ECON322 Game Theory

Part II Cardinal Payoffs
Topic 6 Extensive-Form Games

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Dynamic Games with Probabilistic Outcomes

- In **T5** the framework of static games has been generalized by admitting probabilistic outcomes.
- Formally, lotteries over outcomes have replaced the simple, deterministic outcomes in the notion of strategic form.
- Randomized choices are definable in such a cardinal framework.
- Also, in dynamic games choices can be generalized by admitting randomizations as choice objects.

Outline

- Probabilistic Outcomes in Dynamic Games
- Behavioural Strategies
- Subgame Perfect Equilibrium

Probabilistic Outcomes in Dynamic Games

Two Approaches to Modelling Probabilistic Outcomes in Dynamic Games

1. Generalization of the extensive form via lotteries over basic outcomes *(analogous to extending the strategic form in* ***T5****)*

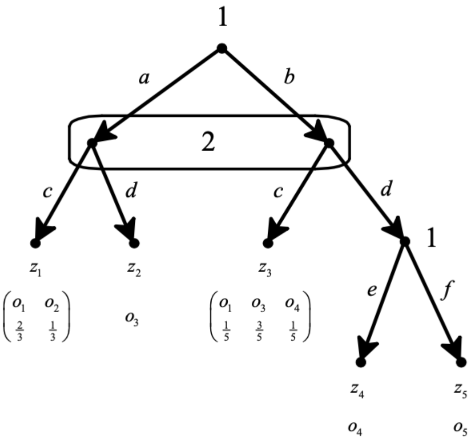
2. Admission of a dummy player – “Nature” – with chance moves while the notion of extensive form is kept unaltered *(cf.* ***T3****)*

Both approaches do require cardinal payoffs of course.

Approach 1: Tweaking the Extensive Form

- In the definition of the extensive-form frame, the function α_O is rendered probabilistic i.e. $\alpha_O : Z \rightarrow \mathcal{L}(O)$.
- Accordingly, α_O assigns a lottery over the basic outcomes to every terminal node (instead of merely a basic outcome).
- In the definition of the extensive-form game, the preferences are then brought into line with vNM's Expected Utility Theory.
- Accordingly, $\succsim_i$ is turned into a preference relation over $\mathcal{L}(O)$ satisfying AXIOMS 1 – 4 for every player $i \in I$.

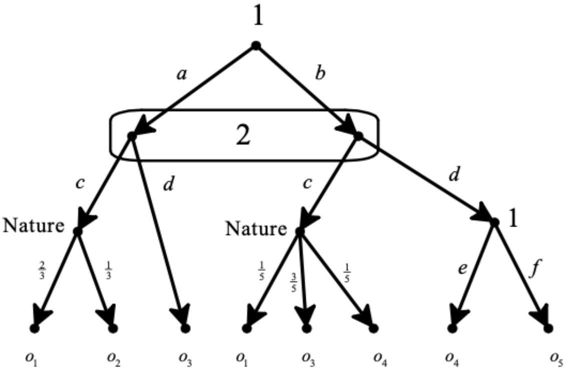
Illustration



Approach 2: Player 'Nature' with Chance Moves

- The definition of extensive-form frame is left untouched.
- Random events are explicitly represented by means of chance moves of a dummy player called 'Nature'.
- In the definition of the extensive-form game, the preferences are then also governed by vNM's Expected Utility Theory.
- Accordingly, $\succsim_i$ is turned into a preference relation over $\mathcal{L}(O)$ satisfying AXIOMS 1 – 4 for every player $i \in I$.

Illustration



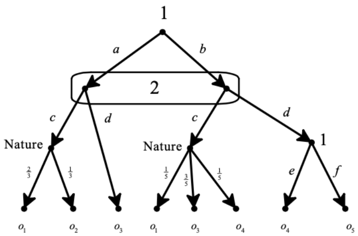
Extensive-Form Games with Cardinal Payoffs

Definition 1

A cardinal extensive-form game is a tuple $\mathcal{G}^{\mathcal{E}} = \langle \mathcal{F}^{\mathcal{E}}, (\succsim_i)_{i \in I} \rangle$, where

- $\mathcal{F}^{\mathcal{E}}$ is an extensive-form frame.
- $\succsim_i$ is a preference relation over $\mathcal{L}(O)$ satisfying AXIOMS 1 – 4 for every player $i \in I$.

