ECON915 Microeconomic Theory
Part A: Introduction to Decision Theory
Lecture 1: Preferences

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ECON915 Module Information: Structure

- Preliminaries: ULMS055 Mathematics Crammer
  (online self-learning module on CANVAS)
- ECON915 Part A *Introduction to Decision Theory*: by CWB
  (weeks 2-4)
- MID-TERM: format to be announced / Part A only
  (week 5)
- ECON915 Part B *General Equilibrium & Social Choice*: by RRR
  (weeks 7-11)
- EXAM: 2 hours exam on campus; closed-book / Part B only
  (January assessment period)

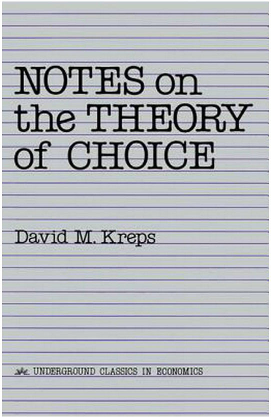
ECON915 Module Information: Organization of Part A

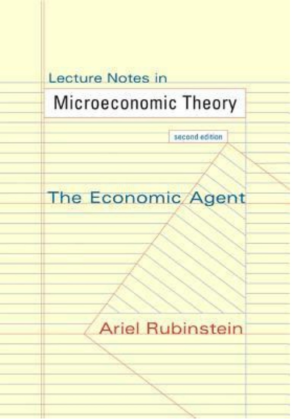
- Topic: introduction to advanced-level **decision theory**
- Three lecture topics:
  - Lecture 1: Preferences
  - Lecture 2: Utility
  - Lecture 3: Choice
- Three lectures (weeks 2, 3, and 4):
  - Tuesdays, at *9am-10.30pm*, in room *REN-SR11*
- Two seminars (in weeks 3 and 4):
  - Thursdays, at *11am-noon*, in room *REN-SR4*

ECON915 Module Information: Questions & Material of Part A

- In case of questions on Part A Decision Theory:
  - Questions are always welcome!
  - during lectures!
  - during seminars!
  - email: cwbach@liv.ac.uk
  - office hours: Thursdays, at *3.30pm-5pm*, in *ULMS-CR2*
- All material for Part A Decision Theory is available on CANVAS

ECON915 Module Information: Optional Background Reading of Part A





Decision Theory: Single Agent and Axiom-based

Typical Approach

1. A set of objects, the choice set X, is identified.
2. Qualitative statements ("AXIOMS") about the agent's **preferences** among elements of X, are proposed.
3. A **utility function** from X to $\mathbb{R}$ is sought such that higher utility corresponds to more preferred items ("REPRESENTATIONS").
4. Are the AXIOMS sufficient *(if the axioms hold, then the representation obtains)*, and are the AXIOMS necessary *(if the representation holds, then the axioms obtain)*? ("REPRESENTATION THEOREMS")
5. Results for uniqueness are sought, which characterize the extent to which two similarly structured REPRESENTATIONS of given preferences can vary *("The representation is unique up to . . .")*.

What Constitutes a "Good" Set of Axioms

- In general, AXIOMS should be
  - basic,
  - primitive,
  - intuitive,
  - qualitative.

("The meaning is rather subjective and controversial").

- Two (rather uncontroversial) formal properties:
  1. **Consistency:** the AXIOMS can be satisfied simultaneously, i.e. there exists some identifiable collection of objects satisfying all of them *("Contradiction-free")*.
  2. **Independency:** no strict subset of the set of all AXIOMS implies the ones left out *("Parsimony")*.

Normative Considerations

- Suppose an agent thinks a given set of AXIOMS is a reasonable guide to choice.
- Corresponding REPRESENTATION THEOREMS guarantee that the agent wants his choice behaviour to conform to the respective quantitative (utility) representation.
- The theory can then aid the agent by inferring his choice from the simpler formulation of the quantitative (utility) representation.

Descriptive Considerations

- Insofar as an agent's preferences (e.g. as revealed by his choices) conform to a set of AXIOMS, his behaviour can be modelled as if he chooses in line with a correspoding quantitative (utility) representation.
- Concerning descriptiveness the obvious question then becomes empirical: to what extent do real-world persons' choices conform to given AXIOMS on preferences?
- It is thus important that the theory delivers testable AXIOMS and testable implications of the AXIOMS.
- Almost no one seriously maintains that real-world persons do conform exactly to the AXIOMATIC SYSTEM of **decision theory**: in fact, there exists a substantial amount of empirical evidence against it.
- At best, real-world behaviour approximates the AXIOMATIC SYSTEM of **decision theory**.
- So what about **Relevance** then? If real-world persons' behaviour is approximately what is modelled, then the model might unveil something about how behaviours interact or intertwine in the real world.

