ECON322 Game Theory

Part I Ordinal Payoffs
Topic 1 Static Games

Christian W. Bach

University of Liverpool & EPICENTER

Welcome to the Course

- Lecturer: Christian Bach
- Website: www.epicenter.name/bach
- Email: cwbach@liv.ac.uk
- Office hours: Thursdays at ULMS-CR2, 3.30pm-5pm
- Questions or Comments always welcome!

Program

Part I: Ordinal Payoffs

- Topic 1 Static Games
- Topic 2: Dynamic Games with Perfect Information
- Topic 3: General Dynamic Games

Part II: Cardinal Payoffs

- Topic 4: Expected Utility Theory
- Topic 5: Strategic-Form Games
- Topic 6: Extensive-Form Games

Part III: Interactive Epistemology

- Topic 7: Knowledge
- Topic 8: Belief
- Topic 9: Rationality

Special Event on Tuesday 14.11.2023

- Founder from leading econonic consulting firm Swiss Econoomics: *Dr Christian Jaag* is visiting our module.

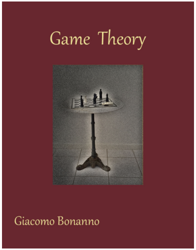
- Case Study to experience how Microeconomics & Game Theory are used in the corporate world.
- Introduction to economic consulting
- Career advice
- Networking event over coffee & biscuits

Organization

- Theory
  - Live Lectures on Campus in REN-LT7
  - Recorded Lectures on Canvas
  - Background Reading
- Exercises
  - Ten 1hour-long on Campus seminars by Tien NGUYEN (also in the same room: REN-LT7)
  - Please attempt the questions by yourselves first!
- Questions on Theory & Lectures: cwbach@liv.ac.uk
- Questions on Exercises & Seminars: hstnguy6@liverpool.ac.uk

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition



available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html

Assessment

- MID-TERM in Week 6:
  - 2 hours homework (online; open-book)
  - Topics covered: only **Part I: Ordinal Payoffs** (T1 – T3)
  - worth 40% of the final grade
- EXAM in the January Exam Period:
  - 2 hours exam (on Campus; closed-book)
  - Topics covered: ALL i.e. **Parts I – III** (T1 – T9)
  - worth 60% of the final grade

What is Game Theory?

- Origin of GAME THEORY as a discipline:

  John von Neumann & Oscar Morgenstern (1944),
  "*Theory of Games and Economic Behavior*", PUP

- GAME THEORY can be viewed as the mathematical theory of interactive decision-making or interactive decision theory.

- It models and analyzes interactive situations, where several entities ("players") take actions that jointly affect the outcome.

- Its range of applications is numerous: *Biology, Computer Science, Economics, Logic, Philosophy, Politics, Physics, . . .*

- The nature of the players depends on the context of application: *animals, artificial intelligence, electrons, firms, governments, human beings, non-thinking living organisms, robots, . . .*

Two Branches of Game Theory

- COOPERATIVE GAME THEORY
  - Players can communicate in binding ways & form coalitions
  - Typical applications in politics (e.g. voting behaviour)
- NON-COOPERATIVE GAME THEORY:
  - Players cannot communicate in binding ways
  - Typical applications in economcis (e.g. competition of firms)
- Here, we exclusively focus on:

NON-COOPERATIVE GAME THEORY

Standard Assumption of Homo Rationalis

- Homo Rationalis Assumption: the players are assumed to be intelligent, sophisticated, and rational.

- Cf. Robert Y. Aumann (1985), “What is Game Theory Trying to Accomplish?”, in Kenneth Arrow & Seppo Honkapohja, eds., *Frontiers in Economics*, Basil Blackwell, 1985, 28–76:

> *“Homo Rationalis is the species that always acts both purposefully and logically, has well-defined goals, is motivated solely by the desire to approach these goals as closely as possible, and has the calculating ability required to do so.” (Aumann, 1985, p. 35)*

Outline

- Games
- Dominance
- Iterated Deletion Procedures
- Nash Equilibrium

Games

Example: Golden Balls

- 2 players, Sarah and Steven, each have to pick one of two balls.
  - Inside one ball: the word “split”
  - Inside one ball: the word “steal”
- Each player is first asked to secretly check which of the two balls in front of them is the split and the steal ball.
- They make their decisions simultaneously.

Possible Outcomes

		Steven			
		Split		Steal	
Sarah	Split	Sarah gets $50,000	Steven gets $50,000	Sarah gets nothing	Steven gets $100,000
	Steal	Sarah gets $100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Remark

- Sarah chooses between the rows.
- Steven chooses between the columns.
- Each cell corresponds to a possible pair of choices and displays the resulting outcome.

Game Frames

Definition 1

A game frame in strategic form is a tuple $\mathcal{F} = \langle I, (S_i)_{i \in I}, O, f \rangle$, where

- I is a set of players,
- S_i is a set of strategies for every player $i \in I$,
- O is a set of outcomes,
- $f : \times_{i \in I} S_i \to O$ is a consequence function associating with every strategy profile $s \in \times_{i \in I} S_i$ and outcome $f(s) \in O$.

Golden Balls as a Game Frame

- $I = \{Sarah, Steven\}$
- $S_{Sarah} = S_{Steven} = \{split, steal\}$
- $O = \{o_1, o_2, o_3, o_4\}$ with
  - o_1 = Sarah gets $50k and Steven gets $50k.
  - o_2 = Sarah gets nothing and Steven gets $100k.
  - o_3 = Sarah gets $100k and Steven gets nothing.
  - o_4 = Sarah gets nothing and Steven gets nothing.
- $f : S_{Alice} \times S_{Bob} \to O$ such that

$$f(split, split) = o_1 \qquad f(split, steal) = o_2$$

$$f(steal, split) = o_3 \qquad f(steal, steal) = o_4$$

Matrix Representation in the Case of Two Players

WIthout loss of generality suppose that $I = \{1, 2\}$.

