ECON322 Game Theory

Part II Cardinal Payoffs
Topic 4 Expected Utility Theory

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Uncertainty

- In general, outcomes can be uncertain and not deterministic.
- Uncertainty is typically modelled by probabilities.
- Probabilistic outcomes are called lotteries (cf. DYNAMIC GAMES WITH CHANCE MOVES).
- In **T4** we explore the basics of Expected Utility Theory, which deals with decision-making under uncertainty.

Outline

Attitudes to Risk

Money Lotteries

- In this section we continue to consider the specific class of money lotteries, where the outcomes are sums of money.

- Recall that a money lottery L is a probability distribution of the form

$$\begin{bmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{bmatrix}$$

where $p_i \geq 0$ for all $i \in \{1, 2, \dots, n\}$ and $p_1 + p_2 + \dots + p_n = 1$

- Its expected value is $\mathbb{E}(L) = x_1 \cdot p_1 + x_2 \cdot p_2 + \dots + x_n \cdot p_n$.

Attitudes to Risk

Definition 1

Let L be a money lottery and consider the choice between playing L and getting $\$\mathbb{E}(L)$ for certain.

- An agent is risk averse, whenever $\$\mathbb{E}(L) \succ L$.
- An agent is risk neutral, whenever $\$\mathbb{E}(L) \sim L$.
- An agent is risk loving, whenever $L \succ \$\mathbb{E}(L)$.

Illustration

- Suppose that a risk neutral agent has transitive preferences over money lotteries and prefers more money to less.

- Consider the following two lotteries

$$L_1 = \begin{bmatrix} \$30 & \$45 & \$90 \\ \frac{1}{3} & \frac{5}{9} & \frac{1}{9} \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} \$5 & \$100 \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

- Note that $\$\mathbb{E}(L_1) = 45$ and $\$\mathbb{E}(L_2) = 43$.

- Then, $L_1 \sim \$45$ and $L_2 \sim \$43$, i.e. $\$43 \sim L_2$ (as $\sim$ is symmetric).

- Since the agent prefers more money to less, $\$45 \succ \$43$, and by transitivity it follows that $L_1 \succ L_2$.

Illustration

- However, for a risk averse agent, knowing him to hold transitive preferences over money lotteries and to prefer more money to less, is **not sufficient** to always predict his choice.
- Similarly, for a risk loving agent, knowing him to hold transitive preferences over money lotteries and to prefer more money to less, is also **not sufficient** to always predict his choice.
- Expected Utility Theory is capable of covering choice under risk aversion and risk lovingness as well as more general lotteries.
- This theory will be developed in the next two sections.

De Gustibus Non Est Disputandum

- A theory of choice should not dictate which attitude to risk to hold.
- An attitude to risk is merely a reflection of individual preferences.
- Generally accepted principle:

  IN MATTERS OF TASTE, THERE CAN BE NO DISPUTES.

- Accordingly, there are no irrational preferences and hence also no irrational attitude to risk.
- Empirically, most people reveal through their choices risk aversion though, at least when the stakes are high.

Axioms

A Theory with General Lotteries

- From now onwards we consider general lotteries, where the outcomes do not need to be sums of money.

- Expected Utility Theory (EUT) was also developed by the founders of game theory in their seminal work:

  John von Neumann & Oscar Morgenstern (1944),
  "*Theory of Games and Economic Behavior*", PUP

- In this section, the assumptions of EUT are expounded, while in the next section the theory's main results are presented.

Notions and Notation

- By O a set of basic outcomes is denoted.

- These can be sums of money, an individual's health state, receiving an award or not, tomorrow's weather possibilities, etc.

- By $\mathcal{L}(O)$ the set of simple lotteries over O is denoted, where O is assumed to be finite i.e. $O = \{o_1, o_2, \ldots, o_m\}$ for some $m \in \mathbb{N}$.

- Thus, an element $L \in \mathcal{L}(O)$ is a probability distribution of the form

$$L = \begin{bmatrix} o_1 & o_2 & \ldots & o_m \\ p_1 & p_2 & \ldots & o_m \end{bmatrix}$$

with $0 \leq p_i \leq 1$ for all $i \in \{1, 2, \ldots, m\}$ and $\sum_{i=1}^{m} p_i = 1$.