Agenda

- Binary Relations
- Strict Preference
- Weak Preference and Indifference

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- **Binary Relations**
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Binary Relations

- Let X, Y, and Z be sets.

- The product set $X \times Y$ of X and Y consists of all ordered pairs (x, y), where x is from X and y is from Y. Formally,

$$X \times Y := \{(x, y) : x \in X, y \in Y\}$$

with special case

$$X^2 := X \times X = \{(x, x') : x, x' \in X\}$$

- A binary relation B on X is a subset of the product set $X \times X$.

$$B \subseteq X \times X$$

- Sometimes $(x, x') \in B$ is also denoted by xBx', and $(x, x') \notin B$ by $\neg(xBx')$ or $x \not{B} x'$.

Brief Digression: N-ary Relations

- Consider finitely many sets $A_1, A_2, \ldots, A_n$.
- The product set $A_1 \times A_2 \times \cdots \times A_n$ of $A_1, A_2, \ldots, A_n$ consists of all ordered tuples $(a_1, a_2, \ldots, a_n)$, where a_i is from A_i for all $i \in \{1, 2, \ldots, n\}$.

$$A_1 \times A_2 \times \cdots \times A_n := \big\{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for all } i \in \{1, 2, \ldots, n\}\big\}$$

- A n-ary relation R on X is a subset of the product set $X^n := X \times X \times \cdots \times X$ of "n-times" the set X.

$$R \subseteq X^n = \big\{(x_1, x_2, \ldots, x_n) : x_i \in X \text{ for all } i \in \{1, 2, \ldots, n\}\big\}$$

Examples

1. $X = \{1, 2, 3\}$, and $B = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}$.

2. X is the set of all people in the world, and B is the binary relation *"shares at least one given name with"*.

3. $X = \mathbb{R}$, and B is the binary relation defined by xBy if $x \geq y$ for all $x, y \in \mathbb{R}$.

4. $X = \mathbb{R}$, and B is the binary relation defined by xBy if $\mid x - y \mid > 1$ for all $x, y \in \mathbb{R}$.

5. $X = \mathbb{R}$, and B is the binary relation defined by xBy if $x - y$ is an integer multiple of 2 for all $x, y \in \mathbb{R}$.

Some Possible Properties of Binary Relations

Let B be a binary relation on some set X and $x, y, z, x_1 \ldots, x_n \in X$.

Reflexivity: xBx.

Irreflexivity: $\neg(xBx)$.

Symmetry: If xBy, then yBx.

Asymmetry: If xBy, then $\neg(yBx)$.

Antisymmetry: If xBy and yBx, then $x = y$.

Completeness (or Connectedness): xBy or yBx.

Weak Connectedness: $x = y$ or xBy or yBx.

Transitivity: If xBy and yBz, then xBz.

Negative Transitivity: If $\neg(xBy)$ and $\neg(yBz)$, then $\neg(xBz)$.

Cyclicity: If $x_1Bx_2, x_2Bx_3, \ldots, x_{n-1}Bx_n$, then x_nBx_1.

Acyclicity: If $x_1Bx_2, x_2Bx_3, \ldots, x_{n-1}Bx_n$, then $\neg(x_nBx_1)$.

Examples

1. $X = \{1, 2, 3\}$, and $B = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 1)\}$. B is weakly connected.

2. X is the set of all people in the world, and B is the binary relation *"shares at least one given name with"*. B is reflexive, and symmetric.

3. $X = \mathbb{R}$, and B is the binary relation defined by xBy if $x \geq y$ for all $x, y \in \mathbb{R}$. B is reflexive, antisymmetric, transitive, negatively transitive, complete, and weakly connected.

4. $X = \mathbb{R}$, and B is the binary relation defined by xBy if $| x - y | > 1$ for all $x, y \in \mathbb{R}$. B is irreflexive, and symmetric.

5. $X = \mathbb{R}$, and B is the binary relation defined by xBy if $x - y$ is an integer multiple of 2 for all $x, y \in \mathbb{R}$. B is reflexive, symmetric, and transitive.

Agenda

- Binary Relations
- **Strict Preference**
- Weak Preference and Indifference

Preferences as Paired Comparisons

- Suppose some set of items X.

