ECON322 Game Theory

Part I Ordinal Payoffs

Topic 3 General Dynamic Games

Christian W. Bach

University of Liverpool & EPICENTER

Imperfect information

- There are many situations where players have to make decisions with only partial information about previous moves.

- Such situations are said do exhibit imperfect information and a general model of dynamic games can account for these.

- The notion of information set is used to represent a player's uncertainty about previous moves by opponents.

- Formally, an information set of a player is a collection of his decision nodes and he is unsure at which node play has arrived.

- Graphically, an information set is reprseented by enclosing the corresponding decision nodes in a rounded rectangle.

Outline

- Formal Structure
- Strategies
- Subgames
- Subgame-Perfect Equilibrium
- Chance Moves

FORMAL STRUCTURE

Some Additional Terminology and Notation

- Given a rooted directed tree $\mathcal{T}$ and two nodes $x, y \in X \cup Z$:
  - y is a successor of x (x predecessor of y), if there is a sequence of directed edges from x to y.
  - y is an immediate succesor of x (y immediate predecessor of x), if there is a single directed edge from x to y.

- A partition of a set H is a collection $\mathcal{H} = \{H_1, \ldots, H_m\}$, where $m \geq 1$ of non-empty subsets of H such that:
  - If $H_j, H_k \in \mathcal{H}$ with $j \neq k$, then $H_j \cap H_k = \emptyset$ ("Disjointness").
  - $H_1 \cup H_2 \cup \cdots \cup H_m = H$ ("Covering").

- The set of decision nodes assigned to player $i \in I$ is typically denoted as X_i and $X_i \subseteq X$ holds.

General Extensive Form Frames

Definition 1

An extensive-form frame is a tuple $\mathcal{F}^{\mathcal{E}} = \langle \mathcal{T}, I, \alpha_I, A, \alpha_A, O, \alpha_O, (\mathcal{D}_i)_{i \in I} \rangle$, where

- $\mathcal{T}$ is a rooted directed tree.
- I is a set of players and $\alpha_I : X \to I$ is a function assigning exactly one player to every decision node.
- A is a set of actions and $\alpha_A : \Sigma \to A$ is a function assigning exactly one action to every directed edge such that no two edges of the same node receive the same action.
- O is a set of outcomes and $\alpha_O : Z \to O$ is a function assigning exactly one outomce to every terminal node.
- $\mathcal{D}_i$ is a partition of X_i for every player $i \in I$. Each element $D_i \in \mathcal{D}_i$ is called an information set of player i such that the actions available at any two nodes in the same information set are the same.

- Note that perfect information obtains as a special case, all information sets are singleton, i.e. $\mid H_i \mid = 1$ for all $H_i \in \mathcal{H}_i$ and for all $i \in I$.

- **Graphical convention**: for simplicity sake, when representing an extensive-form frame graphically, an information set is enclosed in a rounded rectangle, if and only if, it contains at least two decision nodes.

General Extensive Form Games

Definition 2

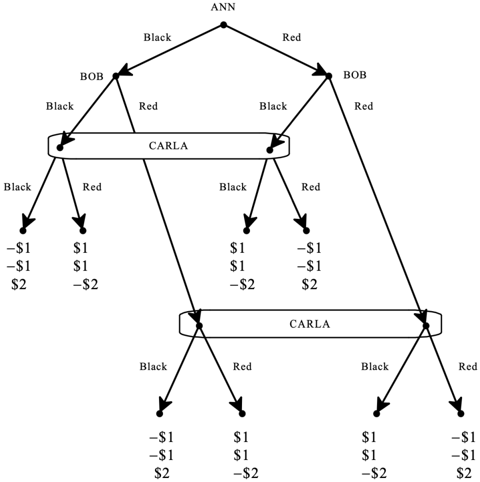
An extensive-form game is a tuple $\mathcal{G}^{\mathcal{E}} = \langle \mathcal{G}^{\mathcal{E}}, (\succsim_i)_{i \in I} \rangle$, where

- $\mathcal{G}^{\mathcal{E}}$ is an extensive form frame.
- $\succsim_i$ is a complete and transitive preference relation over O for every player $i \in I$.

