ECON322 Game Theory

Part III Interactive Epistemology
Topic 8 Belief

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Adding Beliefs to Knowledge

- In **T7**, the notion of knowledge has been treated.
- Since knowledge satisfies TRUTH, there is no uncertainty whatsoever in the epistemic attitude of the agent.
- In **T8**, the weaker idea of **belief** is introduced.
- **Beliefs** are modelled by means of probabilities and they always admit the possibility of error.

Outline

- Probability
- Belief
- Belief Change
- Like-Mindedness
- Agreeing to Disagree

PROBABILITY

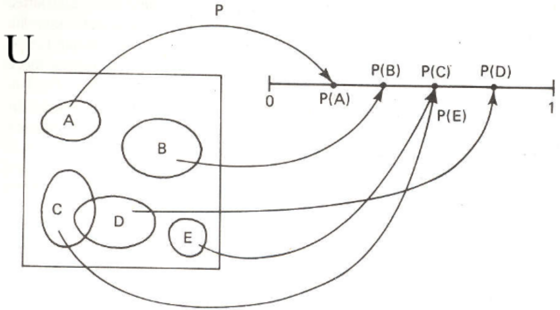
Probability Measures

- The sample space U (also called universal set) contains all objects of interest and the subsets of U are called events.

- A probability measure on U is a function $P : 2^U \rightarrow [0, 1]$ satisfying the following two properties:

  1. $P(U) = 1$,
  2. $P(E \cup F) = P(E) + P(F)$ for all $E, F \in 2^U$ such that $E \cap F = \emptyset$.

Illustration



Properties of Probability Measures

The definition of probability measure implies the following properties:

- $P(\neg E) = 1 - P(E)$ for all $E \in 2^U$
  (This stems from $E \cap \neg E = \emptyset$ and $E \cup \neg E = U$)

- $P(\emptyset) = 0$
  (This stems from the previous property and $\emptyset = \neg U$)

- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ for all $E, F \in 2^U$
  (Intuitively, $P(E \cap F)$ are subtracted to avoid "double-counting")

- For all $E, F \in 2^U$, if $E \subseteq F$, then $P(E) \leq P(F)$
  (This is obtained from **2** *with E and $F \setminus E$)*

- Let $m \geq 2$ be a natural number. If $E_1, \ldots, E_m \in 2^U$ are mutually disjoint events, then $P(E_1 \cup \ldots \cup E_m) = P(E_1) + \ldots + P(E_m)$
  (This is obtained from **2** *via the principle of induction)*

Probability Distributions

- If the sample space U is finite, then a probability distribution on U is a function $p : U \to [0, 1]$ such that $\sum_{z \in U} p(z) = 1$.

- Given a probability distribution p on U, a probability measure P on 2^U can be defined as follows:

$$P(E) = \sum_{z \in E} p(z) \qquad \text{for all } E \in 2^U$$

- Conversely, given a probability measure P on 2^U, a probability distribution p on U can be defined as follows:

$$p(z) = P(\{z\}) \qquad \text{for all } z \in U$$

- In this sense, probability distribution and probability measure are equivalent notions.

Conditional Probability

- Let $A, B \subseteq U$ be two events and P a probability measure on U such that $P(B) > 0$.

- The conditional probability of A given B, denoted by $P(A \mid B)$, is defined as follows:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

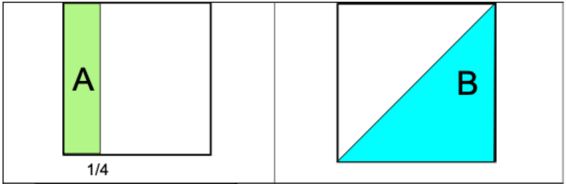
- For example, if $P(A \cap B) = 0.2$ and $P(B) = 0.6$, then

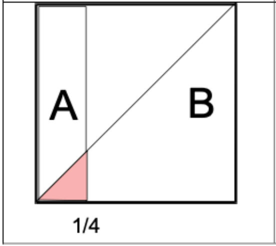
$$P(A \mid B) = \frac{0.2}{0.6} = \frac{1}{3}$$

Geometric Interpretation of Conditional Probability

- A way to visualize conditional probability is to think of U as a geometric shape of area 1.
- Eg.: a square with each side equal to 1 unit of measurement.
- For a subset A of the unit square, $P(A)$ is the area of A.
- If B is another subset of the square, then $A \cap B$ is that part of U that lies in both A and B.
- $P(A \mid B)$ is the area of $A \cap B$ relative to the area of B.
- That is, the area $A \cap B$ as a fraction of the area of B.