Interpretational Remark

- Lotteries are used to represent situations of uncertainty.
- For example, suppose that $m = 4$ and the agent faces L, where

$$L = \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{2}{5} & 0 & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

- The agent then knows that eventually the outcome will be one and only one of o_1, o_2, o_3, o_4.
- However, the agent does not know which one.
- Still, the agent is able to quantify his uncertainty by assigning probabilities to the basic outcomes.

Simplifications

- Degenerate lotteries assign probability 1 to one basic outcome.
- To simplify notation they are typically denoted by the basic outcome they assign positive probability to.
- For instance, $\begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is denoted by o_3.
- Since the degenerate lotteries are also elements of $\mathcal{L}(O)$, a preference relation on $\mathcal{L}(O)$ induces a preference relation on O.
- Moreover, basic outcomes receiving probability 0 are often omitted.
- For instance, $\begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{bmatrix}$ is denoted by $\begin{bmatrix} o_1 & o_3 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

Basic Outcomes: best and worst

- A best basic outcome is denoted by o_{best} and has the property that
$$o_{best} \succsim o$$
for all $o \in O$.

- A worst basic outcome is denoted by o_{worst} and has the property that
$$o \succsim o_{worst}$$
for all $o \in O$.

- Note that there may possibly be several such outcomes.

- It is standard to assume that $o_{best} \succ o_{worst}$, i.e. that the agent is not indifferent among all basic outcomes.

Compound Lotteries

- A compound lottery is a lottery of the form

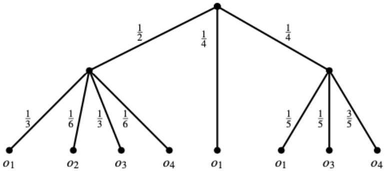
$$\begin{bmatrix} x_1 & x_2 & \dots & x_r \\ p_1 & p_2 & \dots & p_r \end{bmatrix}$$

where each $x_i \in \{O, \mathcal{L}(O)\}$ for all $i \in \{1, 2, \dots, r\}$.

- An example is the following compound lottery C:

$$C = \begin{bmatrix} \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} & o_1 & \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{bmatrix} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

- C can also be viewed graphically as a tree:



Corresponding Simple Lottery

Given a compound lottery $C = \begin{bmatrix} x_1 & x_2 & \dots & x_r \\ p_1 & p_2 & \dots & p_r \end{bmatrix}$ the corresponding simple lottery $L(C) = \begin{bmatrix} o_1 & o_2 & \dots & o_m \\ q_1 & q_2 & \dots & q_m \end{bmatrix}$ is constructed as follow:

- First of all, for every $i \in \{1, 2, \dots, m\}$ and for every $j \in \{1, 2, \dots, r\}$, define

$$o_i(x_j) := \begin{cases} 1 & \text{if } x_j = o_i \\ 0 & \text{if } x_j = o_k \text{ with } k \neq i \\ s_i & \text{if } x_j = \begin{bmatrix} o_1 & \dots & o_{i-1} & o_i & o_{i+1} & \dots & o_m \\ s_1 & \dots & s_{i-1} & s_i & s_{i+1} & \dots & s_m \end{bmatrix} \end{cases}$$

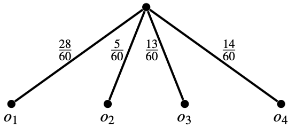
- Then, define $q_i := \sum_{j=1}^{r} p_j \cdot o_i(x_j)$.

Illustration

- Consider $C = \begin{bmatrix} \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} & o_1 & \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{bmatrix} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

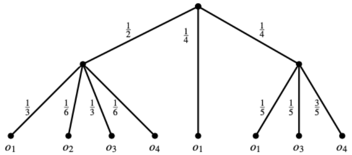
- In this case, $m = 4$, $r = 3$, $x_1 = \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$, $x_2 = o_1$, and $x_3 = \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{5} \end{bmatrix}$, so:
  - $o_1(x_1) = \frac{1}{3}, o_1(x_2) = 1, o_1(x_3) = \frac{1}{5} \implies q_1 = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{1}{5} = \frac{28}{60}$
  - $o_2(x_1) = \frac{1}{6}, o_2(x_2) = 0, o_2(x_3) = 0 \implies q_2 = \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 = \frac{1}{12} = \frac{5}{60}$
  - $o_3(x_1) = \frac{1}{3}, o_3(x_2) = 0, o_3(x_3) = \frac{1}{5} \implies q_3 = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{5} = \frac{13}{60}$
  - $o_4(x_1) = \frac{1}{6}, o_4(x_2) = 0, o_4(x_3) = \frac{3}{5} \implies q_4 = \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{3}{5} = \frac{14}{60}$