Strategies



- A pure strategy is a list of local choices, one for every information set of the respective player.
- In the example:

$$S_1 = \{(a,e),(a,f),(b,e),(b,f)\}$$
$$S_2 = \{c,d\}$$

- A mixed strategy is a probability distribution over the set of pure strategies of the respective player.
- In the example:

$$\Delta(S_1) = \left\{ \begin{pmatrix} (a,e) & (a,f) & (b,e) & (b,f) \\ p & q & r & 1-p-q-r \end{pmatrix} : p,q,r \in [0,1] \text{ and } p+q+r \leq 1 \right\}$$

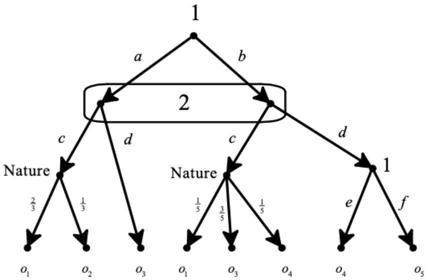
$$\Delta(S_2) = \left\{ \begin{pmatrix} c & d \\ p & 1-p \end{pmatrix} : p \in [0,1] \text{ and } p \leq 1 \right\}$$

BEHAVIOURAL STRATEGIES

Local Randomizations

- Another kind of randomization is conceivable in the tree: a player could locally mix between his choices at a given information set.
- Bundling together such a local randomization for every information set also provides a complete contingent plan.
- A behavioural strategy is a list of probability distributions over the set of local choices, one for every information set of the player.
- The set of behavioural strategies of a player $i \in I$ is denoted by B_i with generic element $\beta_i \in \mathrm{B}_i$.

Illustration



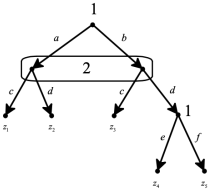
$$\mathrm{B}_1 = \left\{ \left(\begin{pmatrix} a & b \\ p & 1-p \end{pmatrix}, \begin{pmatrix} e & f \\ q & 1-q \end{pmatrix} \right) : p, q \in [0,1] \text{ and } p, q \leq 1 \right\}$$

$$\mathrm{B}_2 = \left\{ \begin{pmatrix} c & d \\ p & 1-p \end{pmatrix} : p \in [0,1] \text{ and } p \leq 1 \right\}$$

Behavioural versus Mixed

- Both behavioural strategies as well as mixed strategies constitute randomized choices.
- In fact, behavioural strategies are the simpler objects.
- In the preceding example, a behavioural strategy for Player 1 requires specifying two parameters (p and q).
- In contrast, a mixed strategy for Player 1 requires specifying three parameters (p, q, as well as r).
- It would thus be convenient to use behavioural strategies rather than mixed strategies: would that always be possible?

Illustration



- Consider the mixed strategy profile $(\sigma_1, \sigma_2) = \left(\begin{pmatrix} (a,e) & (a,f) & (b,e) & (b,f) \\ \frac{1}{12} & \frac{4}{12} & \frac{2}{12} & \frac{5}{12} \end{pmatrix}, \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$.
- The probabilities of reaching the five terminal nodes – if (σ_1, σ_2) is played – can be computed as follows:

$$\text{Prob}(z_1) = \sigma_1(a,e) \cdot \sigma_2(c) + \sigma_1(a,f) \cdot \sigma_2(c) = \frac{1}{12} \cdot \frac{1}{3} + \frac{4}{12} \cdot \frac{1}{3} = \frac{5}{36}$$

