ECON322 Game Theory

Part II Cardinal Payoffs
Topic 5 Strategic-Form Games

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Strategic-Form Games with Random Events

- In **T3** the possibility of incorporating random events in dynamic games was modelled by means of chance moves.
- In static games random events can also occur and players thus face probabilistic outcomes.
- This is modelled by allowing probabilistic outcomes (or lotteries) to be associated with strategy profiles.
- The question of how players rank probabilistic outcomes then has to be addressed.
- Expected Utility Theory from **T4** provides one possible answer.

Outline

- General Strategic Form
- Mixed Strategies
- Mixed Strategy Nash Equilibrium
- Iterated Strict Dominance

General Strategic Form

General Strategic Form Frames

Definition 1

A game frame in strategic form is a quadruple $\mathcal{F} = \langle I, (S_i)_{i \in I}, O, f \rangle$, where

- I is a set of players,
- S_i is a set of strategies for every player $i \in I$,
- O is a set of basic outcomes,
- $f : \times_{i \in I} S_i \to \mathcal{L}(O)$ is a probabilistic consequence function associating with every strategy profile $s \in \times_{i \in I} S_i$ a lottery over the set of basic outcomes $f(s) \in O$.

General Strategic Form Games

Definition 2

A game in strategic form is a pair $\mathcal{G} = \langle \mathcal{F}, (\succsim_i)_{i \in I} \rangle$, where

- $\mathcal{F} = \langle I, (S_i)_{i \in I}, O, f \rangle$ is a game frame in strategic form,
- $\succsim_i$ is a preference relation over $\mathcal{L}(O)$ satisfying AXIOMS 1 – 4 for every player $i \in I$.

General Reduced Strategic Form Games

Definition 3

Let $\mathcal{G} = \left\langle \mathcal{F}, (\succsim_i)_{i\in I} \right\rangle$ be a game in strategic form. Suppose that $U_i : O \to \mathbb{R}$ is an vNM utility function that represents $\succsim_i$ for every player $i \in I$. A reduced game in strategic form is a triple $\mathcal{G}^* = \left\langle I, (S_i)_{i\in I}, (\pi_i)_{i\in I} \right\rangle$, where $\pi_i : S \to \mathbb{R}$ such that

$$\pi_i(s) := \mathbb{E}\Big(U_i\big(f(s)\big)\Big)$$

for all $s \in \times_{j\in I} S_j$ is player i's vNM payoff function for all $i \in I$.

Illustration

- ALICE and BOB simultaneously submit a bid for a painting: either \$100 or \$200 are possible as bids.
- The higher bidder wins and has to pay his own (higher) bid.
- If both bid the same amount, then a fair coin is tossed.
- If the outcome is heads, ALICE wins and has to pay her own bid.
- If the outcome is tails, BOB wins and has to pay his own bid.

Illustration

- o_1: ALICE wins and pays \$100.
- o_2: BOB wins and pays \$100.
- o_3: BOB wins and pays \$200.
- o_4: ALICE wins and pays \$200.

BOB

ALICE		\$100	\$200
	\$100	$\begin{bmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	o_3
	\$200	o_4	$\begin{bmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Illustration

- o_1: ALICE wins and pays \$100.
- o_2: BOB wins and pays \$100.
- o_3: BOB wins and pays \$200.
- o_4: ALICE wins and pays \$200.

- Suppose the following preferences in line with AXIOMS 1 – 4:

$$o_1 \succ_{ALICE} o_4 \succ_{ALICE} o_2 \sim_{ALICE} o_3$$

$$o_2 \succ_{BOB} o_4 \succ_{BOB} o_3 \succ_{BOB} o_1$$

- Represent these preferences by the following vNM utility functions:

$$U_{ALICE}(o_1) = 4,\ U_{ALICE}(o_4) = 2,\ U_{ALICE}(o_2) = U_{ALICE}(o_3) = 1$$
$$U_{BOB}(o_2) = 6,\ U_{BOB}(o_4) = 5,\ U_{BOB}(o_3) = 4,\ U_{BOB}(o_1) = 1$$

- It follows that

$$\mathbb{E}\left(U_{ALICE}\begin{bmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = 2.5 \quad \text{and} \quad \mathbb{E}\left(U_{ALICE}\begin{bmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = 1.5$$

$$\mathbb{E}\left(U_{BOB}\begin{bmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = 3.5 \quad \text{and} \quad \mathbb{E}\left(U_{BOB}\begin{bmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = 4.5$$

Illustration

BOB

		$100	$200
ALICE	$100	$2.5, 3.5$	$1, 4$
	$200	$2, 5$	$1.5, 4.5$

Note that $NE = \emptyset$.