- Thus, $L(C) = \begin{bmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{28}{60} & \frac{5}{60} & \frac{13}{60} & \frac{14}{60} \end{bmatrix}$

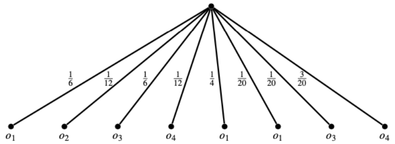


Graphical Illustration

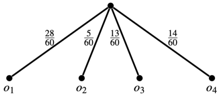
The probabilities in $L(C)$ correspond to multiplying the probabilities along the edges of the tree visualizing C:



leading to an outcome as shown in the following tree:



and then adding up the probabilities of each outcome, resulting in the tree visualizing $L(C)$:



The Four vNM Axioms (1/4): Consistency

Axiom 1 (Consistency)

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of of simple lotteries over O, and $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$. The weak preference relation $\succsim$ is complete and transitive.

The Four vNM Axioms (2/4): Monotonicity

Axiom 2 (Monotonicity)

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of of simple lotteries over O, and $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$.

$$\begin{bmatrix} o_{best} & o_{worst} \\ p & 1-p \end{bmatrix} \succsim \begin{bmatrix} o_{best} & o_{worst} \\ q & 1-q \end{bmatrix}$$

if and only if

$$p \geq q.$$

The Four vNM Axioms (3/4): Continuity

Axiom 3 (Continuity)

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of of simple lotteries over O, and $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$. For every basic outcome $o \in O$ there exists $p_o \in [0, 1]$ such that

$$o \sim \begin{bmatrix} o_{best} & o_{worst} \\ p_o & 1 - p_o \end{bmatrix}.$$

The Four vNM Axioms (4/4): Substitutability

Axiom 4 (Substitutability)

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of of simple lotteries over O, $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$, $o_i \in O$ some basic outcome, and

$$L = \begin{bmatrix} o_1 & \dots & o_{i-1} & o_i & o_{i+1} & \dots & o_m \\ p_1 & \dots & p_{i-1} & p_i & p_{i+1} & \dots & p_m \end{bmatrix}$$

some simple lottery. If $\hat{L} \in \mathcal{L}(O)$ such that $o_i \sim \hat{L}$, then $L \sim M$, where $M \in \mathcal{L}(O)$ denotes the simple lottery that corresponds to the following compound lottery

$$C = \begin{bmatrix} o_1 & \dots & o_{i-1} & \hat{L} & o_{i+1} & \dots & o_m \\ p_1 & \dots & p_{i-1} & p_i & p_{i+1} & \dots & p_m \end{bmatrix}.$$

Main Results

More Notation

- Let $f : O \to \mathbb{R}$ be a function that assigns numbers to the basic outcomes.

- Given a lottery $L = \begin{bmatrix} o_1 & o_2 & \dots & o_m \\ p_1 & p_2 & \dots & p_m \end{bmatrix}$ a transformed lottery $f(L)$ can be formed as follows

$$f(L) = \begin{bmatrix} f(o_1) & f(o_2) & \dots & f(o_m) \\ p_1 & p_2 & \dots & p_m \end{bmatrix}$$

- The expected value of $f(L)$ can then be computed:

$$\mathbb{E}\big(f(L)\big) = \sum_{i=1}^{m} p_i \cdot f(o_i)$$

Numerical Representation of Preferences

Definition 2

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of simple lotteries over O, $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$, and $f : O \to \mathbb{R}$ a function. The function f represents the preference relation $\succsim$, whenever the following property

$$L \succsim L', \text{ if and only if, } \mathbb{E}\big(f(L)\big) \geq \mathbb{E}\big(f(L')\big)$$

holds for all $L, L' \in \mathcal{L}(O)$.