- The strategies for player 1 are the rows.
- The strategies for player 2 are the columns.
- The strategy profiles are the cells.
- Each cell contains an outcome.

What should Sarah do?

It depends on her preferences about the outcomes!

- Scenario 1: Sarah is self-interested only.
- Scenario 2: Sarah is fair-minded and benevolent.

Different Preferences lead to Different Choices

- **Scenario 1**: Sarah is self-interested only and suppose that her ranking is as follows:
  - o_3 preferred to o_1, o_2, o_4
  - o_1 preferred to o_2, o_4
  - indifferent between o_2 and o_4
- Then, her rational choice is steal.

- **Scenario 2**: Sarah is fair-minded and benevolent and suppose that her ranking is as follows:
  - o_1 preferred to o_2, o_3, o_4
  - o_3 preferred to o_2, o_4
  - o_2 preferred to o_4
- Then, her rational choice is split.

Notation for Preference Relations over Outcomes

- Player i considers outcome o at least as good as outcome o':

$$o \succsim_i o'$$

i.e. i weakly prefers o to o'

- Player i considers outcome o better than outcome o':

$$o \succ_i o'$$

i.e. i strictly prefers o to o'

- Player i considers outcome o just as good as outcome o':

$$o \sim_i o'$$

i.e. i is indifferent between o and o'

Weak Preference

- We shall suppose that $\succsim_i$ embodies the preferences over outcomes of player i as a primitive.

- The other two preference relations can then be defined:
  - $o \succ_i o$, whenever $o \succsim_i o'$ and $o' \not\succsim_i o$
  - $o \sim_i o'$, whenever $o \succsim_i o'$ and $o' \succsim_i o$

- Consistency Assumptions:
  - COMPLETENESS: for all $o, o' \in O$ it holds that

$$o \succsim_i o' \text{ or } o' \succsim_i o$$

  - TRANSITIVITY: for all $o, o', o'' \in O$ it holds that

$$\text{if } o \succsim_i o' \text{ and } o' \succsim_i o'', \text{ then } o \succsim_i o''$$

Ordinal Utility Functions

Definition 2

Let $\mathcal{F}$ be a game frame in strategic form and $i \in I$ some player. Suppose that the set of outcomes O is finite and that i holds a complete as well as transitive preference relation $\succsim_i$. An ordinal utility function representing $\succsim_i$ is a function

$$U_i : O \to \mathbb{R},$$

whenever for all $o, o' \in O$ it is the case that

- $U_i(o) > U_i(o')$ if and only if $o \succ_i o'$,
- $U_i(o) = U_i(o')$ if and only if $o \sim o'$.

The real number $U(o)$ is called utility of outcome o.

Remark: Definition 2 implies that $o \succsim_i o'$ if and only if $U_i(o) \geq U_i(o')$.

Non-Uniqueness of Ordinal Utility Functions

- In fact, there are infinitely many ordinal utility functions that represent the same preference relation.
- For example, consider the ranking $o_3 \succ_i o_1 \succ_i o_2 \sim_i o_4$, represented by the following functions:
  - $f(o_1) = 5$ and $f(o_2) = 2$ and $f(o_3) = 10$ and $f(o_4) = 2$
  - $g(o_1) = 0.8$ and $g(o_2) = 0.7$ and $g(o_3) = 1$ and $g(o_4) = 0.7$
  - $g(o_1) = 27$ and $g(o_2) = -1$ and $g(o_3) = 100$ and $g(o_4) = -1$

$$\dots$$

Ordinal Games in Strategic Form

Definition 3

An ordinal game in strategic form is a tuple $\mathcal{O} = \langle \mathcal{F}, (\succsim_i)_{i \in I} \rangle$, where

- $\mathcal{F} = \langle I, (S_i)_{i \in I}, O, f \rangle$ is a game frame in strategic form,
- $\succsim_i$ is a complete and transitive preference relation over O for every player $i \in I$.

Reduced Ordinal Games in Strategic Form

- Ordinal Utility Functions form a particularly convenient way of representing preference relations.
- They enable a more condensed representation of ordinal games:

Definition 4

Let $\mathcal{O} = \langle \mathcal{F}, (\succsim_i)_{i \in I} \rangle$ be an ordinal game in strategic-form. Suppose that $U_i : O \to \mathbb{R}$ is an ordinal utility function that represents $\succsim_i$ for every player $i \in I$. A reduced ordinal game in strategic form is a tuple $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$, where $\pi_i = U_i \circ f$ is player i's payoff function for all $i \in I$.

- "Reduced" because some information is lost, namely the specification of the possible outcomes via the set O, and a particular (among many) utility representation is used.
- Note that $\pi_i(s) = (U_i \circ f)(s) = U_i\big(f(s)\big)$ for all $s \in \times_{j \in I} S_j$.