Mixed Strategies

Extending the Players' Choice Objects

- So far, the choice objects of the players have been their strategies, formally assembled in the set S_i for all $i \in I$.
- The strategies are sometimes also referred to as pure strategies.
- It is possible to extend the choice object space of the players, by also admitting probability distributions over their strategy sets.
- Indeed, a probability distribution over S_i is called a mixed strategy of player i and typically denoted by $\sigma_i \in \Delta(S_i)$.
- Pure strategies can be viewed as degenerate mixed strategies, that assign probability 1 to a single pure strategy.

Interpretation

- Objective Randomization: instead of choosing a strategy himself, a player delegates the choice to a random device.
- Others' Beliefs: the probabilities reflect the opponents' uncertainty about a player's choice.
- If mixed strategies are admitted, then the framework must admit probabilistic outcomes and consequently cardinal payoffs.

Illustration

		BOB	
		\$100	\$200
ALICE	\$100	$\begin{bmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	o_3
	\$200	o_4	$\begin{bmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

- Consider a mixed strategy of ALICE such that $\sigma_{ALICE}(\$100) = \frac{1}{3}$ and $\sigma_{ALICE}(\$200) = \frac{2}{3}$.
- σ_{ALICE} could be interpreted as a decision to let, say, a die determine the bid: ALICE will roll a die and bid $\$100$ if the outcome is 1 or 2, and $\$200$ if the outcome is 3, 4, 5, or 6.
- Suppose that BOB uses a mixed strategy such that $\sigma_{BOB}(\$100) = \frac{3}{5}$ and $\sigma_{BOB}(\$200) = \frac{2}{5}$.
- Since the players rely on independent random devices, the pair $(\sigma_{ALICE}, \sigma_{BOB})$ of mixed strategies gives rise to following probabilistic outcome:

strategy profile:	$(\$100, \$100)$	$(\$100, \$200)$	$(\$200, \$100)$	$(\$200, \$200)$
outcome:	$\begin{bmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	o_3	o_4	$\begin{bmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
probability:	$\frac{1}{3} \cdot \frac{3}{5} = \frac{3}{15}$	$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$	$\frac{2}{3} \cdot \frac{3}{5} = \frac{6}{15}$	$\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$

Illustration

- By AXIOM 4 (SUBSTITUTABILITY), which establishes a relation between simple and compound lotteries, both players are indifferent between the following two lotteries:

$$\begin{bmatrix} \begin{bmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} & o_3 & o_4 & \begin{bmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ \frac{3}{15} & \frac{2}{15} & \frac{6}{15} & \frac{4}{15} \end{bmatrix} \sim \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{3}{30} & \frac{3}{30} & \frac{8}{30} & \frac{16}{30} \end{bmatrix}$$

- Consider again the vNM utility functions previously fixed, i.e.:

$$U_{ALICE}(o_1) = 4,\ U_{ALICE}(o_4) = 2,\ U_{ALICE}(o_2) = U_{ALICE}(o_3) = 1$$

$$U_{BOB}(o_2) = 6,\ U_{BOB}(o_4) = 5,\ U_{BOB}(o_3) = 4,\ U_{BOB}(o_1) = 1$$

- The following expected utilities then follow:

$$\mathbb{E}\left(U_{ALICE}\begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{3}{30} & \frac{3}{30} & \frac{8}{30} & \frac{16}{30} \end{bmatrix}\right) = \frac{3}{30}\cdot 4 + \frac{3}{30}\cdot 1 + \frac{8}{30}\cdot 1 + \frac{16}{30}\cdot 2 = \frac{55}{30}$$

$$\mathbb{E}\left(U_{BOB}\begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{3}{30} & \frac{3}{30} & \frac{8}{30} & \frac{16}{30} \end{bmatrix}\right) = \frac{3}{30}\cdot 1 + \frac{3}{30}\cdot 6 + \frac{8}{30}\cdot 4 + \frac{16}{30}\cdot 5 = \frac{133}{30}$$