Representation Theorem

Theorem 3

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of simple lotteries over O, and $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$. If $\succsim$ satisfies AXIOMS 1 – 4*, then there exists a function $U : O \to \mathbb{R}$, called utility function, that represents the preference relation $\succsim$.*

Illustration of Theorem 3's Usefulness

- Theorem 3 can sometimes be used to predict an agent's preference between two lotteries, if it is known how he ranks two different lotteries.
- For example, suppose that $\succ$ satisfies Axioms 1–4 and that $A \succ B$, where

$$A = \begin{bmatrix} o_1 & o_2 & o_3 \\ 0 & 0.25 & 0.75 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} o_1 & o_2 & o_3 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

- Consider the following two lotteries C and D, where

$$C = \begin{bmatrix} o_1 & o_2 & o_3 \\ 0.8 & 0 & 0.2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} o_1 & o_2 & o_3 \\ 0 & 1 & 0 \end{bmatrix} = o_2$$

- By Theorem 3 there then exists a utility function and let $U(o_1) = a$, $U(o_2) = b$, and $U(o_3) = c$.
- From $A \succ B$ it follows that $\mathbb{E}\big(U(A)\big) > \mathbb{E}\big(U(B)\big)$, i.e.

$$0.25 \cdot b + 0.75 \cdot c > 0.2 \cdot a + 0.8 \cdot c$$

which is equivalent to

$$b > 0.8 \cdot a + 0.2 \cdot c$$

- Since $\mathbb{E}\big(U(C)\big) = 0.8 \cdot a + 0.2 \cdot c$ and $\mathbb{E}\big(U(D)\big) = b$, it follows that $D \succ C$.

Multiplicity of Utility Functions

Theorem 4

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of simple lotteries over O, and $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$ that satisfies AXIOMS 1–4.

(i) *If $U : O \to \mathbb{R}$ represents $\succsim$, then for every $a \in \mathbb{R}^+$ and for every $b \in \mathbb{R}$, the function $V : O \to \mathbb{R}$, defined by $V(o) = a \cdot U(o) + b$ for all $o \in O$, represents $\succsim$.*

(ii) *If $U : O \to \mathbb{R}$ and $V : O \to \mathbb{R}$ both represent $\succsim$, then there exists $a \in \mathbb{R}^+$ and there exists $b \in \mathbb{R}$ such that $V(o) = a \cdot U(o) + b$ for all $o \in O$.*

Affine Transformations

- An affine transformation is a function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = a \cdot x + b$ such that $a, b \in \mathbb{R}$.
- An affine transformation is called positive, whenever $a > 0$.
- Theorem 4 (i) is often stated as follows: a utility function that represents $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ is **unique up to a positive affine transformation**.

An Application of Theorem 4 (i)

Remark: Among the utility functions representing $\succsim$ there is one assigning 1 to the best and 0 to the worst basic outcome(s).

- To see this, consider a utility function $F : O \to \mathbb{R}$ representing $\succsim$ and define $G : O \to \mathbb{R}$ s.t. $G(o) = F(o) - F(o_{worst})$ for all $o \in O$.
- By Theorem 4 (i), with $a = 1$ and $b = -F(o_{worst})$, the function G also is a utility function representing $\succsim$.
- Note that $G(o_{worst}) = F(o_{worst}) - F(o_{worst}) = 0$ (by construction) as well as $G(o_{best}) > 0$ (since $o_{best} \succ o_{worst}$).
- Define $U : O \to \mathbb{R}$ s.t. $U(o) = \frac{G(o)}{G(o_{best})}$ for all $o \in O$.
- By Theorem 4 (i), with $a = \frac{1}{G(o_{best})}$ and $b = 0$, the function U represents $\succsim$ too, where $U(o_{worst}) = 0$ and $U(o_{best}) = 1$ holds.

Illustration

- Let $O = \{o_1, o_2, o_3, o_4, o_5, o_6\}$ and $o_3 \sim o_6 \succ o_1 \succ o_4 \succ o_2 \sim o_5$.
- Fix $o_{best} = o_3$ and $o_{worst} = o_2$.
- Consider some utility function $F : O \to \mathbb{R}$ such that $F(o_1) = 2$, $F(o_2) = -2$, $F(o_3) = 8$, $F(o_4) = 0$, $F(o_5) = -2$, and $F(o_6) = 8$.
- Then, $G : O \to \mathbb{R}$ such that $G(o_1) = 4$, $G(o_2) = 0$, $G(o_3) = 10$, $G(o_4) = 2$, $G(o_5) = 0$, and $G(o_6) = 10$.
- Then, $U : O \to \mathbb{R}$ such that $U(o_1) = 0.4$, $U(o_2) = 0$, $U(o_3) = 1$, $U(o_4) = 0.2$, $U(o_5) = 0$, and $U(o_6) = 1$.