Illustration

- ANN and BOB are in a same room, while CARLA is outside.
- ANN chooses either a red or a black card from a full deck of cards, shows it to BOB and puts it face down on the table.
- CARLA then enters and BOB makes one of the following two statements (he could be lying or telling the truth):
    - “ANN chose a red card.”
    - “ANN chose a black card.”
- CARLA guesses the colour of the card and the card is turned.
    - If her guess was correct, then the others give her each $1.
    - If her guess was false, then she gives each of the others $1.

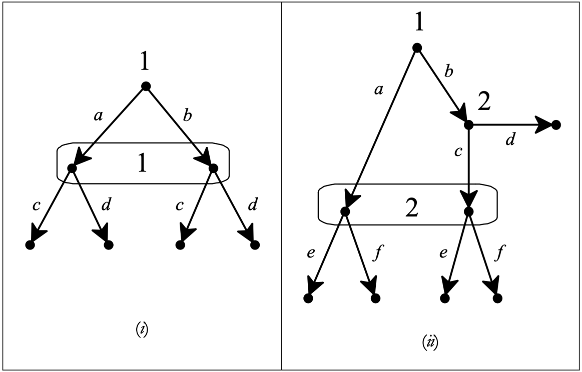
Extensive Form Frame



Perfect Recall

- It is standard to restrict attention to situations of so-called perfect recall.
- Accordingly, a player always remembers what he once knew in the game as well as his own past actions.
- Formally, $\mathcal{F}^{\mathcal{E}}$ satisfies perfect recall, whenever:
  - If any two decision nodes x and y are in a same information set, then it is not the case that one node is a predecessor of the other.
  - Let $D_i, D_i' \in \mathcal{D}_i$ of some player $i \in I$. If $x \in D_i$ is a predecessor of $y \in D_i'$ and a is the action assigned out of x in the sequence of edges leading from x to y, then for every node $v \in D_i'$ there is a predecessor $w \in D_i$ such that the action out of w in the sequence of edges leading from w to v is a.

Examples of Imperfect Recall



- ***(i)*** Player 1 does not remember **what** he chose previously.
- ***(ii)*** At his non-singleton information set, Player 2 does not remember **whether** he choose previously.

STRATEGIES

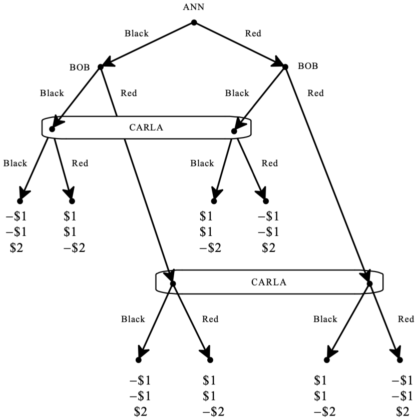
Information Sets as Choice Situations

- A strategy is a complete, contigent plan that covers all possible choice situations the player may face in the dynamic game.
  - Perfect-Information Dynamic Games: a possible choice situation for a player is a decision node of the player.
  - General Dynamic Games: a possible choice situation for a player is an information set of the player.

Definition 3

Let $\mathcal{G}^{\mathcal{E}}$ be an extensive-form game and $i \in I$ some player. A strategy for i is a tuple of choices, which contains one choice for each of i's information set. The set of all strategies of i is denoted by S_i.

Illustration



- Assume that all players are self-interested only and care about money
- The above tree then captures both the extensive-form frame as well as a possible extensive-form game of the underlying sequential strategic interaction.

