{*}f(a_0 + h) = {}^{*}f(a_0) &+ \frac{{}^{*}f'(a_0)}{1!}h + \frac{{}^{*}f''(a_0)}{2!}h^2 + \cdots \\ &+ \frac{{}^{*}f^{(n)}(a_0)}{n!}h^n + \frac{{}^{*}f^{(n+1)}(a_0 + \theta h)}{(n+1)!}h^{n+1}, \end{aligned}$$

where $0 \leqq \theta \leqq 1$. Now on the right hand side of this identity ${}^{*}f^{(k)}(a_0) = f^{(k)}(a_0)$ since a_0 is standard. Also, since $f(x)$ is infinitely differentiable, $f^{(n+1)}(x)$, and hence ${}^{*}f^{(n+1)}(x)$, is bounded by a standard real number in any standard closed subinterval of $\langle a, b\rangle$ and hence, is bounded by a standard number B in the monad of a_0. Hence

$$(5.5) \qquad \left|\frac{{}^{*}f^{(n+1)}(a_0 + \theta h)}{(n+1)!}\, h^{n+1}\right| \leqq B\,|\,h\,|^{n+1}.$$

Let ν be any standard positive number less than ν_1. Then (5.5) together with the fact that $v(\eta) = {}^{0}(\log_{\rho}|\,h\,|) \geqq \nu_1$ shows that

$$\left|\frac{{}^{*}f^{(n+1)}(a_0 + \theta h)}{(n+1)!}\, h^{n+1}\right| < \rho^{(n+1)\nu}.$$

Hence

$$\left|{}^{*}f(a_0 + h) - \sum_{k=0}^{n} \frac{f^{(k)}(a_0)}{k!}\, h^k\right| < \rho^{(n+1)\nu},$$

and so

$$v\left({}^{\rho}f(\xi) - \sum_{k=0}^{n} \frac{f^{(k)}(a_0)}{k!}\eta^k\right) \geqq (n+1)v.$$

This shows that

$${}^{\rho}f(\xi) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a_0)}{k!}\eta^k,$$

proving (5.4).

The identity (5.3) is of interest in itself since it provides a natural justification of Laugwitz' definition within a more comprehensive framework. Beyond that, by relating Laugwitz' theory to that wider framework, we are able to make use of the resources of Nonstandard Analysis in order to provide satisfactory answers to several problems which were left open by Laugwitz. We shall turn to this task in our next section.

6. The intermediate value theorem in L. In view of (5.3), the function ${}^{\rho}f(x)$, with values restricted to ${}^{\rho}L$, behaves in exactly the same way as the function ${}^{L}f(x)$ on a corresponding interval in L. Consider the real valued function $f(x)$ which is defined in the interval $-1 < x < 1$ by

$$f(x) = \begin{cases} e^{-1/|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{6.1}$$

Then $f(x)$ is infinitely differentiable in the entire interval of definition, including $x = 0$. At that point $f^{(n)}(x) = 0$ for $n = 0, 1, 2, \cdots$.

Let $x_1 = 0$, $x_2 = \frac{1}{2}$. Then ${}^{\rho}f(x_1) = f(x_1) = 0$, ${}^{\rho}f(x_2) = f(x_2) = 1/e^2$. If ${}^{\rho}f(x)$ satisfied the intermediate value theorem, there would exist a $\xi \in {}^{\rho}L$, $0 < \xi < \frac{1}{2}$, such that ${}^{\rho}f(\xi) = \bar{\rho}$. We shall show that there is no such ξ.

Suppose first that ξ is infinitely close to 0,

$$\xi = a_0\bar{\rho}^{v_0} + a_1\bar{\rho}^{v_1} + a_2\bar{\rho}^{v_2} + \cdots,\ 0 < v_0 < v_1 < \cdots \to \infty,\ a_0 > 0.$$

Then, by (5.4)

$${}^{\rho}f(\xi) = f(0) + \frac{f'(0)}{1!}\xi + \frac{f''(0)}{2!}\xi^2 + \cdots = 0$$

so ${}^{\rho}f(\xi)$ cannot be equal to $\bar{\rho}$.

Suppose next that ξ is not infinitely close to 0. Then $\xi = a_0 + \eta$, where $0 < a_0 \leqq \frac{1}{2}$, $v(\eta) > 0$ and so, by (5.4), ${}^{\rho}f(\xi) = f(a_0) + \zeta$, where $v(\zeta) > 0$. This shows that ${}^{\rho}f(\xi)$ is infinitely close to $f(a_0)$, which is a standard real number different from 0, and so again ${}^{\rho}f(\xi)$ cannot be equal to $\bar{\rho}$, which is infinitesimal.

