ECON322 Game Theory

Part III Interactive Epistemology
Topic 9 Rationality

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Adding Beliefs to Knowledge

- The interactive epistemology can be linked to games.
- Thereby, it becomes possible to formally define rationality in games and to model the reasoning of players.
- The **epistemic program** in game theory characterizes solution concepts in terms of epistemic conditions.
- The meaning of solution concepts in terms of the players' thinking is thus brought to light.
- For simplicity sake only the epistemic operator of knowledge (and not belief) is used in **T8** to formulate an epistemic condition.

Outline

- Epistemic Model
- Rationality
- Common Knowledge of Rationality

EPISTEMIC MODEL

The Thinking of Players in Games

- A strategic-form frame specifies the choices available to the players.
- In a strategic-form game this is complemented by what motivates the players (i.e. their preferences over the possible outcomes).
- However, an important factor in the determination of the players' choices is left out: their thinking about the opponents.
- Interactive Epistemology can serve to add a specification of the players' knoweldge and beliefs to a game model.
- This determines the context in which a particular game is played.

Epistemic Models

Definition 1

Let $\mathcal{G}$ be a game in strategic form. An epistemic model of $\mathcal{G}$ is a tuple $\mathcal{M}^{\mathcal{G}} = \langle \mathcal{E}^*, (\zeta_i)_{i \in I} \rangle$, where

- $\mathcal{E}^*$ is an epistemic structure with beliefs
- $\zeta_i : \Omega \to S_i$ is a $\mathcal{I}_i$-measurable choice function assigning to every state $\omega \in \Omega$ a strategy of player $i \in I$.

- The interpretation of $s_i = \zeta_i(\omega)$ is that, at state ω, player i chooses strategy s_i.
- $\mathcal{I}_i$-measurability of ζ_i means that at every state $\omega \in \Omega$ it is the case that $\zeta_i(\omega') = \zeta_i(\omega)$ for all $\omega' \in \mathcal{I}_i(\omega)$.
- This implies that player i always knows his own choice.

Comments

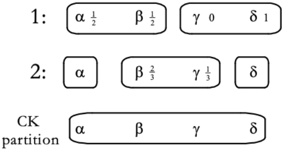
- As a game in Definition 1, a reduced game $\mathcal{G}^*$ can also be used and furnished with an epistemic model.
- To keep things simple, the range of the choice functions run over pure strategies.
- A player's knowledge is encoded by his information partition and his beliefs by the probability distributions at his information sets.
- The choice functions enable the formulation of events about what strategies the players choose.

Illustration

Consider the following reduced game in strategic form

		Player 2		
		L	C	R
Player 1	T	$4,6$	$3,2$	$8,0$
	M	$0,9$	$0,0$	$4,12$
	B	$8,3$	$2,4$	$0,0$

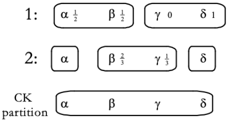
with the following epistemic model of it, where $\Omega = \{\alpha, \beta, \gamma, \delta\}$:



	α	β	γ	δ
ζ_1	B	B	M	M
ζ_2	C	L	L	R

Illustration

		Player 2 L	C	R
Player 1	T	4, 6	3, 2	8, 0
	M	0, 9	0, 0	4, 12
	B	8, 3	2, 4	0, 0



	α	β	γ	δ
ζ_1	B	B	M	M
ζ_2	C	L	L	R

For instance, consider state β, which describes the following situation:

- Player 1 chooses B (since $\zeta_1(\beta) = B$) and Player 2 chooses L (since $\zeta_2(\beta) = L$).
- Player 1 is uncertain (since $\mathcal{I}_1(\beta) = \{\alpha, \beta\}$) as to whether Player 2 chooses C (since $\zeta_2(\alpha) = C$) or L (since $\zeta_2(\beta) = L$); in fact, Player 1 attaches probability $\frac{1}{2}$ to each of these two possibilities.
- Player 2 is uncertain (since $\mathcal{I}_2(\beta) = \{\beta, \gamma\}$) as to whether Player 1 chooses B (since $\zeta_1(\beta) = B$) or M (since $\zeta_1(\gamma) = M$); in fact, Player 1 attaches probability $\frac{2}{3}$ to Player 1 picking B and $\frac{1}{3}$ to him opting for M.

RATIONALITY

The Enriched Framework

- A strategic-form game only offers a partial description of an interactive situation.
- It specifies who the players are, what choices they can make, and how they rank the possible outcomes.
- An epistemic model completes this description.
- It specifies what each player actually does and what he is actually thinking about the opponents.
- The enriched framework with the full description of an interactive situation enables to judge whether a choice is rational or not.

