# ECON813 Game Theory
# Part A: Interactive Reasoning and Choice
## Topic 1 Rationality

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# Welcome to the Course

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- Office hours: Thursdays at ULMS-CR2, 3.30pm-5pm
- Questions or Comments always welcome!

# Program

- ECON813 Game Theory **Part A**
  - Weeks 1–5 run by CW Bach
  - Topic 1 Rationality (T1)
  - Topic 2: Common Belief in Rationality (T2)
  - Topic 3: Correct Beliefs (T3)

- ECON813 Game Theory **Part B**
  - Weeks 7–11 run by M Lombardi
  - Topics to be announced

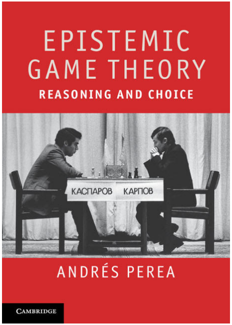
# Organization of Part A (Weeks 1–5)

- Lectures
  - Four ≈90min Lectures on Campus: Thursdays, 9am-11am, BROD-106 in weeks 1, 2, 3, and 4
  - Four accompanying Video Podcasts streamable on Canvas
- Seminars
  - Two ≈50min Seminars on Campus: Thursdays, 1pm-2pm, ULMS-SR3 in weeks 3 and 4
  - Please attempt the questions by yourself first!
- Required Background Reading

# Assessment

- MID-TERM in week 5:
  - 60min test (on campus; closed-book)
  - Topics covered: all of Part A
  - worth 20% of the final grade
- EXAM in the January examination period:
  - 120min exam (on campus; closed-book)
  - Topics covered: all of Part A and all of Part B
  - worth 80% of the final grade

# The Book: Perea (2012)



# Required Background Reading in Perea (2012)

- **Chapter 1**: Introduction
- **Chapter 2**: Belief in the Opponents' Rationality
- **Chapter 3**: Common Belief in Rationality
- **Chapter 4**: Simple Belief Hierarchies

# Two Approaches to Game Theory

- In interactive situations ("games") an agent must make a decision, while knowing that the outcome will not only depend on his choice, but also on the choices of other agents.
- **Fundamental question:** What choices are plausible & why?
- In **classical game theory** a unique answer is sought by refining the solution concept of NASH EQUILIBRIUM.
  - *"towards a single universal solution concept across agents and interactive situations"*
- The more recent discipline of **epistemic game theory** focusses on REASONING and admits different possible answers.
  - *"endorsing the heterogenity/diversity of agents and interactive situations"*
- Characterization results link the two approaches to game theory.

# Rationality as a Point of Departure

- Intuitively, in a game an agent makes a choice that he thinks will yield the best outcome to him.
- It is thus crucial what an agent believes his opponents to do.
- In **epistemic game theory** indeed beliefs become the central objects and some intuitive notions can be defined with them.
- A choice is called optimal for an agent, if it yields the best outcome given *his* belief about his opponents' choices.
- A choice is then said to be **rational**, if it is optimal for *some* belief about his opponents' choices.
- **Rationality** typically serves as the primitive, based on which various reasoning concepts are constructed.

# Example: Going to a Party

**Story:**

- *Alice* and *Bob* are going together to a party tonight.
- *Alice* asks herself what colour she should wear.
- *Alice* prefers *blue* to *green*, *green* to *red*, and *red* to *yellow*.
- However, *Alice* dislikes most to wear the same colour as *Bob*.
- Let *Alice*'s utilities be given as follows:
  - *blue*: $4$
  - *green*: $3$
  - *red*: $2$
  - *yellow*: $1$
  - same colour as *Bob*: $0$
- **Question:** Which colours can *Alice* **rationally** choose for tonight's party?

# Example: Going to a Party

- *Blue* is optimal for *Alice*, if she believes *Bob* to pick any other colour than *blue*.
- *Green* is optimal for *Alice*, if she believes *Bob* to pick *blue*.
- *Red* is optimal for *Alice*, if she believes that with probability $0.6$ *Bob* chooses *blue* and with probability $0.4$ *Bob* chooses *green*.
  - Given this belief *Alice* gets $1.6$ from *blue* and $1.8$ from *green* and $1$ from *yellow*
- The colours *blue*, *green*, and *red* are therefore **rational** for *Alice*.

