ECON322 Game Theory

Part III Interactive Epistemology
Topic 7 Knowledge

Christian W. Bach

University of Liverpool & EPICENTER

A General Framework for Modelling Knowledge

- In dynamic games by means of information sets it can be represented what players **know** about past choices.
- An information set is a collection of decision nodes in the tree, where the respective player's mind satisfies two properties:
  - he **knows** that play has reached the information set,
  - he does **not know** which decision node is the actual one.
- In **T7**, such ideas of information and **knowledge** are generalized and a THEORY OF KNOWLEDGE is developed.

Outline

INDIVIDUAL KNOWLEDGE

Motivating Example

- After examining his patient, a doctor concludes that there can be five possible causes for the patient's symptoms:

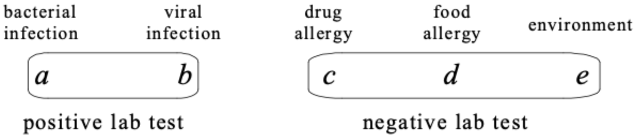
  1. bacterial infection,
  2. viral infection,
  3. allergic reaction to a drug,
  4. allergic reaction to food,
  5. environmental factors.

- The doctor decides to do a lab test.

  - If the lab test turns out to be positive, then the doctor will be able to rule out causes **(3) - (5)**.
  - If the lab test turns out to be negative, then it is indicated that causes **(1) - (2)** can be ruled out.

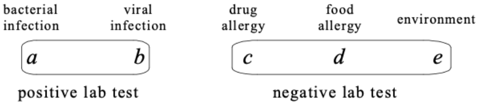
Motivating Example

- The doctor's epistemic states can be represented as follows:



- The set of possible states is $\{a, b, c, d, e\}$, where each state represents a possible cause.
- $\{a, b, c, d, e\}$ can be partitioned into two sets:
  - $\{a, b\}$ corresponds to the epistemic state of the doctor given he is informed of a positive test result.
  - $\{c, d, e\}$ corresponds to the epistemic state of the doctor given he is informed of a negative test result.

Motivating Example



- Consider the proposition:

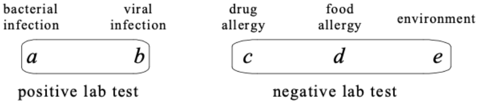
  "The cause of the patient's symptoms is either an infection or environmental factors."

- This proposition can be formally viewed as the set $\{a, b, e\}$.
- After the lab test, at which states would the doctor know this proposition?
- If the lab test is positive, then the doctor only deems possible states a and b – at both these states the proposition holds true and as the consequence the doctor knows the proposition.
- If the lab test is negative, then the doctor only deems possible states c, d, and e – among these there exists a state at which the proposition fails to hold and therefore the doctor does not know the proposition.

Terminology

- Ω is a finite set of states, where each state is to be understood as a complete specification of the relevant facts about the world.
- $\mathcal{I}$ is an information partition of Ω, i.e. a collection of subsets of Ω such that
  - all subsets are pairwise disjoint,
  - the union of all subsets covers Ω in it s entirety.
- An element of the information partition is called information set.
- For every state $\omega \in \Omega$, the agent's epistemic mental set-up – if ω is the actual state – is captured by the information set $\mathcal{I}(\omega)$.
- It is assumed that $\omega \in \mathcal{I}(\omega)$ for all $\omega \in \Omega$, i.e. the agent always considers possible the actual state (Property of ACTUALITY).
- Subsets of Ω are called events and denoted by E.

Illustration



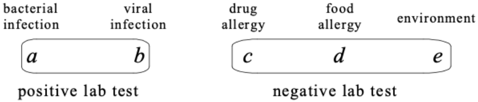
- $\Omega = \{a, b, c, d, e\}$
- $\mathcal{I} = \{\{a, b\}, \{c, d, e\}\}$
- $\mathcal{I}(a) = \mathcal{I}(b) = \{a, b\}$
- $\mathcal{I}(c) = \mathcal{I}(d) = \mathcal{I}(e) = \{c, d, e\}$
- $E = \{a, b, e\}$

Knowledge

Definition 1

Let Ω be a set of states, $\mathcal{I}$ an information partition, $E \subseteq \Omega$ an event, and $\omega \in \Omega$ a state. The agent knows E at state ω, whenever $\mathcal{I}(\omega) \subseteq E$.

