Epistemic Game Theory
Part 3: Conditional Beliefs in Dynamic Games

Andrés Perea

EPICENTER
Research Center for Epistemic Game Theory

Maastricht University

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Outline

- Until now, we investigated static games:
- When a player makes a choice, he has no information about the choices made by other players.
- This will change today, when we study dynamic games:
- Before you make a choice, you may fully or partially observe what your opponents have chosen so far.
- It may happen that your initial belief about the opponents' choices will be contradicted later on.
- Then you must revise your belief about the opponents' choices.
- Belief revision will be at center stage today.

Outline

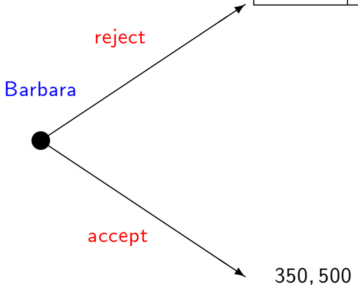
- We will present, formalize and compare two lines of reasoning for dynamic games:
- Common belief in future rationality (backward induction reasoning)
- Common strong belief in rationality (forward induction reasoning)
- We present recursive elimination procedures that characterize the strategies induced by these concepts.
- We show a logical relationship between the two concepts in terms of induced outcomes.

Example: Painting Chris' house

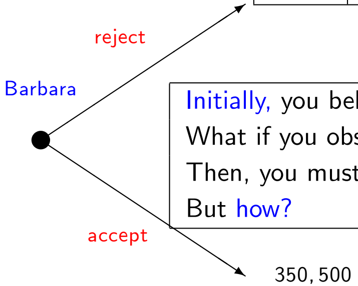
Story

- Chris is planning to paint his house tomorrow, and needs someone to help him.
- You and Barbara are both interested. This evening, both of you must come to Chris' house, and whisper a price in his ear. Price must be either 200, 300, 400 or 500 euros.
- Person with lowest price will get the job. In case of a tie, Chris will toss a coin.
- Before you leave for Chris' house, Barbara gets a phone call from a colleague, who asks her to repair his car tomorrow at a price of 350 euros.
- Barbara must decide whether or not to accept the colleague's offer.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250

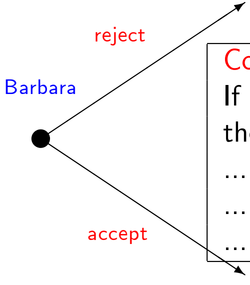


	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Initially, you believe that Barbara accepts the offer.
What if you observe that she has rejected the offer?
Then, you must revise your belief.
But how?

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Common belief in future rationality:

If you observe that Barbara has rejected offer, then you believe that

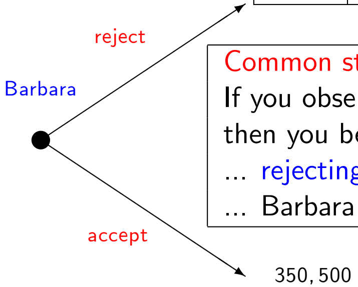
... rejecting offer was a mistake,

... Barbara chooses rationally from now on

... Barbara believes that you choose rationally.

You will choose price 200.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Common strong belief in rationality:

If you observe that Barbara has rejected offer, then you believe that

... rejecting offer is part of a rational strategy,

... Barbara will choose price 400.

You will choose price 300.

Dynamic games

- An information set for player i is a situation where player i must make a choice.
- H_i: collection of information sets for player i.
- At an information set h, more than one player can make a choice.

Definition (Strategy)

A strategy for player i is a function s_i that assigns to each of his information sets $h \in H_i$ some available choice $s_i(h)$, unless h cannot be reached due to some choice $s_i(h')$ at an earlier information set $h' \in H_i$.

In the latter case, no choice needs to be specified at h.

- This is different from the classical definition of a strategy!
- Rubinstein (1991) calls this a plan of action.