Corresponding Strategic Form

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2
	R	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2

CARLA: B , B

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2
	R	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2

CARLA: R , R

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	−1 , −1 , 2	1 , 1 , −2	−1 , −1 , 2	1 , 1 , −2
	R	1 , 1 , −2	−1 , −1 , 2	−1 , −1 , 2	1 , 1 , −2

CARLA: B , R

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	1 , 1 , −2	−1 , −1 , 2	1 , 1 , −2	−1 , −1 , 2
	R	−1 , −1 , 2	1 , 1 , −2	1 , 1 , −2	−1 , −1 , 2

CARLA: R , B

SUBGAMES

Motivation

- A generalization of the solution concept of Backward Induction is needed that can be applied to general dynamic games.
- A first step in such a direction is the notion of subgame.
- Intuitively, a subgame of an extensive-form game is a portion of the game that could be an extensive-form game in itself.

Notion of Subgame

Definition 4

Let $\mathcal{G}^{\mathcal{E}}$ be an extensive-form game. A subgame of $\mathcal{G}^{\mathcal{E}}$ is obtained as follows.

- Take a decision node $x \in X$ whose information set is singleton and enclose in an oval x with all its successors.
- If the oval does not "cut" any information set, i.e. there exists no information set D with $x', x'' \in D$ such that x' is a successor of x while x'' is not, then the content of the oval constitutes a subgame of $\mathcal{G}^{\mathcal{E}}$.

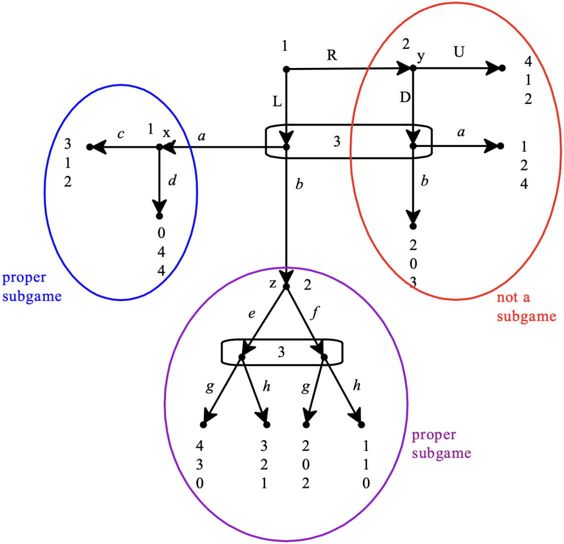
Proper Subgames and Minimal Subgames

Note that an extensive-form game is always a subgame of itself.

Definition 5

Let $\mathcal{G}^{\mathcal{E}}$ be an extensive form game. A proper subgame of $\mathcal{G}^{\mathcal{E}}$ is a subgame that is distinct from $\mathcal{G}^{\mathcal{E}}$ itself.

Illustration

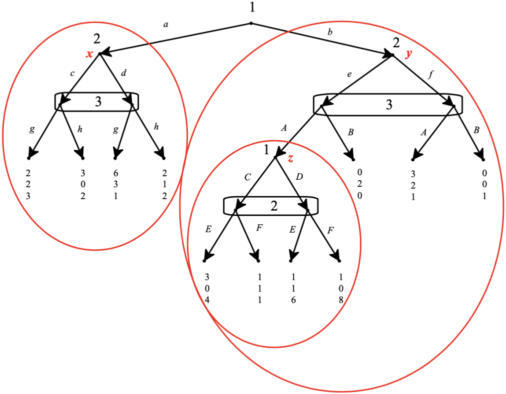


Minimal Subgames

Definition 6

Let $\mathcal{G}^{\mathcal{E}}$ be an extensive-form game. A minimal subgame of $\mathcal{G}^{\mathcal{E}}$ is a subgame that does not contain any proper subgames.

Illustration



- The subgame starting at decision node x is proper and minimal.
- The subgame starting at decision node y is proper.
- The subgame starting at decision node z is proper and minimal.

