ECON813 Game Theory

Part A: Problem Set

Question 1

Suppose you wish to open a supermarket in an area with three little villages: Colmont, Winthagen and Ransdaal. Colmont has 300 *inhabitants, Winthagen has* 200 *inhabitants, and Ransdaal has* 400 *inhabitants. Every inhabitant is a potential customer. There are four possible locations for the supermarket, which we call* a*;* b*;* c *and* d*. Figure 1 provides a map of the area with the scale* $1 : 50.000$*. It shows how the villages and the possible locations are situated. However, there is a competitor who also wishes to open a supermarket in the same area. Once you and your competitor have chosen a location, every inhabitant will always visit the supermarket that is closest to his village. If you happen to choose the same location, you will share the market equally with him.*

(a) Formulate the story as a static game between you and a competitor.

(b) Which locations are rational for you, and which are not? For every rational location, find a belief about the competitor's choice for which this location is optimal. For every irrational location, explain why there can be no belief for which this location is optimal.

(c) Determine those locations you can rationally choose while believing that the competitor chooses rationally as well.

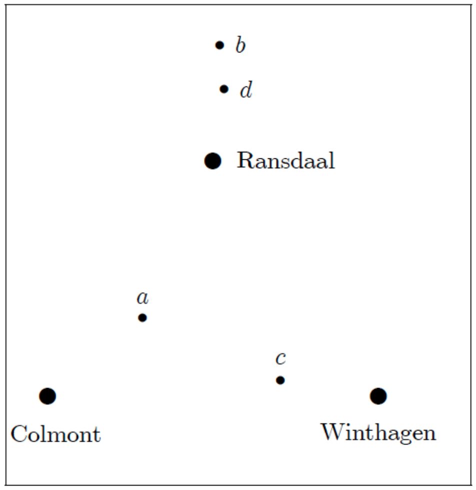

Fig. 1. Map of the area

(d) Use the corresponding algorithm to determine the locations you and your competitor can rationally choose under common belief in rationality.

(e) Construct an epistemic model such that, for each of the locations found in **(d)** there is a type such that:
- the location is optimal for the respective type,
- and the type expresses common belief in rationality.

(f) For each of *your* types in the epistemic model constructed in **(e)**, describe the induced belief hierarchy.

Question 2

This evening Barbara has invited you for dinner. You promised her to bring something to drink, and as usual you either bring some bottles of beer, or a bottle of white wine, or a bottle of red wine. Barbara's favourite dishes are salmon, souvlaki, and nasi goreng. Of course, you want to bring a drink that combines well with the dish that Barbara prepares. Both – you and Barbara – agree that salmon combines reasonably well with beer, combines badly with red wine, but *provides an excellent combination with white wine. You also agree that souvlaki combines reasonably well with beer, combines badly with white wine, but provides an excellent combination with red wine. Finally, you agree that nasi goreng provides a reasonable combination with white wine and red wine. However, you find that nasi goreng combines excellently with beer, whereas according to Barbara nasi goreng only provides a reasonable combination with beer. Suppose that a bad combination gives a utility of* 0*, that a reasonable combination yields a utility of* 1*, and that an excellent combination gives a utility of* 3*.*

(a) Formulate the story as a static game between you and Barbara.

(b) Which drinks are rational for you, and which are not? For every rational drink, find a belief about the Barbara's choice for which this drink is optimal. For every irrational drink, explain why there can be no belief for which this drink is optimal.

(c) Determine those drinks you can rationally choose while believing that Barbara chooses rationally as well.

(d) Use the corresponding algorithm to determine the drinks you and Barbara can rationally choose under common belief in rationality.

(e) Construct an epistemic model such that, for each of the drinks found in **(c)** resp. **(d)** there is a type such that:
- the drink is optimal for the respective type,
- and the type expresses common belief in rationality.

(f) For each of *your* types in the epistemic model constructed in **(e)**, describe the induced belief hierarchy.

Question 3

Suppose you study piano. In two weeks you have an important exam, but you have not been studying too hard for it lately. There are three pieces that you may be asked to play in the exam: a relatively easy piece by Mozart, a somewhat more difficult piece by Chopin and a very tough piece by Rachmaninov. During *the exam, the jury will select two out of these three pieces, but you do not know which. The jury will give you a grade for both pieces and your final grade will be the average of these two grades.*

Since you have only two weeks left, you decide that you will focus on at most two pieces for the remaining time. So, you can dedicate the full two weeks to a single piece or you can dedicate one week to one of the pieces and one week to another. Let x denote the number of weeks you dedicate to a given piece and suppose that your expected grade for the Mozart piece is given by

$$4 + 3 \cdot \sqrt{x},$$

for the Chopin piece is given by

$$4 + 2.5 \cdot x,$$

and for the Rachmaninov piece is given by

$$4 + 1.5 \cdot x^2.$$

The jury wants to see you perform well during the exam, but they prefer listening to Chopin rather than listening to Rachmaninov, and they prefer Rachmaninov to Mozart. More precisely, the jury's utilities for listening to Chopin, Rachmaninov and Mozart are equal to 3, 2 *and* 1*, respectively, and the jury's overall utility is given by the sum of your grade and the utilities they obtain from listening to the two pieces.*

(a) Formulate the story as a static game between you and the jury.

(b) Which practice schedules are rational for you, and which are not? For every rational practice schedule, find a belief about the jury's choice for which it is optimal. For every irrational practice schedule, explain why there can be no belief for which it is optimal.

(c) Which piece selections are rational for the jury, and which are not? For every rational piece selection, find a belief about your choice for which it is optimal. For every irrational piece selection, explain why there can be no belief for which it is optimal.

(d) Determine those practice schedules you can rationally choose while believing that the jury rationally selects pieces as well.