Game Frame in Strategic Form of Golden Balls

Steven

Sarah		Split		Steal	
	Split	Sarah gets $50,000	Steven gets $50,000	Sarah gets nothing	Steven gets $100,000
	Steal	Sarah gets $100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

A First Example of a Reduced Ordinal Game in Strategic Form for Golden Balls

- Suppose that both players are self-interested.
- The following rankings then ensue:

$$o_3 \succ_{Sarah} o_1 \succ_{Sarah} o_2 \sim_{Sarah} o_4$$

$$o_2 \succ_{Steven} o_1 \succ_{Steven} o_3 \sim_{Steven} o_4$$

- Moreoever, suppose that the players' preference relations are represented by the following payoff functions:

$$\pi_{Sarah}(split, split) = 3, \pi_{Sarah}(split, steal) = 2, \pi_{Sarah}(steal, split) = 4, \pi_{Sarah}(steal, steal) = 2$$

$$\pi_{Steven}(split, split) = 3, \pi_{Steven}(split, steal) = 4, \pi_{Steven}(steal, split) = 2, \pi_{Steven}(steal, steal) = 2$$

- Matrix representation of the corresponding reduced ordinal game in strategic form:

		Steven	
		split	*steal*
Sarah	*split*	3, 3	2, 4
	steal	4, 2	2, 2

A Second Example of a Reduced Ordinal Game in Strategic Form for Golden Balls

- Suppose that Sarah is fair-minded and benevolent, while Steven is self-interested.
- The following rankings then ensue:

$$o_1 \succ_{Sarah} o_3 \succ_{Sarah} o_2 \succ_{Sarah} o_4$$

$$o_2 \succ_{Steven} o_1 \succ_{Steven} o_3 \sim_{Steven} o_4$$

- Moreoever, suppose that the players' preference relations are represented by the following payoff functions:

$$\pi_{Sarah}(split, split) = 4, \pi_{Sarah}(split, steal) = 2, \pi_{Sarah}(steal, split) = 3, \pi_{Sarah}(steal, steal) = 1$$

$$\pi_{Steven}(split, split) = 3, \pi_{Steven}(split, steal) = 4, \pi_{Steven}(steal, split) = 2, \pi_{Steven}(steal, steal) = 2$$

- Matrix representation of the corresponding reduced ordinal game in strategic form:

		Steven	
		split	*steal*
Sarah	*split*	4, 3	2, 4
	steal	3, 2	1, 2

Dominance

Solution Concepts

- In GAME THEORY so-called **solution concepts** are devised to predict the players' behaviour.
- Formally, a **solution concept** provides a set of strategies $SC_i \subseteq S_i$ for every player $i \in I$ according to some "reasonable" criterion.
- The prediction then ensues as $SC = \times_{i \in I} SC_i$.
- Most of the remainder of **Topic 1** is devoted to various critera of how to solve ordinal games in strategic form.

Some Further Notation

- Let $s \in \times_{i \in I} S_i$ be a strategy profile.
- $s_{-i} \in \times_{j \in I \setminus \{i\}} S_j$ then denotes the sub-profile consisting of the strategies from s of the players other than i.
- Thus, $s = (s_i, s_{-i})$.
- $S_{-i} = \times_{j \in I \setminus \{i\}} S_j$ is used to denote the set of strategy profiles of the players other than i.

Dominance Notions and Equivalence: Ordinal Games in Strategic Form

Definition 5

Let $\mathcal{O} = \langle \mathcal{F}, (\succsim_i)_{i \in I} \rangle$ be an ordinal game in strategic form, $i \in I$ some player, and $s_i, s'_i \in S_i$ two strategies of player i.

- s_i strictly dominates s'_i (or s'_i is strictly dominated by s_i), whenever $f(s_i, s_{-i}) \succ_i f(s'_i, s_{-i})$ holds for all $s_{-i} \in S_{-i}$.
- s_i weakly dominates s'_i (or s'_i is weakly dominated by s_i), whenever $f(s_i, s_{-i}) \succsim_i f(s'_i, s_{-i})$ holds for all $s_{-i} \in S_{-i}$ and there exists $\bar{s}_{-i} \in S_{-i}$ such that $f(s_i, \bar{s}_{-i}) \succ_i f(s'_i, \bar{s}_{-i})$.
- s_i is equivalent to s'_i, whenever $f(s_i, s_{-i}) \sim_i f(s'_i, s_{-i})$ holds for all $s_{-i} \in S_{-i}$.

Dominance Notions and Equivalence: Reduced Ordinal Games in Strategic Form

Definition 6

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a reduced-form ordinal game in strategic form, $i \in I$ some player, and $s_i, s_i' \in S_i$ two strategies of player i.

- s_i strictly dominates s_i' (or s_i' is strictly dominated by s_i), whenever $\pi_i(s_i, s_{-i}) > \pi_i(s_i', s_{-i})$ holds for all $s_{-i} \in S_{-i}$.
- s_i weakly dominates s_i' (or s_i' is weakly dominated by s_i), whenever $\pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i})$ holds for all $s_{-i} \in S_{-i}$ and there exists $\bar{s}_{-i} \in S_{-i}$ such that $\pi_i(s_i, \bar{s}_{-i}) > \pi_i(s_i', \bar{s}_{-i})$.
- s_i is equivalent to s_i', whenever $\pi_i(s_i, s_{-i}) = \pi_i(s_i', s_{-i})$ holds for all $s_{-i} \in S_{-i}$.

Illustration

Consider the following matrix representation of some reduced ordinal game in strategic form, where Colin's payoffs are omitted.