SUBGAME-PERFECT EQUILIBRIUM

The Idea

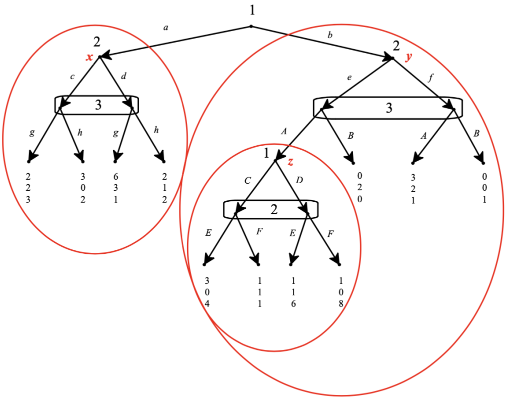
A Subgame-Perfect Equilibrium of an extensive-form game is a Nash Equilibrium of the entire game which remains a Nash Equilibrium in every proper subgame.

Restriction of a Strategy Profile to a Sugame

Definition 7

Let $\mathcal{G}^{\mathcal{E}}$ be an extensive form game, $s \in \times_{i \in I} S_i$ a strategy profile, and G a proper subgame of $\mathcal{G}^{\mathcal{E}}$. The restriction of s to G, denoted by $s\,|_G$, is that part of s which prescribes choices at every information set of G and only at those information sets.

Illustration



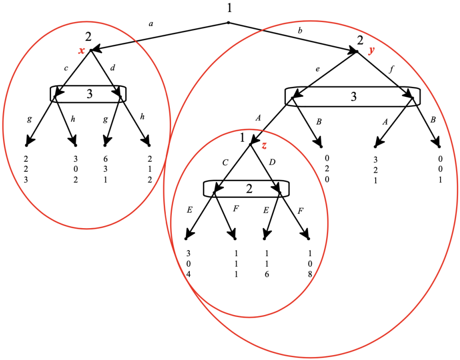
- Consider the strategy profile $((a, C), (d, f, E), (h, B))$ and denote the subgame starting at node y by G.
- Then, $s\,|_G = (C, (f, E), B)$.

A Solution Concept for General Dynamic Games

Definition 8

Let $\mathcal{G}^{\mathcal{E}}$ be an extensive-form game and $s \in \times_{i \in I} S_i$ some strategy profile. The strategy profile s forms a Subgame-Perfect Equilibrium of $\mathcal{G}^{\mathcal{E}}$, whenever for every subgame G of $\mathcal{G}^{\mathcal{E}}$, the restricted strategy profile $s\mid_G$ constitutes a Nash Equilibrium of G. The set of all such strategy profiles is denoted by SPE.

Illustration



- Note that the strategy profile $\big((a,C),(d,f,E),(h,B)\big)$ is a Nash Equilibrium of the entire game:
  - Player 1's payoff is 2 and if he were to switch to any startegy where he plays b his payoff would be 0.
  - Player 2's payoff is 1 and if he were to switch to any startegy where he plays c his payoff would be 0.
  - Player 3's payoff is 2 and if he were to switch to any startegy where he plays g his payoff would be 1.
- However, $\big((a,C),(d,f,E),(h,B)\big)$ is not subgame-perfect, as the proper subgame starting at node z, i.e. (C,E) does not form a Nash Equilibirum there: for Player 2 the unique best response to C is F.

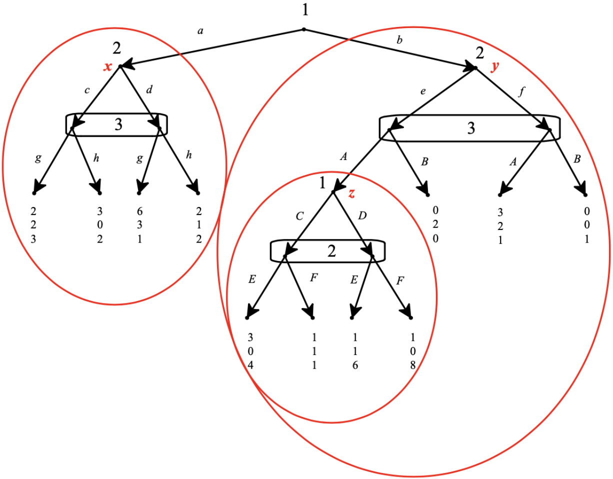
A Procedure to find Subgame-Perfect Equilibria