- Thus, the players' expected payoffs from the mixed strategy profile $(\sigma_{ALICE}, \sigma_{BOB})$ can be constructed as follows, where $\sigma_{ALICE}(\$100) = \frac{1}{3}$, $\sigma_{ALICE}(\$200) = \frac{2}{3}$, $\sigma_{BOB}(\$100) = \frac{3}{5}$, and $\sigma_{BOB}(\$200) = \frac{2}{5}$:

$$\mathbb{E}\pi_{ALICE}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{55}{30} \quad \text{and} \quad \mathbb{E}\pi_{BOB}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{133}{30}$$

Illustration

- The payoffs $\mathbb{E}\pi_{ALICE}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{55}{30}$ and $\mathbb{E}\pi_{BOB}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{133}{30}$ can also be computed in a different – yet equivalent – way based on the corresponding reduced game in strategic form.

		BOB	
		\$100	\$200
ALICE	\$100	$2.5, 3.5$	$1, 4$
	\$200	$2, 5$	$1.5, 4.5$

- Accordingly:

strategy profile:	(\$100, \$100)	(\$100, \$200)	(\$200, \$100)	(\$200, \$200)
expected utilities:	$(2.5, 3.5)$	$(1, 4)$	$(2, 5)$	$(1.5, 4.5)$
probability:	$\frac{1}{3} \cdot \frac{3}{5} = \frac{3}{15}$	$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$	$\frac{2}{3} \cdot \frac{3}{5} = \frac{6}{15}$	$\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$

- The players' expected payoffs from the mixed strategy profile $(\sigma_{ALICE}, \sigma_{BOB})$ then ensue as follows:

$$\mathbb{E}\pi_{ALICE}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{3}{15} \cdot 2.5 + \frac{2}{15} \cdot 1 + \frac{6}{15} \cdot 2 + \frac{4}{15} \cdot 1.5 = \frac{55}{30}$$

$$\mathbb{E}\pi_{BOB}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{3}{15} \cdot 3.5 + \frac{2}{15} \cdot 4 + \frac{6}{15} \cdot 5 + \frac{4}{15} \cdot 4.5 = \frac{133}{30}$$

Notation

- Let $\sigma \in \times_{i\in I}\big(\Delta(S_i)\big)_{i\in I}$ be a mixed strategy profile.
- Let $s \in \times_{i\in I} S_i$ be a (pure) strategy profile.
- The probability $\sigma(s) := \Pi_{i=1}^{n} \sigma_i(s_i) = \sigma_1(s_1)\cdot\sigma_2(s_2)\cdot\ldots\cdot\sigma_n(s_n)$ denotes the product of the probabilities $\sigma_i(s_i)$ for all $i \in I$.
- Let $\mathcal{G}^*$ a reduced game in strategic form and $i \in I$ some player.
- The payoff functions $\pi_i : \times_{j\in I} S_j \to \mathbb{R}$ are then extended to expected payoff functions $\mathbb{E}\pi_i : \times_{j\in I}\big(\Delta(S_j)\big)_{j\in I} \to \mathbb{R}$ for mixed strategies as follows:

$$\mathbb{E}\pi_i(\sigma) := \sum_{s\in\times_{j\in I} S_j} \sigma(s)\cdot\pi_i(s)$$

for all $\sigma \in \times_{i\in I}\big(\Delta(S_i)\big)_{i\in I}$.