By contrast, if $f(x)$ is continuous in an interval $a < x < b$ and if the definition

(5.1) is effective in an interval $x_1 \leqq x \leqq x_2$ where $a < x_1 < x_2 < b$, $x_1, x_2 \in {}^{\rho}R$, then the intermediate value theorem does apply in ${}^{\rho}R$. That is to say, under these conditions:

6.2. THEOREM. *If* ${}^{\rho}f(x_1) < \eta < {}^{\rho}f(x_2)$ for $\eta \in {}^{\rho}R$, *then there exists a* $\xi \in {}^{\rho}R$, $x_1 < \xi < x_2$ *such that* ${}^{\rho}f(\xi) = \eta$.

To see this, we only have to choose elements of ${}^*R$, x_1', x_2', η' such that $\psi x_1' = x_1$, $\psi x_2' = x_2$, $\psi \eta' = \eta$. Then ${}^*f(x_1') < \eta' < {}^*f(x_2')$ and so, by the intermediate value theorem for ${}^*f(x)$ there exists a $\xi' \in {}^*R$, $x_1' < \xi' < x_2'$ such that ${}^*f(\xi') = \eta'$. Putting $\xi = \psi\xi'$ we then have ${}^{\rho}f(\xi) = \psi({}^*f(\xi')) = \psi\eta' = \eta$. This shows that the intermediate value theorem is satisfied in this case.

For the remainder of this section, it will be our main purpose to show that the intermediate value theorem holds also in ${}^{\rho}L$ for functions ${}^{\rho}f(x)$ which are obtained from infinitely differentiable functions $f(x)$ in R—and hence, holds also in L for the corresponding functions ${}^{L}f(x)$—subject to rather mild restrictions, as follows.

6.3. THEOREM. *Let $f(x)$ be a real-valued function which is defined and infinitely differentiable for $a < x < b$, a, $b \in R$ and let ${}^{\rho}f(x)$ be defined by* 5.1. *Suppose that for every $x \in R$, $a < x < b$, there is a positive integer n such that $f^{(n)}(x) \neq 0$. For any $x_1, x_2 \in {}^{\rho}L$, $a < x_1 < x_2 < b$, let a_0 and b_0 be the uniquely defined elements of R which are infinitely close to x_1 and x_2 respectively, i.e.,*

$$x_1 = a_0 + a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots, \quad 0 < \nu_1 < \nu_2 < \cdots \to \infty,$$

$$x_2 = b_0 + b_1\bar{\rho}^{\mu_1} + b_2\bar{\rho}^{\mu_2} + \cdots, \quad 0 < \mu_1 < \mu_2 < \cdots \to \infty,$$

and suppose that $a < a_0 \leqq b_0 < b$. Let η be an element of ${}^{\rho}L$ such that ${}^{\rho}f(x_1) < \eta < {}^{\rho}f(x_2)$.

Then there exists a $\xi \in {}^{\rho}L$, $x_1 < \xi < x_2$, such that ${}^{\rho}f(\xi) = \eta$.

Proof. Comparing 6.3 with 6.2 (which applies to the situation described in 6.3) we see that we only have to show that the $\xi \in {}^{\rho}R$ mentioned in 6.2 belongs more particularly to ${}^{\rho}L$. Choosing x_1', x_2', η' as in the proof of 6.2 such that $\psi x_1' = x_1$, $\psi x_2' = x_2$, $\psi\eta' = \eta$ we have, for some $\xi' \in {}^*R$, $x_1' < \xi' < x_2'$, ${}^*f(\xi') = \eta'$ and hence ${}^{\rho}f(\xi) = \eta$ where $\xi = \psi\xi'$. Now $x_1' < \xi' < x_2'$ implies that ξ' is finite and has a standard part, ${}^{0}\xi' = d_0$, where $a < a_0 \leqq d_0 \leqq b_0 < b$. At the same time, η must be of the form $e_0 + e_1\bar{\rho}^{\lambda_1} + e_2\bar{\rho}^{\lambda_2} + \cdots, 0 < \lambda_1 < \lambda_2 < \cdots \to \infty$ since it is in ${}^{\rho}L$ and finite. Hence, ${}^{0}\eta' = e_0$ and $f(d_0) = e_0$.