Illustration



- $\mathcal{I} = \{\{a, b\}, \{c, d, e\}\}$
- $E = \{a, b, e\}$
- At states a and b, the doctor knows E, since $\mathcal{I}(a) = \mathcal{I}(b) = \{a, b\} \subseteq \{a, b, e\} = E$.
- At states c, d, and e, the doctor does not know E, since $\mathcal{I}(c) = \mathcal{I}(d) = \mathcal{I}(e) = \{c, d, e\} \not\subseteq \{a, b, e\} = E$
- It is possible that there is no state where the agents knows a given event.
- For instance, the doctor never knows the event $F = \{a, c\}$.

Event Space

- The event space is the set of all events, i.e. all subsets of Ω.
- It is denoted by 2^{Ω}.
- If Ω contains n elements i.e. states, then there exist 2^n subsets of Ω, i.e. events.
- For example, if
$$\Omega = \{a, b, c\},$$
then
$$2^{\Omega} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$
- Note that $\mid \Omega \mid = 3$ and $\mid 2^{\Omega} \mid = 8 = 2^3$.

Knowledge Operator

Definition 2

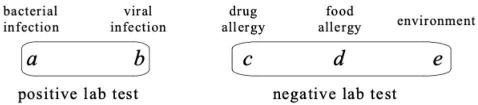
Let Ω be a set of states and $\mathcal{I}$ an information partition. The knoweldge operator is the function $K : 2^\Omega \to 2^\Omega$ such that

$$KE := \{\omega \in \Omega : \mathcal{I}(\omega) \subseteq E\}$$

for all $E \in 2^\Omega$.

The knowledge operator *K* turns any event E as input into the event KE defined as the set of states at which the agent knows E as output.

Illustration

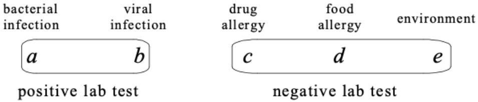


- $\mathcal{I} = \big\{\{a, b\}, \{c, d, e\}\big\}$
- If $E = \{a, b, d, e\}$, then $KE = \{a, b\}$.
- If $F = \{a, c\}$, then $KF = \emptyset$.

Negation of Events

- Given an event $E \in 2^{\Omega}$, the complement of E contains the states not in E and is denoted by $\neg E$.
- E.g. if $\Omega = \{a, b, c, d, e\}$ and $E = \{a, b, d\}$, then $\neg E = \{c, e\}$.
- Thus, while KE is the event that the agent knows E, the event that the agent does not know E is denoted by $\neg KE$.
- KKE is the event that the agent knows that he knows E.
- $K\neg KE$ is the event that the agent knows that he does not know E.

Illustration



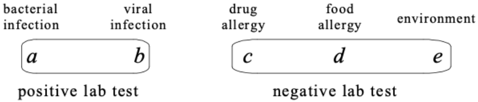
- $\mathcal{I} = \big\{\{a,b\},\{c,d,e\}\big\}$
- If $E = \{a,b,d,e\}$, then $KE = \{a,b\}$ and $\neg E = \{c\}$.
- Thus, $\neg KE = \{c,d,e\}$ and $K\neg E = \emptyset$.
- Also, $KKE = K\{a,b\} = \{a,b\}$ and $K\neg KE = K\{c,d,e\} = \{c,d,e\}$.
- If $F = \{a,c\}$, then $KF = \emptyset$ and $\neg F = \{b,d,e\}$.
- Thus, $\neg KF = \Omega$ and $K\neg F = \emptyset$.
- Also, $KKF = K\emptyset = \emptyset$ and $K\neg KF = K\Omega = \Omega$.