Illustration





The shaded area, representing $A \cap B$, is $\frac{1}{2}$ of a small square with sides of length $\frac{1}{4}$ so that $P(A \cap B) = \frac{1}{2} \times \left(\frac{1}{4} \times \frac{1}{4}\right) = \frac{1}{32}$

$P(B) = \frac{1}{2}$ and thus

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{32}}{\frac{1}{2}} = \frac{1}{16}$$

Bayes' Rule (Version 1)

- Let $E, F \in 2^U$ be events such that $P(E) > 0$ and $P(F) > 0$.
- Then, $P(E \mid F) = \frac{P(E \cap F)}{P(F)}$ as well as $P(F \mid E) = \frac{P(F \cap E)}{P(E)}$
- Thus, $P(E \cap F) = P(F \mid E) \cdot P(E)$, since $E \cap F = F \cap E$.
- Consequently, the following property ensues:

Bayes' Rule (Version 1)

Let $E, F \in 2^U$ be events such that $P(E) > 0$ and $P(F) > 0$. Then,

$$P(E \mid F) = \frac{P(F \mid E) \cdot P(E)}{P(F)}$$

Illustration

- A doctor examines a patient who complains about lower back pain and the doctor knows that 25% of the persons in the same age group as the patient suffer from lower back pain.
- There are various causes of lower back pain: one of them is chronic inflammation of the kidneys, which affects 4% in the considered age group.
- Among those who suffer from chronic inflammation of the kidneys, 85% complain of lower back pain.
- What is the conditional probability then that the patient has a chronic inflammation of the kidneys?
- Let I denote inflammation of kidneys and L denote lower back pain.
- The doctor's information can be summarized as follows: $P(I) = \frac{4}{100}$, $P(L) = \frac{25}{100}$, and $P(L \mid I) = \frac{85}{100}$.
- Then, by Bayes' Rule (Version 1):

$$P(I \mid L) = \frac{P(L \mid I) \cdot P(I)}{P(L)} = \frac{\frac{85}{100} \cdot \frac{4}{100}}{\frac{25}{100}} = 0.136 = 13.6\%$$

Bayes' Rule (Version 2)

- Let $E, F \in 2^U$ be events such that $P(E) > 0$ and $P(F) > 0$.
- Then, by Bayes' Rule (Version 1), $P(E \mid F) = \frac{P(F|E) \cdot P(E)}{P(F)}$.
- Since $F = (F \cap E) \cup (F \cap \neg E)$ and $(F \cap E) \cap (F \cap \neg E) = \emptyset$, it follows that

$$P(F) = P(F \cap E) + P(F \cap \neg E)$$

- By the definition of conditional probability, $P(F \cap E) = P(F \mid E) \cdot P(E)$ and $P(F \cap \neg E) = P(F \mid \neg E) \cdot P(\neg E)$ hold.
- Consequently, the following property ensues:

Bayes' Rule (Version 2)

Let $E, F \in 2^U$ be events such that $P(E) > 0$ and $P(F) > 0$. Then,

$$P(E \mid F) = \frac{P(F \mid E) \cdot P(E)}{P(F \mid E) \cdot P(E) + P(F \mid \neg E) \cdot P(\neg E)}$$

Illustration

- Enrolment in a Game Theory module is as follows: 60% economics majors (E) and 40% other majors $(\neg E)$.
- According to past data, 80% of the economics majors passed and 65% of the other majors passed.
- A student utters proudly that he has passed (denoted by A): what is the conditional probability that he is an economics major?
- With Bayes' Rule (Version 2) it follows that:

$$P(E \mid A) = \frac{P(A \mid E) \cdot P(E)}{P(A \mid E) \cdot P(E) + P(A \mid \neg E) \cdot P(\neg E)}$$

$$= \frac{\frac{80}{100} \cdot \frac{60}{100}}{\frac{80}{100} \cdot \frac{60}{100} + \frac{65}{100} \cdot \frac{40}{100}} = \frac{24}{37} = 64.86\%$$

Bayes' Rule (Version 3)

- Bayes' Rule (Version 2) can be generalized.
- Let $E_1, \ldots, E_n \in 2^U$ be events such that they form a partition of the sample space U and consider some event $F \in 2^U$.
- It follows that
$$P(F) = P(F \cap E_1) + P(F \cap E_2) + \ldots + P(F \cap E_n)$$
and thus by the definition of conditional probability,
$$P(F) = P(F \mid E_1) \cdot P(E_1) + P(F \mid E_2) \cdot P(E_2) + \ldots + P(F \mid E_n) \cdot P(E_n).$$
- Consequently, the following property ensues:

Bayes' Rule (Version 3)

Let $E_1, \ldots, E_n, F \in 2^U$ be events such that $P(E_i) > 0$ for all $i\{1, \ldots, n\}$ and $E_1, \ldots, E_n$ form a partition of U, as well as $P(F) > 0$. Then,

$$P(E_i \mid F) = \frac{P(F \mid E_i) \cdot P(E_i)}{P(F \mid E_1) \cdot P(E_1) + P(F \mid E_2) \cdot P(E_2) + \ldots + P(F \mid E_n) \cdot P(E_n)}$$

holds for all $i \in \{1, \ldots, n\}$

Illustration

- Enrolment in a Game Theory module is as follows: 40% economics majors (E), 35% statistics majors (S), and 25% mathematics majors (M).

- With A denoting the event "pass the module", the following past data is available:
  - $P(A \mid E) = 60\%$
  - $P(A \mid S) = 50\%$
  - $P(A \mid M) = 75\%$

- A student utters proudly that he has passed: what is the conditional probability that he is an economics major?