Suppose now that $f'(d_0) \neq 0$. Then the inversion theorem is applicable. It follows that there exist $h_1 > 0$, $h_2 > 0$, $k_1 > 0$, $k_2 > 0$, such that $f(x)$ is a one-to-one mapping of the interval D defined by $d_0 - h_1 < x < d_0 + h_2$ on the interval E defined by $e_0 - k_1 < y < e_0 + k_2$ such that the inverse function $g(y) = f^{-1}(y)$ is infinitely differentiable on E. Passing to ${}^*R$, we find that the infinitely differentiable function ${}^*f(x)$ maps ${}^*D$ in one-to-one correspondence on ${}^*E$ such that ${}^*g(y)$ is the inverse of this mapping and is infinitely differentiable as well (in the sense of $^*R$). Hence, $^*f(\xi')=\eta'$ entails $^*g(\eta')=\xi'$ and so

$$\xi = \psi\xi' = \psi(^*g(\eta')) = {}^\rho g(\eta) \in {}^\rho L,$$

proving our assertion in this case.

Dropping the restriction that $f'(d_0) \neq 0$ (but not excluding this case) we put $F(x) = f(x) - f(d_0)$ and define $H(x)$ for $a < x < b$ by

$$H(x) = \begin{cases} \dfrac{F(x)}{x - d_0} & \text{for } x \neq d_0 \\ F'(d_0) & \text{for } x = d_0. \end{cases}$$

Also, on the assumption of our theorem, there is an $n \geqq 0$ such that

$$F(d_0) = F'(d_0) = \cdots = F^{(n)}(d_0) = 0, \qquad F^{(n+1)}(d_0) \neq 0.$$

Then $F(x) = H(x)\,(x - d_0)$, and so

$$F'(x) = H(x) + H'(x)\,(x - d_0) \qquad \text{for} \quad x \neq d_0 \tag{6.4}$$

and, more generally,

$$F^{(k)}(x) = kH^{(k-1)}(x) + H^{(k)}(x)\,(x - d_0) \tag{6.5}$$

for $k = 1, 2, \cdots$, $x \neq d_0$, $a < x < b$, by induction.

We now wish to show that, for $x \neq d_0$,

$$H^{(\lambda)}(x) = \frac{F^{(\lambda+1)}(d_0)}{\lambda+1} + \frac{F^{(\lambda+2)}(d_0)}{1!(\lambda+2)}(x - d_0) + \cdots + \frac{F^{(\lambda+m)}(d_0)}{(m-1)!\,(\lambda+m)}(x - d_0)^{m-1} + G_{\lambda,m}(x - d_0)^m \tag{6.6}$$

provided $\lambda \geqq 1$, $m \geqq 1$, where $G_{\lambda,m}$ is a linear combination with rational coefficient of values of $F^{(\lambda+m+1)}(x)$ taken at points x' in the interval $\langle d_0, x\rangle$.

For $\lambda = 1$, we have the Taylor expansion for $F'(x)$

$$F'(x) = F'(d_0) + \frac{F''(d_0)}{1!}(x - d_0) + \cdots + \frac{F^{(1+m)}(d_0)}{m!}(x - d_0)^m + \frac{F^{(2+m)}(d_0 + \theta_1(x - d_0))}{(m+1)!}(x - d_0)^{m+1}$$

where $0 \leqq \theta_1 \leqq 1$, while the Taylor expansion for $F(x)$ yields

$$H(x) = F'(d_0) + \frac{F''(d_0)}{2!}(x - d_0) + \cdots + \frac{F^{(1+m)}(d_0)}{(m+1)!}(x - d_0)^m + \frac{F^{(2+m)}(d_0 + \theta_0(x - d_0))}{(m+2)!}(x - d_0)^{m+1}, \tag{6.7}$$

where $0 \leqq \theta_0 \leqq 1$. Hence, from (6.4),

$$H'(x) = \frac{F'(x) - H(x)}{(x - d_0)}$$

$$= \frac{F''(d_0)}{2} + \cdots + \frac{F^{(1+m)}(d_0)}{(m-1)!\,(1+m)}(x - d_0)^{m-1} + G_{1,m}(x - d_0)^m,$$

where

$$G_{1,m} = \frac{F^{(2+m)}(d_0 + \theta_1(x - d_0))}{(m+1)!} - \frac{F^{(2+m)}(d_0 + \theta_0(x - d_0))}{(m+2)!},$$

as required.