Normalization of Utility Functions

Definition 5

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of simple lotteries over O, $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$ that satisfies AXIOMS 1–4, and $U : O \to \mathbb{R}$ a utility function representing $\succsim$. The utility function U is normalized, whenever $U(o_{worst}) = 0$ and $U(o_{best}) = 1$.

Construction of Utility Functions

- While Theorem 3 guarantees the existence of a utility function that represents $\succsim$, Theorem 4 characterizes the set of such functions.

- It is always possible to construct a utility function that represents $\succsim$ by asking the agent at most $(m-1)$ questions, where $|\ O\ | = m$.

  - 1st Q: "what is your preference over O?"

  - Then, construct the normalized utility function by setting $U(o_{worst}) = 0$ and $U(o_{best}) = 1$, leaving $m-2$ values to fix.

  - By Axiom 3 (Continuity), for every $o \in O$ there exists $p_o \in [0,1]$ such that

$$o \sim \begin{bmatrix} o_{best} & o_{worst} \\ p_o & 1-p_o \end{bmatrix}.$$

  - Q for every yet utility-unfixed o: "what is your value of p_o such that $o \sim \begin{bmatrix} o_{best} & o_{worst} \\ p_o & 1-p_o \end{bmatrix}$?"

  - Then, set $U(o) = p_o$, since

$$\mathbb{E}\left(\begin{bmatrix} o_{best} & o_{worst} \\ p_o & 1-p_o \end{bmatrix}\right) = p_o \cdot U(o_{best}) + (1-p_o) \cdot U(o_{worst}) = p_o \cdot 1 + (1-p_o) \cdot 0 = p_o$$

Illustration

- Let $O = \{o_1, o_2, o_3, o_4, o_5\}$ and suppose the agent states the following ranking:

$$o_2 \succ o_1 \sim o_5 \succ o_3 \sim o_4$$

- Then, $U(o_2) = 1$ and $U(o_3) = U(o_4) = 0$ can be assigned.
- The agent is subsequently asked what $p \in [0, 1]$ for him satisfies:

$$o_1 \sim \begin{bmatrix} o_2 & o_3 \\ p & 1 - p \end{bmatrix}$$

- Suppose that the agent answers 0.4.
- Then, the agent's normalized utility function $U : O \to \mathbb{R}$ is as follows:
  - $U(o_1) = U(o_5) = 0.4$
  - $U(o_2) = 1$
  - $U(o_3) = U(o_4) = 0$

PROOFS

Representation Theorem

Theorem 3

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of simple lotteries over O, and $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$. If $\succsim$ satisfies AXIOMS 1 – 4*, then there exists a function $U : O \to \mathbb{R}$, called utility function, that represents the preference relation $\succsim$.*

Proof of Theorem 3

- To simplify notation, assume that the basic outcomes have been renumbered such that $o_{best} = o_1$ and $o_{worst} = o_m$, where $| O | = m$.

- For every basic outcome $o \in O$ fix $q_o \in [0, 1]$ such that $o \sim \begin{bmatrix} o_1 & o_m \\ q_o & 1 - q_o \end{bmatrix}$, which exists by AXIOM 3 (Continuity).

- Note that $q_{o_{best}} = 1$ and $q_{o_{worst}} = 0$.