Colin

Rowena		*e*	*f*	*g*
	a	3, ·	2, ·	1, ·
	b	2, ·	1, ·	0, ·
	c	3, ·	2, ·	1, ·
	d	2, ·	0, ·	0, ·

- *a* strictly dominates *b*
- *a* and c are equivalent
- *a* strictly dominates *d*
- *b* is strictly dominated by *c*
- *b* weakly (but not strictly) dominates *d*
- *c* strictly dominates *d*

Strict Dominance and Weak Dominance

- If s_i strictly dominates s_i', then s_i weakly dominates s_i'.
- However,s_i weakly dominating s_i' does not imply s_i strictly dominating s_i' (e.g. *b* and *d* in the game on slide 34).
- Throughout the module we typically make the **convention** that the statement "s_i weakly dominates s_i'" means "s_i weakly dominates s_i' but s_i does not strictly dominate s_i'".

Strictly (Weakly) Dominant Strategies

Definition 7

Let $\mathcal{O} = \langle \mathcal{F}, (\succsim_i)_{i \in I} \rangle$ be an ordinal game in strategic form (or $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a reduced ordinal game in strategic form), $i \in I$ some player, and $s_i \in S_i$ a strategy of player i.

- s_i is a strictly dominant strategy, whenever for all $s_i' \in S_i \setminus \{s_i\}$ it is the case that s_i strictly dominates s_i'.
- s_i is a weakly dominant strategy, whenever for all $s_i' \in S_i \setminus \{s_i\}$ it is the case that s_i weakly dominates s_i' or s_i is equivalent to s_i'.

Remark: formally, strictly & weakly dominating are binary relations over S_i, while strictly & weakly dominant are unary relations over S_i.

Illustration

Consider the following matrix representation of some reduced ordinal game in strategic form, where Colin's payoffs are omitted.

Colin

Rowena		*e*	*f*	*g*
	a	3, ·	2, ·	1, ·
	b	2, ·	1, ·	0, ·
	c	3, ·	2, ·	1, ·
	d	2, ·	0, ·	0, ·

- *a* and *c* are both weakly dominant
- There exists no strictly dominant strategy for Rowena.

Some Observations

- If a player has two (or more) strategies that are weakly dominant, then any two of those must be equivalent.
- There can be at most one strictly dominant strategy.
- The definition of weakly dominant strategy is **equivalent** to the following statement given $\mathcal{O}$ and $\mathcal{R}$ as frameworks, respectively:

$$f(s_i, s_{-i}) \succsim_i f(s_i', s_{-i}) \text{ for all } s_i' \in S_i \text{ and for all } s_{-i} \in S_{-i}$$

$$\pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i \text{ and for all } s_{-i} \in S_{-i}$$

Interpretational Remarks

- The expression "s_i strictly dominates s_i'" can be understood as "s_i is better than s_i'".
- The expression "s_i weakly dominates s_i'" can be understood as "s_i is at least as good as s_i'".
- The expression "s_i is strictly dominant" can be understood as "s_i is best".
- The expression "s_i is weakly dominant" can be understood as "s_i is among the best".
- Analogous to the earlier **convention** (cf. page 35), the statement "s_i is a weakly dominant strategy" means "s_i is a weakly dominant but not a strictly dominant strategy" by default.

Dominant Strategy Profile

Definition 8

Let $\mathcal{O} = \langle \mathcal{F}, (\succsim_i)_{i \in I} \rangle$ be an ordinal game in strategic form (or $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a reduced ordinal game in strategic form), and $s \in \times_{i \in I} S_i$ some strategy profile.

- s forms a strictly dominant strategy profile, whenever for all $i \in I$ it is the case that s_i is a strictly dominant strategy.
- s forms a weakly dominant strategy profile, whenever for all $i \in I$ it is the case that s_i is a weakly dominant strategy

Illustration

- Consider the reduced ordinal game in strategic form of *Golden Balls* from slide 27:

		Steven *split*	Steven *steal*
Sarah	*split*	3, 3	2, 4
	steal	4, 2	2, 2

- *steal* is a weakly dominant strategy for both players.
- Thus, *(steal,steal)* forms a weakly dominant strategy profile.

Illustration

- Consider the reduced ordinal game in strategic form of *Golden Balls* from slide 28:

		Steven	
		split	*steal*
Sarah	*split*	4, 3	2, 4
	steal	3, 2	1, 2

- *split* is a strictly dominant strategy for Sarah and *steal* is a weakly dominant strategy for Steven.

- Thus, *(split,steal)* forms a weakly dominant strategy profile.

Prisoner's Dilemma

- The so-called **Prisoner's Dilemma (PD)** is an example of a game with a strictly dominant strategy profile.
- An instance of it is the following situation:
  - Alice has a red car but would prefer a blue one, while Bob has a blue car but would prefer a red one.
  - Both players prefer two cars to any one and either of the car to none at all.
  - They are each asked without the other present to choose between keeping the car they have or giving it to the other.

The PD as an Ordinal Game in Strategic Form

- The set of outcome $O = \{o_1, o_2, o_3, o_4\}$ is as follows:
  - o_1 = Alice has a blue car and Bob has a red car.
  - o_2 = Alice has no car and Bob has two cars.
  - o_3 = Alice has two cars and Bob has no car.
  - o_4 = Alice has a red car and Bob has a blue car.

- Matrix representation of the game:

Alice \ Bob	*give*	*keep*
give	o_1	o_2
keep	o_3	o_4

- Supposing that both players are self-interested, the following preference relations over O ensue:
  - $o_3 \succ_{Alice} o_1 \succ_{Alice} o_4 \succ_{Alice} o_2$
  - $o_2 \succ_{Bob} o_1 \succ_{Bob} o_4 \succ_{Bob} o_3$

- It follows that *keep* is a strictly dominant strategy for both players.

- Hence, *(keep,keep)* forms a strictly dominant strategy profile.

