Mini-course on Epistemic Game Theory
Lecture 2: Nash Equilibrium

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Introduction

- **Nash equilibrium** has dominated game theory for many years.
- Many people have taken Nash equilibrium for granted, without critically studying its (implicit) assumptions.
- Some people have even argued that Nash equilibrium is a logical **consequence** of **common belief in rationality**.
- This is absolutely false!

We will see that ...

- ... "Nash equilibrium = common belief in rationality + **extra conditions**",
- ... these **extra conditions** are rather **implausible**,
- ... Nash equilibrium may rule out some perfectly reasonable choices in games.

Nash equilibrium

- Consider for every player i a probability distribution σ_i on i's choices.

Definition (Nash (1950, 1951))

The combination $(\sigma_1, ..., \sigma_n)$ is a **Nash equilibrium** if for every player j, the probability distribution σ_j only assigns positive probability to choices c_j that are optimal under σ_{-j}.

Interpretation of $(\sigma_1, ..., \sigma_n)$ from player i's perspective?

- For every opponent j, the probability distribution σ_j is i's belief about j's choice.
- And σ_{-j} is i's belief about j's belief about his opponents' choices.

Theorem (Nash equilibrium implies common belief in rationality)

Consider a finite static game Γ, and some Nash equilibrium $(\sigma_1, \ldots, \sigma_n)$ in that game.

For every player i, consider the set of types $T_i = \{t_i^*\}$, where t_i^* only considers possible type t_j^* for every opponent j, and where t_i^* holds belief σ_j about j's choice.

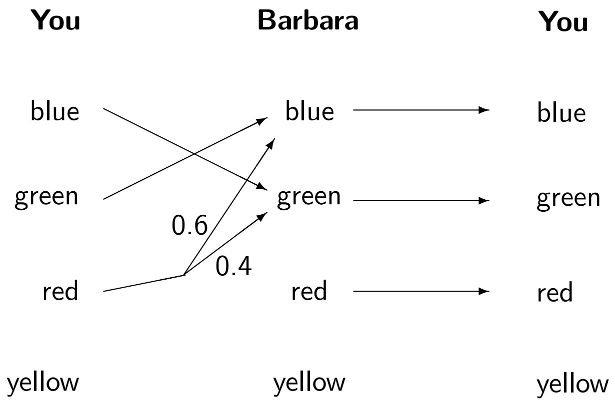
Then, every such type t_i^* expresses **common belief in rationality.**

Proof.

- Every type t_i^* **believes in his opponents' rationality.**
- Hence, every type in the epistemic model expresses **common belief in rationality.** ■

- But does **common belief in rationality** imply Nash equilibrium? **No!**
- Some choices are possible under **common belief in rationality,** but not under Nash equilibrium.
- Yet, these choices may be **perfectly reasonable!**

Example: Going to a party



	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	4	3	2	1	5

You can rationally choose *blue*, *green* and *red* under **common belief in rationality.**

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	4	3	2	1	5

- You can rationally choose *blue, green* and *red* under **common belief in rationality.**

- However, there is only one Nash equilibrium (σ_1, σ_2) in this game, namely

$$\sigma_1 = (\frac{1}{2}green + \frac{1}{2}red) \text{ and } \sigma_2 = (\frac{2}{3}blue + \frac{1}{3}green).$$

- So, when "reasoning in accordance with Nash equilibrium", you can only rationally choose *green* and *red,* but **not blue!**

Correct Beliefs

- We have seen that **Nash equilibrium** implies **common belief in rationality**, but not *vice versa.*
- So, "Nash equilibrium = common belief in rationality **+ extra conditions".**
- What are these **extra conditions?**
- How reasonable are these **extra conditions?**

