ECON322 Game Theory

Part I Ordinal Payoffs
Topic 2 Dynamic Games with Perfect Information

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Sequential Interactions

- Often interactions are not simultaneous but sequential.
  - An example is Chess, where the two players *White* and *Black* take turns moving pieces on the board.
- Such games are called Dynamic Games or Games in Extensive Form or Extensive-Form Games.
- In **T2** we consider the subclass of dynamic games with perfect information.
- The property of perfect information states that, whenever it is his turn to move, a player knows all the preceding moves.
  - Again Chess is an example, as each player entertains full knowledge of all past moves throughout the game.
- Dynamic Games with Perfect Information can be represented by means of rooted directed trees.

Outline

- Formal Structure
- Backward Induction
- Strategies
- Relationship between Backward Induction and Nash Equilibrium

Formal Structure

Trees

Definition 1

A rooted directed tree is a pair $\mathcal{T} = \langle X, \Sigma \rangle$, where X is a set of nodes and $\Sigma \subseteq X \times X$ is set of directed edges connecting nodes.

- The root of the tree has no directed edges leading to it, while every other node has exactly one directed edge leading to it.
- There exists a unique path (i.e. unique sequence of directed edges) leading from the root to any other node.
- A node that has no directed edges out of it is called terminal node, while every other node is called a decision node.
- $X = D \cup Z$, where D is the set of decision nodes and Z is the set of terminal nodes.

Frames

Definition 2

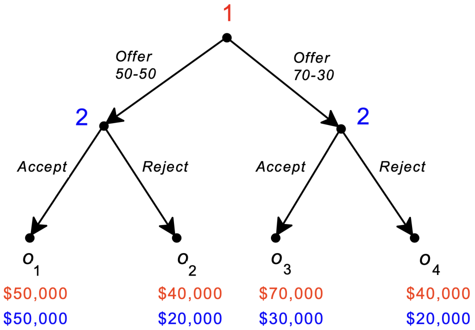
An extensive-form frame with perfect information is a tuple $\mathcal{F}_{PI}^{\mathcal{E}} = \langle \mathcal{T}, I, \alpha_I, A, \alpha_A, O, \alpha_O \rangle$, where

- $\mathcal{T}$ is a rooted directed tree.
- I is a set of players and $\alpha_I : X \to I$ is a function assigning exactly one player to every decision node.
- A is a set of actions and $\alpha_A : \Sigma \to A$ is a function assigning exactly one action to every directed edge such that no two edges of the same node receive the same action.
- O is a set of outcomes and $\alpha_O : Z \to O$ is a function assigning exactly one outomce to every terminal node.

Illustration

- Alice (Player 1) and Bob (Player 2) have decided to dissolve a business partnership whose assets have been valued at $100k.
- According to their charter, the senior partner, Alice, makes an offer about the assets division to the junior partner, Bob.
- The junior partner can
  - *Accept*, in which case the proposal is implemented,
  - *Reject*, in which case the division goes to litigation.
- A litigation costs $20k in legal fees per partner while the typical verdict assigns $60k to the senior and $40k to the junior partner.
- For simplicity sake: there is no uncertainty about the verdict and Alice can only propose two possible offers: 50-50 or 70-30 splits.

Representation as an Extensive Form Frame with Perfect Information



Games

Definition 3

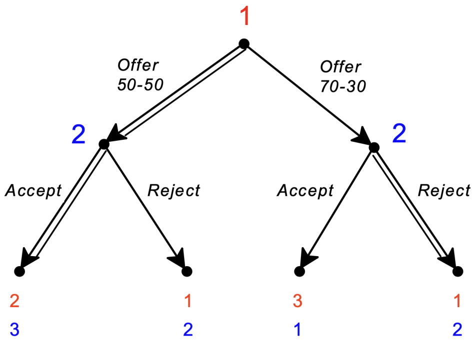
An extensive-form game with perfect information is a tuple $\mathcal{G}^{\mathcal{E}}_{PI} = \langle \mathcal{F}^{\mathcal{E}}_{PI}, (\succsim_i)_{i \in I} \rangle$, where

- $\mathcal{F}^{\mathcal{E}}_{PI}$ is an extensive form frame with perfect information.
- $\succsim_i$ is a complete and transitive preference relation over O for every player $i \in I$.