Suppose now that the assertion has been proved up to some $\lambda \geqq 1$, for all $m \geqq 1$. In order to prove the corresponding formula for $\lambda + 1$, we write down the appropriate Taylor expansion for $F^{(\lambda+1)}(x)$, so

$$F^{(\lambda+1)}(x) = F^{(\lambda+1)}(d_0) + \frac{F^{(\lambda+2)}(d_0)}{1!}(x - d_0) + \cdots + \frac{F^{(\lambda+m+1)}(d_0)}{m!}(x - d_0)^m$$

$$+ \frac{F^{(\lambda+m+2)}(d_0 + \theta_{\lambda+1}(x - d_0))}{(m+1)!}(x - d_0)^{m+1},$$

where $0 \leqq \theta_{\lambda+1} \leqq 1$. Then, by (6.5) and (6.6) (with $m + 1$ for m)

$$H^{(\lambda+1)}(x) = \frac{F^{(\lambda+1)}(x) - (\lambda+1)H^{(\lambda)}(x)}{x - d_0}$$

$$= \frac{F^{(\lambda+2)}(d_0)}{\lambda+2} + \cdots + \frac{F^{(\lambda+m+1)}(d_0)}{(m-1)!\,(\lambda+1+m)}(x - d_0)^{m-1} + G_{\lambda+1,m}(x - d_0)^m,$$

where

$$G_{\lambda+1,m} = \frac{F^{(\lambda+m+2)}(d_0 + \theta_{\lambda+1}(x - d_0))}{(m+1)!} - (\lambda+1)G_{\lambda,m+1}.$$

This proves (6.6). We now obtain immediately, for $\lambda \geqq 1$

$$\lim_{x \to d_0} H^{(\lambda)}(x) = \frac{F^{(\lambda+1)}(d_0)}{\lambda+1} \tag{6.8}$$

and this is true also for $\lambda = 0$, by (6.7). On the other hand, we may calculate the derivatives of $H(x)$ *at* d_0. We have, from (6.7), which is valid also for $m = 0$,

$$H'(d_0) = \lim_{x \to d_0} \frac{H(x) - H(d_0)}{x - d_0} = \lim_{x \to d_0} \frac{H(x) - F'(d_0)}{x - d_0} = \lim_{x \to d_0} \frac{F''(d_0 + \theta_0(x - d_0))}{2}$$

$$= \frac{F''(d_0)}{2},$$

where θ_0 may depend on x. Thus, $H(x)$ has a continuous derivative everywhere in its interval of definition.

Suppose now that we have proved that $H(x)$ has continuous derivatives up to order $\lambda \geqq 1$ in the entire interval of definition $a < x < b$ such that $H^{(\lambda)}(d_0) = F^{(\lambda+1)}(d_0)/(\lambda + 1)$. We then make use of (6.6) for $m = 2$, where we observe that $G_{\lambda,2}$ (as a linear combination with fixed rational coefficients of values of $F^{(\lambda+3)}$ for arguments x' in the interval $\langle d_0, x \rangle$) remains bounded in the neighborhood of x_0. Hence, for such x,

$$H^{(\lambda)}(x) = \frac{F^{(\lambda+1)}(d_0)}{\lambda + 1} + \frac{F^{(\lambda+2)}(d_0)}{\lambda + 2}(x - d_0) + O(x - d_0)^2$$

and so

$$\lim_{x \to d_0} \frac{H^{(\lambda)}(x) - H^{(\lambda)}(d_0)}{x - d_0} = \frac{F^{(\lambda+2)}(d_0)}{\lambda + 2} = \lim_{x \to d_0} H^{(\lambda+1)}(x).$$

This shows that $H(x)$ possesses continuous derivatives of all orders in its interval of definition. In particular

$$H^{(\lambda)}(d_0) = F^{(\lambda+1)}(d_0)/(\lambda + 1), \qquad \lambda = 0, 1, \cdots \tag{6.9}$$

and so

$$H(d_0) = H'(d_0) = \cdots = H^{(n-1)}(d_0) = 0, \qquad H^{(n)}(d_0) = \frac{F^{(n+1)}(d_0)}{n + 1} \neq 0.$$