A Possible Reduced Ordinal Game in Strategic Form Representation of the PD

		Bob	
		give	*keep*
Alice	*give*	2, 2	0, 3
	keep	3, 0	1, 1

- Suppose that both players are self-interested
- It follows that *keep* is a strictly dominant strategy for both players.
- Hence, *(keep,keep)* forms a strictly dominant strategy profile.

Individual Rationality versus Collective Rationality

- Whenever a player has a strictly dominant strategy, it would be irrational for him to choose any other strategy, since he would then get a lower payoff no matter what the opponents do.
- In the **PD**, individual rationality thus leads to *(keep,keep)*, yet both players would be better off if each were to pick *give*.
- A binding agreement to choose *give* is not possible in the framework of NON-COOPERATIVE GAME THEORY.
- A non-binding agreement to choose *give* would not be viable: it would be beneficial to deviate from it ex-post.
- The **PD** illustrates a conflict between individual rationality and collective rationality: while *(keep,keep)* is the individually rational strategy profile, *(give,give)* would be the collectively rational one.

Pareto Superiority

Definition 9

Let $\mathcal{O} = \langle \mathcal{F}, (\succsim_i)_{i \in I} \rangle$ be an ordinal game in strategic form and $o, o' \in O$ two outcomes.

- o is strictly Pareto superior to o', whenever $o \succ_i o'$ for all $i \in I$.
- o is wekly Pareto superior to o', whenever $o \succsim_i o'$ for all $i \in I$ and there exists $j \in I$ such that $o \succ_j o'$.

Definition 10

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a reduced ordinal game in strategic form, and $s, s' \in \times_{i \in I} S_i$ two strategy profiles.

- s is strictly Pareto superior to s', whenever $\pi_i(s) > \pi_i(s')$ for all $i \in I$.
- s is wekly Pareto superior to s', whenever for all $\pi_i(s) \geq \pi_i(s')$ for all $i \in I$ and there exists $j \in I$ such that $\pi_j(s) > \pi_j(s')$.

For example, in the **PD**, outcome o_1 is strictly Pareto superior to o_4, or equivalently (in terms of strategy profiles), *(give,give)* is strictly Pareto superior to *(keep,keep)*.

Iterated Deletion Procedures

General Idea behind Iterated Deletion

- Fix some reasonability criterion about choices (e.g. dominance).
- For every player any unreasonable choice is discarded.
- A reduced game then obtains with possibly smaller strategy sets.
- Also in this reduced game, for every player any unreasonable choice is eliminated from consideration.
- Yet a further reduced game ensues and the deletion argument is applied again, etc.
- Once a reduced game is reached, where every choice satisfies the reasonability criterion, the procedure stops.

Iterated Strict Dominance

Definition 11

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a finite reduced ordinal game in strategic form.

- Let $\mathcal{R}^1_{SD}$ be the game obtained by removing from $\mathcal{R}$, for every player $i \in I$, all those strategies of i (if any) that are strictly dominated in $\mathcal{R}$ by some other strategy.
- Let $\mathcal{R}^2_{SD}$ be the game obtained by removing from $\mathcal{R}^1_{SD}$, for every player $i \in I$, all those strategies of i (if any) that are strictly dominated in $\mathcal{R}^1_{SD}$ by some other strategy.
- Etc.

The final output is called Iterated Strict Dominance and denoted by $\mathcal{R}^\infty_{SD}$. The set if strategy profiles surviving step $k \geq 1$ is denoted by SD^k and the set of those that are contained in the final output by ISD.

Remarks

- Since the initial game $\mathcal{R}$ is assumed to be finite, the output $\mathcal{R}^\infty_{SD}$ will be obtained in finitely many steps.
- Henceforth we will focus on reduced ordinal games to keep the exposition shorter and "non-repetitive".
- A simpler (related) solution concept is Strict Dominance, which is formally denoted by SD in terms of its output and which is a **special case** of Iterated Strict Dominance (indeed $SD = SD^1$).
  - Accordingly, for every player $i \in I$, all those strategies $s_i \in S_i$ are removed that are strictly dominated in $\mathcal{R}$ by some other strategy.

Illustration

$\mathcal{R}$:

		Bob			
		e	*f*	*g*	*h*
Alice	*a*	6, 3	4, 4	4, 1	3, 0
	b	5, 4	6, 3	0, 2	5, 1
	c	5, 0	3, 2	6, 1	4, 0
	d	2, 0	2, 3	3, 3	6, 1

In $\mathcal{R}$, the strategy *h* is strictly dominated by *g*.

Illustration

$\mathcal{R}^1_{SD}$:

		Bob		
		e	*f*	*g*
Alice	*a*	6, 3	4, 4	4, 1
	b	5, 4	6, 3	0, 2
	c	5, 0	3, 2	6, 1
	d	2, 0	2, 3	3, 3

In $\mathcal{R}^1_{SD}$, the strategy *d* is strictly dominated by *c*.

Illustration

$\mathcal{R}^2_{SD}$:

		Bob		
		e	f	g
	a	6, 3	4, 4	4, 1
Alice	b	5, 4	6, 3	0, 2
	c	5, 0	3, 2	6, 1

In $\mathcal{R}^2_{SD}$, the strategy g is strictly dominated by f.

Illustration

$\mathcal{R}^3_{SD}$:

		Bob	
		e	*f*
Alice	*a*	6, 3	4, 4
	b	5, 4	6, 3
	c	5, 0	3, 2

In $\mathcal{R}^3_{SD}$, the strategy *c* is strictly dominated by *a*.