Optimal Behaviour given Beliefs about the Opponents' Behaviour

Intuitively, a player is rational, whenever he picks a choice which is "best" given what he believes about the opponents' choices.

Some Terminology

- Let $\omega \in \Omega$ be some state and $i \in I$ some player.

- $\zeta_{-i}(\omega)$ denotes the profile of strategies at ω chosen by i's opponents i.e.:

$$\zeta_{-i}(\omega) = \left(\zeta_j(\omega)\right)_{j \in I \setminus \{i\}}$$

- $\zeta(\omega)$ denotes the profile of strategies at ω chosen by all players:

$$\zeta(\omega) = \left(\zeta_i(\omega)\right)_{j \in I}$$

- Recall that $P_i^{\mathcal{I}_i(\omega)}$ denotes i's beliefs about events at the state ω.

- By definition, $\mathcal{I}_i$-measurability also holds for i's beliefs, i.e. if $\omega' \in \mathcal{I}_i(\omega)$, then $P_i^{\mathcal{I}_i(\omega')} = P_i^{\mathcal{I}_i(\omega)}$.

Rationality

Definition 2

Let $\mathcal{G}^*$ be a reduced game in strategic form, $\mathcal{M}^{\mathcal{G}^*}$ an epistemic model of it, $i \in I$ some player, and $\omega \in \Omega$ some state. Player i is rational at state ω, whenever

$$\sum_{\omega' \in \mathcal{I}_i(\omega)} p_i^{\mathcal{I}_i(\omega)}(\omega') \cdot U_i\big(\zeta_i(\omega), \zeta_{-i}(\omega')\big) \geq \sum_{\omega' \in \mathcal{I}_i(\omega)} p_i^{\mathcal{I}_i(\omega)}(\omega') \cdot U_i\big(s_i, \zeta_{-i}(\omega')\big)$$

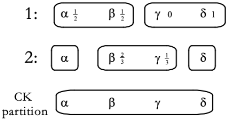
holds for all $s_i \in S_i$. The event of player i being rational is

$$R_i := \{\omega \in \Omega : \text{player } i \text{ is rational at } \omega\}.$$

Illustration

Player 2

Player 1		L	C	R
	T	4, 6	3, 2	8, 0
	M	0, 9	0, 0	4, 12
	B	8, 3	2, 4	0, 0

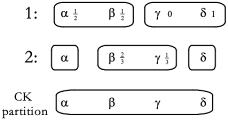


	α	β	γ	δ
ζ_1	B	B	M	M
ζ_2	C	L	L	R

- At state β Player 1 is rational.
- Indeed, given his beliefs and his choice of $\zeta_1(\beta) = B$, Player 1's expected payoff is $\pi_1(B, p_1^{\mathcal{I}_1(\beta)}) = \frac{1}{2} \cdot U_1(B, C) + \frac{1}{2} \cdot U_1(B, L) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 8 = 5$.
- This is maximal since $\pi_1(M, p_1^{\mathcal{I}_1(\beta)}) = \frac{1}{2} \cdot U_1(M, C) + \frac{1}{2} \cdot U_1(M, L) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$ as well as $\pi_1(T, p_1^{\mathcal{I}_1(\beta)}) = \frac{1}{2} \cdot U_1(T, C) + \frac{1}{2} \cdot U_1(T, L) = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 4 = 3.5$.

Illustration

		Player 2		
		L	C	R
Player 1	T	4, 6	3, 2	8, 0
	M	0, 9	0, 0	4, 12
	B	8, 3	2, 4	0, 0



	α	β	γ	δ
ζ_1	B	B	M	M
ζ_2	C	L	L	R

- At state β Player 2 is also rational.
- Indeed, given his beliefs and his choice of $\zeta_2(\beta) = L$, Player 2's expected payoff is $\pi_2(L, p_1^{\mathcal{I}_2(\beta)}) = \frac{2}{3} \cdot U_2(B, L) + \frac{1}{3} \cdot U_2(M, L) = \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 9 = 5$.
- This is maximal since $\pi_2(C, p_2^{\mathcal{I}_2(\beta)}) = \frac{2}{3} \cdot U_2(B, C) + \frac{1}{3} \cdot U_2(M, C) = \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 0 = \frac{8}{3}$ as well as $\pi_2(R, p_2^{\mathcal{I}_2(\beta)}) = \frac{2}{3} \cdot U_2(B, R) + \frac{1}{3} \cdot U_2(M, R) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 12 = 4$.