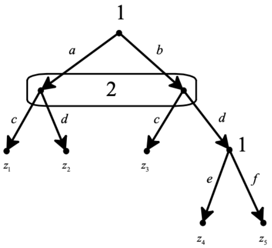
$$\text{Prob}(z_2) = \sigma_1(a,e) \cdot \sigma_2(d) + \sigma_1(a,f) \cdot \sigma_2(d) = \frac{1}{12} \cdot \frac{2}{3} + \frac{4}{12} \cdot \frac{2}{3} = \frac{10}{36}$$

$$\text{Prob}(z_3) = \sigma_1(b,e) \cdot \sigma_2(c) + \sigma_1(b,f) \cdot \sigma_2(c) = \frac{2}{12} \cdot \frac{1}{3} + \frac{5}{12} \cdot \frac{1}{3} = \frac{7}{36}$$

$$\text{Prob}(z_4) = \sigma_1(b,e) \cdot \sigma_2(d) = \frac{2}{12} \cdot \frac{2}{3} = \frac{4}{36}$$

$$\text{Prob}(z_5) = \sigma_1(b,f) \cdot \sigma_2(d) = \frac{5}{12} \cdot \frac{2}{3} = \frac{10}{36}$$

Illustration



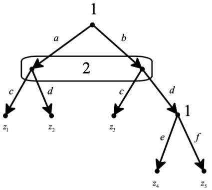
Thus, the mixed strategy profile

$$(\sigma_1, \sigma_2) = \left(\begin{pmatrix} (a,e) & (a,f) & (b,e) & (b,f) \\ \frac{1}{12} & \frac{4}{12} & \frac{2}{12} & \frac{5}{12} \end{pmatrix}, \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$$

induces the following probability distribution over terminal nodes:

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36} \end{pmatrix}$$

Illustration



- Consider the behavioural strategy profile

$$(\beta_1, \beta_2) = \left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7} \end{pmatrix} \right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$$

- The probabilities of reaching the five terminal nodes – if (β_1, β_2) is played – can be computed as follows:

$$\text{Prob}(z_1) = \beta_1(a) \cdot \beta_2(c) = \frac{5}{12} \cdot \frac{1}{3} = \frac{5}{36}$$

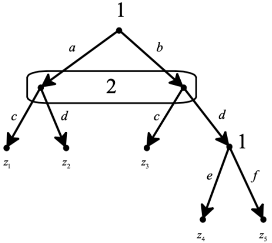
$$\text{Prob}(z_2) = \beta_1(a) \cdot \beta_2(d) = \frac{5}{12} \cdot \frac{2}{3} = \frac{10}{36}$$

$$\text{Prob}(z_3) = \beta_1(b) \cdot \beta_2(c) = \frac{7}{12} \cdot \frac{1}{3} = \frac{7}{36}$$

$$\text{Prob}(z_4) = \beta_1(b) \cdot \beta_2(d)\beta_1(e) = \frac{7}{12} \cdot \frac{2}{3} \cdot \frac{2}{7} = \frac{4}{36}$$

$$\text{Prob}(z_5) = \beta_1(b) \cdot \beta_2(d) \cdot \beta_1(f) = \frac{7}{12} \cdot \frac{2}{3} \cdot \frac{5}{7} = \frac{10}{36}$$

Illustration



Thus, the mixed strategy profile

$$(\sigma_1, \sigma_2) = \left(\begin{pmatrix} (a,e) & (a,f) & (b,e) & (b,f) \\ \frac{1}{12} & \frac{4}{12} & \frac{2}{12} & \frac{5}{12} \end{pmatrix}, \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$$

and the behavioural strategy profile

$$(\beta_1, \beta_2) = \left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7} \end{pmatrix} \right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$$

induce the **same** probability distribution over terminal nodes:

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36} \end{pmatrix}$$

General Equivalence between Behavioural and Mixed Strategies whenever Perfect Recall holds

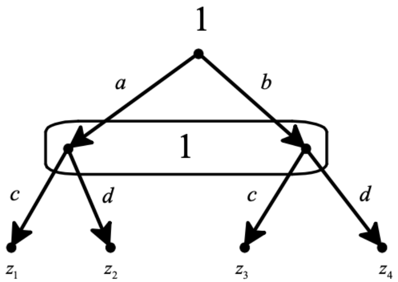
Theorem 2 (Kuhn, 1953)