Illustration

$\mathcal{R}_{SD}^4$:

		Bob	
		e	*f*
Alice	*a*	6, 3	4, 4
	b	5, 4	6, 3

- In $\mathcal{R}_{SD}^4$, no strategy is strictly dominated by another for neither player.
- Thus $\mathcal{R}_{SD}^4 = \mathcal{R}_{SD}^\infty$ and Iterated Strict Dominance stops.
- As a **solution** of the game

  $$ISD = ISD_{Alice} \times ISD_{Bob} = \{a, b\} \times \{e, f\} = \{(a, e), (a, f), (b, e), (b, f)\}$$

  ensues.

Iterated Weak Dominance

Definition 12

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a finite reduced ordinal game in strategic form.

- Let $\mathcal{R}^1_{WD}$ be the game obtained by removing from $\mathcal{R}$, for every player $i \in I$, all those strategies of i (if any) that are weakly dominated in $\mathcal{R}$ by some other strategy.
- Let $\mathcal{R}^2_{WD}$ be the game obtained by removing from $\mathcal{R}^1_{WD}$, for every player $i \in I$, all those strategies of i (if any) that are weakly dominated in $\mathcal{R}^1_{WD}$ by some other strategy.
- Etc.

The final output is called Iterated Weak Dominance and denoted by $\mathcal{R}^{\infty}_{WD}$. The set of strategy profiles surviving step $k \geq 1$ is denoted by WD^k and the set of those that are contained in the final output by IWD.

Remarks

- Since the initial game $\mathcal{R}$ is assumed to be finite, the output $\mathcal{R}^{\infty}_{WD}$ will be obtained in finitely many steps.
- Iterated Weak Dominance can be seen as a refinement of iterated strict dominance in that it allows the deletion also of weakly dominated strategies.
- Formally, $IWD \subseteq ISD$.
- A simpler (related) solution concept is Weak Dominance, which is formally denoted by WD in terms of its output and which is a **special case** of Iterated Weak Dominance (indeed $WD = WD^1$).
    - Accordingly, for every player $i \in I$, all those strategies $s_i \in S_i$ are removed that are weakly dominated in $\mathcal{R}$ by some other strategy.

Illustration

$\mathcal{R}$:

		Bob	
		e	*f*
Alice	*a*	4, 0	0, 0
	b	3, 2	2, 2
	c	1, 1	0, 0
	d	0, 0	1, 1

In $\mathcal{R}$, the strategies *c* and *d* are each strictly dominated by *b*.

Illustration

$\mathcal{R}^1_{WD}$:

		Bob e	Bob f
Alice	a	4, 0	0, 0
Alice	b	3, 2	2, 2

- In $\mathcal{R}^1_{WD}$, no strategy is strictly (or weakly) dominated by another for neither player.
- Thus $\mathcal{R}^1_{WD} = \mathcal{R}^\infty_{WD}$ and Iterated Weak Dominance stops.
- As a **solution** of the game

  $$IWD = IWD_{Alice} \times IWD_{Bob} = \{a, b\} \times \{e, f\} = \{(a, e), (a, f), (b, e), (b, f)\}$$

  ensues.

Order Dependence of Iterated Deletion Procedures

- **ISD** satisfies a monotonicity property: if a strategy is strictly dominated by another, then this relation remains to hold if the opponents' strategy sets were to be reduced.
- It follows that **ISD** is order independent in the sense that any deletion sequence leads to the same output eventually.
- In contrast, **IWD** can be sensitive to the order of eliminiation.
- When using **IWD** it is therefore crucial to always delete whatever possible at every step of the procedure.

Illustration

$\mathcal{R}$:

Alice (rows) / Bob (columns)

	e	f
a	4, 0	0, 0
b	3, 2	2, 2
c	1, 1	0, 0
d	0, 0	1, 1

- In $\mathcal{R}$, the strategies c and d are each strictly dominated by b.
- Suppose that only c were to be deleted.

Illustration

$\mathcal{R}$:

		Bob	
		e	*f*
Alice	*a*	4, 0	0, 0
	b	3, 2	2, 2
	d	0, 0	1, 1

- Now, *e* is weakly dominated by *f*.
- Thus, delete *e*.

Illustration

$\mathcal{R}$:

		Bob
		f
	a	$0, 0$
Alice	*b*	$2, 2$
	d	$1, 1$

- *a* and *d* are each strictly dominated by *b*.
- Deleting them both yields as **solution** of the game

$$\{(b,f)\} \neq \{(a,e),(a,f),(b,e),(b,f)\} = IWD$$

NASH EQUILIBRIUM

General Idea behind Nash Equilibrium

- Every player chooses a strategy that is optimal given the opponents' strategies.
- In other words, the players' strategies are mutually best responses to each other.

Formal Definition

Definition 13

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a finite reduced ordinal game in strategic form and $s \in \times_{i \in I} S_i$ some strategy profile. The strategy profile s forms a Nash Equilibrium, whenever

$$\pi_i(s) \geq \pi_i(s'_i, s_{-i})$$

holds for all $s'_i \in S_i$ and for all $i \in I$. The set of all such strategy profiles is denoted by NE.

Best Response Terminology

- Given an opponents' strategy combination s_{-i}, a strategy s_i of player i is called best response (or best reply) to s_{-i}, whenever

$$\pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i})$$

for all $s_i' \in S_i$.

- All best responses of player i to s_{-i} are formally assembled in the following set

$$BR_i(s_{-i}) = \{s_i \in S_i : \pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}$$

- The definition of Nash Equilibrium can thus be reformulated as:

Definition 14

Let $\mathcal{R} = \langle I, (S_i)_{i\in I}, (\pi_i)_{i\in I}\rangle$ be a finite reduced ordinal game in strategic form and $s \in \times_{i\in I} S_i$ some strategy profile. The strategy profile s forms a Nash Equilibrium, whenever $s_i \in BR_i(s_{-i})$ for all $i \in I$.