Mixed Strategy Nash Equilibrium

Generalizing the Idea of NE with Mixed Strategies

Definition 4

Let $\mathcal{G}^* = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a reduced game in strategic form and $\sigma \in \times_{i \in I} \big(\Delta(S_i)\big)_{i \in I}$ be some mixed strategy profile. The mixed strategy profile σ forms a Nash Equilibrium, whenever

$$\mathbb{E}\pi_i(\sigma) \geq \mathbb{E}\pi_i(\sigma_i', \sigma_{-i})$$

holds for all $\sigma_i' \in \Delta(S_i)$ and for all $i \in I$. The set of all such strategy profiles is denoted by NE.

- Nash Equilibrium with pure strategies obtains as a special case, if attention is restricted to degenerate mixed strategies.
- The set of mixed strategy Nash Equilibria is also denoted by $MSNE$ and the set of pure strategy Nash Equilibria by $PSNE$.

Illustration

		BOB	
		\$100	\$200
ALICE	\$100	25, 35	10, 40
	\$200	20, 50	15, 45

- Does $(\sigma_{ALICE}, \sigma_{BOB})$ with $\sigma_{ALICE} = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $\sigma_{BOB} = \begin{pmatrix} \$100 & \$200 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$ form a NE?
- Note that
$$\mathbb{E}\pi_{ALICE}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{3}{15} \cdot 25 + \frac{2}{15} \cdot 10 + \frac{6}{15} \cdot 20 + \frac{4}{15} \cdot 15 = \frac{55}{3}.$$
- However, if ALICE switches to $\hat{\sigma}_{ALICE} = \begin{pmatrix} \$100 & \$200 \\ 1 & 0 \end{pmatrix}$, then her payoff becomes
$$\mathbb{E}\pi_{ALICE}(\hat{\sigma}_{ALICE}, \sigma_{BOB}) = \frac{3}{5} \cdot 25 + 0 \cdot 20 + \frac{2}{5} \cdot 10 + 0 \cdot 15 = 19.$$
- As
$$\mathbb{E}\pi_{ALICE}(\sigma_{ALICE}, \sigma_{BOB}) = \frac{55}{3} < 19 = \mathbb{E}\pi_{ALICE}(\sigma'_{ALICE}, \sigma_{BOB}),$$
the pair $(\sigma_{ALICE}, \sigma_{BOB})$ does not form a NE.

Illustration

		BOB	
		\$100	\$200
ALICE	\$100	25, 35	10, 40
	\$200	20, 50	15, 45

- Now, consider $(\sigma^*_{ALICE}, \sigma^*_{BOB})$ with $\sigma^*_{ALICE} = \sigma^*_{BOB} = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
- Note that $\mathbb{E}\pi_{ALICE}(\sigma^*_{ALICE}, \sigma^*_{BOB}) = \frac{1}{4} \cdot 25 + \frac{1}{4} \cdot 10 + \frac{1}{4} \cdot 20 + \frac{1}{4} \cdot 15 = \frac{70}{4} = 17.5$.
- Could ALICE possibly obtain a larger payoff with some other mixed strategy $\sigma_{ALICE} = \begin{pmatrix} \$100 & \$200 \\ p & 1-p \end{pmatrix}$ such that $p \in [0, 1] \setminus \{\frac{1}{2}\}$?

$$\mathbb{E}\pi_{ALICE}(\sigma_{ALICE}, \sigma^*_{BOB}) = \frac{1}{2} \cdot p \cdot 25 + \frac{1}{2} \cdot p \cdot 10 + \frac{1}{2} \cdot (1-p) \cdot 20 + \frac{1}{2} \cdot (1-p) \cdot 15$$

$$= p \cdot (\frac{1}{2} \cdot 25 + \frac{1}{2} \cdot 10) + (1-p) \cdot (\frac{1}{2} \cdot 20 + \frac{1}{2} \cdot 15) = \frac{35}{2} = 17.5.$$

- Thus, against σ^*_{BOB} any mixed strategy of ALICE yields the same expected payoff, and consequently all mixed strategies of ALICE are best resonses to σ^*_{BOB}.
- It can be verified that the same applies to BOB: any mixed strategy of his yields same expected payoff against σ^*_{ALICE}, and consequently all mixed strategies of ALICE are best resonses to σ^*_{ALICE}.
- Therefore, $(\sigma^*_{ALICE}, \sigma^*_{BOB}) \in NE$.