Let $\mathcal{G}^{\mathcal{E}} = \langle \mathcal{F}^{\mathcal{E}}, (\succsim_i)_{i \in I} \rangle$ be a cardinal extensive-form game with perfect recall and $i \in I$ some player. Consider an arbitrary strategy profile x_{-i} of i's opponents, where for every $j \in I \setminus \{i\}$ it is the case that $x_j \in \Delta(S_j) \cup \mathrm{B}_j$. Then, for every mixed strategy $\sigma_i \in \Delta(S_i)$ of player i there exists a behavioural strategy $\beta_i \in \mathrm{B}_i$ of player i such that (σ_i, x_{-i}) and (β_i, x_{-i}) induce the same probability distribution over Z.

- In words, behavioural and mixed strategies are **equivalent**, in the sense that, every mixed strategy can be mimicked by a behavioural strategy to yield the **same** probability distribution over terminal nodes.
- Thus, attention can be restricted to the simpler objects of behavioural strategies in the case of perfect recall.

Without Perfect Recall the Equivalence Collapses

- Consider the following extensive-form frame without perfect recall:



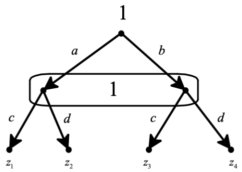
- The mixed strategy

$$\begin{pmatrix} (a,c) & (a,d) & (b,c) & (b,d) \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

induces as probability distribution over terminal nodes:

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Without Perfect Recall the Equivalence Collapses



- Let $\left(\begin{pmatrix} a & b \\ p & 1-p \end{pmatrix}, \begin{pmatrix} c & d \\ q & 1-q \end{pmatrix} \right)$ be a an arbitrary behavioural strategy.
- Its induced probability distribution over terminal nodes is as follows:

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ p \cdot q & p \cdot (1-q) & (1-p) \cdot q & (1-p) \cdot (1-q) \end{pmatrix}$$

- In order to have $\text{Prob}(z_2) = 0$ it must be the case that either $p = 0$ or $q = 1$.
- However, if $p = 0$, then $\text{Prob}(z_1) = 0$. And, if $q = 1$, then $\text{Prob}(z_4) = 0$.
- Therefore, the probability distribution over terminal nodes $\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ of the mixed strategy $\begin{pmatrix} (a,c) & (a,d) & (b,c) & (b,d) \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ cannot be obtained with any behavioural strategy.

Representation in terms of Utilities

- As usual, it is convenient to represent preferences that are in line with the vNM axioms by means of vNM utility functions.
- The basic outcomes in the tree can then be replaced by vectors of utilities, one utility for every player.
- The ensuing framework can then be pinned down as reduced cardinal extensive-form games:

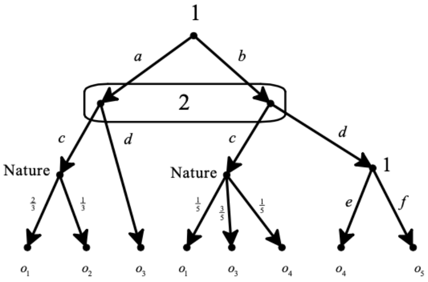
Definition 3

Let $\mathcal{G}^{\mathcal{E}} = \langle \mathcal{F}^{\mathcal{E}}, (U_i)_{i \in I} \rangle$ be cardinal extensive-form game. Suppose that $U_i : O \to \mathbb{R}$ is a vNM utility function that represents $\succsim_i$ for every player $i \in I$. A reduced cardinal extensive-form game is a tuple $\mathcal{G}^{\mathcal{E}*} = \langle \mathcal{T}, I, \alpha_I, A, \alpha_A, (D_i, \pi_i)_{i \in I} \rangle$, where $\pi_i : Z \to \mathbb{R}$ such that

$$\pi_i(z) := \mathbb{E}\Big(U_i\big(\alpha_O(z)\big)\Big)$$

for all $z \in Z$ is player i's vNM payoff function for all $i \in I$.