Illustration

- Assume that Alice is self-interested, while Bob is above all concerned with fairness, giving rise to the following rankings:
  - $o_3 \succ_1 o_1 \succ_1 o_2 \sim_1 o_4$
  - $o_1 \succ_2 o_2 \sim_2 o_4 \succ_2 o_3$
- These preferences can be represented by means of ordinal utility functions $U_i : O \to \mathbb{R}$ for $i \in \{1, 2\}$, e.g.:
  - $U_1(o_1) = 2$ and $U_1(o_2) = U_1(o_4) = 1$ and $U_1(o_2) = 3$
  - $U_2(o_1) = 3$ and $U_2(o_2) = U_2(o_4) = 2$ and $U_2(o_3) = 1$
- The outcomes in the previous extensive-form frame can then be replaced by a corresponding pair of utilities.

Reasoning about the Ensuing Extensive Form Game



Backward Induction

Towards A Solution Concept for Finite Dynamic Games with Perfect Information

Definition 4

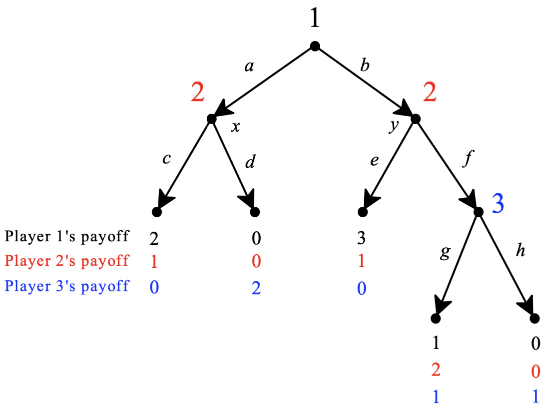
Let $\mathcal{G}_{PI}^{\mathcal{E}}$ be a finite extensive-form game with perfect information. The following marking procedure is called Backward Induction:

1. Let every terminal node $z \in Z$ be called marked.
2. Select a decision node $x \in X$ whose immediate successors are all marked. Let $i \in I$ be the player who moves at x. Select a choice $a \in A$ that leads to an immediate successor of x with the highest utility (among the imimediate successors of x) for i. Mark x with the utility vector associated with the node that follows a.
3. Repeat step **1** until every decision node $x \in X$ has been marked.

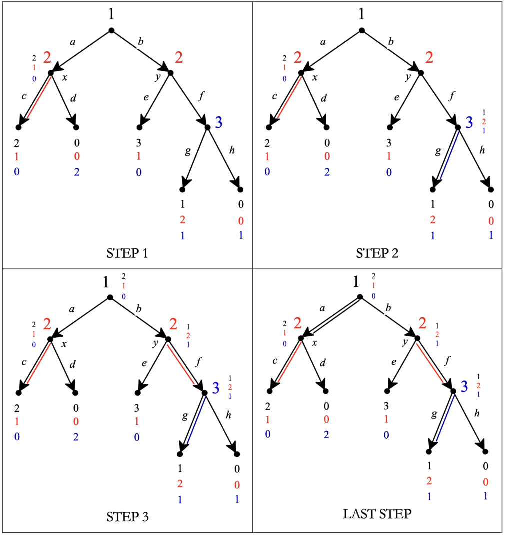
Some Comments on Backward Induction

- By the assumption of finiteness, the procedure of Backward Induction is well-defined.
- In general, at a decision node there may be several choices that maximize the utility of the moving player at that node.
    - In that case Backward Induction requires that one such choice be selected.
    - This arbitrary selection may lead to the existence of multiple solutions in line with Backward Induction.

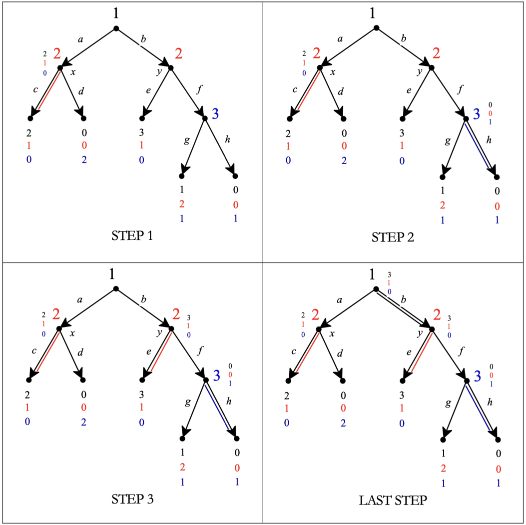
Illustration



Possibility 1



Possibility 2



Ways to graphically implement Backward Induction

- At every step, the selected choices are shown by double edges.
- In the previous figures the node marking is done explicitly, but a more succint approach merely highlights the selected choices.