- With Bayes' Rule (Version 3) it follows that:

$$P(E \mid A) = \frac{P(A \mid E) \cdot P(E)}{P(A \mid E) \cdot P(E) + P(A \mid S) \cdot P(S) + P(A \mid M) \cdot P(M)}$$

$$= \frac{\frac{60}{100} \cdot \frac{40}{100}}{\frac{60}{100} \cdot \frac{40}{100} + \frac{50}{100} \cdot \frac{35}{100} + \frac{75}{100} \cdot \frac{25}{100}} = \frac{96}{241} = 39.83\%$$

Belief

Individual Possibility

- What the agent deems possible is what he cannot rule out given his information about the universal set.
- Within the framework of **T7**, individual possibility is captured by an information partition $\mathcal{I}$ of the set of all states Ω.
- More precisely, at a state $\omega \in \Omega$, the agent considers all states in his information set $\mathcal{I}(\omega)$ to be possible.
- Yet, among the possible states, the agents might still deem some more likely than others and even dismiss some as implausible.

Illustration

- Consider a module with only three students: Ann, Bob, and Carla.
- The lecturer tells them that in the last exam one of them got 95 points, another 78, and the third 54.
- A state can be thought of as a triple (a, b, c), where a is Ann's score, b is Bob's score, and c is Carla's score.
- Based on the lecturer's information, Ann must consider all of the following six states possible:
  - $(95, 78, 54)$
  - $(95, 54, 78)$
  - $(78, 95, 54)$
  - $(78, 54, 95)$
  - $(54, 95, 78)$
  - $(54, 78, 95)$
- Suppose, that in all the previous exams Ann and Bob always obtained a higher score than Carla: then, Ann might consider states $(95, 78, 54)$ and $(78, 95, 54)$ much more likely than $(78, 54, 95)$ and $(54, 78, 95)$.
- Moreover, suppose that often Ann also outperformed Bob in the past: then, Ann might also consider states $(95, 78, 54)$ more likely than all the other states.

Beliefs as Probabilities

- Judgements of likelihood are represented by beliefs which are formally defined as probability distributions.
- An information set is equipped with a probability distribution over the set of states, where all the states out of it get probability 0.
- **Beliefs**, **Knowledge**, and **Possibility**:
  - The probability distribution express what the agent **believes**.
  - The information set captures what the agent **knows** and what he deems **possible**.

Illustration

- Based on the lecturer's information, Ann must consider all of the following six states $(95, 78, 54)$, $(95, 54, 78)$, $(78, 95, 54)$, $(78, 54, 95)$, $(54, 95, 78)$, and $(54, 78, 95)$ possible.

- Suppose, that in all the previous exams Ann and Bob always obtained a higher score than Carla: then, Ann might consider states $(95, 78, 54)$ and $(78, 95, 54)$ much more likely than $(78, 54, 95)$ and $(54, 78, 95)$.

- Moreover, suppose that often Ann also outperformed Bob in the past: then, Ann might also consider states $(95, 78, 54)$ more likely than all the other states.

- Ann's beliefs could be described by the following probability distribution:

$$\blacklozenge = \begin{pmatrix} (95,78,54) & (95,54,78) & (78,95,54) & (54,95,78) & (78,54,95) & (54,78,95) \\ \frac{9}{16} & \frac{4}{16} & \frac{2}{16} & \frac{1}{16} & 0 & 0 \end{pmatrix}$$

- According to these beliefs:
  - Ann considers it very likely that she got the highest score.
  - Ann is willing to dismiss the possibility that Carla received the highest score as extremely unlikely.
  - Ann deems it much more likely that she – rather than Bob – received the highest score.

Beliefs about Events

- Recall that propositions of interest are represented by events and let $\omega^* \in \Omega$ be some state.

- The information set $\mathcal{I}(\omega^*)$ is equipped with a probability distribution $p : \Omega \to [0, 1]$ such that $p(\omega) = 0$ for all $\omega \notin \mathcal{I}(\omega^*)$.

- The induced probability measure P on 2^Ω on the full event space is $P : 2^\Omega \to [0, 1]$ such that

$$P(E) = \sum_{\omega \in E} p(\omega)$$

  for all $E \in 2^\Omega$.

- P formally represents the agent's beliefs about events.

- Beliefs represented by probabilities are also called probabilistic beliefs or graded beliefs.

Probabilistic Beliefs, Exclusion, and Certainty

- Let $\alpha \in [0, 1]$. An agent is said to believe an event E with probability α, whenever $P(E) = \alpha$.

- The extreme cases of $\alpha = 0$ and $\alpha = 1$ get special names:
  - An agent excludes an event E, whenever $P(E) = 0$.
  - An agent is certain of an event E, whenever $P(E) = 1$.

Exclusion versus Impossibility and Certainty versus Knowledge

- If an agent excludes an event, then he may still deem it possible.
- E.g.: $\mathcal{I}(\omega_1) = \{\omega_1, \omega_2\}$, $p_{\mathcal{I}(\omega_1)} = \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & 1 \end{pmatrix}$, and $E = \{\omega_1\}$.
- If an agent knows an event, then the event is true, i.e. $KE \subseteq E$ holds for all $E \in 2^\Omega$, due to **T7**, Proposition 3 (TRUTH).
- However, an agent can be certain of an event that is false.
- E.g.: $\mathcal{I}(\omega_1) = \{\omega_1, \omega_2\}$, $p_{\mathcal{I}(\omega_1)} = \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & 1 \end{pmatrix}$, and $E = \{\omega_2\}$.