# Example: Going to a Party

- What about the colour *yellow*?
- To see that there is actually no belief such that *yellow* is optimal for *Alice* distinguish two exhaustive cases.
- **Case 1:** Suppose *Alice*'s belief assigns probability of less than $0.5$ to *Bob* choosing *blue*. Then, *Alice* expects utility of at least $2$ from *blue*, hence *yellow* is not optimal.
- **Case 2:** Suppose *Alice*'s belief assigns probability of at least $0.5$ to *Bob* choosing *blue*. Then, *Alice* expects utility of at least $1.5$ from *green*, hence *yellow* is not optimal.
- Therefore, *yellow* is **irrational** for *Alice*.

# Outline

- Rationality
- Strict Dominance
- Pearce's Lemma

# RATIONALITY

# Games

## Definition 1

A *static game* is a tuple

$$\Gamma = \big(I, (C_i, U_i)_{i \in I}\big),$$

where

- $I$ denotes the finite set of *players*,
- $C_i$ denotes the finite set of *choices* of player $i$,
- $U_i : \times_{j \in I} C_j \to \mathbb{R}$ denotes the *utility function* of player $i$.

# Beliefs

**Definition 2**

Let $S$ be some space of uncertainty. A *belief*

$$p : S \to [0; 1]$$

is a probability measure on $S$.

# Conjectures

**Definition 3**

Let $\Gamma$ be a static game, and $i$ be a player. A *conjecture* for player $i$ is a belief

$$\beta_i : C_{-i} \rightarrow [0;1]$$

about his opponents' choices, where $C_{-i} := \times_{j \in I \setminus \{i\}} C_j$.

# Expected utility

## Definition 4

Let $\Gamma$ be a static game, and $i$ be a player with utility function $U_i$. Suppose that player $i$ entertains conjecture $\beta_i$ and chooses $c_i$. The *expected utility* for player $i$ is

$$u_i(c_i, \beta_i) := \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot U_i(c_i, c_{-i}),$$

where $(c_i, c_{-i}) := (c_1, \dots, c_n) \in \times_{j \in I} C_j$.

# Optimality

**Definition 5**

Let $\Gamma$ be a static game, and $i$ be a player with utility function $U_i$. Suppose that player $i$ entertains conjecture $\beta_i$. A choice $c_i$ for player $i$ is *optimal* given conjecture $\beta_i$, if

$$u_i(c_i, \beta_i) \geq u_i(c_i', \beta_i)$$

holds for all choices $c_i' \in C_i$ of player $i$.

# Rationality

**Definition 6**

Let $\Gamma$ be a static game, and $i$ be a player with utility function $U_i$. A choice $c_i$ for player $i$ is *rational*, if there exists a conjecture $\beta_i$ such that $c_i$ is optimal.

# Illustration

| | | Bob: $L$ | Bob: $R$ |
|---|---|---|---|
| Alice | $U$ | $10, 5$ | $0, 3$ |
| | $M$ | $0, 2$ | $10, 2$ |
| | $D$ | $7, -3$ | $7, 1$ |

All three choices for *Alice* are **rational**.

- *U* is optimal for *Alice*, if she believes *Bob* to choose *L*.
- *M* is optimal for *Alice*, if she believes *Bob* to choose *R*.
- *D* is optimal for *Alice*, if she believes with probability $0.5$ *Bob* to choose *L* and with probability $0.5$ *Bob* to choose *R*.

# Illustration

| | | | *Bob* | |
|---|---|---|---|---|
| | | | $c$ | $d$ |
| *Alice* | | $a$ | $1,2$ | $2,2$ |
| | | $b$ | $0,-3$ | $1,1$ |

Only choice $a$ is **rational** for *Alice*.

- For instance, *a* is optimal for *Alice*, if she believes *Bob* to choose *c*.
- However, *b* is not optimal for *Alice* for any belief about Bob´s choices: both against *c* as well as against *d* – and thus also against all convex combinations of *c* and *d* – choice *a* is better.

# STRICT DOMINANCE

# Randomizing

## Definition 7

Let $\Gamma$ be a static game, and $i$ be a player. A *mixed choice* for player $i$ is a probability measure

$$r_i : C_i \to [0; 1]$$

over the set $C_i$ of player $i$'s choices

**Remark:**

- It seems unnatural that people randomize when taking serious decisions.
- In **epistemic game theory** it is typically assumed that players make definite decisions also called pure choices – and so do we.
- However, mixed choices are still used as technical tools for identifying the rational (pure) choices in games.