STRATEGIES

Objects of Choice in Dynamic Games

- A strategy for a player in a dynamic game is a complete, contingent plan on how to act in any situation that may emerge.

- Formally:

Definition 5

Let $\mathcal{G}_{PI}^{\mathcal{E}}$ be an extensive-form game with perfect information and $i \in I$ some player. A strategy for i is a tuple of choices, which contains one choice for each of i's decision nodes. The set of all strategies of i is denoted by S_i.

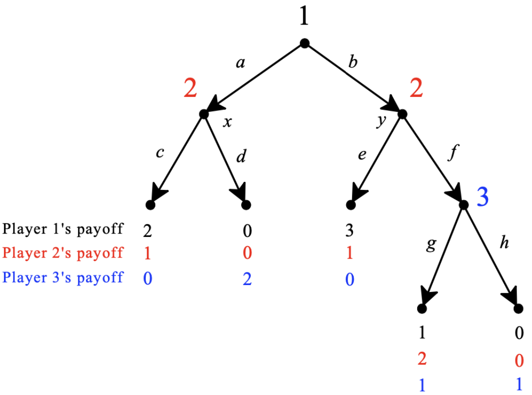
Illustration

- For example, suppose that Alice has three decision nodes:
  - at node a she has three possible choices a_1, a_2, a_3,
  - at node b she has two possible choices b_1, b_2,
  - at node c she has four possible choices c_1, c_2, c_3, c_4,
- A strategy can be thought of as a way of filling in three blanks:

$$\left(\underbrace{\quad\quad\quad\quad}_{\text{one of } a_1, a_2, a_3}, \underbrace{\quad\quad\quad\quad}_{\text{one of } b_1, b_2}, \underbrace{\quad\quad\quad\quad}_{\text{one of } c_1, c_2, c_3, c_4} \right)$$

- In total, there are thus $3 \times 2 \times 4 = 24$ possible strategies for Alice.

Illustration



- $S_1 = \{(a), (b)\}$
- $S_2 = \{(c, e), (c, f), (d, e), (d, f)\}$
- $S_3 = \{(g), (h)\}$

Interpretational Remarks

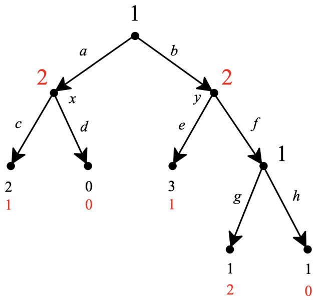
- The notion of strategy involves redundancies.
- Indeed, a player is requried to plan for nodes that are excluded by his own earlier choices.
- Possible justifications:
  - A player is so cautious that he wants his plan to also cover the possibility of making a mistake in implementing his plan.
  - A strategy as a set of instructions to a third party; thus the player may indeed worry about implementation mistakes.
  - A strategy as a belief in the mind of the opponents about what the player would do.

Strategic Form of a Dynamic Game

- For every dynamic game it is possible to construct a corresponding strategic-form game.
- A strategy profile determines a unique terminal node reached if the players act accordingly and thus a unique utility vector.
- Indeed, a set of players, a set of strategies for every player, and a utility function for every player define a game in strategic form.
- Because of the discussed redundancies above, the strategic form of a dynamic game can also display redundancies. *("identical rows or identical columns")*
- The strategic-form game corresponding to a given dynamic game $\mathcal{G}^{\mathcal{E}}_{PI}$ can be denoted by $\mathcal{R}(\mathcal{G}^{\mathcal{E}}_{PI})$.

Illustration

Consider the following Extensive Form...



