ECON915 Microeconomic Theory
Part A: Introduction to Decision Theory
Lecture 3: Choice

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Behavioural Foundation

- **Preferences** can be seen as a summary of the agent's mental attitude towards a set of alternatives.
- In contrast, **choices** of an agent are his observable actions.
- In comparison, **preferences** can be expressed, however **choices** are being made.
- Thus, from a descriptive point of view **choice** is more basic than **preferences**, since it can directly be observed.
- Observed **choice** behaviour can be connected to the agent's **preferences** which are never directly expressed.
  - Descriptively, an agent's **choice** behaviour reveals his **preferences**.
  - Normatively, it can be asked how should **choices** be made given some underlying **preferences** of the agent.

Choice Problem

- A model of agent behaviour does not only refer to his actual choices, but to a full description of his behaviour in all scenarios he might confront in a certain context.

- Given a set X of alternatives, a choice problem is a non-empty subset $A \subseteq X$ of X.

- A choice from A specifies an element $a \in A$.

- Assumption by modelling a choice problem as a set: the choice does not depend on the way the alternatives are presented.

- For instance, if the alternatives appear in a list, the agent ignores the order in which they are presented and the number of times an alternative appears in the list.

Context

- In some contexts, not all choice problems are relevant.
- Consequently, the agent's behaviour could also be defined only on a set $\mathcal{D} \subseteq \mathcal{P}(X) \setminus \emptyset$ of non-empty subsets of X.
- A pair $(X, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{P}(X) \setminus \emptyset$, is called a context.
- For simplicity sake the subsequent formal development restricts attention to choice problems, where $\mathcal{D} = \mathcal{P}(X) \setminus \emptyset$.

Illustration

- Suppose a student who has to select which university to attend.
- Let $X = \{x_1, x_2, \ldots, x_N\}$ be the set of universities to which the student has applied.
- The student's choice problem $A \subseteq X$ is the set of universities to which he has been admitted.
- If the admission to universities is independent, then $\mathcal{D}$ contains all $2^N - 1$ non-empty subsets of X as the context.
- However, suppose that the universities are listed according to decreasing admittance difficulty, and admittance to x_k implies admittance to all less prestigious universities with $l > k$.
- Then, the set $\mathcal{D}$ consists of the N sets $A_1, A_2, \ldots, A_N$, where $A_k = \{x_k, x_{k+1}, \ldots, x_N\}$.

Agenda

- Introduction
- Choice Functions
- Weak Axiom of Revealed Preference

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- **Choice Functions**
- Weak Axiom of Revealed Preference

Choice Function

Definition

Let X be a finite set of alternatives. A choice function

$$c : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$$

such that $c(A) \subseteq A$ and $c(A) \neq \emptyset$ for all $A \in \mathcal{P}(X) \setminus \emptyset$.

Interpretation: If the agent is offered as possible alternatives the set A, then he considers all alternatives in the set $c(A)$ as fine.

Undominatedness Function

- If the agent's preferences are given by the strict preference relation $\succ \subseteq X \times X$, it is natural to suppose that he chooses from a set A anything that is undominated.

- Then, an undominatedness function can be defined as follows:

$$c_{\succ} : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$$

such that for all $A \in \mathcal{P}(X) \setminus \emptyset$ it is the case that

$$c_{\succ}(A) = \{x \in A : y \not\succ x \text{ for all } y \in A\}.$$

Unrelatedness Function

- More generally, given some binary relation $\star \subseteq X \times X$, an unrelatedness function can be defined as follows:

$$c_\star : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$$

such that for all $A \in \mathcal{P}(X) \setminus \emptyset$ it is the case that

$$c_\star(A) = \{x \in A : y \not\star x \text{ for all } y \in A\}.$$

- Is an unrelatedness function always a choice function?

- Note that an undominatedness function is a special case of an unrelatedness function.

Normative and Descriptive Perspectives

- Normative question: given a relation $\star$, under what conditions is the induced unrelatedness function $c_\star$ a choice function?
- Descriptive question: given a choice function c, under what conditions does there exist a relation $\star$ with an induced unrelatedness function $c_\star$ such that $c = c_\star$?

Characterization of Choice Functions with Acyclic Binary Relations

Proposition 8

Let X be a finite set of alternatives, and $\star \subseteq X \times X$ be a binary relation on X. The binary relation $\star$ is acyclic, if and only if, the unrelatedness function $c_\star : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$ is a choice function.