(e) Use the corresponding algorithm to determine the prachtice schedules you can rationally choose under common belief in rationality and the piece selections the jury can rationally choose under common belief in rationality.

(f) Construct an epistemic model such that, for each of the practice schedules found in **(e)** there is a type such that:
- the practice schedule is optimal for the respective type,
- and the type expresses common belief in rationality.

(g) For each of your types in the epistemic model constructed in **(f)**, describe the induced belief hierarchy.

(h) For each of the jury's types in the epistemic model constructed in **(f)**, describe the induced belief hierarchy.

Question 4

Let Γ be a game and $\mathcal{M}^\Gamma$ an epistemic model of it. For every player $i \in I$ consider the set $T_i^* = \{t_i \in T_i : t_i \text{ believes in rationality}\}$ and suppose that $T_i^* \neq \emptyset$ for all $i \in I$. Show that if $\operatorname{supp}\big(\operatorname{marg}_{T_{-i}} b_i(t_i)\big) \subseteq T_{-i}^*$ for all $t_i \in T_i^*$ and for all $i \in I$, then t_i expresses common belief in rationality for all $t_i \in T_i^*$ and for all $i \in I$.

Question 5

This evening there will be a party in the village and you as well as Barbara are invited. The problem is that you don't know whether to go or not, and if you go, which colour to wear. Assume that you only have white and black suits in your wardrobe, and the same is true for Barbara. You and Barbara have conflicting interests when it comes to wearing clothes: You strongly dislike it when Barbara wears the same colour as you, whereas Barbara prefers to wear the same colour as you. Also, you know that you will only have a good time at the party if Barbara goes, and similarly for Barbara. More precisely, your utilities are as follows:

- *Staying at home gives you a utility of* 2.
- *If you go to the party and Barbara shows up wearing a different colour to you, your utility will be* 3.
- *In all other cases, your utility is* 0.

For Barbara the utilities are similar. The only difference is that she gets a utility of 3, if she goes to the party and you show up wearing the same colour as her.

(a) Formulate the story as a game between you and Barbara.

(b) Which choices can each of the players rationally make under common belief in rationality?

(c) Construct an epistemic model such that, for each of the choices identified in **(b)** there is a type such that:
- the choice is optimal for the respective type,
- and the type expresses common belief in rationality.

(d) Which types in your epistemic model believe that the opponent has correct beliefs? Which of these types believe that the opponent believes that these types have correct beliefs too? Which types have a simple belief hierarchy and which do not?

(e) Compute all Nash equilibria in this game. For every Nash equilibrium, write down the simple belief hierarchy it implies for you.

(f) Compute all choices you can rationally make under simple belief hierarchy and up to 2-fold belief in rationality.

Question 6

Your friend Deborah bought a ticket for an air balloon ride over the city of Liverpool. However, she recently discovered that she actually is afraid of height, which does of course pose a problem for such an endeavour. Since you, Barbara, and Chris would all love to see Liverpool from the air, Deborah decides to auction her air ballon ride ticket among the three of you.

The rules of the auction are as follows: the three of you each secretly write down a price on a piece of paper and hand it to Deborah. The price must be 10, 20, 30, 40, or 50 pounds. The person with the highest price shall get the ticket and has to pay the price he wrote down. If two persons write down the same highest price, Deborah will toss a coin to decide who gets the ticket. If all three persons write down the same price, Deborah will throw a dice, and each person *will get the ticket with probability $\frac{1}{3}$. Of course, a person only pays his noted price in case he gets the ticket.*

Suppose that you and Barbara value the air balloon ride ticket at 31 pounds, and that Chris values it at 21 pounds only. The utilities are as follows:

- *If you win the ticket, your utility is your valuation minus the price your pay.*
- *If you do not win the ticket, your utility is zero.*

Similarly for Barbara and Chris.

(a) Which prices are rational for you, and which are not? For every rational price, find a belief about the opponents's choices for which that price is optimal. For every irrational price, find another price – or randomization over prices – that strictly dominates it.

(b) Which prices can you, Barbara, and Chris rationally make under common belief in rationality?

(c) Construct an epistemic model such that, for each of the choices identified in **(b)** there is a type such that:
- the choice is optimal for the respective type,
- and the type expresses common belief in rationality.

(d) Compute all Nash equilibria in this game.

(e) Compute all choices you can rationally make under simple belief hierarchy and up to 2-fold belief in rationality.

Question 7

Alice, Bob, and Claire must decide where to go on summer holiday. After a long discussion there seem to be two options left, Spain and Iceland, but there is no way to reach an unanimous agreement on either of these two destinations. The three friends therefore agree on the following procedure: Each person will write down a destination on a piece of paper. The holiday destination will be the country chosen by the majority and only the persons who actually voted for it will go. That is, somebody who votes for a country that receives the minority of votes will stay at home. The utilities assigned by the three friends to the three possible outcomes are given in the following table.

	Alice	*Bob*	*Claire*
Spain	2	1	1
Iceland	1	2	0
staying at home	0	0	2

(a) Formulate the story as a game between *Alice*, *Bob*, and *Claire*.

(b) Which choices can each of the players rationally make under common belief in rationality?

(c) Construct an epistemic model such that, for each of the choices identified in **(b)** there is a type such that:
- the choice is optimal for the respective type,
- and the type expresses common belief in rationality.

(d) Show that there exists no Nash equilibrium that consists soley of probability 1 marginal conjectures.

(e) Show that there exists a Nash equilibrium in which *Alice's* marginal conjecture assigns probability 1 to Spain.

(f) Show that the Nash equilibrium found in **(e)** constitutes the unique Nash equilibrium of this game.

(g) Compute all the choices each of the players can rationally make under simple belief hierarchy and up to 2-fold belief in rationality.