The Event of Everyone being Rational

Definition 3

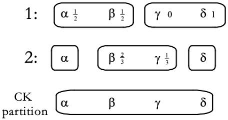
Let $\mathcal{G}^*$ be a reduced game in strategic form and $\mathcal{M}^{\mathcal{G}^*}$ an epistemic model of it. The event

$$R := \cap_{i \in I} R_i$$

is called rationality.

Illustration

		Player 2 L	Player 2 C	Player 2 R
Player 1	T	4, 6	3, 2	8, 0
Player 1	M	0, 9	0, 0	4, 12
Player 1	B	8, 3	2, 4	0, 0



	α	β	γ	δ
ζ_1	B	B	M	M
ζ_2	C	L	L	R

- It can be shown that $R_1 = \{\alpha, \beta\}$ as well as $R_2 = \{\alpha, \beta, \gamma, \delta\}$.
- Therefore, $R = R_1 \cap R_2 = \{\alpha, \beta\} \cap \{\alpha, \beta, \gamma, \delta\} = \{\alpha, \beta\}$.

Knowledge of own Rationality coincides with own Rationality

Proposition 4

Let $\mathcal{G}^*$ be a reduced game in strategic form, $\mathcal{M}^{\mathcal{G}^*}$ an epistemic model of it, and $i \in I$ some player. Then, $K_i R_i = R_i$.

Proof

- Let $\omega \in \Omega$ be some state such that $\omega \in K_i R_i$.
- By **T7** Proposition 3 (TRUTH), it follows that $\omega \in R_i$.
- Conversely, let $\omega \in \Omega$ be some state such that $\omega \in R_i$.
- Then, i is rational at ω, i.e. for all $s_i \in S_i$ it is the case that

$$\sum_{\omega' \in \mathcal{I}_i(\omega)} p_i^{\mathcal{I}_i(\omega)}(\omega') \cdot U_i\big(\zeta_i(\omega), \zeta_{-i}(\omega')\big) \geq \sum_{\omega' \in \mathcal{I}_i(\omega)} p_i^{\mathcal{I}_i(\omega)}(\omega') \cdot U_i\big(s_i, \zeta_{-i}(\omega')\big).$$

- The $\mathcal{I}_i$-measurability of ζ_i and P_i implies that $\zeta_i(\hat{\omega}) = \zeta_i(\omega)$ and $p_i^{\mathcal{I}_i(\hat{\omega})} = p_i^{\mathcal{I}_i(\hat{\omega})}$ for all $\hat{\omega} \in \mathcal{I}_i(\omega)$.
- Consequently, for all $\hat{\omega} \in \mathcal{I}_i(\omega)$ and for all $s_i \in S_i$ it also holds that

$$\sum_{\omega' \in \mathcal{I}_i(\hat{\omega})} p_i^{\mathcal{I}_i(\hat{\omega})}(\omega') \cdot U_i\big(\zeta_i(\hat{\omega}), \zeta_{-i}(\omega')\big) \geq \sum_{\omega' \in \mathcal{I}_i(\hat{\omega})} p_i^{\mathcal{I}_i(\hat{\omega})}(\omega') \cdot U_i\big(s_i, \zeta_{-i}(\omega')\big).$$

- Therefore, i is rational at every state $\hat{\omega} \in \mathcal{I}_i(\omega)$, thus $\mathcal{I}_i(\omega) \subseteq R_i$ and $\omega \in K_i R_i$ obtains.

Common Knowledge of Rationality

Common Knowledge of Rationality

- Since rationality is an event, the knowledge operator and the common knowledge operator can be applied to it.
- Mutual knowledge of rationality is the event KR and common knowledge of rationality is the event CKR.
- It follows via **T7** Proposition 3 (TRUTH) that $KR \subseteq R$ as well as $CKR \subseteq R$.
- It turns out that common knowledge of rationality characterizes the solution concept of iterated strict dominance.
- Thus, the meaning of *ISD* in terms of reasoning is *CKR*.

Epistemic Foundation

Theorem 5

Let $\mathcal{G}^*$ be a finite reduced game in strategic form, $\mathcal{M}^{\mathcal{G}^*}$ an epistemic model of it, and $\omega \in \Omega$ some state. If $\omega \in CKR$, then $\zeta(\omega) \in ISD$.