Knowing the Negation and Not Knowing the Event

- $K\neg E \subseteq \neg KE$ holds.
  - If $\omega \in K\neg E$, then $\mathcal{I}(\omega) \subseteq \neg E$.
  - It follows that $\mathcal{I}(\omega) \cap E = \emptyset$ and thus $\mathcal{I}(\omega) \not\subseteq E$.
  - Consequently, $\omega \notin KE$ and therefore, $\omega \in \neg KE$.

- However, $\neg KE \subseteq K\neg E$ does not hold.
  - If $\omega \in \neg KE$, then there exists $\omega' \in \mathcal{I}(\omega)$ such that $\omega' \in \neg E$.
  - Yet, it could be possible that there exists another world $\omega'' \neq \omega'$ such that $\omega'' \in \mathcal{I}(\omega) \cap E$.
  - The agent thus neither knows E nor $\neg E$ but considers both events possible, and it follows in particular that $\omega \notin K\neg E$.

Illustration



- $\mathcal{I} = \{\{a, b\}, \{c, d, e\}\}$
- Consider $F = \{a, c\}$ and $\neg F = \{b, d, e\}$.
- Then, $KF = \emptyset$ and $K\neg F = \emptyset$.
- For instance, if the true state is a, then the doctor considers F possible, since $\mathcal{I}(a) \cap F = \{a\} \neq \emptyset$.
- However, the doctor then also considers $\neg F$ possible, since $\mathcal{I}(a) \cap \neg F = \{b\} \neq \emptyset$.

Properties of Knowledge

Truth

Proposition 3

Let Ω be a set of states, $\mathcal{I}$ an information partition, and $E \in 2^{\Omega}$ an event. Then,

$$KE \subseteq E$$

Proof

- Let $\omega \in \Omega$ be some state such that $\omega \in KE$.
- Then, $\mathcal{I}(\omega) \subseteq E$.
- Since $\omega \in \mathcal{I}(\omega)$ by ACTUALITY, it follows that $\omega \in E$.

Consistency

Proposition 4

Let Ω be a set of states, $\mathcal{I}$ an information partition, and $E \in 2^{\Omega}$ an event. Then,

$$KE \cap K\neg E = \emptyset$$

Proof

- Towards a contradiction, suppose that there exists a state $\omega \in \Omega$ such that $\omega \in KE \cap K\neg E$.
- It follows that $\omega \in KE$ and thus $\mathcal{I}(\omega) \subseteq E$ as well as $\omega \in K\neg E$ and thus $\mathcal{I}(\omega) \subseteq \neg E$.
- Consequently, $\mathcal{I}(\omega) \subseteq E \cap \neg E$.
- Since $\mathcal{I}(\omega) \subseteq E \cap \neg E = \emptyset$, it then follows that $\mathcal{I}(\omega) = \emptyset$ which contradicts the fact that $\omega \in \mathcal{I}(\omega)$.

Positive Introspection

Proposition 5

Let Ω be a set of states, $\mathcal{I}$ an information partition, and $E \in 2^{\Omega}$ an event. Then,

$$KE \subseteq KKE$$

Proof

- Let $\omega \in \Omega$ be some state such that $\omega \in KE$.
- Then, $\mathcal{I}(\omega) \subseteq E$.
- For every state $\omega' \in \mathcal{I}(\omega)$ it holds that $\mathcal{I}(\omega') = \mathcal{I}(\omega)$.
- It follows that $\mathcal{I}(\omega') \subseteq E$ and thus $\omega' \in KE$ for all $\omega' \in \mathcal{I}(\omega)$.
- As a consequence, $\mathcal{I}(\omega) \subseteq KE$ and therefore $\omega \in KKE$.

Negative Introspection

Proposition 6

Let Ω be a set of states, $\mathcal{I}$ an information partition, and $E \in 2^{\Omega}$ an event. Then,

$$\neg KE \subseteq K\neg KE$$

Proof

- Let $\omega \in \Omega$ be some state such that $\omega \in \neg KE$.
- Then, $\mathcal{I}(\omega) \not\subseteq E$ and thus $\mathcal{I}(\omega) \cap \neg E \neq \emptyset$.
- For every state $\omega' \in \mathcal{I}(\omega)$ it holds that $\mathcal{I}(\omega') = \mathcal{I}(\omega)$.
- It follows that $\mathcal{I}(\omega') \cap \neg E \neq \emptyset$ and thus $\omega' \in \neg KE$ for all $\omega' \in \mathcal{I}(\omega)$.
- As a consequence, $\mathcal{I}(\omega) \subseteq \neg KE$ and therefore $\omega \in K\neg KE$.