- The agent is asked to express his preferences among these items by making paired comparisons of the form

  "I strictly prefer x to y"

  for $x, y \in X$, which is written as

  $$x \succ y.$$

- Formally, strict preference is a binary relation on the set X, i.e.

  $$\succ \subseteq X \times X.$$

Preferences

Definition 1

The strict preference relation on some set X is a binary relation $\succ \subseteq X \times X$ such that $\succ$ is asymmetric and negative transitive.

Some Properties of Strict Preference

Proposition 1

Let X be some set and $\succ \subseteq X \times X$ a strict preference relation.

- **(i)** $\succ$ is irreflexive.
- **(ii)** $\succ$ is transitive.
- **(iii)** $\succ$ is acyclic.

Proof of Proposition 1 (i)

Observe that asymmetry directly implies irreflexivity.

Proof of Proposition 1 (ii)

- Let $x, y, z \in X$ such that $x \succ y$ and $y \succ z$.
- By asymmetry, $y \not\succ x$ and $z \not\succ y$.
- By contraposition of negative transitivity applied to $y \succ z$, it follows that
$$y \succ x \text{ or } x \succ z.$$
- Since $y \not\succ x$, it must be the case that $x \succ z$.
- Therefore, $\succ$ is transitive.

Proof of Proposition 1 (iii)

- Let $x_1, \ldots, x_n \in X$ such that $x_1 \succ x_2, x_2 \succ x_3, \ldots, x_{n-1} \succ x_n$.
- By $(n-2)$-times using transitivity via Proposition 1 (ii), it follows that $x_1 \succ x_n$.
- By asymmetry, $x_n \not\succ x_1$ obtains.
- Therefore, $\succ$ is acyclic.

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Two Further Preference Relations

- Given the strict preference relation $\succ$, two further preference relations can be defined.
- The binary relation
$$\succsim \subseteq X \times X$$
is called weak preference relation and defined as follows:
$$x \succsim y, \text{ if and only if } , y \not\succ x$$
for all $x, y \in X$.
- The binary relation
$$\sim \subseteq X \times X$$
is called indifference relation and defined as follows:
$$x \sim y, \text{ if and only if, } x \not\succ y \text{ and } y \not\succ x$$
for all $x, y \in X$.

Interpretation

- Weak preference

  $$\succsim$$

  expresses the absence of strict preference in one direction.

- Indifference

  $$\sim$$

  expresses the absence of strict preference in either direction.

Some Properties of Weak Preference

Proposition 2

Let X be some set and $\succsim \subseteq X \times X$ a weak preference relation.

(i) $\succsim$ is complete.

(ii) $\succsim$ is transitive.

Proof of Proposition 2

(i) Let $x, y \in X$.

- It is the case that $x \succ y$ (Case 1) or $x \not\succ y$ (Case 2).
- Case 1: by asymmetry, $y \not\succ x$, and thus, by definition, $x \succsim y$.
- Case 2: by definition, $y \succsim x$.
- Consequently, $x \succsim y$ or $y \succsim x$ obtains.
- Therefore, $\succsim$ is complete.

(ii) Let $x, y, z \in X$ such that $x \succsim y$ and $y \succsim z$.

- By definition, $y \not\succ x$ and $z \not\succ y$.
- By negative transitivity, $z \not\succ x$.
- By definition, $x \succsim z$ follows.
- Therefore, $\succsim$ is transitive.

Some Properties of Indifference

Proposition 3

Let X be some set and $\sim \subseteq X \times X$ an indifference relation.

(i) $\sim$ is reflexive.

(ii) $\sim$ is symmetric.

(iii) $\sim$ is transitive.

Proof of Proposition 3

(i) Let $x \in X$.

- By irreflexivity via Proposition 1 (i), $x \not\succ x$.
- By definition, $x \sim x$ follows.
- Therefore, $\sim$ is reflexive.

(ii) Let $x, y \in X$ such that $x \sim y$.

- By definition, $x \not\succ y$ and $y \not\succ x$.
- Of course it directly also holds that $y \not\succ x$ and $x \not\succ y$.
- By definition, $y \sim x$ follows.
- Therefore, $\sim$ is symmetric.

(iii) Let $x, y, z \in X$ such that $x \sim y$ and $y \sim z$.

- By definition, $x \not\succ y$ and $y \not\succ x$, as well as, $y \not\succ z$ and $z \not\succ y$.
- By negative transitivity, $x \not\succ z$ and $z \not\succ x$.
- By definition, $x \sim z$ follows.
- Therefore, $\sim$ is transitive.

Some "Interactive" Properties

Proposition 4

Let X be some set and $x, y \in X$.

(i) Exactly one of the following three relations holds:

- **(a)** $x \succ y$,
- **(b)** $y \succ x$,
- **(c)** $x \sim y$.