Definition 9

Let $\mathcal{G}^{\mathcal{E}}$ be a finite extensive-form game. The following procedure is called SPE Procedure:

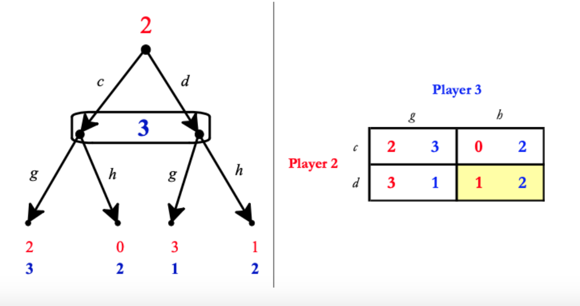
1. Select a minimal subgame and pick a Nash Equilibrium of it.
2. Delete the selected subgame and replace it with the utility vector associated with the picked Nash Equilibrium, while making a note of the strategies constituting the Nash Equilibrium. This yields a smaller extensive-form game.
3. Repeat Steps **1** and **2** in the smaller game thus obtained.

Illustration



Illustration

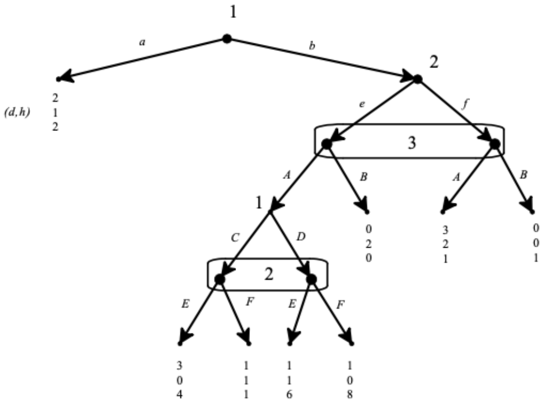
- Begin with the minimal subgame starting at decision node x.
- Since this is a game only between Player 2 and Player 3, merely these players' payoffs are shown in the below tree and matrix.



		Player 3	
		g	h
Player 2	c	2 3	0 2
	d	3 1	1 2

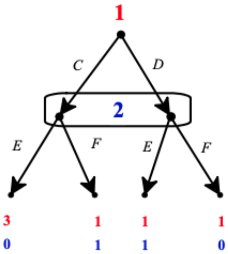
Illustration

Delete the subgame by turning the decision node x into a terminal node with the full utility vector following history (a, d, h) from the full game, which results in the following reduced game:



Illustration

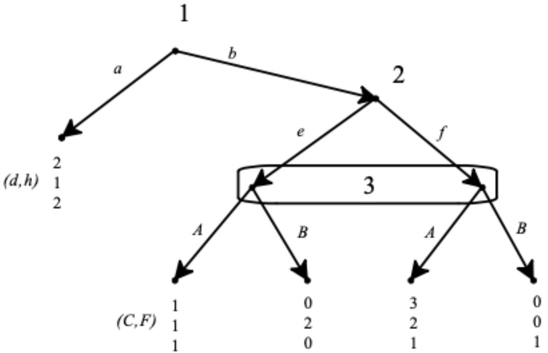
Select the (only) minimal subgame, namely the one starting at the bottom decision node of Player 1, shown in the below tree and matrix:



		Player 2	
		E	F
Player 1	C	3 0	1 1
	D	1 1	1 0

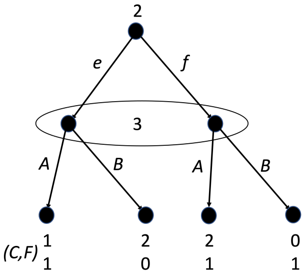
Illustration

Replace the subgame by a terminal node with the full utility vector following history (b, e, A, C, F) from the full game, which results in the following reduced game:



Illustration

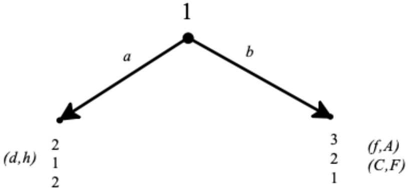
Select the (only) minimal subgame, namely the one starting at the decision node of Player 2, shown in the below tree and matrix:



		Player 3	
		A	*B*
Player 2	*e*	**1 1**	**2 0**
	f	**2 1**	**0 1**

Illustration

- Replace the subgame by a terminal node with the full utility vector following history (b, f, A) from the full game, which results in the following reduced game:



- The (unique) Nash Equilibrium is b.
- Patching together the choices selected during the application of the SPE Procedure:

$$\big((b, C), (d, f, F), (h, A)\big) \in SPE$$

Comments

- Possibly, with the SPE Procedure one encounters a subgame or a reduced with multiple Nash Equilibria.
  - A single Nash Equilibrium has to be selected, in order to continue the procedure, and in the end a single Subgame-Perfect Equilibrium is obtained.
  - The procedure then has to be repeated by selecting a different Nash Equilibrium, and in the end a different Subgame-Perfect Equilibrium is obtained.
  - Etc.
  - **Remark**: This is similar to what happens with Backward Induction in perfect-information games.
- Possibly, with the SPE Procedure one encounters a subgame or a reduced game with no Nash Equilibrium at all: it follows that then $SPE = \emptyset$.
- Since, when applied to perfect-information games, it holds that $SPE = BI$, the solution concept of Subgame-Perfect Equilibrium constitutes a generalization of Backward Induction.
- For extensive-form games with no proper subgames, it holds that $SPE = NE$.
  - In general, however, the solution concept of Subgame-Perfect Equilibrium constitutes a refinement of Nash Equilibrium, i.e. $SPE \subseteq NE$.

CHANCE MOVES

Uncertain Outcomes

- So far only games have been considered where the outcomes are certain.
- Chance moves are a way to incorporate uncertain, probabilistic events in the extensive form.

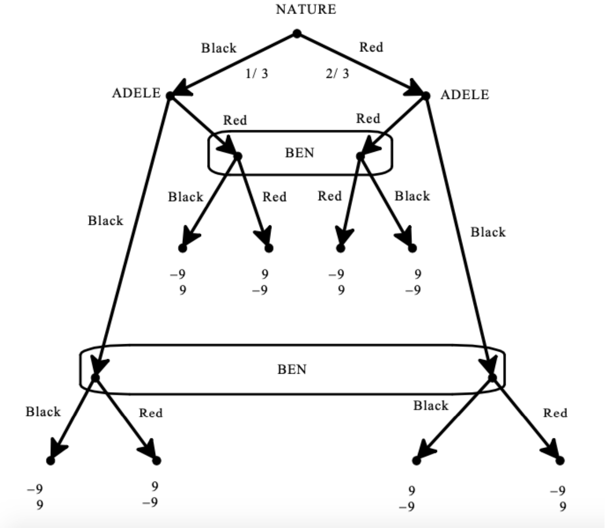
Illustration

- There are three cards: one black and two red.
- They are shuffled well and put as a pile face down on the table.
- ADELE picks the top card, checks it secretly, and then tells BEN:
  - either “the top card is black”
  - or “the top card is red”
- ADELE could be telling the truth or could be lying.
- BEN has to guess the colour of the top card.
  - If he guesses correctly, then he gets \$9 from ADELE.
  - If he guesses falsely, then ADELE gets \$9 from him.

Illustration

- Whether the top card is black or red is not affected by a player's decision, but the result of a random event, namely the shuffling.
- This is captured by introducing a fictitious player called NATURE (or CHANCE): probabilities are assigned to his "choices".
- Since one card is black and two are red, the probability of a black top card is $\frac{1}{3}$ and the probability of a red top card is $\frac{2}{3}$.
- No payoffs are assigned to NATURE who is a "dummy" player while the only "real" players are ADELE and BEN.