- Consider an arbitrary simple lottery

$$L_1 = \begin{bmatrix} o_1 & \dots & o_m \\ p_1 & \dots & p_m \end{bmatrix}$$

Proof of Theorem 3 (continued)

- First, it is shown that, by a repeated application of AXIOM 4 (Substitutability), the following indifference holds:

$$L_1 = \begin{bmatrix} o_1 & \dots & o_m \\ p_1 & \dots & p_m \end{bmatrix} \sim \begin{bmatrix} o_1 & o_m \\ \sum_{i=1}^m p_i \cdot q_{o_i} & 1 - \sum_{i=1}^m p_i \cdot q_{o_i} \end{bmatrix} \qquad (\star)$$

- Recall that, by construction, $o_2 \sim \begin{bmatrix} o_1 & o_m \\ q_{o_2} & 1 - q_{o_2} \end{bmatrix}$ and consider the compound lottery C_2, where

$$C_2 = \begin{bmatrix} o_1 & \begin{bmatrix} o_1 & o_m \\ q_{o_2} & 1 - q_{o_2} \end{bmatrix} & o_3 & \dots & o_m \\ p_1 & p_2 & p_3 & \dots & p_m \end{bmatrix}$$

- The simple lottery corresponding to C_2, which actually omits the basic outcome o_2, is

$$L(C_2) = \begin{bmatrix} o_1 & o_3 & \dots & o_{m-1} & o_m \\ p_1 + p_2 \cdot q_{o_2} & p_3 & \dots & p_{m-1} & p_m + p_2 \cdot (1 - q_{o_2}) \end{bmatrix}$$

- By AXIOM 4 (Substitutability), it follows that $L_1 \sim L(C_2)$.

Proof of Theorem 3 (continued)

- Recall that, by construction, $o_3 \sim \begin{bmatrix} o_1 & o_m \\ q_{o_3} & 1 - q_{o_3} \end{bmatrix}$ and consider the compound lottery C_3, where

$$C_3 = \begin{bmatrix} o_1 & \begin{bmatrix} o_1 & o_m \\ q_{o_3} & 1 - q_{o_3} \end{bmatrix} & o_4 & \dots & o_m \\ p_1 + p_2 \cdot q_{o_2} & p_2 & p_4 & \dots & p_m + p_2 \cdot (1 - q_{o_2}) \end{bmatrix}$$

- The simple lottery corresponding to C_3, which actually omits the basic outcomes o_2 and o_3, is

$$L(C_3) = \begin{bmatrix} o_1 & o_4 & \dots & o_{m-1} & o_m \\ p_1 + p_2 \cdot q_{o_2} + p_3 \cdot q_{o_3} & p_4 & \dots & p_{m-1} & p_m + p_2 \cdot (1 - q_{o_2}) + p_3 \cdot (1 - q_{o_3}) \end{bmatrix}$$

- By Axiom 4 (Substitutability), it follows that $L(C_2) \sim L(C_3)$.
- By transitivity, it follows that $L_1 \sim L(C_3)$.
- By analogously repeating this argument, it follows that $L_1 \sim L(C_{m-1})$, where

$$L(C_{m-1}) = \begin{bmatrix} o_1 & o_m \\ p_1 + \sum_{i=2}^{m-1} p_i \cdot q_{o_i} & p_m + \sum_{i=2}^{m-1} p_i \cdot (1 - q_{o_i}) \end{bmatrix}$$

Proof of Theorem 3 (continued)

- Since $q_{o_1} = 1$ and $q_{o_m} = 0$, it is the case that

$$p_1 + \sum_{i=2}^{m-1} p_i \cdot q_{o_i} = \sum_{i=1}^{m} p_i \cdot q_{o_i}$$

as well as

$$p_m + \sum_{i=2}^{m-1} p_i \cdot (1 - q_{o_i}) = \sum_{i=2}^{m} p_i - \sum_{i=2}^{m-1} p_i \cdot q_{o_i} = p_1 - p_1 + \sum_{i=2}^{m} p_i - \sum_{i=2}^{m-1} p_i \cdot q_{o_i}$$

$$= \sum_{i=1}^{m} p_i - p_1 \cdot q_{o_1} - \sum_{i=2}^{m-1} p_i \cdot q_{o_i} = \sum_{i=1}^{m} p_i - p_1 \cdot q_{o_1} - p_m \cdot q_{o_m} - \sum_{i=2}^{m-1} p_i \cdot q_{o_i}$$

$$= \sum_{i=1}^{m} p_i - \sum_{i=1}^{m} p_i \cdot q_{o_i} = 1 - \sum_{i=1}^{m} p_i \cdot q_{o_i}$$

- Therefore,

$$L(C_{m-1}) = \begin{bmatrix} o_1 & o_m \\ \sum_{i=1}^{m} p_i \cdot q_{o_i} & 1 - \sum_{i=1}^{m} p_i \cdot q_{o_i} \end{bmatrix}.$$

- Since $L_1 \sim L(C_{m-1})$, it follows that $\star$ holds.