NE with Mixed Strategies Always Exist

Theorem 5 (Nash, 1951)

Let $\mathcal{G}^* = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a reduced game in strategic form such that S_i is finite for all $i \in I$. Then, $NE \neq \emptyset$.

How To Find the NE in a given Game?

- First, all pure strategies ruled out by ISD can be discarded.
- Attention can thus be restricted to the reduced game given by $ISD \subseteq \times_{i \in I} S_i$.
- The NE of the reduced game will also be NE of the original game where all strategies outside the set ISD receive zero probability.
- The Principle of Indifference (PI) can then be used to identify the NE in the reduced game.
- **Remark:** The support ("supp") of a probability distribution is a set containing all objects that receive positive probability.

Principle of Indifference

Principle of Indifference (PI)

Let $\mathcal{G}^* = \langle I, (S_j)_{j \in I}, (\pi_j)_{j \in I} \rangle$ *be a reduced game in strategic form,* $(\sigma_j^*)_{j \in I} \in NE$ *some mixed strategy Nash equilibrium, and* $i \in I$ *some player. Then,*

$$\mathbb{E}\pi_i(s_i, \sigma_{-i}^*) = \mathbb{E}\pi_i(\sigma^*)$$

for all $s_i \in \mathrm{supp}(\sigma_i^*)$*.*

Intuition

- Towards a contradiction, let $s_i, s_i' \in \text{supp}(\sigma_i^*)$ such that $\mathbb{E}\pi_i(s_i, \sigma_{-i}^*) > \mathbb{E}\pi_i(s_i', \sigma_{-i}^*)$.
- Player i can then increase his expected payoff by reducing $\sigma_i^*(s_i')$ to zero and adding that value to $\sigma_i^*(s_i)$.
- Indeed, define a mixed strategy $\hat{\sigma}_i$ by $\hat{\sigma}_i(s_i) := \sigma_i^*(s_i) + \sigma_i^*(s_i')$, $\hat{\sigma}_i(s_i') := 0$, and $\hat{\sigma}_i(s_i'') := \sigma_i^*(s_i'')$ for all $s_i'' \in S_i \setminus \{s_i, s_i'\}$.
- It follows that $\mathbb{E}\pi_i(\hat{\sigma}_i, \sigma_{-i}^*) > \mathbb{E}\pi_i(\sigma^*)$, contradicting that $\sigma^* \in NE$.
- Therefore, all $s_i \in \text{supp}(\sigma_i^*)$ induces the same expected payoff.
- It follows that σ_i^* as a convex combination of these same expected payoffs also induces this same expected payoff.

Illustration

		Colin		
		E	F	G
Rowena	A	$2, 4$	$3, 3$	$6, 0$
	B	$4, 0$	$2, 4$	$4, 2$
	C	$3, 3$	$4, 2$	$3, 1$
	D	$3, 6$	$1, 1$	$2, 6$

- First of all, note that $PSNE = \emptyset$ and that $ISD = \{B, C\} \times \{E, F\}$.

		Colin	
		E	F
Rowena	B	$4, 0$	$2, 4$
	C	$3, 3$	$4, 2$

- Note that in the reduced game $PSNE = \emptyset$ also holds.
- Next, $p, q \in (0, 1)$ have to be deternined such that

$$(\sigma^*_{Rowena}, \sigma^*_{Colin}) = \left(\begin{pmatrix} B & C \\ p & 1-p \end{pmatrix}, \begin{pmatrix} E & F \\ q & 1-q \end{pmatrix} \right) \in NE$$