Illustration



- Suppose that the players satisfy the vNM axioms and hold the following preferences:

$$o_1 \succ_1 o_5 \succ_1 o_2 \succ_1 o_4 \succ_1 o_3$$

$$o_2 \succ_2 o_4 \succ_2 o_3 \succ_2 o_1 \succ_2 o_5$$

- Represent these by vNM utility functions as follows:

$$U_1(o_1) = 5, U_1(o_5) = 3, U_1(o_2) = 2, U_1(o_4) = 1, U_1(o_3) = 0$$

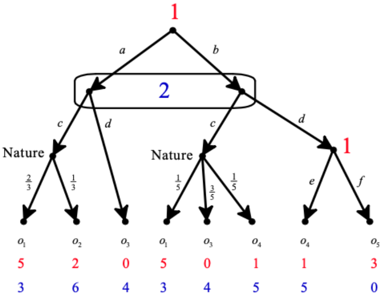
$$U_2(o_2) = 6, U_2(o_4) = 5, U_2(o_3) = 4, U_2(o_1) = 3, U_2(o_5) = 0$$

Computing Payoffs with Behavioural Strategies

- Given a cardinal extensive-form game, associated with every behavioural strategy profile is a lottery over basic outcomes.

- Via the vNM utility functions, a payoff for every player ensues.

Illustration

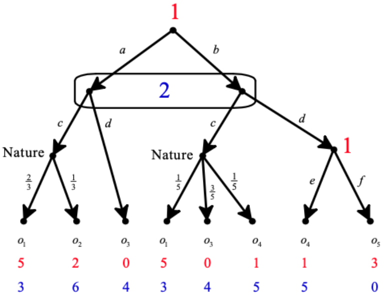


- The behavioural strategy profile $\left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12} \end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7} \end{pmatrix}\right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right)$ gives rise to the lottery $\begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ \frac{71}{540} & \frac{25}{540} & \frac{213}{540} & \frac{81}{540} & \frac{150}{540} \end{pmatrix}$.

- For example, the probability of the basic outcome o_1 is computed as follows:

$$\text{Prob}(o_1) = \text{Prob}(a) \cdot \text{Prob}(c) \cdot \frac{2}{3} + \text{Prob}(b) \cdot \text{Prob}(c) \cdot \frac{1}{5} = \frac{5}{12} \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{7}{12} \cdot \frac{1}{3} \cdot \frac{1}{5} = \frac{71}{540}$$

Illustration

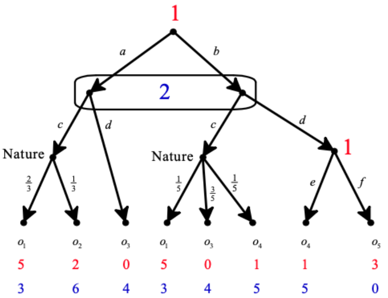


- The behavioural strategy profile $\left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12}\end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7}\end{pmatrix}\right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\right)$ gives rise to the lottery $\begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ \frac{71}{540} & \frac{25}{540} & \frac{213}{540} & \frac{81}{540} & \frac{150}{540}\end{pmatrix}$.
- Consequently,

$$\mathbb{E}\pi_1\left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12}\end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7}\end{pmatrix}\right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\right) = \frac{71}{540}\cdot 5 + \frac{25}{540}\cdot 2 + \frac{213}{540}\cdot 0 + \frac{81}{540}\cdot 1 + \frac{150}{540}\cdot 3 = \frac{936}{540}$$

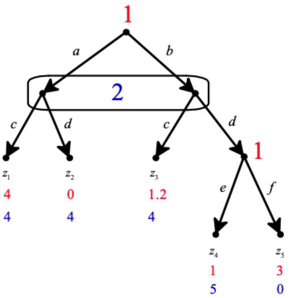
$$\mathbb{E}\pi_2\left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12}\end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7}\end{pmatrix}\right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\right) = \frac{71}{540}\cdot 3 + \frac{25}{540}\cdot 6 + \frac{213}{540}\cdot 4 + \frac{81}{540}\cdot 5 + \frac{150}{540}\cdot 0 = \frac{1620}{540}$$