...its corresponding Strategic Form is

1 \ 2	(c,e)	(c,f)	(d,e)	(d,f)
(a,g)	2, 1	2, 1	0, 0	0, 0
(a,h)	2, 1	2, 1	0, 0	0, 0
(b,g)	3, 1	1, 2	3, 1	1, 2
(b,h)	3, 1	1, 0	3, 1	1, 0

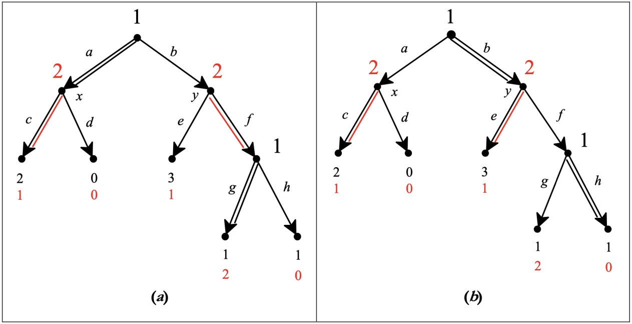
Formulating Backward Induction in terms of Strategies

- From the definition of Backward Induction it follows that the procedure selects a unique choice at every decision node.

- Consequently, Backward Induction yields a strategy profile for the entire game.

- The output of the solution concept Backward Induction is typically denoted as BI and note that $BI \subseteq \times_{i \in I} S_i$ holds.

Illustration



(a) $\{((a,g),(c,f))\} \in BI$

(b) $\{((b,h),(c,e))\} \in BI$

Relationship between Backward Induction and Nash Equilibrium

Backward Induction implies Nash Equilibrium

Theorem 6

Let $\mathcal{G}^{\mathcal{E}}_{PI}$ be an extensive-form game with perfect information. Consider the corresponding strategic-form game $\mathcal{R}(\mathcal{G}^{\mathcal{E}}_{PI})$. Then, $BI \subseteq NE$.

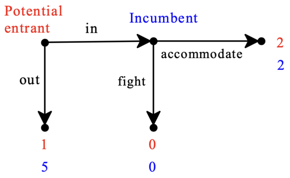
Proof of Theorem 6

- Towards a contradiction, suppose that there exists $s^* \in \times_{i \in I} S_i$ such that $s^* \in BI$ but $s^* \notin NE$ in the strategic-form game $\mathcal{R}(\mathcal{G}_{PI}^{\mathcal{E}})$.
- Then, there exists some player $i \in I$ such that s_i^* is not a best response to s_{-i}^*, i.e. there exists some strategy $s_i' \in S_i$ such that $\pi_i(s_i', s_{-i}^*) > \pi_i(s_i^*, s_{-i}^*)$.
- Consequently, applied to $\mathcal{G}_{PI}^{\mathcal{E}}$, the strategy $s_i' \in S_i$ leads to a terminal node with higher utility than s_i^* fixing the opponents' choices throughout the tree in line with s_{-i}^*.
- It follows that s_i^* does not prescribe optimal choices at all of i's decision nodes given the opponents' choices prescribed at all other decision odes according to s_{-i}^*.
- Therefore, $s^* \notin BI$, which is a contradiction.

Incredible Threats

- Nash Equilibria that are not backward inductive often involve incredible threats.
- To see this, consider the so-called **Entry Game**.
- An industry is currently a monopoly and the incumbent is making a profit of $5m, while a potential entrant is thinking about entry.
  - In the case of staying out, another investement yields $1m.
  - In the case of entering, the incumbent can fight with a price war resulting in both firms making zero profits.
  - Alternatively, the incumbent can accommodate by sharing the market resulting in both firms making $2m.
- It is assumed that both firms care about their own profits only.

The Entry Game



		Chain store	
		fight	accommodate
Potential entrant	In	0 , 0	2 , 2
	Out	1 , 5	1 , 5

(a) $BI = \{(in, accommodate)\}$

(a) $NE = \{((in, accommodate), (out, fight))\}$

The Entry Game

- In fact, the NE $(out, fight)$ involves an incredible threat on the part of the incumbent, namely to fight if entry occurs.
- If the potential entrant believes the threat, then he is better off staying out, however he should ignore the threat.
- Indeed – when faced with the fait accompli of entry – the incumbent would not want to carry out the threat.
- BI filters out the “more plausible solution” in the **Entry Game**.
- In general, BI can be seen as a refinement of NE.

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition

- Chapter 3: **Perfect-Information Games**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html