If $n > 0$, we may repeat our procedure, obtaining from $H(x)$ a function $H_1(x)$ in the same way in which we obtained $H(x)$ from $F(x)$. Thus, putting

$$H_1(x) = \begin{cases} H(x)/(x - d_0) = F(x)/(x - d_0)^2 & \text{for} \quad x \neq d_0 \\ H'(d_0) & \text{for} \quad x = d_0, \end{cases}$$

we find that $H_1(x)$ is infinitely differentiable for $a < x < b$ and

$$H_1(d_0) = H_1'(d_0) = \cdots = H_1^{(n-2)}(d_0) = 0, \ H_1^{(n-1)}(d_0) = \frac{F^{(n+1)}(d_0)}{n(n + 1)} \neq 0.$$

Continuing in this way, we obtain after $n - 1$ more steps the function

$$H_n(x) = \begin{cases} H_{n-1}(x)/(x - d_0) = F(x)/(x - d_0)^{n+1} = \dfrac{f(x) - f(d_0)}{(x - d_0)^{n+1}}, \text{ for } x \neq d_0 \\ \dfrac{f^{(n+1)}(d_0)}{(n + 1)!} \neq 0, \text{ for } x = d_0 \end{cases}$$

which is infinitely differentiable for $a < x < b$.

Suppose first that n is even, $n + 1$ is odd. Then the function $w^{1/(n+1)}$, with the determination that $(H_n(d_0))^{1/(n+1)}$ be real, is infinitely differentiable in the neighborhood of $H_n(d_0)$ and so the function $P(x) = (H_n(x))^{1/(n+1)}$ is infinitely differentiable in some neighborhood of $x = d_0$, for $d_0 - h < x < d_0 + h$, say. The function

$$Q(x) = P(x)(x - d_0) = (f(x) - f(d_0))^{1/(n+1)}$$

therefore is also infinitely differentiable in the same interval, and

$$Q'(x) = P(x) + P'(x)\,(x - d_0),\ Q'(d_0) = P(d_0) \neq 0.$$

Passing to $^*R$ we see that, for $x = \xi'$,

$$^*Q(\xi') = (f(\xi') - f(d_0))^{1/(n+1)} = (\eta' - e_0)^{1/(n+1)}.$$

Hence

$$^\rho Q(\xi) = \psi((\eta' - e_0)^{1/(n+1)}) = (\eta - e_0)^{1/(n+1)} \in {}^\rho L$$

since L, and hence $^\rho L$, is real closed (see section 3 above). Hence, applying the inversion theorem to $Q(x)$ at $x = d_0$ (exactly as we applied it earlier to $f(x)$ on the assumption that $f'(d_0) \neq 0$), and letting $S(y)$ be the inverse function to $Q(x)$ at $x = d_0$, $y = 0$, we obtain

$$\xi = \psi\xi' = \psi\,(^*S((\eta' - e_0)^{1/(n+1)})) = {}^\rho S(\eta - e_0)^{1/(n+1)} \in {}^\rho L.$$

This disposes of the case that n is even.

Suppose finally that n is odd, $n + 1$ is even. We may assume without loss of generality that $H_n(d_0) = f^{(n+1)}(d_0)/(n + 1)!$ is positive, otherwise we consider $-f(x)$ in place of $f(x)$. Then $f(x) - f(d_0) = H_n(x)\,(x - d_0)^{n+1}$ must be positive, for $x \neq d_0$ in a sufficiently small neighborhood of d_0. Introducing $P(x) = (H_n(x))^{1/(n+1)}$ with the postiive determination for $(H_n(d_0))^{1/(n+1)}$, and $Q(x) = P(x)\ (x - d_0)$ we then have again that $P(x)$ and $Q(x)$ are infinitely differentiable in a neighborhood of $x = d_0$, and that $Q'(d_0) = P(d_0) \neq 0$. Also,

$$^*Q(\xi') = \pm\,(f(\xi') - f(d_0))^{1/(n+1)} = \pm\,(\eta' - e_0)^{1/(n+1)}$$

leading to $^\rho Q(\xi) = \pm\,(\eta - e_0)^{1/(n+1)}$, which is again an element of $^\rho L$. Finally, introducing the inverse function $S(y)$ of $Q(x)$ with $S(0) = d_0$, as before, we have

$$\xi = \psi\xi' = \psi\,(S(\pm\,(\eta' - e_0)^{1/(n+1)})) = {}^\rho S(\pm\,(\eta - e_0)^{1/(n+1)}) \in {}^\rho L.$$

The proof of Theorem 6.3 is now complete.

Although the counterexample given at the beginning of the section shows that some restriction on the behavior of the derivatives of $f(x)$ is required, the particular set of conditions given in 6.3, is not strictly necessary. Thus, if $f(x) = \text{const.}$, then the conditions of the theorem are not satisfied but its conclusion is, trivially. Nevertheless, 6.3 includes a large number of interesting cases, e.g., all non-constant real analytic functions $f(x)$.