Illustration



- Further simplifications are possible.
- Since $\mathbb{E}\left(U_1\begin{pmatrix} o_1 & o_2 \\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}\right) = \mathbb{E}\left(U_2\begin{pmatrix} o_1 & o_2 \\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}\right) = 4$, the first "decision node" by Nature can be replaced by the payoff vector $(4, 4)$.
- Since $\mathbb{E}\left(U_1\begin{pmatrix} o_1 & o_2 & o_3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5}\end{pmatrix}\right) = 1.2$ and $\mathbb{E}\left(U_2\begin{pmatrix} o_1 & o_2 & o_3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5}\end{pmatrix}\right) = 4$, the second "decision node" by Nature can be replaced by the payoff vector $(1.2, 4)$.

Illustration



- The behavioural strategy profile $\left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12}\end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7}\end{pmatrix}\right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\right)$ gives rise to the lottery $\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36}\end{pmatrix}$.

- Consequently,

$$\mathbb{E}\pi_1\left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12}\end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7}\end{pmatrix}\right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\right) = \frac{5}{36}\cdot 4 + \frac{10}{36}\cdot 0 + \frac{7}{36}\cdot 1.2 + \frac{4}{36}\cdot 1 + \frac{10}{36}\cdot 3 = \frac{936}{540}$$

$$\mathbb{E}\pi_2\left(\left(\begin{pmatrix} a & b \\ \frac{5}{12} & \frac{7}{12}\end{pmatrix}, \begin{pmatrix} e & f \\ \frac{2}{7} & \frac{5}{7}\end{pmatrix}\right), \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\right) = \frac{5}{36}\cdot 4 + \frac{10}{36}\cdot 4 + \frac{7}{36}\cdot 4 + \frac{4}{36}\cdot 5 + \frac{10}{36}\cdot 0 = \frac{1620}{540}$$

Subgame Perfect Equilibrium

Existence

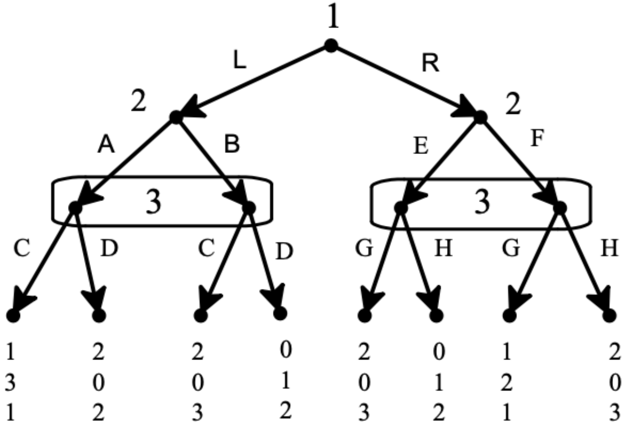
- With ordinal payoffs, a SPE may fail to exist (cf. **T3**).
- Indeed, the entire game or some proper subgame could possibly have no PSNE.
- With cardinal payoffs, it is possible to use randomized choices and then Nash's Existence Theorem applies to all subgames.
- Consequently, a SPE always exists too in finite dynamic games with cardinal payoffs.

SPE with Randomized Strategies Always Exist

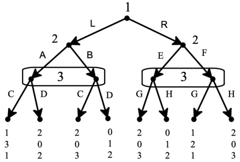
Theorem 4 (Selten, 1965)

Let $\mathcal{G}^{\mathcal{E}} = \left\langle \mathcal{F}^{\mathcal{E}}, (\succsim_i)_{i \in I} \right\rangle$ be a finite cardinal extensive-form game with perfect recall. Then, $SPE \neq \emptyset$.