Example: Teaching a lesson

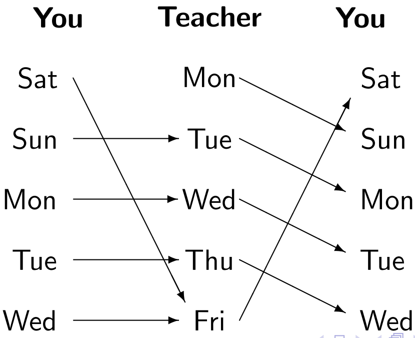
Story

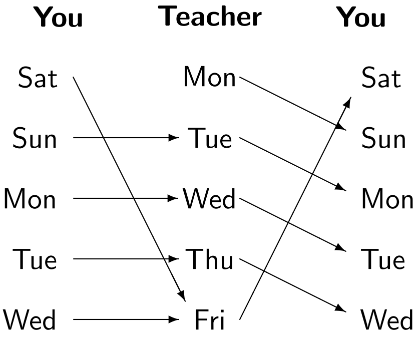
- It is Friday, and your biology teacher tells you that he will give you a **surprise exam** next week.
- You must decide on what day you will start preparing for the exam.
- In order to **pass** the exam, you must study for **at least two days.**
- To write the **perfect exam,** you must study for **at least six days.** In that case, you will get a **compliment** by your father.
- Passing the exam increases your utility by 5.
- Failing the exam increases the teacher's utility by 5.
- Every day you study decreases your utility by 1, but increases the teacher's utility by 1.
- A compliment by your father increases your utility by 4.

Teacher

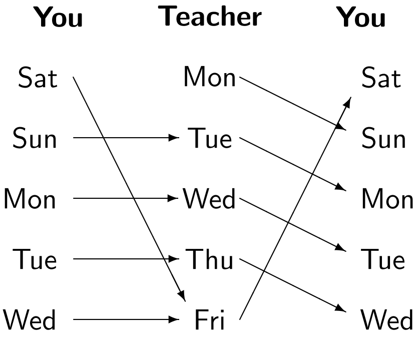
You

	Mon	Tue	Wed	Thu	Fri
Sat	$3, 2$	$2, 3$	$1, 4$	$0, 5$	$3, 6$
Sun	$-1, 6$	$3, 2$	$2, 3$	$1, 4$	$0, 5$
Mon	$0, 5$	$-1, 6$	$3, 2$	$2, 3$	$1, 4$
Tue	$0, 5$	$0, 5$	$-1, 6$	$3, 2$	$2, 3$
Wed	$0, 5$	$0, 5$	$0, 5$	$-1, 6$	$3, 2$

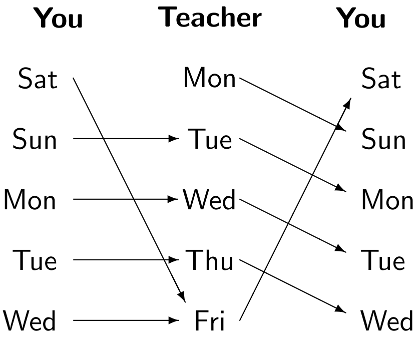




- Under **common belief in rationality,** you can rationally choose **any** day to start studying.
- However, in every Nash equilibrium (σ_1, σ_2) of this game we have $\sigma_2 = Fri$.
- So, under a **Nash equilibrium**, you can only rationally start studying on *Sat* and *Wed*.



- The belief hierarchy starting at your choice *Sat* is generated by the Nash equilibrium (*Sat*, *Fri*).
- In that belief hierarchy, you believe that the teacher is **correct about your beliefs.**
- You also believe that the teacher believes that you are **correct about his beliefs.**



- The belief hierarchy starting at your choice *Sun* is not generated by any Nash equilibrium.
- In that belief hierarchy, you believe that the teacher is **wrong about your beliefs.**
- But there is **nothing wrong** with this belief hierarchy!

$$\begin{tikzcd}
 & t_j \arrow[dr] & \\
t_i \arrow[ur] \arrow[dr] & & t_i \\
 & t_j' \arrow[ur] &
\end{tikzcd}$$

Definition (Correct beliefs)

Type t_i believes that his opponents are **correct about his beliefs** if t_i only assigns positive probability to opponents' types t_j which assign probability 1 to i's actual type t_i.

Belief hierarchies generated by Nash equilibrium

Definition (Belief hierarchy generated by a Nash equilibrium)

Consider a type t_i in some epistemic model. We say that t_i's belief hierarchy is **generated by some Nash equilibrium** $(\sigma_1, ..., \sigma_n)$ if

- t_i's belief about the opponents' choices is σ_{-i},

- t_i believes that, with probability 1, opponent j has belief σ_{-j} about his opponents' choices,

- t_i believes that, with probability 1, opponent j believes that, with probability 1, opponent k has belief σ_{-k} about his opponents' choices,

and so on.

Epistemic characterization for two players

Theorem (Nash equilibrium for two players)

Consider a finite static game with two players. Consider a type t_i in some epistemic model.

Then, t_i's belief hierarchy is **induced by a Nash equilibrium,** *if and only if,*

type t_i expresses **common belief in rationality**, *believes that j is* **correct** *about his beliefs, and believes that j believes that i is* **correct** *about his beliefs.*

- Based on Perea (2007).
- Similar results can be found in Tan and Werlang (1988), Brandenburger and Dekel (1987 / 1989), Aumann and Brandenburger (1995), Polak (1999) and Asheim (2006).