Monotonicity

Proposition 7

Let Ω be a set of states, $\mathcal{I}$ an information partition, and $E, F \in 2^{\Omega}$ events. If

$$E \subseteq F,$$

then

$$KE \subseteq KF.$$

Proof

- Let $\omega \in \Omega$ be some state such that $\omega \in KE$.
- Then, $\mathcal{I}(\omega) \subseteq E$.
- Since $E \subseteq F$ holds, it follows that $\mathcal{I}(\omega) \subseteq F$.
- Therefore, $\omega \in KF$.

Conjunction

Proposition 8

Let Ω be a set of states, $\mathcal{I}$ an information partition, and $E, F \in 2^{\Omega}$ events. Then,

$$KE \cap KF = K(E \cap F)$$

Proof

- Let $\omega \in \Omega$ be some state such that $\omega \in KE \cap KF$.
- Then, $\omega \in KE$ and thus $\mathcal{I}(\omega) \subseteq E$ as well as $\omega \in KF$ and thus $\mathcal{I}(\omega) \subseteq F$.
- Consequently, $\mathcal{I}(\omega) \subseteq E \cap F$ and thus $\omega \in K(E \cap F)$.
- Conversely, let $\omega \in \Omega$ be some state such that $\omega \in K(E \cap F)$.
- Then, $\mathcal{I}(\omega) \subseteq E \cap F$ and thus $\mathcal{I}(\omega) \subseteq E$ as well as $\mathcal{I}(\omega) \subseteq F$.
- It follows that $\omega \in KE$ as well as $\omega \in KF$, and hence $\omega \in KE \cap KF$.

Interactive Knowledge

Reasoning About Others' Knowledge

- The analysis is now extended to the case of several agents.
- In particular, the reasoning realm is rendered interactive.
- Not only knowledge about relevant facts but also interactive knowledge is considered.
- E.g. what does an agent know about what other agents know.
- The possible states of mind of an agent are represented by an information partition and knowledge operators individualized.

Epistemic Structures

Definition 9

An epistemic structure is a tuple $\mathcal{E} = \langle \Omega, I, (\mathcal{I}_i)_{i \in I} \rangle$, where

- Ω is a set of states,
- I is a set of agents,
- $\mathcal{I}_i$ is an information partition of player $i \in I$.

Knowledge Operators for every Agent

Definition 10

Let $\mathcal{E}$ be an epistemic structure and $i \in I$ an agent. The knowledge operator of agent i is the function $K_i : 2^{\Omega} \to 2^{\Omega}$ such that

$$K_i E := \{\omega \in \Omega : \mathcal{I}_i(\omega) \subseteq E\}$$

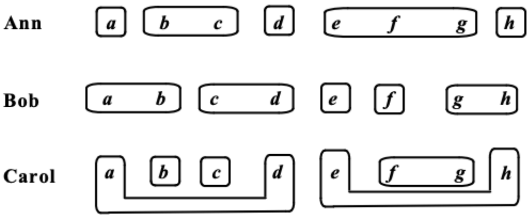
for all $E \in 2^{\Omega}$.

- Consider an event $E \in 2^{\Omega}$ and three distinct agents $i, j, k \in I$.
- Since $K_i E$ forms an event, the event $K_j K_i E$ can be computed.
- Further interactive knowledge events can be constructed, for instance: $K_k K_j K_i E$ and $K_i K_k K_j K_i E$, etc.