Equivalence of Rationality and Strict Dominance

Theorem 6

Let $\mathcal{G}^*$ be a finite reduced game in strategic form, $i \in I$ some player, and $s_i \in S_i$ some strategy of player i. There exists a belief $\rho_i : S_{-i} \to [0,1]$ about i's opponents' strategies such that s_i is optimal given p_i (i.e. $\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) \cdot \rho_i(s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} U_i(s_i', s_{-i}) \cdot \rho_i(s_{-i})$ for all $s_i' \in S_i$), if and only if, $s_i \in SD_i^1$

- Intuitively, Theorem 6 states that a choice being rational is equivalent to it not being strictly dominated.
- This result – also known as PEARCE'S LEMMA – has been established for the 2 player case by Pearce (1984, Lemma 3).
- It has been generalized by Perea (2012, Theorem 2.5.3) with any finite number of players.

Proof of Theorem 5

- By induction on $m \in \mathbb{N}$, it will be shown that for every state $\omega' \in \mathcal{I}_{CK}(\omega)$ and for every player $i \in I$ it is the case that $\zeta_i(\omega') \in SD_i^m$.
- **Induction Basis** $m = 1$:
  - Consider some state $\omega' \in \mathcal{I}_{CK}(\omega)$ and some player $i \in I$.
  - Since $\omega \in CKR$, it holds that $\mathcal{I}_{CK}(\omega) \subseteq R = \cap_{j \in I} R_j \subseteq R_i$ and thus $\omega' \in R_i$.
  - Consequently, $\zeta_i(\omega')$ is optimal given belief $p_i^{\mathcal{I}_i(\omega')}$ and therefore, by Theorem 6, $\zeta_i(\omega') \in SD_i^1$.
- **Induction Basis** $m > 1$:
  - Assume that the inductive hypothesis holds, i.e. for for every state $\omega' \in \mathcal{I}_{CK}(\omega)$ and for every player $i \in I$ it is the case that $\zeta_i(\omega') \in SD_i^k$ for all $k \leq m - 1$.
  - Consider some state $\omega' \in \mathcal{I}_{CK}(\omega)$ and some player $i \in I$.
  - As above, $\omega \in CKR$ implies that $\omega' \in R_i$ and thus $\zeta_i(\omega')$ is optimal given belief $p_i^{\mathcal{I}_i(\omega')}$.
  - Now, by the inductive hypothesis, $\zeta(\omega'') = \left(\zeta_j(\omega'')\right)_{j \in I} \in (SD_j^{m-1})_{j \in I} = SD^{m-1}$ for all $\omega'' \in \mathcal{I}_{CK}(\omega) = \mathcal{I}_{CK}(\omega')$ and hence, since $\mathcal{I}_i(\omega') \subseteq \mathcal{I}_{CK}(\omega')$, the relation $\operatorname{supp}(p_i^{\mathcal{I}_i(\omega')}) \subseteq S_{-i}^{m-1}$ obtains.
  - It then follows, by Theorem 6 applied to the reduced game ${\mathcal{G}^*}_{SD}^{m-1}$, that $\zeta_i(\omega') \in SD_i^{(m-1)+1} = SD_i^m$.
- As $\cap_{m \in \mathbb{N}} \left((SD_i^m)_{i \in I}\right) = \cap_{m \in \mathbb{N}} SD^m = ISD$ as well as $\omega \in \mathcal{I}_{CK}(\omega)$, the desired conclusion $\zeta(\omega) = \left(\zeta_i(\omega)\right)_{i \in I} \in ISD$ ensues.

Existence

Theorem 7

Let $\mathcal{G}^*$ be a finite reduced game in strategic form and $s \in (S_i)_{i \in I}$ a strategy profile. If $s \in ISD$, then there exists an epistemic model $\mathcal{M}^{\mathcal{G}^*}$ with a state $\omega \in \Omega$ such that $\zeta(\omega) = s$ and $\omega \in CKR$.

Proof of Theorem 7

- Construct an epistemic model of $\mathcal{G}^*$ such that
  - $\Omega := ISD$,
  - $\mathcal{I}_i(\omega^s) := \{s' \in \Omega : s'_i = s_i\}$ for every player $i \in I$ and for every state $s \in \Omega$,
  - $\zeta_i(s) = s_i$ for every player $i \in I$,
  - for every player $i \in I$ and for every state $s \in \Omega$ define i's probability distribution $p_i^{\mathcal{I}_i(s)}$ as follows:
    - By definition of ISD, for every $j \in I$, if $s_j \in ISD_j$, then it is not strictly dominated in $\mathcal{G}^{*\infty}_{SD}$.
    - For every player $j \in I$, it follows by Theorem 6 applied to $\mathcal{G}^{*\infty}_{SD}$ that there exists a probability distribution $\rho_j^{s_j}$ over ISD_{-j} such that s_j is optimal given $\rho_j^{s_j}$.
    - Take one such probability distribution $\rho_i^{s_i}$ as i's probability distribution $p_i^{\mathcal{I}_i(s)}$.
- Now, consider some $s \in ISD$.
- Take the corresponding state $s \in \Omega$ in the epistemic model.
- It holds at state s that $\zeta(s) = \big(\zeta_i(s)\big)_{i \in I} = (s_i)_{i \in I}$.
- Moreover, at every state $s' \in \Omega$ it is the case that $\zeta_i(s') = s'_i$ is optimal given $\rho_i^{s'_i} = p_i^{\mathcal{I}_i(s')}$ for every player $i \in I$ and consequently $s' \in \cap_{i \in I} R_i = R$.
- Since $s' \in \cap_{i \in I} R_i = R$ holds for all $s' \in \Omega$, it directly follows that $R = \Omega$.
- Since $\mathcal{I}_{CK}(\omega) \subseteq \Omega$, it is the case that $\omega \in CKR$.