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Proof. Suppose that t_i's belief hierarchy is induced by some Nash equilibrium (σ_i, σ_j). Then,

- type t_i believes that j is correct about his beliefs,
- type t_i believes that j believes that i is correct about his beliefs, and
- type t_i expresses common belief in rationality.

Proof continued. Now, suppose that type t_i expresses **common belief in rationality**, believes that j is **correct** about his beliefs, and believes that j believes that i is **correct** about his beliefs.

To show: Type t_i's belief hierarchy is generated by a Nash equilibrium (σ_i, σ_j).

Step 1. Type t_i assigns probability 1 to a **single** type t_j for player j.

Suppose that t_i would assign positive probability to two different types t_j and t_j' for player j.

$$
\begin{tikzcd}
 & t_j \arrow[dr] & & t_j \\
t_i \arrow[ur] \arrow[dr] & & t_i \arrow[ur] \arrow[dr] & \\
 & t_j' \arrow[ur] & & t_j'
\end{tikzcd}
$$

Then, t_j would not believe that i is correct about j's beliefs. **Contradiction.**

Step 2. Type t_i's complete belief hierarchy is generated by a pair (σ_i, σ_j), where $\sigma_i \in \Delta(C_i)$ and $\sigma_j \in \Delta(C_j)$.

- From step 1, we know that t_i assigns probability 1 to some type t_j for player j, and t_j assigns probability 1 to t_i.
- Let σ_j be the belief that t_i has about j's choice, and let σ_i be the belief that t_j has about i's choice.

$$t_i \xrightarrow[\sigma_j]{} t_j \xrightarrow[\sigma_i]{} t_i$$

But then, t_i's belief hierarchy is generated by (σ_i, σ_j).

Step 3. Type t_i's belief hierarchy is generated by some Nash equilibrium (σ_i, σ_j).

- From step 2, we know that t_i's belief hierarchy is generated by some pair (σ_i, σ_j).
- As t_i believes in j's rationality, we have that $\sigma_j(c_j) > 0$ only if c_j is optimal under σ_i.
- As t_i believes that j believes in i's rationality, we have that $\sigma_i(c_i) > 0$ only if c_i is optimal under σ_j.
- Hence, (σ_i, σ_j) is a Nash equilibrium. ■

- Hence, in two-player games,

  Nash equilibrium = common belief in rationality **+ correct beliefs.**

- But the **correct beliefs assumption** is **not** a plausible condition!
- Why should you believe that the opponent is correct about your beliefs?

More than two players

- In a game with **more than two players,**

  Nash equilibrium $\neq$ common belief in rationality **+ correct beliefs.**

- More conditions are needed in order to arrive at Nash equilibrium!

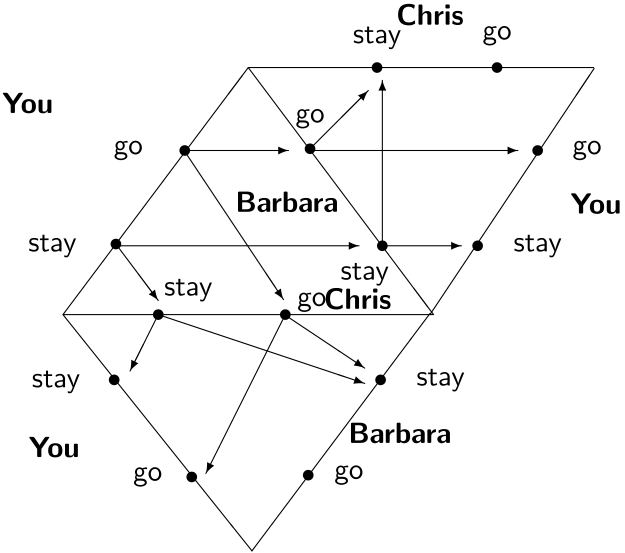
Consider a Nash equilibrium $(\sigma_1, \sigma_2, \sigma_3)$ in a three-player game. Then,

- player 1's belief about 2's choice is **independent** from 1's belief about 3's choice,
- player 1 holds belief σ_3 about 3's choice, but also believes that 2 holds the **same** belief about 3's choice. So, player 1 believes that player 2 **shares** his belief about player 3.

Example: Movie or party?