# Utility with randomizing

## Definition 8

Let $\Gamma$ be a static game, and $i$ be a player with utility function $U_i$. Suppose that player $i$ chooses $r_i$, and that his opponents choose according to $c_{-i}$. The *randomizing-utility* for player $i$ is

$$V_i(r_i, c_{-i}) := \sum_{c_i \in C_i} r_i(c_i) \cdot U_i(c_i, c_{-i}).$$

# Expected utility with randomizing

## Definition 9

Let $\Gamma$ be a static game, and $i$ be a player with utility function $U_i$. Suppose that player $i$ entertains conjecture $\beta_i$ and chooses $r_i$. The *expected randomizing-utility* for player $i$ given conjecture $\beta_i$ is

$$v_i(r_i, \beta_i) := \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot V_i(r_i, c_{-i})$$

$$= \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot \Big( \sum_{c_i \in C_i} r_i(c_i) \cdot U_i(c_i, c_{-i}) \Big).$$

# Conceptual Interlude: Randomizing is not Necessary

- **Indifference Principle:** if a mixed choice is optimal for some conjecture, then the expected utilities of all pure choices in its support are identical.
- **Intuition:** if the support contains two pure choices with distinct expected utilites, then the player could improve by reassigning weight from the *"weaker"* pure choice to the *"stronger"* one.
- The **Indifference Principle** implies that, if a mixed choice is optimal for some conjecture, then its expected randomizing-utility equals the expected utility of any pure choice in its support.
- In this sense, a player cannot gain anything from randomizing.
- Phrased differently, picking a mixed choice can never be superior to all pure choices.

# Indifference Principle

**Theorem 10 (Indifference Principle)**

*Let $\Gamma$ be a static game, $i$ be a player, $\beta_i$ be a conjecture of player $i$, and $r_i$ be a mixed choice for player $i$ that is optimal. Then,*

$$u_i(c_i, \beta_i) = u_i(c_i', \beta_i)$$

*for all $c_i, c_i' \in \text{supp}(r_i)$.*

# Proof

- Towards a contradiction, suppose that there exists $c_i, c_i' \in \text{supp}(r_i)$ such that $u_i(c_i, \beta_i) \neq u_i(c_i', \beta_i)$, and without loss of generality that $u_i(c_i, \beta_i) > u_i(c_i', \beta_i)$.
- Define a mixed strategy $r_i^* : C_i \to [0, 1]$ for player $i$ as follows:

$$r_i^*(c_i'') = \begin{cases} r_i(c_i'') & \text{if } c_i'' \notin \{c_i, c_i'\}, \\ 0 & \text{if } c_i'' = c_i', \\ r_i(c_i) + r_i(c_i') & \text{if } c_i'' = c_i. \end{cases}$$

- Observe that

$$v_i(r_i^*, \beta_i) = \sum_{c_i'' \in C_i} r_i^*(c_i'') \cdot u_i(c_i'', \beta_i)$$

$$= \left( \sum_{c_i'' \in C_i \setminus \{c_i, c_i'\}} r_i(c_i'') \cdot u_i(c_i'', \beta_i) \right) + \Big( r_i(c_i) + r_i(c_i') \Big) \cdot u_i(c_i, \beta_i) + 0 \cdot u_i(c_i', \beta_i)$$

$$> \left( \sum_{c_i'' \in C_i \setminus \{c_i, c_i'\}} r_i(c_i'') \cdot u_i(c_i'', \beta_i) \right) + r_i(c_i) \cdot u_i(c_i, \beta_i) + r_i(c_i') \cdot u_i(c_i', \beta_i)$$

$$= \sum_{c_i'' \in C_i} r_i(c_i'') \cdot u_i(c_i'', \beta_i) = v_i(r_i, \beta_i)$$

which contradicts the optimality of $r_i$.

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# A Consequence of the Indifference Principle

## Corollary 11

*Let $\Gamma$ be a static game, $i$ be a player, $\beta_i$ a conjecture of player $i$, and $r_i$ be a mixed choice for player $i$ that is optimal given conjecture $\beta_i$. Then,*

$$v_i(r_i, \beta_i) = u_i(c_i, \beta)$$

*for all $c_i \in$ supp$(r_i)$.*

# Proof

- By Theorem 10, it follows that $u_i(c_i', \beta_i) = u_i(c_i'', \beta_i)$ for all $c_i', c_i'' \in \text{supp}(r_i)$.

- Hence, there exists $a \in \mathbb{R}$ such that $u_i(c_i, \beta_i) = a$ for all $c_i \in \text{supp}(r_i)$.