Illustration

$$\blacklozenge = \begin{pmatrix} (95,78,54) & (95,54,78) & (78,95,54) & (54,95,78) & (78,54,95) & (54,78,95) \\ \frac{9}{16} & \frac{4}{16} & \frac{2}{16} & \frac{1}{16} & 0 & 0 \end{pmatrix}$$

- Let $E = \{(95,78,54),(78,95,54),(54,95,78)\}$ be the event *"Bob's score is higher than Carla's score"*.
- With Ann's beliefs given by $\blacklozenge$, it follows then that:

$$P(E) = p(95,78,54) + p(78,95,54) + p(54,95,78) = \frac{9}{16} + \frac{2}{16} + \frac{1}{16} = 75\%$$

- Let $F = \{(95,78,54),(95,54,78),(78,95,54),(54,95,78)\}$ be the event *"Carla did not receive the highest score"*.
- With Ann's beliefs given by $\blacklozenge$, it follows then that Ann is certain of F (yet does not know F as $(78,54,95),(54,78,95) \notin F$ but $(78,54,95)$ and $(54,78,95)$ are both in her information set):

$$P(F) = p(95,78,54) + p(95,54,78) + p(78,95,54) + p(54,95,78) = \frac{9}{16} + \frac{4}{16} + \frac{2}{16} + \frac{1}{16} = 100\%$$

- Let $G = \{(78,54,95),(54,78,95)\}$ be the event *"Carla received the highest score"*.
- With Ann's beliefs given by $\blacklozenge$, it follows then that Ann excludes G (yet deems G possible as the states $(78,54,95)$ and $(54,78,95)$ are both in her information set):

$$P(G) = p(78,54,95) + p(54,78,95) = 0 + 0 = 0\%$$

BELIEF CHANGE

How to Respond to New Information?

- Consider an agent who holds beliefs about a universal set U embodied by a probability measure $P : 2^U \to [0, 1]$.
- Suppose that the agent receives a piece of information represented by a set $F \in 2^U$.
- Two distinct situations may arise:
  - **Belief Updating**
    - The item of information was not ruled out by the initial beliefs, in the sense that $P(F) > 0$.
    - Information might still be somewhat surprising (small $P(F)$), but it is not completely unexpected.
  - **Belief Revision**
    - The item of information was initially dismissed, in the sense that $P(F) = 0$.
    - The received Information is completely surprising.

Belief Updating via Conditional Probability

- The initial probability measure is conditioned on the received information by means of conditional probability.
- Such a belief modification is called **belief updating** (or Bayesian updating) – it assumes the information to carry positive measure.
- Formally, given information $F \in 2^U$ such that $P(F) > 0$, the changed beliefs are given by P_{new}:
  - reduce the probability of every state in $\neg F$ to zero,
  - set $P_{new}(\{\omega\}) = P(\{\omega\} \mid F)$ for every state $\omega \in F$.

The new Belief

Consequently, for every state $\omega \in U$,

$$P_{new}(\{\omega\}) = P(\{\omega\} \mid F) = \begin{cases} 0 & \text{if } \omega \notin F \\ \frac{P(\{\omega\})}{P(F)} & \text{if } \omega \in F \end{cases}$$

and for every event $E \in 2^U$,

$$P_{new}(E) = \sum_{\omega \in E} P_{new}(\{\omega\}) = \sum_{\omega \in E} P_{new}(\{\omega\} \mid F) = P(E \mid F)$$

Illustration

- Recall the story about the lecturer and the three students Ann, Bob, and Carla.
- The information given by the lecturer could be represented as follows:

$$U = \{(95,78,54),(95,54,78),(78,95,54),(78,54,95),(54,95,78),(54,78,95)\}$$

- Based on this information Ann has formed the following probabilistic beliefs:

$$\blacklozenge = \begin{pmatrix} (95,78,54) & (95,54,78) & (78,95,54) & (54,95,78) & (78,54,95) & (54,78,95) \\ \frac{9}{16} & \frac{4}{16} & \frac{2}{16} & \frac{1}{16} & 0 & 0 \end{pmatrix}$$

- Suppose that the lecturer makes the additional remark "Surprisingly, this time, Ann did not get the highest score": this announcement informs the students that the true state is neither $(95,78,54)$ nor $(95,54,78)$.
- Thus, the new piece of information is the event $F = \{(78,95,54),(78,54,95),(54,95,78),(54,78,95)\}$.
- Conditioning Ann's beliefs on the event F yields the following updated beliefs:

$$\spadesuit = \begin{pmatrix} (95,78,54) & (95,54,78) & (78,95,54) & (54,95,78) & (78,54,95) & (54,78,95) \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix}$$

Belief Revision

- How can beliefs be changed upon receiving completely surprising information in the sense that $P(F) = 0$?
- For example, this is relevant for dynamic games, if a player faces an information set he initially excluded.
- The players needs to form a new belief assigning positive probability to the information set being reached.
- The best known theory of **belief revision** is the so-called AGM THEORY due to Alchourrón, Gärdenfors, and Makinson (1985).
- Only a glimpse into AGM THEORY can be offered in the remainder of this section.

Belief Revision Function

Definition 1

Let U be a universal set and $\mathfrak{E} \subseteq 2^U$ a collection of events such that $U \in \mathfrak{E}$ and $\emptyset \notin \mathfrak{E}$. A belief revision function is a function $f : \mathfrak{E} \to 2^U$ such that:

- $f(E) \subseteq E$ for all $E \in \mathcal{E}$,
- $f(E) \neq \emptyset$ for all $E \in \mathcal{E}$.