Illustration



Lotteries

- In games with chance moves the outcomes are probabilistic.
- Generally, probabilistic outcomes are called lotteries.
- It thus needs to be specified how the players rank the lotteries.

Expected Value of a Money Lottery

- Lotteries whose outcomes are sums of money are called money lotteries.
- Consider a money lottery with outcomes

$$\begin{bmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{bmatrix}$$

where $p_i \geq 0$ for all $i \in \{1, 2, \dots, n\}$ and $p_1 + p_2 + \cdots + p_n = 1$.

- The expected value of the lottery is the following sum of money:

$$x_1 \cdot p_1 + x_2 \cdot p_2 + \cdots + x_n \cdot p_n$$

Risk Neutrality

- One possible way (not the only one though!) to convert a (money) lottery into a payoff is risk neutrality.
- A player is said to be risk neutral, whenever he considers a lottery to be equally good as its expected value.
- Consequently, a risk neutral agent ranks lotteries according to their expected value.

Illustration

- Consider the following three money lotteries:

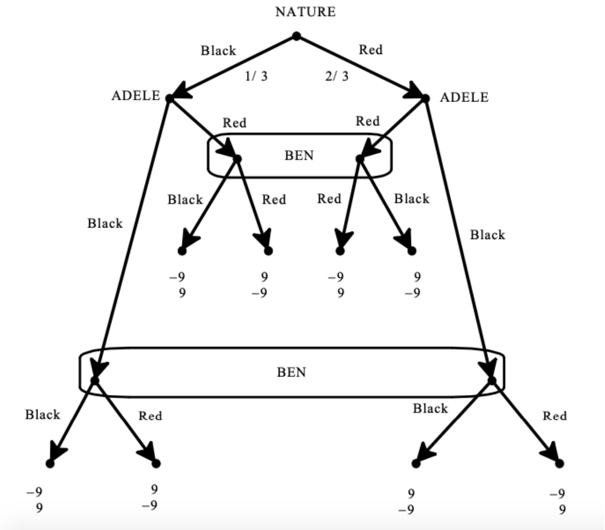
$$L_1 = \begin{bmatrix} \$5 & \$15 & \$25 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}, L_2 = \begin{bmatrix} \$16 \\ 1 \end{bmatrix}, L_3 = \begin{bmatrix} \$0 & \$32 & \$48 \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

- The expected values of L_1, L_2, and L_3 are $\$17$, $\$16$, and $\$16$, respectively.

- Thus, a risk-neutral agent would entertain the following preferences:

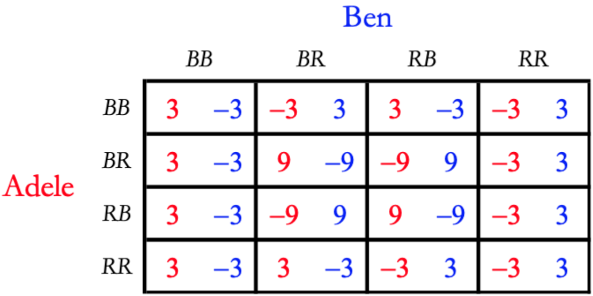
$$L_1 \succ L_2 \sim L_3$$

Illustration



Illustration

- Suppose that Adele and Ben are risk neutral.
- The corresponding strategic-form game then ensues as follows:



		Ben			
		BB	*BR*	*RB*	*RR*
Adele	*BB*	3 –3	–3 3	3 –3	–3 3
	BR	3 –3	9 –9	–9 9	–3 3
	RB	3 –3	–9 9	9 –9	–3 3
	RR	3 –3	3 –3	–3 3	–3 3

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2^{nd} Edition

- Chapter 4: **General Dynamic Games**

available at:

`http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html`