Illustration

		Colin E	Colin F
Rowena	B	4, 0	2, 4
	C	3, 3	4, 2

- By PI, it needs to be the case that $\mathbb{E}\pi_{Rowena}(B, \sigma^*_{Colin}) = \mathbb{E}\pi_{Rowena}(C, \sigma^*_{Colin})$, i.e.:

$$\mathbb{E}\pi_{Rowena}(B, \sigma^*_{Colin}) = 4 \cdot q + 2 \cdot (1-q) = 3 \cdot q + 4 \cdot (1-q) = \mathbb{E}\pi_{Rowena}(C, \sigma^*_{Colin})$$

$$q = \frac{2}{3}$$

- Thus, B and C as well as any mixture between B and C yield an expected payoff of $\frac{10}{3}$ to Rowena: consequently any mixed strategy is a best response for Rowena against $\sigma*_{Colin} = \begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.
- By PI, it needs to be the case that $\mathbb{E}\pi_{Colin}(\sigma^*_{Rowena}, E) = \mathbb{E}\pi_{Colin}(\sigma^*_{Rowena}, F)$, i.e.:

$$\mathbb{E}\pi_{Colin}(\sigma^*_{Rowena}, E) = 0 \cdot p + 3 \cdot (1-p) = 4 \cdot p + 2 \cdot (1-p) = \mathbb{E}\pi_{Colin}(\sigma^*_{Rowena}, F)$$

$$p = \frac{1}{5}$$

- Thus, E and F as well as any mixture between E and F yield an expected payoff of $\frac{12}{5}$ to Colin: consequently any mixed strategy is a best response for Colin against $\sigma*_{Rowena} = \begin{pmatrix} B & C \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$.
- It follows that $(\sigma^*_{Rowena}, \sigma^*_{Colin}) = \left(\begin{pmatrix} A & B & C & D \\ 0 & \frac{1}{5} & \frac{4}{5} & 0 \end{pmatrix}, \begin{pmatrix} E & F & G \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \right) \in NE$ in the original game.

The Principle of Indifference is a Necessary but Not Sufficient Condition for MSNE

- NE implies PI (“necessary condition”).
- However, PI does not imply NE (“sufficient condition”).

Illustration

		Colin	
		D	E
Rowena	A	3, 0	0, 2
	B	0, 2	3, 0
	C	2, 0	2, 1

- Consider the mixed strategy profile $\sigma = (\sigma_{Rowena}, \sigma_{Colin}) = \left(\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right)$

- Given σ_{Rowena},Colin is indifferent between D and E, as both these pure strategies induce an expected payoff of 1, which is also the expected payoff induced by the mixed strategy σ_{Colin}.

- Given σ_{Colin}, Rowena is indifferent between A and B, as both these pure strategies induce an expected payoff of 1.5, which is also the expected payoff induced by the mixed strategy σ_{Rowena}.

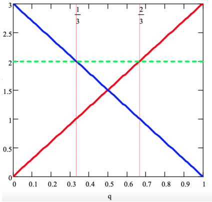
- However, σ does not form a Nash Equilibrium, as Rowena could get an expected payoff of 2 by switching to $\hat{\sigma}_{Rowena} = \begin{pmatrix} A & B & C \\ 0 & 0 & 1 \end{pmatrix}$.

Illustration

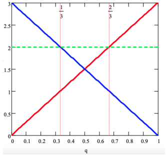
		Colin D	Colin E
Rowena	A	3, 0	0, 2
Rowena	B	0, 2	3, 0
Rowena	C	2, 0	2, 1

- What are the Nash Equilibria of this game then?
- Against an arbitrary mixed strategy $\sigma_{Colin} = \begin{pmatrix} D & E \\ q & 1-q \end{pmatrix}$, Rowena's expected payoffs for her pure strategies are as follows:

$$\mathbb{E}\pi_{Rowena}(A, \sigma_{Colin}) = 3 \cdot q + 0 \cdot (1-q) = 3 \cdot q \quad \text{(solid red line)}$$
$$\mathbb{E}\pi_{Rowena}(B, \sigma_{Colin}) = 0 \cdot q + 3 \cdot (1-q) = 3 - 3 \cdot q \quad \text{(solid blue line)}$$
$$\mathbb{E}\pi_{Rowena}(C, \sigma_{Colin}) = 2 \cdot q + 2 \cdot (1-q) = 2 \quad \text{(dashed green line)}$$