Proof of the *only if* ($\Rightarrow$) Direction of Proposition 8

- Note that $c_\star(A) = \{x \in A : y \not\star x \text{ for all } y \in A\} \subseteq A$ for all $A \in \mathcal{P}(X) \setminus \emptyset$ directly holds.
- Hence, it still has to be established that $c_\star(A) = \{x \in A : y \not\star x \text{ for all } y \in A\} \neq \emptyset$ for all $A \in \mathcal{P}(X) \setminus \emptyset$.
- Towards a contradiction, let $c_\star(A) = \emptyset$ for some $A \in \mathcal{P}(X) \setminus \emptyset$.
- Then, for every $x \in A$ there exists some $y \in A$ such that $y \star x$.
- Consider some $x_1 \in A$.
- Then there exists $x_2 \in A$ such that $x_2 \star x_1$.
- Then there exists $x_3 \in A$ such that $x_3 \star x_2$.
- etc.

Proof of the *only if* ($\Rightarrow$) Direction of Proposition 8

- Consider the sequence $(x_1, x_2, x_3, \ldots)$ such that

$$\cdots \star x_3 \star x_2 \star x_1.$$

- As A is of finite cardinality there exists $m, n \in \mathbb{N}$ such that $m > n$ and $x_m = x_n$.

- This forms a cycle, however, $\star$ is acyclic, a contradiction.

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Proof of the *if* ($\Leftarrow$) Direction of Proposition 8

- As $c_\star$ is a choice function, it is the case that $c_\star(A) \neq \emptyset$ for all $A \in \mathcal{P}(X) \setminus \emptyset$.
- Towards a contradiction, suppose that $\star$ is not acyclic.
- Then, there exists a cycle $x_1 \star x_2 \star \cdots \star x_n \star x_1$, where $\{x_1, x_2, \ldots, x_n\} \in \mathcal{P}(X) \setminus \emptyset$.
- Consequently, for every $x \in \{x_1, x_2, \ldots, x_n\}$ there exists $x' \in \{x_1, x_2, \ldots, x_n\}$ such that $x' \star x$.
- Therefore, $c_\star\big(\{x_1, x_2, \ldots, x_n\}\big) = \emptyset$, however, $c_\star$ is a choice function, a contradiction.

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Summary

The binary relation $\star$ is acyclic.

$$\Longleftrightarrow$$

There exists a choice function c such that $c = c_{\star}$.

Agenda

- Choice Functions
- **Weak Axiom of Revealed Preference**

Characterization of Strict Preferences with Choice Functions

- Properties of choice functions are sought that characterize strict preferences.
- Thereby, foundations for preferences in terms of choice behaviour are given.
- Indeed, the key property is Houthakker's axiom, which is also called weak axiom of revealed preference.

Weak Axiom of Revealed Preference

Weak Axiom of Revealed Preference (WARP)

Let X be a finite set of alternatives, $A, B \in \mathcal{P}(X) \setminus \emptyset$ be choice problems, and $c : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$ be a choice function. If $x, y \in A \cap B$ as well as $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$.

- Intuitively, if x is chosen from A when y is available, then whenever y is chosen and x is available, x is also chosen.
- WARP is broken into two pieces by Sen.

Sen's Two Properties: α

Sen's Property α

Let X be a finite set of alternatives, $A, B \in \mathcal{P}(X) \setminus \emptyset$ be choice problems, and $c : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$ be a choice function. If $x \in B \subseteq A$ and $x \in c(A)$, then $x \in c(B)$.

Illustration:
Suppose that vodka is chosen when whisky and wine is available. Then, according to α, vodka is also chosen if only whisky is available.

Sen's Two Properties: β

SEN'S PROPERTY β

Let X be a finite set of alternatives, $A, B \in \mathcal{P}(X) \setminus \emptyset$ be choice problems, and $c : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$ be a choice function. If $x, y \in c(A)$, and $A \subseteq B$, as well as $y \in c(B)$, then $x \in c(B)$.

Illustration:
Suppose that vodka and whisky are chosen from a drinks menu and whisky is chosen when tea is added to the drinks menu. Then, according to β, vodka is also chosen from this extended drinks menu.