Interpretation:

- $f(U)$ represents the initial beliefs: the set of states that the agent initially considers possible.
- The universal set U can be thought of as representing minimum information: all beliefs are possible.
- For every event $E \subseteq \mathfrak{E}$, the set of states $f(E)$ is considered possible by the agent if informed that the true states belongs to E.
- Thus, $f(E)$ captures the agent's revised beliefs after receiving information E.

Arrow's Axiom

- An important condition that can be derived in the AGM Theory is an axiom due to Arrow from a different context (choice theory):

Arrow's Axiom

Let U be a finite universal set, $\mathfrak{E} \subseteq 2^U$ a collection of events such that $U \in \mathfrak{E}$ and $\emptyset \notin \mathfrak{E}$, $E, F \in \mathfrak{E}$ two events, as well as $f : \mathfrak{E} \to 2^U$ a belief revision function. If $E \subseteq F$ and $E \cap f(F) \neq \emptyset$, then $f(E) = E \cap f(E)$.

- Suppose that information E implies information F and that there exist states in E considered possible upon receiving F.
- Then, the states that the agent would deem possible upon receiving information E are precisely those in both E and $f(F)$.

Plausibility

Definition 2

Let U be a finite universal set. A plausibility order on U is a binary relation $\trianglerighteq \subseteq U \times U$ that is complete and transitive.

- $\omega \trianglerighteq \omega'$: the agent considers ω at least as plausible as ω'.
- $\omega \triangleright \omega'$: the agent considers ω more plausible than ω'.
- $\omega - \omega'$: the agent considers ω just as plausible as ω'.
- $\triangleright$ and $-$ can be defined in terms of $\trianglerighteq$:
  - $\omega \triangleright \omega'$, whenever $\omega \trianglerighteq \omega'$ and $\omega' \not\trianglerighteq \omega$.
  - $\omega - \omega'$, whenever $\omega \trianglerighteq \omega'$ and $\omega' \trianglerighteq \omega$.

AGM Axiom System and Plausibility Order

Theorem 3 (Grove, 1988)

Let U be a finite universal set, $\mathfrak{E} \subseteq 2^U$ a collection of events such that $U \in \mathfrak{E}$ and $\emptyset \notin \mathfrak{E}$, as well as $f : \mathfrak{E} \to 2^U$ a belief revision function. The belief revision function f is compatible with the AGM axioms, if and only if, there exists a plausibility order $\trianglerighteq \subseteq U \times U$ such that for every $E \in \mathfrak{E}$, $f(E)$ forms the set of most plausible states in E, i.e. $f(E) = \{\omega \in E : \omega \trianglerighteq \omega'$ for all $\omega' \in E\}$.

Adding Probabilities to the Picture

- Let U be a finite universal set and $P : 2^U \to [0, 1]$ represent the initial beliefs.

- $P_E : 2^U \to [0, 1]$ then denote the updated beliefs upon receiving information E, if $P(E) > 0$.

- By **belief updating**, it follows that:

$$\text{If } E \cap \text{supp}(p) \neq \emptyset, \text{ then } \text{supp}(P_E) = E \cap \text{supp}(P).$$

- This is called qualitative belief updating (or qualitative Bayes' rule).

- It can be shown that qualitative belief updating is built into AGM THEORY.

Dealing with completely surprising Information

- Yet, a belief revision function needs to go beyond **belief updating**, as it also encodes new beliefs, if $P(E) = 0$.

- To this end, let $P_\circ : U \to [0, 1]$ be some full-support probability measure on U.

- Then, for every possible piece of information $E \in \mathfrak{E}$, let $P_E : 2^U \to [0, 1]$ be the probability measure obtained by conditioning $P_\circ$ on $f(E)$ (note: **not** on E):

$$P_E(\{\omega\}) = P_\circ\big(\{\omega\} \mid f(E)\big) = \begin{cases} \frac{P_\circ(\{\omega\})}{\sum_{\omega' \in f(E)} P_\circ(\{\omega'\})} & \text{if } \omega \in f(E) \\ 0 & \text{if } \omega \notin f(E) \end{cases}$$

- Accordingly, P_U gives the initial probabilistic beliefs and, for every other $E \in \mathfrak{E} \setminus U$, P_E gives the revised probabilistic beliefs after receiving information E.

- The collection $\{P_E\}_{E \in \mathfrak{E}}$ thus obtained forms the agent's **probabilistic belief revision policy**, while the belief revision function $f : \mathfrak{E} \to 2^U$ constitutes the agent's **qualitative belief revision policy**.

Like-Mindedness

Interactive Reasoning with Beliefs

- In epistemic structures, a probability distribution is added for every information set of every player.
- These probability distributions are formed over the respective information sets.
- Yet, they can be viewed as probability distributions over Ω too by assigning 0 to every state outside the respective information set.

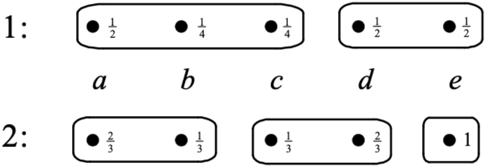
Epistemic Structures with Beliefs

Definition 4

An epistemic structure with beliefs is a tuple $\mathcal{E}^* = \langle \mathcal{E}, \big((p_i^{S_i})_{S_i \in \mathcal{I}_i} \big)_{i \in I} \rangle$, where

- $\mathcal{E}$ is an epistemic structure,
- $p_i^{S_i} \in \Delta(S_i)$ is a probability distribution over information set $S_i \in \mathcal{I}_i$ of player $i \in I$.