- It then follows that

$$v_i(r_i, \beta_i) = \sum_{c_i \in C_i} r_i(c_i) \cdot u_i(c_i, \beta_i) = \left( \sum_{c_i \in \text{supp}(r_i)} r_i(c_i) \cdot u_i(c_i, \beta_i) \right) + 0$$

$$= \sum_{c_i \in \text{supp}(r_i)} r_i(c_i) \cdot a = a \cdot \sum_{c_i \in \text{supp}(r_i)} r_i(c_i) = a \cdot 1 = a.$$

- Therefore, $v_i(r_i, \beta_i) = u_i(c_i, \beta_i)$ for all $c_i \in \text{supp}(r_i)$.

# The Classical Solution Concept of Strict Dominance

## Definition 12

Let $\Gamma$ be a static game, and $i$ be a player. A choice $c_i$ for player $i$ is *strictly dominated*, if there exists some mixed choice $r_i \in \Delta(C_i)$ of player $i$ such that

$$U_i(c_i, c_{-i}) < V_i(r_i, c_{-i})$$

holds for every opponents' choice combination $c_{-i} \in C_{-i}$.

- A special pure case of strict dominance occurs, if $r_i$ only assigns positive probability to a unique pure choice, say $\hat{c}_i$, i.e. $r_i(\hat{c}_i) = 1$.
- Then, it is also said that $c_i$ is strictly dominated by $\hat{c}_i$.

# Example: Going to a Party

- Neither *blue*, nor *green*, nor *red* are strictly dominated for *Alice*:
  - $U_{Alice}(blue, green) \geq U_{Alice}(c_{Alice}, green)$ for all $c_{Alice} \in \{blue, green, red, yellow\}$,
  - $U_{Alice}(green, blue) \geq U_{Alice}(c_{Alice}, blue)$ for all $c_{Alice} \in \{blue, green, red, yellow\}$,
  - $U_{Alice}(red, blue) > U_{Alice}(blue, blue)$, $U_{Alice}(red, green) > U_{Alice}(green, green)$, and $U_{Alice}(red, yellow) > U_{Alice}(yellow, yellow)$, thus no pure choice of *Alice* is better than *red* against all of *Bob*'s ones.
- *yellow* is strictly dominated by $0.5 \cdot blue + 0.5 \cdot green$ for *Alice*, as

$$U_{Alice}(yellow, c_{Bob}) < V_{Alice}(0.5 \cdot blue + 0.5 \cdot green, c_{Bob})$$

for all $c_{Bob} \in \{blue, green, red, yellow\}$.

- Hence, $SD_{Alice} = \{blue, green, red\}$.

# Pearce's Lemma

# A Characterization of Rationality (Pearce, 1986)

**Pearce's Lemma:**
*The rational choices in a static game are exactly those choices that are not strictly dominated.*

# Application

## Four ways to rationality:

1. Identify all rational choices:
   find a conjecture such that the respective choice is optimal.
2. Identify all irrational choices:
   show that the respective choice is not optimal for any conjecture.
3. Identify all choices that are not strictly dominated:
   show that there exists no randomized choice such that for all opponents' choice-combination it is better than the respective choice.
4. Identify all choices that are strictly dominated:
   show that the respective choice fares worse than some mixed choice (or some other pure choice) for all opponents' choice-combinations.

## Note:

- For rational choices it is often easier to find a supporting belief.
- For irrational choices it is often easier to show strict dominance.

# A basic lemma

## Lemma 13

*Let $I$ be some index set, $0 \leq \alpha_i \leq 1$ for all $i \in I$ such that $\sum_{i \in I} \alpha_i = 1$, $x \in \mathbb{R}$, and $y_i \in \mathbb{R}$ for all $i \in I$. If*

$$x < \sum_{i \in I} \alpha_i y_i,$$

*then there exists $i^* \in I$ such that*

$$x < y_{i^*}.$$

**Proof:**

- By contraposition, suppose that $x \geq y_i$ for all $i \in I$.
- Then, $\alpha_i x \geq \alpha_i y_i$ holds for all $i \in I$.
- It directly follows that $1 \cdot x = \sum_{i \in I} \alpha_i x \geq \sum_{i \in I} \alpha_i y_i$.