Illustration



- The maximum expected payoff is given by the blue line up to $q = \frac{1}{3}$, then by the green line up to $q = \frac{2}{3}$, and then by the red line.
- Thus, the best response function of Rowena is as follows:

$$BR_{Rowena}(\sigma_{Colin}) = \begin{cases} B & \text{if } 0 \leq q \leq \frac{1}{3} \\ \begin{pmatrix} B & C \\ p & 1-p \end{pmatrix} \text{ for all } p \in [0,1] & \text{if } q = \frac{1}{3} \\ C & \text{if } \frac{1}{3} < q < \frac{2}{3} \\ \begin{pmatrix} A & C \\ p & 1-p \end{pmatrix} \text{ for all } p \in [0,1] & \text{if } q = \frac{2}{3} \\ A & \text{if } \frac{2}{3} \leq q \leq 1 \end{cases}$$

- Consequently, a Nash Equilibrium takes one of the following two forms:

$$\left(\begin{pmatrix} A & B & C \\ 0 & p & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right) \text{ or } \left(\begin{pmatrix} A & B & C \\ p & 0 & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \right)$$

Illustration

		Colin D	Colin E
Rowena	A	3, 0	0, 2
	B	0, 2	3, 0
	C	2, 0	2, 1

- $\left(\begin{pmatrix} A & B & C \\ p & 0 & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right)$ cannot be a Nash Equilibrium for any $p \in [0,1]$, because if $\sigma_{Rowena}(B) = 0$, then E strictly dominates D and thus $\left(\begin{pmatrix} D & E \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right)$ is not a best response for Colin.
- Consequently, the only candidate for a Nash Equilibrium is of the form

$$\left(\begin{pmatrix} A & B & C \\ 0 & p & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right).$$

- By PI, Colin needs to be indifferent between D and E, i.e.:

$$2 \cdot p + 0 \cdot (1-p) = 0 \cdot p + 1 \cdot (1-p)$$

$$p = \frac{1}{3}$$

- Therefore,

$$\left(\begin{pmatrix} A & B & C \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right) \in NE$$

ITERATED STRICT DOMINANCE WITH MIXED STRATEGIES

Best Response & Strict Dominance with 2 Players

- In n player games, a strategy is a **best response** to a profile of opponents' mixed strategies, if it maximizes the expected payoff and the latter is computed via the **product** of the opponents' mixed strategies.

- Formally,

$$\mathbb{E}\pi_i(\sigma_i, \sigma_{-i}) := \sum_{s_i \in S_i} \sigma_i(s_i) \cdot \sum_{s_{-i} \in S_{-i}} \pi_i(s_i, s_{-i}) \cdot \Pi_{j \in I \setminus \{i\}} \sigma_j(s_j)$$

  and

$$BR_i(\sigma_{-i}) := \{\sigma_i \in \Delta(S_i) : \mathbb{E}\pi_i(\sigma_i, \sigma_{-i}) \geq \mathbb{E}\pi_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(S_i)\}$$

- In two player games, the profile of opponents' mixed strategies reduces to a single mixed strategy and the definition of expected payoff simplifies as follows:

$$\mathbb{E}\pi_i(\sigma_i, \sigma_j) := \sum_{s_i \in S_i} \sigma_i(s_i) \cdot \sum_{s_j \in S_j} \pi_i(s_i, s_j) \cdot \sigma_j(s_j)$$

- In two player games, it is the case that, if a pure strategy is a **best response** to a mixed strategy of the opponent, then it is not strictly dominated by another pure strategy.

- However, the converse does not hold.

Illustration

Colin

Rowena	D	E
A	0, 1	4, 0
B	1, 2	1, 4
C	2, 0	0, 1

- B is not strictly dominated by another pure strategy for Rowena, yet it cannot be a best response to any mixed strategy of Colin.
- To see this, consider an arbitrary mixed strategy $\sigma_{Colin} = \begin{pmatrix} D & E \\ q & 1-q \end{pmatrix}$ with $q \in [0, 1]$ of Colin.
- Observe that

$$\mathbb{E}\pi_{Rowena}(B, \sigma_{Colin}) = 1$$

and

$$\mathbb{E}\pi_{Rowena}\left(\begin{pmatrix} A & B & C \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}, \sigma_{Colin} \right) = \frac{1}{3} \cdot 4 \cdot (1-q) + \frac{2}{3} \cdot 2 \cdot q = \frac{4}{3}$$

- Since

$$\mathbb{E}\pi_{Rowena}\left(\begin{pmatrix} A & B & C \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}, \sigma_{Colin} \right) = \frac{4}{3} > 1 = \mathbb{E}\pi_{Rowena}(B, \sigma_{Colin})$$

the pure strategy B is not a best response to σ_{Colin}.