7. The mean value theorem. Suppose the function $f(x)$ is continuously differentiable for $a < x < b$. Let D be the set of points $\xi \in {}^{\rho}R$ such that ξ is infinitely close to a point a_0 *in the interior* of that interval, $a < a_0 < b$. Then $f'(x)$ is bounded in any closed subinterval $a' \leqq x \leqq b'$ of $a < x < b$ and so $f(x)$ satisfies a Lipschitz condition in that interval. Taking $a' < a_0 < b'$ we see, therefore, that the definition 5.1 is effective. We claim, moreover, that the resulting function ${}^{\rho}f(x)$ is continuous, in the sense of the metric of ${}^{\rho}R$, at all points $\xi \in D$.

To see this, let $\{\xi_n\}$ be a sequence of elements of D such that $\lim_{n\to\infty} \xi_n = \xi$ and choose a number ξ' and a sequence $\{\xi_n'\}$ in ${}^*R$ such that $\psi\xi' = \xi$, $\psi\xi_n' = \xi_n$, $n=0, 1, 2, \cdots$. Since $\lim \xi_n = \xi$, there exist a', $b' \in R$ such that $a' < \xi < b'$, $a' < \xi' < b'$, $a' < \xi_n < b'$, $a' < \xi_n' < b'$, $n = 0, 1, 2, \cdots$. Let m be a bound for $f'(x)$ in the closed interval $a' \leqq x \leqq b'$ within R and, hence within ${}^*R$. Then

$$^*f(\xi_n') - {}^*f(\xi') = {}^*f'(\xi' + \theta(\xi_n' - \xi'))(\xi_n' - \xi')$$

for some $0 \leqq \theta \leqq 1$ and, hence

$$\left| {}^{\rho}f(\xi_n) - {}^{\rho}f(\xi) \right| \leqq m \left| \xi_n - \xi \right|.$$

This, together with $\lim \xi_n = \xi$ implies $\lim {}^{\rho}f(\xi_n) = {}^{\rho}f(\xi)$, proving our assertion.

Suppose next that $f(x)$ is twice continuously differentiable for $a < x < b$. In this case, we propose to show that ${}^{\rho}f(x)$ is differentiable in D (in the sense of the metric of ${}^{\rho}R$) and that on D,

$$\frac{d}{dx}\,{}^{\rho}f(x) = {}^{\rho}(f'(x)). \tag{7.1}$$

For ξ in D and $\eta \neq 0$ such that $\xi + \eta$ also belongs to D, choose ξ' and η' for which $\psi\xi' = \xi$, $\psi\eta' = \eta$. Then there exists a $\theta' \in {}^*R$, $0 \leqq \theta' \leqq 1$, such that

$$\frac{f(\xi' + \eta') - f(\xi')}{\eta'} = f'(\xi' + \theta'\eta'). \tag{7.2}$$

Applying the mapping ψ to (7.2), we obtain

$$\frac{{}^{\rho}f(\xi + \eta) - {}^{\rho}f(\xi)}{\eta} = ({}^{\rho}f'(x))_{x=\xi+\theta\eta}, \tag{7.3}$$

where $\theta = \psi\theta'$. Now let η tend to zero. Then the right hand side of (7.3) tends to $({}^{\rho}f'(x))_{x=\xi}$ since ${}^{\rho}f'(x)$ is continuous on D. This proves (7.1).

In particular, if $f(x)$ is infinitely differentiable, then ${}^{\rho}f(\xi)$ and $({}^{\rho}f'(x))_{x=\xi}$ belong to ${}^{\rho}L$ for $\xi \in {}^{\rho}L$. It follows that in that case ${}^{\rho}f(x)$ is defined and infinitely differentiable in $D \cap {}^{\rho}L$. Accordingly, the same is true of the function

$${}^{L}f(x) \text{ for } x = a_0 + a_1 t^{\nu_1} + a_2 t^{\nu_2} + \cdots,\ 0 < \nu_1 < \nu_2 < \cdots \to \infty, \qquad a < a_0 < b.$$

(7.3), in combination with (7.1) shows also that the mean value theorem holds in

${}^{\rho}R$ under the stated conditions, more particularly for infinitely differentiable $f(x)$. However, here again we may show that the mean value theorem breaks down, for certain infinitely differentiable functions, both in ${}^{\rho}L$ and in L. The function (6.1) which provided an example for the breakdown of the intermediate value theorem, will do also for the present issue as can be seen by considering the ratio of increments $(f(\xi_2)-f(\xi_1))/(\xi_2-\xi_1)$ for $\xi_2 = 1/(2+\bar{\rho})$, $\xi_1 = -\frac{1}{2}$. There is no $\xi_3 \in {}^{\rho}L$ in the closed interval from ξ_1 to ξ_2 such that $({}^{\rho}f(x))'$ is equal to that ratio for $x = \xi_3$.