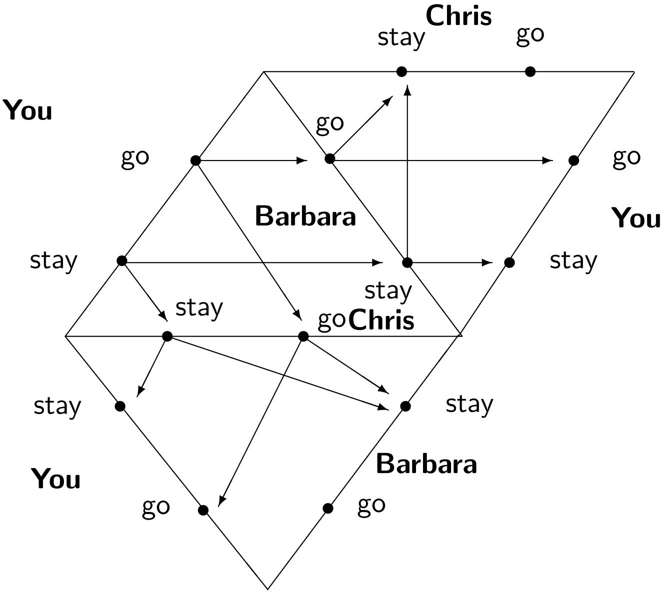
Story

- You have been invited to a party this evening, together with Barbara and Chris. But this evening, your favorite movie *Once upon a time in America*, starring *Robert de Niro*, will be on TV.
- Having a **good time** at the party gives you utility **3**, watching the **movie** gives you utility **2**, whereas having a **bad time** at the party gives you utility **0**. Similarly for Barbara and Chris.
- You will only have a good time at the party if Barbara and Chris both join.
- Barbara and Chris had a fierce discussion yesterday. Barbara will only have a good time at the party if you join, but not Chris.
- Chris will only have a good time at the party if you join, but not Barbara.



Under **common belief in rationality,** you can *go to the party* or *stay at home*.

But in your belief hierarchy starting at *go*, you believe that Barbara has a **different** belief about Chris than you do!



There is only **one** Nash equilibrium: (*stay*, *stay*, *stay*).

Under **Nash equilibrium**, you can only rationally choose to *stay at home*.

Theorem (Nash equilibrium for more than two players)

Consider a game with more than two players. Consider a type t_i in an epistemic model. Then, t_i's belief hierarchy is **generated by a Nash equilibrium** $(\sigma_1, ..., \sigma_n)$,

if and only if,

(1) t_i expresses **common belief in rationality**,

(2) t_i believes that his opponents are **correct** *about his beliefs,*
(3) t_i believes that k **shares** *his belief about j's choice,*
(4) t_i's belief about j's choice is **independent** *from t_i's belief about k's choice,*
(5) t_i believes that all opponents satisfy properties (2), (3) and (4).

- Based on Perea (2007).
- Similar results can be found in Tan and Werlang (1988), Brandenburger and Dekel (1987 / 1989), Aumann and Brandenburger (1995) and Polak (1999).

Conclusion

- The concept of **Nash equilibrium** is based on some very **implausible** epistemic assumptions, beyond common belief in rationality.
- In **classical** game theory, these assumptions remain somewhat **hidden.**
- But in **epistemic** game theory, these assumptions are finally made **explicit**.

G.B. Asheim, *The Consistent Preferences Approach to Deductive Reasoning in Games* (Theory and Decision Library, Springer, Dordrecht, The Netherlands, 2006)

R.J. Aumann and A. Brandenburger, 'Epistemic conditions for Nash equilibrium', *Econometrica,* 63 (1995), 1161–1180

A. Brandenburger and E. Dekel, 'Rationalizability and correlated equilibria', *Econometrica,* 55 (1987), 1391–1402

A. Brandenburger and E. Dekel, 'The role of common knowledge assumptions in game theory', in F. Hahn (ed.), *The Economics of Missing Markets, Information and Games* (Oxford University Press, Oxford, 1989), pp. 46–61

- J.F. Nash, 'Equilibrium points in N-person games', *Proceedings of the National Academy of Sciences of the United States of America*, 36 (1950), 48–49
- J.F. Nash, 'Non-cooperative games', *Annals of Mathematics*, 54 (1951), 286–295
- A. Perea, 'A one-person doxastic characterization of Nash strategies', *Synthese*, 158 (2007a), 251–271 (*Knowledge, Rationality and Action* 341–361)
- B. Polak, 'Epistemic conditions for Nash equilibrium, and common knowledge of rationality', *Econometrica*, 67 (1999), 673–676
- T. Tan and S.R.C. Werlang, 'The bayesian foundations of solution concepts of games', *Journal of Economic Theory*, 45 (1988), 370–391