Once Mixed Strategies enter Stage, an Equivalence Result for Two Player Games ensues

Theorem 6 (Pearce, 1984)

Let $\mathcal{G}^* = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a reduced game in strategic form such that $\mid I \mid = 2$, $i \in I$ some player, and $s_i \in I$ some strategy of player i. The strategy s_i is not a best response to any mixed strategy of i's opponent, if and only if, s_i is strictly dominated by a mixed strategy of player i.

ISD with Mixed Strategies

Definition 7

Let $\mathcal{G}^* = \langle I, (S_i)_{i\in I}, (\pi_i)_{i\in I}\rangle$ be a reduced game in strategic form.

- Let ${\mathcal{G}^*}^1_{SD}$ be the game obtained by removing from $\mathcal{G}^*$, for every player $i \in I$, all those strategies of i (if any) that are strictly dominated in $\mathcal{G}^*$ by some mixed strategy.
- Let ${\mathcal{G}^*}^2_{SD}$ be the game obtained by removing from ${\mathcal{G}^*}^1_{SD}$, for every player $i \in I$, all those strategies of i (if any) that are strictly dominated in ${\mathcal{G}^*}^1_{SD}$ by some mixed strategy.
- Etc.

The final output is called Iterated Strict Dominance and denoted by ${\mathcal{G}^*}^\infty_{SD}$. The set of strategy profiles surviving step $k \geq 1$ is denoted by SD^k and the set of those that are contained in the final output by ISD.

In two player games and in a cardinal framework including mixed strategies, it can be shown that

$$NE \subseteq ISD.$$

Illustration

$\mathcal{G}^*$:

		Bob D	Bob E	Bob F
Alice	A	3, 4	2, 1	1, 2
Alice	B	0, 0	1, 3	4, 1
Alice	C	1, 4	1, 4	2, 6

In $\mathcal{G}^*$, Alice's pure strategy C is strictly dominated by $\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Illustration

${\mathcal{G}^*}^1_{SD}$:

		Bob		
		D	E	F
Alice	A	3, 4	2, 1	1, 2
	B	0, 0	1, 3	4, 1

In ${\mathcal{G}^*}^1_{SD}$, Bob's pure strategy F is strictly dominated by $\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Illustration

$\mathcal{G}^{*2}_{SD}$:

		Bob	
		D	E
Alice	A	3, 4	2, 1
	B	0, 0	1, 3

In $\mathcal{G}^{*2}_{SD}$, Alice's pure strategy B is strictly dominated by A.

Illustration

$\mathcal{G}^{*3}_{SD}$:

		Bob	
		D	E
Alice	A	3, 4	2, 1

In $\mathcal{G}^{*3}_{SD}$, Bob's pure strategy E is strictly dominated by D.

Illustration

$\mathcal{G}^{*4}_{SD}$:

		Bob
		D
Alice	A	$3, 4$

- In $\mathcal{G}^{*4}_{SD}$, no strategy is strictly dominated by any mixed strategy for neither player.
- Thus $\mathcal{G}^{*4}_{SD} = \mathcal{G}^{*\infty}_{SD}$ and Iterated Strict Dominance stops.
- The **solution** of the game then obtains as:

$$ISD = ISD_{Alice} \times ISD_{Bob} = \{A\} \times \{D\} = \{(A, D)\}$$

- Note that, since $NE \subseteq ISD$ holds with mixed strategies in the case of two players, it follows that $NE = \{(A, D)\}$.

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition

- Chapter 6: **Strategic-Form Games**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html