Illustration

$\mathcal{R}$:

		Bob left	Bob middle	Bob right
Alice	*top*	3, 2	0, 0	1, 1
Alice	*middle*	3, 0	1, 5	4, 4
Alice	*bottom*	1, 0	2, 3	3, 0

- *top* is optimal given *left* and *left* is optimal given *top*.
- *bottom* is optimal given *middle* and *middle* is optimal given *bottom*.
- Therefore,

$$NE = \{(top, left), (bottom, middle)\}$$

Interpretational Remarks

- **No Regret**: no player regrets his own choice ex-post.
- **Self-Enforcing Agreement**: no player has an incentive to deviate from a Nash Equilibrium (non-bindingly agreed ex-ante).
- **Viable Recommendation**: no player has an incentive to deviate from a Nash Equilibrium (publicly recommended by a third party ex-ante).

Relationship to Strictly Dominant Strategy Profile

Proposition 15

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a finite reduced ordinal game in strategic form and $s^* \in \times_{i \in I}$ some strategy profile. If s^* forms a strictly dominant strategy profile, then s^* forms a Nash Equilibrium.

Proof

- If s^* is a strictly dominant strategy profile, then s_i^* is a strictly dominant strategy for every player $i \in I$.
- Hence, s_i^* strictly dominates s_i for all $s_i \in S_i \setminus \{s_i^*\}$ and for all $i \in I$.
- Consequently, $\pi_i(s_i^*, s_{-i}) > \pi_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, for all $s_i \in S_i \setminus \{s_i^*\}$, and for all $i \in I$.
- It then follows in particular that, $\pi_i(s_i^*, s_{-i}^*) > \pi_i(s_i, s_{-i}^*)$ for all $s_i \in S_i \setminus \{s_i^*\}$ and for all $i \in I$.
- Thus, $\pi_i(s^*) \geq \pi_i(s_i', s_{-i}^*)$ for all $s_i' \in S_i$ and for all $i \in I$.
- Therefore, s^* constitutes a Nash Equilibrium.

Relationship to Weakly Dominant Strategy Profile

Proposition 16

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a finite reduced ordinal game in strategic form and $s^* \in \times_{i \in I}$ some strategy profile. If s^* forms a weakly dominant strategy profile, then s^* forms a Nash Equilibrium.

Proof

- If s^* is a weakly dominant strategy profile, then s_i^* is a weakly dominant strategy for every player $i \in I$.
- Let $i \in I$ be some player and $s_i \in S_i$ some strategy of i. Then, s_i^* weakly dominates s_i (**Case 1**) or is equivalent to s_i (**Case 2**).
- **Case 1**: Consequently, $\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, and in particular $\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_{-i}^*)$ holds.
- **Case 2**: Consequently, $\pi_i(s_i^*, s_{-i}) = \pi_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, and in particular $\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i, s_{-i}^*)$ holds.
- It follows that, $\pi_i(s^*) \geq \pi_i(s_i, s_{-i}^*)$.
- Since i and s_i have been chosen arbitrarily, $\pi_i(s^*) \geq \pi_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$ and for all $i \in I$ ensues.
- Therefore, s^* constitutes a Nash Equilibrium.

Relationship to Iterated Strict Dominance

Proposition 17

Let $\mathcal{R} = \langle I, (S_i)_{i \in I}, (\pi_i)_{i \in I} \rangle$ be a finite reduced ordinal game in strategic form. Then, $NE \subseteq ISD$.

Proof

- Let $ISD^0 = \times_{i \in I} S_i$ and ISD^k denote the set of surviving strategy profiles after step $k \geq 1$ of iterated strict dominance. The proof proceeds by induction.
- **Induction Basis**:
  - It directly holds that $NE \subseteq \times_{i \in I} S_i = ISD^0$.
- **Induction Step**:
  - Suppose that $(s_i^*)_{i \in I} \in NE$
  - By the induction hypothesis, $(s_i^*)_{i \in I} \in ISD^{k-1}$ follows.
  - For all $i \in I$, by Nash Equilibrium, $s_i^* \in BR_i(s_{-i}^*)$ and hence s_i^* is not strictly dominated in $\mathcal{R}^{k-1}$.
  - Hence, $(s_i^*)_{i \in I} \in ISD^k$ and thus $NE \subseteq ISD^k$.
- Therefore, by **induction**, $NE \subseteq ISD^k$ for all $k \geq 0$.
- It then follows that $NE \subseteq \cap_{k \geq 1} ISD^k = ISD$, which is the desired conclusion.

Weak Dominance does not imply Nash Equilibrium

		Bob: *c*	Bob: *d*
Alice	*a*	$1, 1$	$0, 0$
Alice	*b*	$0, 0$	$1, 1$

Observe that:

- $NE = \{(a, c), (b, d)\}$
- $IWD = WD = \{(a, c), (a, d), (b, c), (b, d)\}$
- Thus, (a, d) and (b, c) do not form Nash Equilibria but survive Iterated Weak Dominance (and a fortiori also Weak Dominance).

Nash Equilibrium does not imply Weak Dominance

		Bob	
		c	*d*
Alice	*a*	$1, 1$	$0, 0$
	b	$0, 0$	$0, 0$

Observe that:

- $NE = \{(a, c), (b, d)\}$
- $WD = \{(a, c)\}$
- Thus, (b, d) forms a Nash Equilibrium but is eliminated by Weak Dominance (and a fortiori also by Iterated Weak Dominance).