Remark

- WARP concerns choice problems A and B such that $x, y \in A \cap B$.
- α specializes to the case $B \subseteq A$, and β specializes to the case $A \subseteq B$.
- α is sometimes referred to as independence of irrelevant alternatives.
- The idea is that the choice out of a larger set of options does not change when some of the (non-chosen, hence) irrelevant alternatives in the set are removed.

From Preferences to WARP via α and β

Proposition 9

Let X be a finite set of alternatives. If the binary relation $\succ \subseteq X \times X$ is a strict preference relation, then the choice function $c_{\succ} : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$ satisfies the weak axiom of revealed preference.

Proof of Proposition 9

- Recall that $c_{\succ}(A) = \{x \in A : y \not\succ x \text{ for all } y \in A\}$ for all $A \in \mathcal{P}(X) \setminus \emptyset$.
- Consider $x, y \in X$ such that $x, y \in A \cap B$, and $x \in c_{\succ}(A)$ as well as $y \in c_{\succ}(B)$.
- Since $x \in c_{\succ}(A)$ and $y \in A$, it holds that $y \not\succ x$.
- As $y \in c_{\succ}(B)$ it is the case for all $z \in B$ that $z \not\succ y$.
- Thus, it follows by negative transitivity of $\succ$ that $z \not\succ x$ for all $z \in B$.
- Therefore, $x \in c_{\succ}(B)$.

From WARP to Preferences via α and β

Proposition 10

Let X be a finite set of alternatives. If a choice function $c : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X)$ satisfies the weak axiom of revealed preference, then there exists a strict preference relation $\succ \subseteq X \times X$ such that $c = c_{\succ}$.

Proof of Proposition 10

- Define $\succ \subseteq X \times X$ as follows:

$$x \succ y \text{ whenever } x \neq y \text{ and } c\big(\{x,y\}\big) = \{x\}$$

  for all $x, y \in X$.

- By definition the binary relation $\succ$ is asymmetric.

- It still needs to be shown that $\succ$ also is negatively transitive, and that $c_{\succ}(A) = c(A)$ for all $A \in \mathcal{P}(X) \setminus \emptyset$.

- To establish the identity of $c_{\succ}$ and c is addressed first.

- Let $A \in \mathcal{P}(X) \setminus \emptyset$ be some decision problem, and $x \in A$ be some alternative in decision problem A.

- Then, $x \in c(A)$ or $x \notin c(A)$.

Proof of Proposition 10

- Firstly, suppose that $x \in c(A)$.
- Then, $z \not\succ x$ for all $z \in A$. (for if $z \succ x$, then $c\big(\{x, z\}\big) = \{z\}$ by def. of $\succ$, contradicting α, which would imply that $x \in c\big(\{x, z\}\big)$.)
- Thus, $x \in c_{\succ}(A)$.
- Secondly, suppose that $x \notin c(A)$.
- Let $z \in c(A)$, which implies $z \in c\big(\{x, z\}\big)$ by α.
- Then, $c\big(\{z, x\}\big) = \{z\}$. (for otherwise β would be violated via contraposition using $x \notin c(A)$ as antecedent.)
- Thus, $z \succ x$, and consequently $x \notin c_{\succ}(A)$.
- Therefore, $c_{\succ}(A) = c(A)$ for all $A \in \mathcal{P}(X) \setminus \emptyset$.

Proof of Proposition 10

- Finally, consider that $x \not\succ y$ and $y \not\succ z$ for some $x, y, z \in X$.
- Towards a contradiction, suppose that $x \succ z$.
- Then, $\{x\} = c(\{x,z\})$ by def. of $\succ$, thus $z \notin c(\{x,y,z\})$ by α via contraposition using $z \notin c(\{x,z\})$ as antecedent.
- Since $z \in c(\{y,z\})$, it follows that $y \notin c(\{x,y,z\})$. (by contraposition of α if $y \notin c(\{y,z\})$ *(Case 1)* and by contraposition of β using $z \notin c(\{x,y,z\})$ as antecedent if $y \in c(\{y,z\})$ *(Case 2)*.)
- Similarly, since $y \in c(\{x,y\})$, it then follows that $x \notin c(\{x,y,z\})$.
- Therefore, $c(\{x,y,z\}) = \emptyset$, however, c is a choice function, a contradiction.

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Summary

The choice function c satisfies WARP.

$$\Longleftrightarrow$$

The choice function c satisfies α and β.

$$\Longleftrightarrow$$

There exists a strict preference relation $\succ$ such that $c_{\succ} = c$.