(ii) $x \succsim y$, if and only if, $x \succ y$ or $x \sim y$.

(iii) $x \succsim y$ and $y \succsim x$, if and only if, $x \sim y$.

Proof of Proposition 4 (i)

- It is the case that $x \succ y$ (Case 1) or $x \not\succ y$ (Case 2).
- Case 1: **(a)** obtains.
- Case 2: Either $y \succ x$ (Case 2.1) or $y \not\succ x$ (Case 2.1) holds.
- Case 2.1: **(b)** obtains.
- Case 2.2: Since $x \not\succ y$ and $y \not\succ x$, it follows by definition that $x \sim y$ and thus **(c)** obtains.

- If **(a)** holds, then by asymmetry and definition it follows that neither **(b)** nor **(c)** can hold.
- If **(b)** holds, then by asymmetry and definition it follows that neither **(a)** nor **(c)** can hold.
- If **(c)** holds, then by definition it follows that neither **(a)** nor **(b)** can hold.
- Therefore, exactly one of **(a)**, **(b)**, and **(c)** obtains.

Proof of Proposition 4 (ii)

"⇒"-direction:

- Suppose that $x \succsim y$.
- By definition, $y \not\succ x$.
- It is then possible that $x \succ y$ or $x \not\succ y$.
- If $x \not\succ y$, then it follows by definition that $x \sim y$.
- Therefore, $x \succ y$ or $x \sim y$.

"⇐"-direction:

- Suppose that $x \succ y$ or $x \sim y$.
- If $x \succ y$, then by asymmetry $y \not\succ x$ and by definition $x \succsim y$ obtains.
- If $x \sim y$, then by definition $y \not\succ x$ and by definition $x \succsim y$ obtains.
- Therefore, $x \succsim y$.

Proof of Proposition 4 (iii)

"$\Rightarrow$"-direction:

- Suppose that $x \succsim y$ and $y \succsim x$.
- By definition $y \not\succ x$ and $x \not\succ y$ then hold.
- By definition, $x \sim y$ obtains.

"$\Leftarrow$"-direction:

- Suppose that $x \sim y$.
- By definition $x \not\succ y$ and $y \not\succ x$ then hold.
- By definition it follows that $y \succsim x$ and $x \succsim y$.
- Therefore, $x \succsim y$ and $y \succsim x$.

Basic and Derived Concepts

- The strict preference relation served as the basis, departing from which weak preference and indifference were defined.
- It would also be possible to ask the agent to express weak preferences about the elements in the choice set X, and to then derive the respective two other concepts of preference.
- In fact, it is also common that the weak preference relation $\succsim$ is taken as the primitive.
- Both approaches – fortunately – lead to the same mathematical results.

Weak Preference as the Primitive Notion

Definition 2

The weak preference relation on some set X is a binary relation $\succsim \subseteq X \times X$ such that $\succsim$ is a weak order (i.e. $\succsim$ is complete and transitive).

Proposition 5

Let X be some some set and $\succsim \subseteq X \times X$ a weak preference relation. Consider a binary relation $\succ \subseteq X \times X$ such that

$$x \succ y, \text{ if and only if }, y \not\succsim x$$

for all $x, y \in X$. Then, $\succ$ is a strict preference relation.

Proof of Proposition 5

- *It has to be shown that $\succ$ is asymmetric and negative transitive.*

- **First of all**, let $x, y \in X$ such that $x \succ y$.
- Thus, $y \not\succsim x$ by the definition of $\succ$, and by completeness $x \succsim y$ obtains.
- Due to the way $\succ$ is defined, it consequently follows that $y \succ x$ cannot hold (as this would require $x \not\succsim y$).
- Therefore, $y \not\succ x$ and $\succ$ is asymmetric.

- **Now**, consider $x, y, z \in X$ such that $x \not\succ y$ and $y \not\succ z$.
- By the definition of $\succ$, it is thus the case that $y \succsim x$ as well as $z \succsim y$.
- By transitivity it follows that $z \succsim x$.
- Due to the way $\succ$ is defined, it can thus not be the case that $x \succ z$ (as this would require $z \not\succsim x$).
- Therefore, $x \not\succ z$ and $\succ$ is negative transitive.

The Same Theory Based on Different Primitives

- Due to Propositions 2 & 5, it does not matter whether the basis is a strict preference relation

  $\succ$ that is asymmetric & negatively transitive,

  or a weak preference relation

  $\succsim$ that is complete & transitive.

- The ensueing **theory of preferences** will be the same!