We shall prove, as our principal positive result in this area:

7.4. THEOREM. *Let $f(x)$ be a real valued function which is defined and infinitely differentiable for $a < x < b$; $a, b \in R$* and let *${}^{\rho}f(x)$ be defined by* 5.1. *Suppose that for every x, $a < x < b$, there is an integer $n \geqq 2$ such that $f^{(n)}(x) \neq 0$. For any $x_1, x_2 \in {}^{\rho}L$, $a < x_1 < x_2 < b$, let a_0 and b_0 be the uniquely defined elements of R which are infinitely close to x_1 and x_2 respectively, i.e.,*

$$x_1 = a_0 + a_1\bar{\rho}^{\nu_1} + a_2\bar{\rho}^{\nu_2} + \cdots,\ 0 < \nu_1 < \nu_2 < \cdots \to \infty,$$

$$x_2 = b_0 + b_1\bar{\rho}^{\mu_1} + b_2\bar{\rho}^{\mu_2} + \cdots,\ 0 < \mu_1 < \mu_2 < \cdots \to \infty,$$

and suppose that $a < a_0 \leqq b_0 < b$.

Then there exists a $\xi \in {}^{\rho}L$, $x_1 \leqq \xi \leqq x_2$ such that

$$\frac{{}^{\rho}f(x_2) - {}^{\rho}f(x_1)}{x_2 - x_1} = \left(\frac{d}{dx}\,{}^{\rho}f(x)\right)_{x=\xi}$$

Here again there is an exactly corresponding theorem for the function ${}^{L}f(x)$ in L. The conditions of the theorem are not necessary since they exclude all functions of constant slope, for which the conclusion of the theorem is obviously correct. However, the theorem is nevertheless of a rather general character, including, for example, all other real analytic functions.

For the proof, we require the following auxiliary consideration.

Assume that the conditions of (7.4) are satisfied and choose $x_1' \in \psi^{-1}x_1$, $x_2' \in \psi^{-1}x_2$. Then we claim that ${}^*f'(x)$ attains its maximum in the interval $x_1' \leqq x \leqq x_2'$ either at x_1' or at x_2' or at some *standard* point x_0, $x_1' \leqq x_0 \leqq x_2'$ (but, possibly, also elsewhere).

Suppose that ${}^*f(x)$ attains its maximum neither at x_1' nor at x_2' but at a point $\bar{x}, x_1' < \bar{x} < x_2'$. Let x_0 be the standard part of $\bar{x}$, $x_0 = {}^0\bar{x}$. Suppose that $x_0 < x_1$ (so that $x_0 = a_0$). Depending on whether the first non-vanishing derivative of $f'(x)$ at x_0 is either positive or negative, ${}^*f'(x)$ will be either strictly increasing or strictly decreasing in some interval $x_0 \leqq x \leqq x_0 + h$, where h is standard and positive. Since $\bar{x}$ and x_1' belong to that interval, the latter case would involve ${}^*f'(x_1') > {}^*f'(\bar{x})$, contrary to our choice of $\bar{x}$. Accordingly, we have to assume that ${}^*f'(x)$ increases strictly for $x_0 \leqq x \leqq x_0 + h$. Now x_2' cannot belong to that interval for then ${}^*f'(x_2') > {}^*f'(\bar{x})$, which is again impossible. It follows that $\bar{x} < x_0 + h < x_2'$ and

${}^*f'(x_0+h) > f(\bar{x})$ which is also impossible. We therefore conclude that $x_0 \geqq x_1$ and, by similar reasoning, $x_0 \leqq x_2$. The discussion of the variation of ${}^*f'(x)$ in the neighborhood of x_0 shows that we must exclude both $x_0 < \bar{x}$ and $x_0 > \bar{x}$ and so we conclude that $\bar{x} = x_0$.