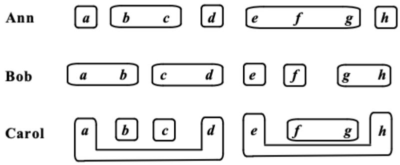
Illustration

Consider $\mathcal{E}$ such that

- $\Omega = \{a, b, c, d, e, f, g, h\}$,
- $I = \{\text{Ann}, \text{Bob}, \text{Carol}\}$,
- the information partitions of the three agents are as follows:



Illustration



Let $E = \{a, b, c, f, g\}$ be a given event. Then:

- $K_{Ann}E = \{a, b, c\}$
- $K_{Bob}E = \{a, b, f\}$
- $K_{Carol}E = \{b, c, f, g\}$
- $K_{Carol}K_{Ann}E = \{b, c\}$
- $K_{Bob}K_{Carol}K_{Ann}E = \emptyset$
- $K_{Ann}\neg K_{Bob}K_{Carol}E = \{a, b, c, d, h\}$

Mutual Knowledge

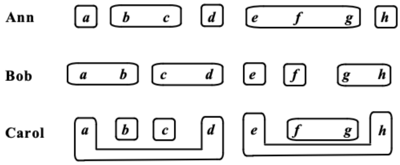
Definition 11

Let $\mathcal{E}$ be an epistemic structure. The mutual knowledge operator is the function $K : 2^{\Omega} \to 2^{\Omega}$ such that

$$KE := \cap_{i \in I} K_i E$$

for all $E \in 2^{\Omega}$.

Illustration



Let $E = \{a, b, c, f, g\}$ be a given event. Then:

- $K_{Ann}E = \{a, b, c\}$
- $K_{Bob}E = \{a, b, f\}$
- $K_{Carol}E = \{b, c, f, g\}$
- $KE = K_{Ann}E \cap K_{Bob}E \cap K_{Carol}E = \{b\}$

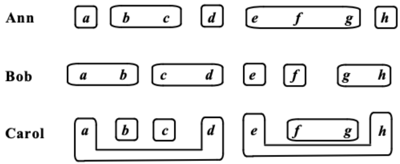
Iterated Mutual Knowledge

Definition 12

Let $\mathcal{E}$ be an epistemic structure and $m \in \mathbb{N} \setminus \{1\}$. The m-th-order mutual knowledge operator is the function defined inductively as follows:

- $K^1 : 2^\Omega \to 2^\Omega$ such that $K^1 E := KE$ for all $E \in 2^\Omega$,
- $K^m : 2^\Omega \to 2^\Omega$ such that $K^m E := KK^{m-1} E$ for all $E \in 2^\Omega$.

Illustration



Let $E = \{a, b, c, f, g\}$ be a given event. Then:

- $KE = K_{Ann}E \cap K_{Bob}E \cap K_{Carol}E = \{b\}$
- $K_{Ann}KE = \emptyset$
- $K_{Bob}KE = \emptyset$
- $K_{Carol}KE = \{b\}$
- $K^2E = KKE = \emptyset$

Common Knowledge

Definition 13

Let $\mathcal{E}$ be an epistemic structure. The common knowledge operator is the function $CK : 2^{\Omega} \rightarrow 2^{\Omega}$ such that

$$CKE := \cap_{m \in \mathbb{N}} K^m E$$

for all $E \in 2^{\Omega}$.

- Common knowledge is the strongest form of interactive knowledge.
- Accordingly, everyone knows that everyone knows that everyone knows that ... that everyone knows the event in question.

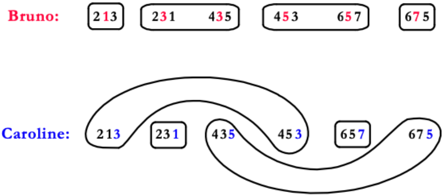
Illustration

Should Bruno and Caroline accept the following proposal?

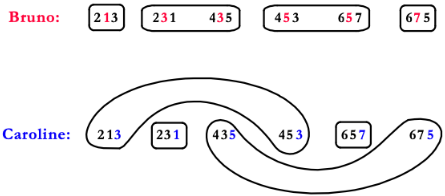
- They will be put in two rooms without communication possibility.
- A random number $n \in \{2, 4, 6\}$ will be picked and two pieces of paper produced, one with $n - 1$ on it and one with $n + 1$ on it.
- Randomly one piece is given to Bruno and one to Caroline.
- After viewing the number on the respective piece, each agent provides a pair of numbers.
- If both agents provide the same pair of numbers **and** at least one of the two numbers equals the one given to Bruno, then:
  - both agents receive \$1000 each,
  - otherwise each agent pays \$1000.