Proof of Theorem 3 (continued)

- Next define the utility function $U : O \to \mathbb{R}$ such that $U(o) = q_o$ for all $o \in O$.
- Consider two arbitrary simple lotteries $L = \begin{bmatrix} o_1 & \cdots & o_m \\ p_1 & \cdots & p_m \end{bmatrix}$ and $L' = \begin{bmatrix} o_1 & \cdots & o_m \\ p'_1 & \cdots & p'_m \end{bmatrix}$
- Note that $\mathbb{E}\big(U(L)\big) = \sum_{i=1}^m p_i \cdot q_{o_i}$ and $\mathbb{E}\big(U(L')\big) = \sum_{i=1}^m p'_i \cdot q_{o_i}$
- Since $\star$ has been established for any simple lottery,

$$L \sim M := \begin{bmatrix} o_1 & o_m \\ \sum_{i=1}^m p_i \cdot q_{o_i} & 1 - \sum_{i=1}^m p_i \cdot q_{o_i} \end{bmatrix}$$

  as well as

$$L' \sim M' := \begin{bmatrix} o_1 & o_m \\ \sum_{i=1}^m p'_i \cdot q_{o_i} & 1 - \sum_{i=1}^m p'_i \cdot q_{o_i} \end{bmatrix}$$

- By transitivity, it follows that $L \succsim L'$, if and only if, $M \succsim M'$.
- By Axiom 2 (Monotonicity), $M \succsim M'$, if and only if, $\sum_{i=1}^m p_i \cdot q_{o_i} \geq \sum_{i=1}^m p'_i \cdot q_{o_i}$.
- Therefore,

$$L \succsim L',$$

  if and only if,

$$\mathbb{E}\big(U(L)\big) = \sum_{i=1}^m p_i \cdot q_{o_i} \geq \sum_{i=1}^m p'_i \cdot q_{o_i} = \mathbb{E}\big(U(L')\big),$$

  which concludes the proof.

Multiplicity of Utility Functions

Theorem 4

Let O be a set of basic outcomes, $\mathcal{L}(O)$ the set of simple lotteries over O, and $\succsim \subseteq \mathcal{L}(O) \times \mathcal{L}(O)$ a weak preference relation over $\mathcal{L}(O)$ that satisfies AXIOMS 1–4*.*

(i) *If $U : O \to \mathbb{R}$ represents $\succsim$, then for every $a \in \mathbb{R}^+$ and for every $b \in \mathbb{R}$, the function $V : O \to \mathbb{R}$, defined by $V(o) = a \cdot U(o) + b$ for all $o \in O$, represents $\succsim$.*

(ii) *If $U : O \to \mathbb{R}$ and $V : O \to \mathbb{R}$ both represent $\succsim$, then there exists $a \in \mathbb{R}^+$ and there exists $b \in \mathbb{R}$ such that $V(o) = a \cdot U(o) + b$ for all $o \in O$.*

Proof of Theorem 4 (i)

- Let $a, b \in \mathbb{R}$ such that $a > 0$ be two real numbers and $L, L' \in \mathcal{L}(O)$ arbitrary two simple lotteries.
- Since U represents $\succsim$, it holds that:
$$L \succsim L',$$
if and only if,
$$\sum_{i=1}^{m} p_i \cdot U(o_i) = \mathbb{E}\big(U(L)\big) \geq \mathbb{E}\big(U(L')\big) = \sum_{i=1}^{m} p'_i \cdot U(o_i)$$
- Manipulating both sides by multiplication of $a > 0$ and subsequent addition of b, the latter inequality is equivalent to
$$b + a \cdot \sum_{i=1}^{m} p_i \cdot U(o_i) \geq b + a \cdot \sum_{i=1}^{m} p'_i \cdot U(o_i)$$
which in turn is equivalent to
$$\sum_{i=1}^{m} p_i \cdot \big(a \cdot U(o_i) + b\big) = b \cdot \underbrace{\sum_{i=1}^{m} p_i}_{=1} + \sum_{i=1}^{m} p_i \cdot a \cdot U(o_i) \geq b \cdot \underbrace{\sum_{i=1}^{m} p'_i}_{=1} + \sum_{i=1}^{m} p'_i \cdot a \cdot U(o_i) = \sum_{i=1}^{m} p'_i \cdot \big(a \cdot U(o_i) + b\big)$$
- Setting $V : O \to \mathbb{R}$ such that $V(o_i) := a \cdot U(o_i) + b$ for all $i \in \{1, 2, \ldots, m\}$, it follows that $L \succsim L'$, if and only, if $\mathbb{E}\big(V(L)\big) = \sum_{i=1}^{m} p_i \cdot \big(a \cdot U(o_i) + b\big) \geq \sum_{i=1}^{m} p'_i \cdot \big(a \cdot U(o_i) + b\big) = \mathbb{E}\big(V(L')\big)$
- Therefore, the function V represents $\succsim$, which concludes the proof.