Illustration

Player 2

Player 1		L	C	R
	T	4, 6	3, 2	8, 0
	M	0, 9	0, 0	4, 12
	B	8, 3	2, 4	0, 0

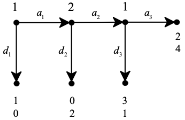
- For Player 1, the strategy M is strictly dominated by any mixed strategy $\begin{pmatrix} T & B \\ p & 1-p \end{pmatrix}$ with $p > \frac{1}{2}$.
- After deletion of M, for Player 2, the strategy R is strictly dominated by either of his other two strategies.
- It follows that $ISD = \{T, B\} \times \{L, C\}$.
- These are the only strategy profiles in this game that are compatible with common knowledge of rationality.
- Indeed, by Theorem 5, at a state in an epistemic model of this game where there is common knowledge of rationality, the players can play only one of these strategy profiles.
- Moreover, by Theorem 7, any of these four strategy profiles can in fact be played in a situation where there is common knowledge of rationality.
- Each of these profiles can be supported by an epistemic model of the game with a state at which the respective profile is actually played and the players reason in line with common knowledge of rationality.

Epistemic Conditions for Nash Equilibrium

- Common knowledge of rationality does not imply Nash Equilibrium.
- Since $PSNE \subseteq ISD$, the solution concept of Nash Equilibrium is compatible with common knowledge of rationality.
- The crucial ingredient in any **epistemic foundation** for Nash Equilibrium is a correct beliefs assumption.
- A correct beliefs assumption entails knowledge of the others' strategies and an independent belief about the others' strategies.
- A correct beliefs assumption plus mutual knowledge of rationality imply Nash Equilibrium.

A Glimpse at Reasoning in Dynamic Games

- A strategy in dynamic games is a complete, contingent plan.
- This raises potential frictions with $\zeta_i(\omega)$ being the actual choice of player i at state ω.
- For example, consider $\zeta_1(\omega) = (d_1, a_3)$ in the following dynamic game:

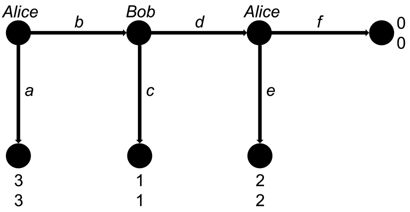


- Only the first part of $\zeta_1(\omega) = (d_1, a_3)$, namely d_1, can be interpreted as 1's actual behaviour at state ω.
- If 1 picks d_1, then he knows he will not make any further choices and the second part, namely a_3, seems meaningless: $\zeta_1(\omega) = (d_1, a_3)$ can thus not be interpreted as *"At state ω, player 1 chooses (d_1, a_3)"*.
- A possibility could be to interpret these counterfactuals strategy ingredients (e.g. a_3 above) as others' beliefs about what the respective player would do if the respective information set were to be reached.

Modelling Hypothetical Reasoning

- In dynamic games players may have to choose several times.
- In between they may learn about opponents' choices or the outcomes of random events.
- Consequently, they may want to update or revise their beliefs.
- A richer notion of an epistemic model is thus needed to capture **hypothetical reasoning** as well as **counterfactual reasoning**.
- More refined reasoning notions are needed that also make explicit the belief revision policy of players.
- For instance, the inherently static notion of *CKR* may not be satisfiable at some situations within a dynamic game.

Illustration



- It is uniquely rational in this game for *Alice* to immediately terminate the game by choosing a at her first information set.
- Consequently, at his information set, *Bob* cannot believe or know rationality and a fortiori not hold *CKR* even if he initially did.
- How would he revise his thinking about *Alice*?
  - Would *Alice* nonetheless be "locally rational" later on?
  - Or would *Alice* also act "locally irrationally" later on?

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition

- Chapter 10: **Rationality**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html