Illustration

- The situation can be formalized by an epistemic structure with $\Omega = \{213, 231, 435, 453, 657, 675\}$ and $I = \{\text{Bruno}, \text{Caroline}\}$.
- A state is described as a triple abc.
- a is the random number drawn, b is the number given to Bruno (b), and c is the number given to Caroline.
- the information partitions of the two agents are as follows:



Illustration



- Consider $E = \{213, 453, 657\}$, that is, *"Bruno gets a 1 or a 5"*.
  - $K_{Bruno}E = E$ as well as $K_{Caroline}E = E$, and thus $KE = E$.
  - $K_{Bruno}KE = E$ as well as $K_{Caroline}KE = E$, and thus $KKE = K^2E = E$.
  - $K_{Bruno}KKE = E$ as well as $K_{Caroline}KKE = E$, and thus $KKKE = K^3E = E$.
  - It follows inductively that $K^mE = E$ for all $m \in \mathbb{N}$, and thus $CKE = E$.
- Consider $F = \{231, 435, 675\}$, that is, *"Bruno gets a 3 or a 7"*.
  - $K_{Bruno}F = F$ as well as $K_{Caroline}F = F$, and thus $KF = F$.
  - $K_{Bruno}KF = F$ as well as $K_{Caroline}KF = F$, and thus $KKF = K^2F = F$.
  - $K_{Bruno}KKF = F$ as well as $K_{Caroline}KKF = F$, and thus $KKKF = K^3F = F$.
  - It follows inductively that $K^mF = F$ for all $m \in \mathbb{N}$, and thus $CKF = F$.

Illustration

- A successful pair of strategies for the two agents must be based on events that, when they occur, are commonly known.

- Consider the following "coordinated" strategy $\bigstar$:

$$\bigstar = \begin{cases} \text{if } E, & \text{then write } (1,5) \\ \text{if } F, & \text{then write } (3,7) \end{cases}$$

- One of the conditioning events of $\bigstar$ must occur, since $E \cup F = \Omega$.

- In the case of both E and F both agents will obtain \$1000 each.

- Consequently, $\bigstar$ is indeed a successful "coordinated" strategy for Bruno and Caroline in this situation.

Reachability

Definition 14

Let $\mathcal{E}$ be an epistemic structure and $\omega, \omega' \in \Omega$.

- ω' is reachable from ω in 1 step, if there exists a player $i \in I$ such that $\omega' \in \mathcal{I}_i(\omega)$.
- Let $m \in \mathbb{N}$ be a natural number. ω' is reachable from ω in m steps, if there exists a sequence $(\omega_1, \omega_2, \omega_3, \ldots, \omega_m) \in \Omega^m$ of states such that
  - **(i)** $\omega_1 = \omega$,
  - **(ii)** $\omega_m = \omega'$,
  - **(iii)** for every $k \in \{2, 3, \ldots, m\}$ it is the case that ω_k is reachable from ω_{k-1} in 1 step.
- ω' is reachable from ω, if there exists a natural number $m \in \mathbb{N}$ such that ω' is reachable from ω in m steps.

Common Knowledge Partition

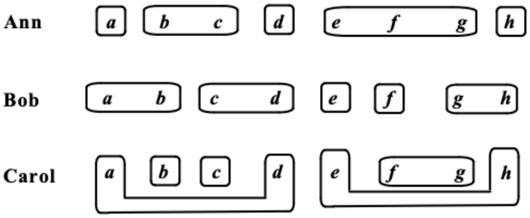
Definition 15

Let $\mathcal{E}$ be an epistemic structure and $\omega, \omega' \in \Omega$.