Illustration



Illustration



- Consider the minimal subgame starting at Player 2's decision node on the left and construct its corresponding strategic form:

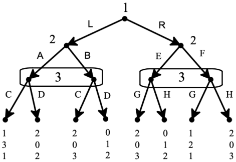
		Player 3	
		C	D
Player 2	A	3, 1	0, 2
	B	0, 3	1, 2

- Since $PSNE = \emptyset$, the SPE algorithm would halt in a framework with ordinal payoffs and spit out $SPE = \emptyset$.
- Assuming cardinal payoffs, $MSNE = \left\{ \left(\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right) \right\}$ can be obtained using PI by the following computations:

$$1 \cdot p + 3 \cdot (1-p) = 2 \cdot p + 2 \cdot (1-p) \quad \text{that is} \quad p = \frac{1}{2}$$

$$3 \cdot q + 0 \cdot (1-q) = 0 \cdot q + 1 \cdot (1-q) \quad \text{that is} \quad q = \frac{1}{4}$$

Illustration



Player 3

Player 2		C	D
	A	3, 1	0, 2
	B	0, 3	1, 2

In the MSNE $\left(\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right)$ the payoffs of all **three** players are as follows:

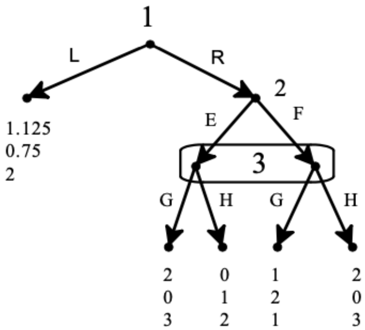
$$\pi_1\left(\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right) = \frac{1}{2}\cdot\frac{1}{4}\cdot 1 + \frac{1}{2}\cdot\frac{3}{4}\cdot 2 + \frac{1}{2}\cdot\frac{1}{4}\cdot 2 + \frac{1}{2}\cdot\frac{3}{4}\cdot 0 = 1.125$$

$$\pi_2\left(\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right) = \frac{1}{2}\cdot\frac{1}{4}\cdot 3 + \frac{1}{2}\cdot\frac{3}{4}\cdot 0 + \frac{1}{2}\cdot\frac{1}{4}\cdot 0 + \frac{1}{2}\cdot\frac{3}{4}\cdot 1 = 0.75$$

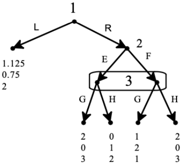
$$\pi_3\left(\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right) = \frac{1}{2}\cdot\frac{1}{4}\cdot 1 + \frac{1}{2}\cdot\frac{3}{4}\cdot 2 + \frac{1}{2}\cdot\frac{1}{4}\cdot 3 + \frac{1}{2}\cdot\frac{3}{4}\cdot 2 = 2$$

Illustration

The tree thus simplifies as follows:



Illustration



- Next consider the minimal subgame starting at Player 2's decision node on the right and construct its corresponding strategic form:

		Player 3 G	Player 3 H
Player 2	E	0, 3	1, 2
	F	2, 1	0, 3

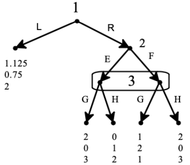
- Since again $PSNE = \emptyset$, the SPE algorithm would halt in a framework with ordinal payoffs and spit out $SPE = \emptyset$.

- Assuming cardinal payoffs however, $MSNE = \left\{ \left(\begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right) \right\}$ can be obtained using PI by the following computations:

$$3 \cdot p + 1 \cdot (1-p) = 2 \cdot p + 3 \cdot (1-p) \quad \text{that is} \quad p = \frac{2}{3}$$

$$0 \cdot q + 1 \cdot (1-q) = 2 \cdot q + 0 \cdot (1-q) \quad \text{that is} \quad q = \frac{1}{3}$$

Illustration



		Player 3 G	Player 3 H
Player 2	E	0, 3	1, 2
	F	2, 1	0, 3

In the MSNE $\left(\begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$ the payoffs of all **three** players are as follows:

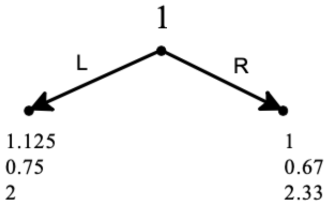
$$\pi_1\left(\begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right) = \frac{2}{3} \cdot \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{2}{3} \cdot 2 = 1$$

$$\pi_2\left(\begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right) = \frac{2}{3} \cdot \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot \frac{2}{3} \cdot 0 = 0.67$$

$$\pi_3\left(\begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right) = \frac{2}{3} \cdot \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{2}{3} \cdot 3 = 2.33$$

Illustration

- The tree thus simplifies as follows:



- The unique optimal choice for Player 1 then is L.