Illustration

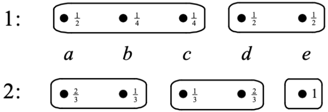


- At every state the two agents hold different beliefs.
- For example, consider the event $E = \{b, c\}$ and state $a \in \Omega$.
- Then, $P_1^{\{a,b,c\}}(E) = p_1^{\{a,b,c\}}(b) + p_1^{\{a,b,c\}}(c) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $P_2^{\{a,b\}}(E) = p_2^{\{a,b\}}(b) = \frac{1}{3}$.

Beliefs and Information

- If agents have different information, then it is not surprising that they can have different beliefs.
- Two agents are said to be like-minded, whenever they would have the same beliefs if they were to have the same information.

Illustration



- At state a, it is in line with agent 1's information that the true state is either a, b, or c.
- In contrast, agent 2 considers only a and b possible: thus, agent 2 holds finer information than agent 1.
- Hypothetical question: if agent 1 had the same information as agent 2, would he agree with agent 2's assessment that the probability of $E = \{b, c\}$ is $\frac{1}{3}$?
- Suppose that agent 1 were to be provided with the information that the true state is either a or b.
- By **belief updating**, he would then change his beliefs from $\begin{pmatrix} a & b & c \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ to $\begin{pmatrix} a & b \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ by means of conditional probability and thus hold the same beliefs as agent 2.

The Common Prior Assumption

- The idea of like-mindedness will now be carved out more precisely and formally.

- To this end the following property is needed:

Definition 5

Let $\mathcal{E}^*$ be an epistemic structure with beliefs and $p \in \Delta(\Omega)$ a probability distribution over Ω with corresponding probability measure $P \in \Delta(2^\Omega)$ over 2^Ω. The probability distribution p is called a common prior, whenever for every agent $i \in I$ and for every state $\omega \in \Omega$ it is the case that:

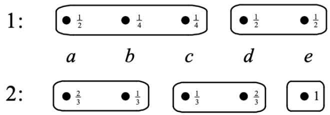
- $P\big(\mathcal{I}_i(\omega)\big) > 0$,
- $P\big(\{\omega'\} \mid \mathcal{I}_i(\omega)\big) = p_i^{\mathcal{I}_i(\omega)}(\omega')$ for all $\omega' \in \mathcal{I}_i(\omega)$.

Like-Mindedness or Harsanyi Consistency

Definition 6

An epistemic structure with beliefs $\mathcal{E}^*$ satisfies Harsanyi Consistency, whenever there exists a common prior. The agents are then called like-minded.

Illustration



- For this particular epistemic structure with beliefs, a common prior does exist.

- Consider $p = \begin{pmatrix} a & b & c & d & e \\ \frac{2}{8} & \frac{1}{8} & \frac{1}{8} & \frac{2}{8} & \frac{2}{8} \end{pmatrix}$.

- All beliefs can be obtained from p by means of conditional probability applied to P, e.g.:

$$P(\{a\} \mid \{a, b, c\}) = \frac{\frac{2}{8}}{\frac{2}{8} + \frac{1}{8} + \frac{1}{8}} = \frac{1}{2} = p_1^{\{a,b,c\}}(a)$$

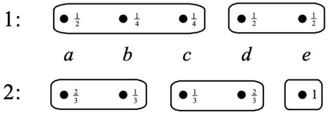
- Indeed, it can be verified that updating P on each information set in this epistemic structure with beliefs yields the probability distribution attached to the respective information set.

- Consequently, the agents are like-minded here.

How to determine whether a Common Prior exists?

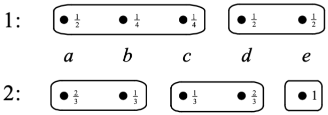
The issue of existence of a common prior can be reduced to the issue of whether a system of equations has a solution.

Illustration



- Assume that $p = \begin{pmatrix} a & b & c & d & e \\ p_a & p_b & p_c & p_d & p_e \end{pmatrix}$ is a common prior.
- Updating on information set $\{a, b, c\}$ of agent 1 then needs to yield $\frac{p_b}{p_a+p_b+p_c} = \frac{1}{4}$ as well as $\frac{p_c}{p_a+p_b+p_c} = \frac{1}{4}$, which together imply that $p_b = \frac{1}{4} \cdot (p_a + p_b + p_c) = p_c$, i.e. $p_b = p_c$.
- Updating on information set $\{d, e\}$ of agent 1 then needs to yield $\frac{p_d}{p_d+p_e} = \frac{1}{2}$, which implies that $p_d = \frac{1}{2} \cdot p_d + \frac{1}{2} \cdot p_e$, i.e. $p_d = p_e$.
- Updating on information set $\{a, b\}$ of agent 2 then needs to yield $\frac{p_a}{p_a+p_b} = \frac{2}{3}$, which implies that $p_a = \frac{2}{3} \cdot p_a + \frac{2}{3} \cdot p_b$, i.e. $p_a = 2 \cdot p_b$.
- Updating on information set $\{c, d\}$ of agent 2 then needs to yield $\frac{p_c}{p_c+p_d} = \frac{1}{3}$, which implies that $p_c = \frac{1}{3} \cdot p_c + \frac{1}{3} \cdot p_d$, i.e. $2 \cdot p_c = p_d$.
- Moreover, it needs to hold that $p_a + p_b + p_c + p_d + p_e = 1$.