Thus we have shown than ${}^*f'(x)$ attains its maximum at x_1' or at x_2' or at some standard point $x_1' < x_0 < x_2'$ (although several of these cases may occur simultaneously). Accordingly ${}^*f'(x)$ attains its maximum in the interval $x_1' \leqq x \leqq x_2'$ in all cases at a point ζ_1' such that $\psi\zeta_2' = \zeta_2 \in {}^\rho L$. By a similar argument (or, by applying the conclusion to $-f(x)$) we find that ${}^*f'(x)$ attains its minimum in the same interval at a point ζ_2' such that $\psi\,\zeta_1' = \zeta_1 \in {}^\rho L$. Passing from ${}^*R$ to ${}^\rho R$, we then conclude that ${}^\rho(f'(x))$ attains its maximum and minimum in $x_1 \leqq x \leqq x_2$ at points ζ_1 and ζ_2 which belong to ${}^\rho L$.

By a well-known formula of the Integral Calculus, which can be transferred from R to ${}^*R$, we have

$$ {}^*f'(\zeta_2')(x_2' - x_1') \leqq \int_{x_1'}^{x_2'} {}^*f'(x)dx \leqq {}^*f'(\zeta_1')(x_2' - x_1'), $$

i.e.,

$$ {}^*f'(\zeta_2')(x_2' - x_1') \leqq {}^*f(x_2') - {}^*f(x_1') \leqq {}^*f'(\zeta_1')(x_2' - x_1'). $$

We apply the mapping ψ to this chain of inequalities and obtain

$$ {}^\rho(f'(\zeta_2))(x_2 - x_1) \leqq {}^\rho f(x_2) - {}^\rho f(x_1) \leqq {}^\rho(f'(\zeta_1))(x_2 - x_1) $$

or, equivalently

$$ {}^\rho(f'(\zeta_2)) \leqq \frac{{}^\rho f(x_2) - {}^\rho f(x_1)}{x_2 - x_1} \leqq {}^\rho(f'(\zeta_1)). $$

But this shows that $({}^\rho f(x_2) - {}^\rho f(x_1))/(x_2 - x_1)$ is intermediate between ${}^\rho(f'(\zeta_2))$ and ${}^\rho(f'(\zeta_1))$. Hence, by the intermediate value Theorem 6.3, there exists a $\xi \in {}^\rho L$ which belongs to the closed interval with endpoint ζ_1 and ζ_2 and, hence, belongs to $x_1 \leqq x \leqq x_2$, such that

$$ \frac{{}^\rho f(x_2) - {}^\rho f(x_1)}{x_2 - x_1} = {}^\rho(f'(\xi)) $$

and this is the same as

$$ \frac{{}^\rho f(x_2) - {}^\rho f(x_1)}{x_2 - x_1} = \left(\frac{d}{dx}\,{}^\rho f(x)\right)_{x=\xi}, $$

by (7.1). The proof of 7.4 is now complete.

8. Conclusion. As Laugwitz points out, his method for extending a function $f(x)$ from R to L applies only in the infinitesimal neighborhood of a point at which $f(x)$ is infinitely differentiable and hence, possesses at least a formal Taylor series. However, if we consider points in the infinitesimal neighborhood of the endpoints of the interval of definition $a < x < b$ of $f(x)$, e.g., $x = a + a_1 t^{\nu_1} + a_2 t^{\nu_2} + \cdots$, $0 < \nu_1 < \nu_2 < \cdots$, $a_1 > 0$, then we can still define ${}^L f$ at x, provided f possesses an asymptotic expansion at $x = a$ as x tends to a from the right. Similarly, if $f(x)$ is defined in a semi-infinite interval, for $x > a$ say, we can define ${}^L f(x)$ for positive infinite x provided $f(x)$ possesses an asymptotic expansion as $x \to +\infty$. In all of these cases, ${}^L f(x)$ can again be obtained "automatically" as $\Phi^{-1}({}^\rho f(\Phi x))$ (see (5.3) above). However, ${}^\rho f(x)$ exists also in many cases where no asymptotic expansion as a generalized power series is available, e.g., ${}^\rho \log x$ exists for positive infinitesimal and infinite x. Conversely, we may investigate the asymptotic expansion of a function $f(x)$ at a singular point (even when it contains logarithmic terms, as happens frequently in the theory of ordinary differential equations) by means of the function ${}^\rho f(x)$. Going further in the direction of concrete applications, ${}^\rho R$ also provides us with a convenient framework for the discussion of matched asymptotic expansions for the solution of singular perturbation problems.

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