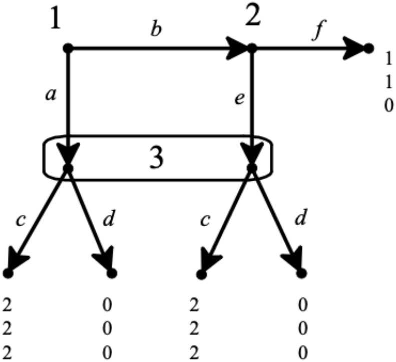
- Expressed in behavioural strategies, it follows that

$$SPE = \Big\{ \Big(\underbrace{\Big(\begin{pmatrix} L & R \\ 1 & 0 \end{pmatrix} \Big)}_{\beta_1^*}, \underbrace{\Big(\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \Big)}_{\beta_2^*}, \underbrace{\Big(\begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \Big)}_{\beta_3^*} \Big) \Big\}$$

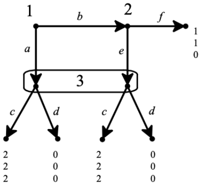
SPE not Fine Enough as a NE Refinement

- SPE constitutes a **refinement** of NE.
- In the context of perfect information, the solution concept of SPE eliminates some "unreasonable" NE involving incredible threats.
- However, in the context of imperfect information, it is possible that SPE admits "unreasonable" strategy profiles as solutions.
- After all, SPE is not fine (or strong) enough as a solution concept for imperfect information games.
- Stronger notions exist that address the deficiencies of SPE: discussing these reaches beyond our ECON322 scope though.

Selten's Horse



Selten's Horse



- There exists no proper subgame in **Selten's horse** and consequently $SPE = NE$.

c

	e	*f*
a	2,2,2	2,2,2
b	2,2,2	1,1,0

d

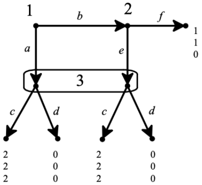
	e	*f*
a	0,0,0	0,0,0
b	0,0,0	1,1,0

- From the strategic form of **Selten's horse** it can be readily concluded that

$$PSNE = \{(a, e, c), (a, f, c), (b, e, c), (b, d, f)\}$$

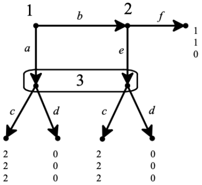
- However, neither (a, f, c) nor (b, f, d) can be considered "reasonable" solutions.

Selten's Horse



- First of all, consider the strategy profile (a, f, c).
- Player 2's plan to play f is only "reasonable" in the very limited sense that, given Player 1 chooses a it is totally irrelevant what Player 2 plans to do, as his information set is not reached.
- However, if Player 2's plan is taken seriously as to what he hypothetically were to do, if he had to move, e would be strictly better than f given Player 3 chooses c.
- Consequently, (a, e, c) qualifies as "reasonable" while (a, f, c) does not.

Selten's Horse



- Next, consider the strategy profile (b, f, d).
- Player 3's plan to play d is only "reasonable" in the very limited sense that, given Player 1 chooses a and Player 2 picks f, it is totally irrelevant what Player 3 plans to do, as his information set is not reached.
- However, if Player 3's plan is taken seriously as to what he hypothetically were to do, if he had to move, c would be strictly better than d: in fact d is strictly dominated by c at his information set locally.
- The reason that d can still be part of a NE is that it is strictly dominated by c conditional on Player 3's information set being reached, but not as a plan formulated before the actual play of the game.
- In other words, d is strictly dominated by c as a choice locally but not as a strategy globally.
- It follows that (b, f, d) does not qualify as "reasonable".

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2^{nd} Edition

- Chapter 7: **Extensive-Form Games**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html