Illustration



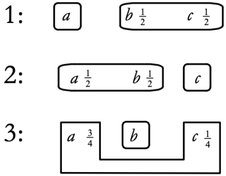
- Assume that $p = \begin{pmatrix} a & b & c & d & e \\ p_a & p_b & p_c & p_d & p_e \end{pmatrix}$ is a common prior.
- The following five conditions thus need to be satisfied by p:
  - *i)* $p_b = p_c$
  - *ii)* $p_d = p_e$
  - *iii)* $p_a = 2 \cdot p_b$
  - *iv)* $2 \cdot p_c = p_d$
  - *v)* $p_a + p_b + p_c + p_d + p_e = 1$
- This is a system of five equations in five unknowns which consequently admits a solution.
- It can be verified that this solution is as follows:

$$p = \begin{pmatrix} a & b & c & d & e \\ \frac{2}{8} & \frac{1}{8} & \frac{1}{8} & \frac{2}{8} & \frac{2}{8} \end{pmatrix}$$

Violating Harsanyi Consistency

It is possible to have epistemic structures with beliefs with agents that are not like-minded.

Illustration



- Assume that $p = \begin{pmatrix} a & b & c \\ p_a & p_b & p_c \end{pmatrix}$ is a common prior.
- Updating on information set $\{b, c\}$ of agent 1 then needs to yield $\frac{p_b}{p_b + p_c} = \frac{1}{2}$, which implies that $p_b = \frac{1}{2} \cdot p_b + \frac{1}{2} \cdot p_c$, i.e. $p_b = p_c$.
- Updating on information set $\{a, b\}$ of agent 2 then needs to yield $\frac{p_a}{p_a + p_b} = \frac{1}{2}$, which implies that $p_a = \frac{1}{2} \cdot p_a + \frac{1}{2} \cdot p_b$, i.e. $p_a = p_b$.
- It follows that $p_a = p_c$.
- However, updating on information set $\{a, c\}$ of agent 3, $\frac{p_a}{p_a + p_c} = \frac{3}{4}$ need to ensue, which implies that $p_a = \frac{3}{4} \cdot p_a + \frac{3}{4} \cdot p_c$, i.e. $p_a = 3 \cdot p_c$, which is a contradiction.
- Therefore, this epistemic structure with beliefs does violate Harsanyi Consistency and represents agents that are not like-minded.

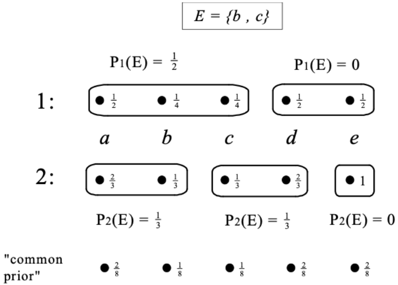
Agreeing to Disagree

An intriguing Question

- Can two like-minded agents agree to disagree?
- It is certainly quite possible for two agents to hold different beliefs about a particular event and to thus disagree about it.
- Indeed, they might have different information.
- However, can they acknowledge such a disagreement in the sense of it being common knowledge among them?

Illustration

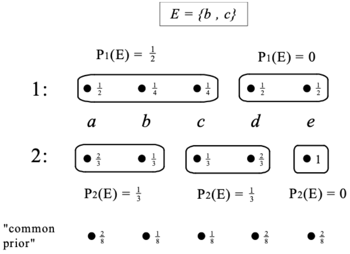
- Consider the following epistemic structure with beliefs, which models like-minded agents:



- Observe that $P_1^{\{a,b,c\}}(E) = \frac{\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}+\frac{1}{4}} = \frac{1}{2}$ and $P_2^{\{a,b\}}(E) = \frac{\frac{1}{3}}{\frac{2}{3}+\frac{1}{3}} = \frac{1}{3}$.

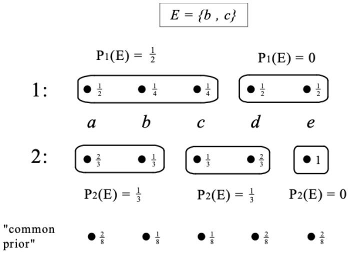
- Consequently, at state a, the agents disagree about E.

Illustration



- The agents also know at state a that they disagree about E.
- To see this, let $\|P_1(E) = \frac{1}{2}\|$ denote the event "agent 1 believes event E with probability $\frac{1}{2}$" and $\|P_2(E) = \frac{1}{3}\|$ the event "agent 2 believes event E with probability $\frac{1}{3}$"
- Then, $\|P_1(E) = \frac{1}{2}\| = \{a, b, c\}$, $\|P_2(E) = \frac{1}{3}\| = \{a, b, c, d\}$, and thus $\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\| = \{a, b, c\}$.
- It follows that $K_1\left(\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\|\right) = \{a, b, c\}$, $K_2\left(\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\|\right) = \{a, b\}$, and thus $K\left(\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\|\right) = \{a, b\}$.
- Since $a \in K\left(\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\|\right)$, the agents do not only disagree about E at state a, but they also know that they disagree.