- Let $\omega \in \Omega$ a state. The common knowledge cell containing ω, denote by $\mathcal{I}_{CK}(\omega)$, is the set of states reachable from ω.
- The common knowledge partition is the collection of all common knowledge sets.

Illustration

- Information partitions . . .



- . . . and the corresponding common knowledge partition

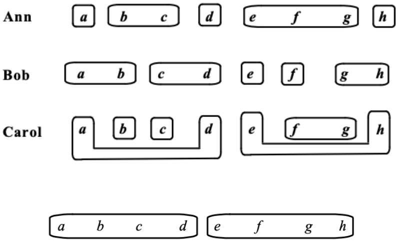
A Convenient Way to compute Common Knowledge of an Event

Theorem 16 (Aumann, 1976)

Let $\mathcal{E}$ be an epistemic structure and $E \in 2^{\Omega}$ an event. Then,

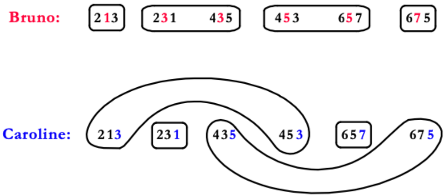
$$CKE = \{\omega \in \Omega : \mathcal{I}_{CK}(\omega) \subseteq E\}$$

Illustration

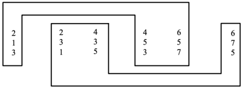


- Let $E = \{a,b,c,d,e,f\}$. Then, $CKE = \{a,b,c,d\}$.
- Let $F = \{a,b,f,g,h\}$. Then, $CKF = \emptyset$.
- Let $x \in \{a,b,c,d\}$. Then, the smallest event that is common knowledge at state x is $\mathcal{I}_{CK}(x) = \{a,b,c,d\}$.
- Let $y \in \{e,f,g,h\}$. Then, the smallest event that is common knowledge at state y is $\mathcal{I}_{CK}(y) = \{e,f,g,h\}$.

Back to Bruno and Caroline



- Observe that 213 is reachable from itself in 1 step via Bruno or Caroline, 453 is reachable in 1 step from 213 via Caroline, and 657 is reachable in 2 steps from 213 (first to 453 via Caroline and then from 453 via Bruno).
- Consequently, $\mathcal{I}_{CK}(213) = \mathcal{I}_{CK}(453) = \mathcal{I}_{CK}(657) = \{213, 453, 657\}$.
- Observe that 231 is reachable from itself in 1 step via Bruno or Caroline, 435 is reachable in 1 step from 231 via Bruno, and 675 is reachable in 2 steps from 231 (first to 435 via Bruno and then from 435 via Caroline).
- Consequently, $\mathcal{I}_{CK}(231) = \mathcal{I}_{CK}(435) = \mathcal{I}_{CK}(675) = \{231, 435, 675\}$.
- Therefore, the common knowledge partition ensues as follows:



Back to Bruno and Caroline

- $\mathcal{I}_{CK}(213) = \mathcal{I}_{CK}(453) = \mathcal{I}_{CK}(657) = \{213, 453, 657\}$.
- $\mathcal{I}_{CK}(231) = \mathcal{I}_{CK}(435) = \mathcal{I}_{CK}(675) = \{231, 435, 675\}$.
- Recall the event $E = \{213, 453, 657\}$ (*"Bruno gets a 1 or a 5"*).
- $\mathcal{I}_{CK}(213) = \mathcal{I}_{CK}(453) = \mathcal{I}_{CK}(657) = \{213, 453, 657\} = E$ holds.
- Thus, $CKE = E$, i.e. at any state where Bruno gets a 1 or a 5, this fact is common knowledge between Bruno and Caroline.
- Recall the event $F = \{231, 435, 675\}$ (*"Bruno gets a 3 or a 7"*).
- $\mathcal{I}_{CK}(231) = \mathcal{I}_{CK}(435) = \mathcal{I}_{CK}(675) = \{231, 435, 675\} = F$ holds.
- Thus, $CKF = F$, i.e. at any state where Bruno gets a 3 or a 7, this fact is common knowledge between Bruno and Caroline.

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition

- Chapter 8: **Common Knowledge**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html