Proof of Theorem 4 (ii)

- Let $U^* : O \to \mathbb{R}$ be the normalization of U and $V^* : O \to \mathbb{R}$ be the normalization of V.
- First of all, it is shown that $U^* = V^*$, i.e. $U^*(o) = V^*(o)$ for all $o \in O$.
- Towards a contradiction, suppose that there exists some $\hat{o} \in O$ such that $U^*(\hat{o}) \neq V^*(\hat{o})$ and, without loss of generality, assume that $U^*(\hat{o}) > V^*(\hat{o})$.
- Since U^* is normalized, $U^*(o) \in [0, 1]$ for all $o \in O$, and the following simple lottery can thus be defined:

$$L = \begin{bmatrix} o_{best} & o_{worst} \\ U^*(\hat{o}) & 1 - U^*(\hat{o}) \end{bmatrix}$$

- Then, $\mathbb{E}\big(U^*(L)\big) = U^*(\hat{o}) = \mathbb{E}\big(V^*(L)\big)$, as both U^* and V^* are normalized.
- By Theorem 3, it follows that $\hat{o} \sim L$.
- By Theorem 3 and the fact that $U^*(\hat{o}) > V^*(\hat{o})$ as well as $\mathbb{E}\big(V^*(L)\big) = U^*(\hat{o})$, it follows that $L \succ \hat{o}$.
- However, $\hat{o} \sim L$ and $L \succ \hat{o}$ is impossible, which yields the desired contradiction.

Proof of Theorem 4 (ii), continued

- Now define $a_1 := \frac{1}{U(o_{best})-U(o_{worst})}$ as well as $b_1 := -\frac{U(o_{worst})}{U(o_{best})-U(o_{worst})}$, and note that $a_1 > 0$.
- It follows for all $o \in O$ that

$$U^*(o) := \frac{U(o) - U(o_{worst})}{U(o_{best}) - U(o_{worst})} = \frac{1}{U(o_{best}) - U(o_{worst})} \cdot U(o) - \frac{U(o_{worst})}{U(o_{best}) - U(o_{worst})} = a_1 \cdot U(o) + b_1$$

- Consequently, U can be transformed positive-affinely into U^* and, since $U^* = V^*$, also into V^*.
- Similarly, define $a_2 := \frac{1}{V(o_{best})-V(o_{worst})}$ as well as $b_2 := -\frac{V(o_{worst})}{V(o_{best})-V(o_{worst})}$, and note that $a_2 > 0$.
- It follows for all $o \in O$ that

$$V^*(o) := \frac{V(o) - V(o_{worst})}{V(o_{best}) - V(o_{worst})} = \frac{1}{V(o_{best}) - V(o_{worst})} \cdot V(o) - \frac{V(o_{worst})}{V(o_{best}) - V(o_{worst})} = a_2 \cdot V(o) + b_2$$

- The latter equation is equivalent to: $V(o) = \frac{1}{a_2} \cdot V^*(o) - \frac{b_2}{a_2}$ for all $o \in O$, where $\frac{1}{a_2} > 0$.
- Consequently, V^* can be transformed positive-affinely into V.
- The composition of the positive affine transformation of U into V^* and the one of V^* into V yields a positive affine transformation of U into V as follows:

$$V(o) = \frac{a_1}{a_2} \cdot U(o) + \frac{b_1 - b_2}{a_2}$$

for all $o \in O$, where $\frac{a_1}{a_2} > 0$, which concludes the proof.

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition

- Chapter 5: **Expected Utility Theory**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html