# Connecting Strict Dominance to Conjectures

## Lemma 14

*If a choice $c_i$ is strictly dominated by $r_i$, then*

$$u_i(c_i, \beta_i) < v_i(r_i, \beta_i)$$

*for all conjectures $\beta_i \in \Delta(C_{-i})$.*

**Proof:**

- By definition of strict dominance, $U_i(c_i, c_{-i}) < V_i(r_i, c_{-i})$ holds for all $c_{-i} \in C_{-i}$.
- Let $\beta_i \in \Delta(C_{-i})$ be some conjecture of player $i$.
- Then,

  $$\beta_i(c'_{-i}) \cdot U_i(c_i, c'_{-i}) < \beta_i(c'_{-i}) \cdot V_i(r_i, c'_{-i}) \text{ for all } c'_{-i} \in \text{supp}(\beta_i),$$

  and

  $$\beta_i(c'_{-i}) \cdot U_i(c_i, c'_{-i}) = 0 = \beta_i(c'_{-i}) \cdot V_i(r_i, c'_{-i}) \text{ for all } c'_{-i} \notin \text{supp}(\beta_i).$$

- Note that $\{c'_{-i} \in C_{-i} : c'_{-i} \in \text{supp}(\beta_i)\} \cup \{c'_{-i} \in C_{-i} : c'_{-i} \notin \text{supp}(\beta_i)\} = C_{-i}$.
- Hence, $u_i(c_i, \beta_i) = \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot U_i(c_i, c_{-i}) < \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot V_i(r_i, c_{-i}) = v_i(r_i, \beta_i)$.

# Pearce's Lemma

**Theorem 15 (Pearce's Lemma)**

*Let $\Gamma$ be a static game, $i$ be a player, and $c_i$ be a choice of player $i$.*

*$c_i$ is rational, if and only if, $c_i$ is not strictly dominated.*

# Epistemic Characterizations of Solution Concepts

- EPISTEMIC CHARACTERIZATIONS of (classical) of solution concepts

$$EPCO \Leftrightarrow SC$$

  have two directions:

- **Epistemic Foundation:** if agents satisfy certain epistemic conditions, then they play in line with the corresponding solution concept.

- **Existence:** if agents play according to some solution concept, then their behavior can be supported by the corresponding epistemic conditions.

# Proof for: *Only If* Direction (Epistemic Foundation) *"Rational Implies Not Strictly Dominated"*

- Let $c_i$ be strictly dominated by $r_i$.

- Lemma 14 then implies that
$$u_i(c_i, \beta_i) < v_i(r_i, \beta_i)$$
holds for all conjectures $\beta_i \in \Delta(C_{-i})$.

- Observe that by associativity, commutativity, and distributivity it follows that
$$v_i(r_i, \beta_i) = \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot \Big( \sum_{c_i' \in C_i} r_i(c_i') \cdot U_i(c_i', c_{-i}) \Big)$$
$$= \sum_{c_i' \in C_i} r_i(c_i') \cdot \Big( \sum_{c_{-i} \in C_{-i}} \beta_i(c_{-i}) \cdot U_i(c_i', c_{-i}) \Big)$$
$$= \sum_{c_i' \in C_i} r_i(c_i') \cdot u_i(c_i', \beta_i).$$

- Hence,
$$u_i(c_i, \beta_i) < \sum_{c_i' \in C_i} r_i(c_i') \cdot u_i(c_i', \beta_i)$$
holds for all conjectures $\beta_i \in \Delta(C_{-i})$.

# Proof for: *Only If* Direction (Epistemic Foundation) *"Rational Implies Not Strictly Dominated"*

- Let $\hat{\beta}_i \in \Delta(C_{-i})$ be some conjecture.
- As $0 \leq r_i(c_i') \leq 1$ for all $c_i' \in C_i$, the inequality

$$u_i(c_i, \hat{\beta}_i) < \sum_{c_i' \in C_i} r_i(c_i') \cdot u_i(c_i', \hat{\beta}_i)$$

  implies – by Lemma 13 – that there exists some choice $\hat{c}_i \in C_i$ such that $u_i(c_i, \hat{\beta}_i) < u_i(\hat{c}_i, \hat{\beta}_i)$.

- Therefore, $c_i$ cannot be optimal given conjecture $\hat{\beta}_i$.
- As this conjecture $\hat{\beta}_i$ has been chosen arbitrarily, $c_i$ cannot be optimal for any conjecture and thus is irrational.

# References


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- PEARCE, D. (1984): Rationalizable Strategic Behavior and the Problem of Perfection. *Econometrica* 52, 1029–1050.
- PEREA, A. (2012): *Epistemic Game Theory: Reasoning and Choice*. Cambridge University Press.