An Ordinal Game with Three Players

Claire: *e*

Alice \ Bob	*c*	*d*
a	0,0,0	2,8,6
b	5,3,2	3,4,2

Claire: *f*

Alice \ Bob	*c*	*d*
a	0,0,0	1,2,5
b	1,6,1	0,0,1

- Alice: *b* is a best response to *(c,e)*, to *(d,e)*, and to *(c,f)*, while *a* is a best response to *(d,f)*.
- Bob: *d* is a best response to *(a,e)*, to *(b,e)*, and to *(a,f)*, while *c* is a best response to *(b,f)*.
- Claire: *e* and *f* are both best responses to *(a,c)*, while *e* is a best response to *(a,d)*, to *(b,c)*, and to *(b,d)*.
- Therefore,

$$NE = \{(b, d, e)\}$$

Some General Remarks about Games with more than Three Players

- Whenever the game has more than three players, a convenient matrix representation will not work any longer.
- Also, there then does not exist any “quick procedure” to identify the Nash Equilibria.
- One must reason by applying the definition of Nash Equilibrium.

An Ordinal Game with more than Three Players

- There are 50 players and a benefactor asks them to simultaneously & secretly write on a piece of paper a request, which must be a multiple of \$10 up to a maximum of \$100.
- Each player's set of strategies is thus $\{\$10, \$20, \$30, \$40, \$50, \$60, \$70, \$80, \$90, \$100\}$
- The benefactor will then proceed as follows:
  - if not more than 10% of the players ask for \$100, then he will grant every player's request,
  - otherwise he will give nothing to every player.
- Suppose that every player is self-interested only (thus cares about his money only & prefers more to less).
- There are several Nash Equilibria in this game:
  - Every startegy profile where 7 or more players request \$100.
  - Every strategy profile where exactly 5 players request \$100 and the remaining players \$90.
- Observe that any other strategy profile does not form a Nash Equilibrium:
  - If exactly 6 request \$100, then a player asking \$100 can profitably switch to, for instance, \$90.
  - If fewer than 5 request \$100, then a player asking less than \$100 can profitably switch to \$100.
  - If exactly 5 request \$100 and someone does not \$90, then the latter can profitably switch to \$90.

An Ordinal Game with no Nash Equilibrium

		Bob	
		heads	*tails*
Alice	*heads*	$1, 0$	$0, 1$
	tails	$0, 1$	$1, 0$

- Alice: *heads* is a (unique) best response to *heads* and *tails* is a (unique) best response to *tails*.
- Bob: *heads* is a (unique) best response to *tails* and *tails* is a (unique) best response to *heads*.
- Therefore,

$$NE = \emptyset$$

Ordinal Games with infinite Strategy Sets

- Games where the strategy set of one (or more) of the players is infinite cannot be given a matrix representation.
- Nonetheless, the definitions of all the concepts introduced so far can still be applied.

Illustration

- The set of players is $I = \{1, 2\}$ and each player has to write down a real number greater than or equal to 1.
- Each player's set of strategies is thus $\{x \in \mathbb{R} : x \geq 1\}$
- The payoffs are defined as follows (where x denotes 1's choice and y denotes 2's choice):

$$\pi_1(x, y) = \begin{cases} x - 1 \text{ if } x < y, \\ 0 \text{ if } x \geq y. \end{cases} \quad \text{and} \quad \pi_2(x, y) = \begin{cases} y - 1 \text{ if } y < x, \\ 0 \text{ if } x \leq y. \end{cases}$$

- In fact, this game has a unique Nash Equilibrium, which is $(1, 1)$.
- To see that $(1, 1) \in NE$ indeed holds, observe that neither player has any beneficial deviation potential:
  - If player 1 switchtes to some $x > 1$, then her payoff remains 0, i.e. $\pi_1(x, 1) = 0$ for all $x > 1$.
  - If player 2 switchtes to some $y > 1$, then his payoff remains 0, i.e. $\pi_2(1, y) = 0$ for all $y > 1$.
- Next we show that there exists no $(x, y) \in NE$ such that $(x, y) \neq (1, 1)$:
  - Consider some pair (x, y) such that $x = y > 1$. Then, $\pi_1(x, y) = 0$, but $\pi_1(x', y) = x' - 1 > 0$ for all $x' \in \mathbb{R}$ such that $1 < x' < x$. Consequently, player 1 has a beneficial deviation potential.
  - Consider some pair (x, y) such that $x < y$. Then, $\pi_1(x, y) = x - 1$, but $\pi_1(x', y) = x' - 1 > x - 1$ for all $x' \in \mathbb{R}$ such that $x < x' < y$. Consequently, player 1 has a beneficial deviation potential.
  - Consider some pair (x, y) such that $y < x$. Then, $\pi_2(x, y) = y - 1$, but $\pi_2(x, y') = y' - 1 > y - 1$ for all $y' \in \mathbb{R}$ such that $y < y' < x$. Consequently, player 2 has a beneficial deviation potential.
- Observe that the strategy 1 actually is a weakly dominated strategy for both players: thus the unqiue Nash Equilibrium of this game actually exhibits the property of being in weakly dominated strategies.

Background Reading

GIACOMO BONANNO (2018): *Game Theory*, 2nd Edition

- Chapter 1: **Introduction**
- Chapter 2: **Ordinal Games in Strategic Form**

available at:

http://faculty.econ.ucdavis.edu/faculty/bonanno/GT_Book.html