Illustration



- However, the agents' disagreement is not common knowledge at state a.
- Actually, they already fail to attain 2nd-order mutual knowledge of it.
- Indeed, $K_1K\big(\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\|\big) = \emptyset$, $K_2K\big(\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\|\big) = \{a, b\}$, and thus $KK\big(\|P_1(E) = \frac{1}{2}\| \cap \|P_2(E) = \frac{1}{3}\|\big) = \emptyset$.
- Consequently, it is nowhere – and, in particular, not at state a – common knowledge that agent 1's beliefs about E are $\frac{1}{2}$ and 2's are $\frac{1}{3}$.

Impossibility of Agreeing to Disagree

- It turns out that the opinions of two like-minded agents can never be in disagreement and, at the same time, commonly known!

- The following result, which is known as the AGREEMENT THEOREM, establishes this impossibility formally:

Agreement Theorem (Aumann, 1976)

Let $\mathcal{E}^*$ be an epistemic structure with beliefs satisfying Harsanyi consistency with two agents 1 and 2, $E \in 2^\Omega$ some event, and $p, q \in [0,1]$ two numbers. If $CK\big(\|P_1(E) = p\| \cap \|P_2(E) = q\|\big) \neq \emptyset$, then $p = q$.

- In other words, two like-minded agents cannot agree to disagree about the probability of an event.

Towards establishing the Agreement Theorem

Lemma 7

Let U be a finite universal set, $P \in \Delta(U)$ a probability measure on U, $E, F \in 2^U$ two events such that $P(F) > 0$, $m \in \mathbb{N}$ a natural number, and $q \in [0,1]$ a real number. If $\{F_1, \ldots, F_m\}$ forms a partition of F and $P(E \mid F_j) = q$ for all $j \in \{1, \ldots, m\}$, then $P(E \mid F) = q$.

Proof:

- For every $j \in \{1, \ldots, m\}$, since $P(E \mid F_j) = q$ and by conditional probability $P(E \mid F_j) = \frac{P(E \cap F_j)}{P(F_j)}$, it follows that

$$P(E \cap F_j) = q \cdot F(F_j).$$

- By the pairwise disjointness of the elements in $\{F_1, \ldots, F_m\}$, finite additivity of P, and the covering of F through $\{F_1, \ldots, F_m\}$,

$$\sum_{j \in \{1,\ldots,m\}} P(E \cap F_j) = P\big(\cup_{j \in \{1,\ldots,m\}} (E \cap F_j)\big) = P\big(E \cap (\cup_{j \in \{1,\ldots,m\}} F_j)\big) = P(E \cap F)$$

- By the disjointness of the elements in the collection $\{F_1, \ldots, F_m\}$ and finite additivity of P,

$$\sum_{j \in \{1,\ldots,m\}} q \cdot P(F_j) = q \cdot \sum_{j \in \{1,\ldots,m\}} P(F_j) = q \cdot P(\cup_{j \in \{1,\ldots,m\}} F_j) = q \cdot P(F)$$

■ Therefore, $P(E \cap F) = \sum_{j \in \{1,\ldots,m\}} P(E \cap F_j) = \sum_{j \in \{1,\ldots,m\}} q \cdot F(F_j) = q \cdot P(F)$, which since $P(F) > 0$ implies that

$$P(E \mid F) = q.$$

Proof of the Agreement Theorem

- Since the epistemic structure with beliefs satisfies Harsanyi Consistency, there exists a common prior $p \in \Delta(\Omega)$ with corresponding probability measure $P \in \Delta(2^{\Omega})$.
- As $CK\big(\|P_1(E) = p\| \cap \|P_2(E) = q\|\big) \neq \emptyset$, there exists a state $\omega \in CK\big(\|P_1(E) = p\| \cap \|P_2(E) = q\|\big)$.
- Consider $\mathcal{I}_{CK}(\omega)$, which is the common knowledge cell containing state ω.
- Then, there exists $m_1 \in \mathbb{N}$ such that $\mathcal{I}_{CK}(\omega) = \cup_{i \in \{1,...,m_1\}} S_1^i$, where $S_1^i \in \mathcal{I}_1$ is an information sets of agent 1 for all $i \in \{1, \ldots, m_1\}$.
- There also exists $m_2 \in \mathbb{N}$ such that $\mathcal{I}_{CK}(\omega) = \cup_{j \in \{1,...,m_2\}} S_2^j$, where $S_2^j \in \mathcal{I}_2$ is an information sets of agent 2 for all $j \in \{1, \ldots, m_2\}$.
- It holds that $S_1^i \subseteq \|P_1(E) = p\|$ for all $i \in \{1, \ldots, m_1\}$ as well as $S_2^j \subseteq \|P_2(E) = q\|$ for all $j \in \{1, \ldots, m_2\}$.
- By Harsanyi Consistency, $P(E \mid S_1^i) = P_1(E) = p$ for all $i \in \{1, \ldots, m_1\}$ as well as $P(E \mid S_2^j) = P_2(E) = q$ for all $j \in \{1, \ldots, m_2\}$.
- By Lemma 7, it then follows that $P\big(E \mid \mathcal{I}_{CK}(\omega)\big) = p$ as well as $P\big(E \mid \mathcal{I}_{CK}(\omega)\big) = q$.
- Therefore, $p = q$.

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition

- Chapter 9: **Adding Beliefs to Knowledge**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html