Epistemic model

- In a dynamic game, you do not only hold a belief once, but you hold a new, conditional belief at each of your information sets.
- You may revise your belief as the game proceeds.
- We would like to model hierarchies of conditional beliefs.
- That is, we want to model
- the conditional belief that player i has, at every information set $h \in H_i$, about his opponents' strategy choices,
- the conditional belief that player i has, at every information set $h \in H_i$, about the conditional belief that opponent j has, at every information set $h' \in H_j$, about the opponents' strategy choices,
- and so on.

- Hence, in a conditional belief hierarchy you hold, at each of your information sets, a conditional belief about
- the opponents' strategy choices, and
- the opponents' conditional belief hierarchies.
- Like before, call a (conditional) belief hierarchy a type.
- Then, a type for you holds, at each of your information sets, a conditional belief about
- the opponents' strategy choices, and
- the opponents' types.
- This leads to an epistemic model.

Definition (Epistemic model)

An epistemic model for a dynamic game specifies for every player i a set T_i of possible types.

Moreover, every type t_i for player i specifies at every information set $h \in H_i$ a probabilistic belief $b_i(t_i, h)$ over the set $S_{-i}(h) \times T_{-i}$ of opponents' strategy-type combinations.

- Based on Ben-Porath (1997) and Battigalli and Siniscalchi (1999).
- From the epistemic model, we can deduce the complete belief hierarchy for every type.
- A type may revise his belief about the opponents' strategies during the game.
- A type may also revise his beliefs about the opponents' beliefs during the game.

Common belief in future rationality

- Type t_i believes at h that opponent j chooses rationally at h' if his conditional belief $b_i(t_i, h)$ only assigns positive probability to strategy-type pairs (s_j, t_j) for player j where strategy s_j is optimal for type t_j at information set h'.

Definition (Belief in the opponents' future rationality)

Type t_i believes at h in opponent j's future rationality if t_i believes at h that j chooses rationally at every information set h' for player j that weakly follows h.

Type t_i believes in the opponents' future rationality if t_i believes, at every information set h for player i, in every opponent's future rationality.

- Based on Perea (2014). Similar ideas appear in Baltag, Smets and Zvesper (2009) and Penta (2015).

Definition (Common belief in future rationality)

(Induction start) Type t_i expresses 1-fold belief in future rationality if t_i believes in the opponents' future rationality.

(Induction step) For every $k \geq 2$, type t_i expresses k-fold belief in future rationality if t_i assigns, at every information set $h \in H_i$, only positive probability to opponents' types that express $(k-1)$-fold belief in future rationality.

Type t_i expresses common belief in future rationality if t_i expresses k-fold belief in future rationality for every k.

- Based on Perea (2014).
- Similar concepts can be found in Baltag, Smets and Zvesper (2009), Penta (2015), Dekel, Fudenberg and Levine (1999, 2002) and Asheim and Perea (2005).
- Equilibrium analogues are subgame perfect equilibrium (Selten (1965)) and sequential equilibrium (Kreps and Wilson (1982)). See Perea and Predtetchinski (2019).

200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250

Barbara

reject

accept

350, 500

Types	$T_1 = \{t_1\},\ T_2 = \{t_2\}$
Beliefs for Barbara	$b_1(t_1, \varnothing) = (200, t_2)$ $b_1(t_1, h_1) = (200, t_2)$
Beliefs for you	$b_2(t_2, h_1) = ((reject, 200), t_1)$

Both types express common belief in future rationality.

Recursive Procedure

- Fix an information set h for player i.
- The full decision problem for player i at h is $\Gamma^0(h) = (S_i(h), S_{-i}(h))$, where $S_i(h)$ is the set of strategies for player i that lead to h, and $S_{-i}(h)$ is the set of opponents' strategy combinations that lead to h.
- A reduced decision problem for player i at h is $\Gamma(h) = (D_i(h), D_{-i}(h))$, where $D_i(h) \subseteq S_i(h)$ and $D_{-i}(h) \subseteq S_{-i}(h)$.
- By Pearce's lemma, a strategy is optimal for player i for some belief in the reduced decision problem $\Gamma(h) = (D_i(h), D_{-i}(h))$, if and only if, it is not strictly dominated there.

Definition (Backward dominance procedure)

Consider a finite dynamic game.

(Round 0) For every information set $h \in H$, create the full decision problem $\Gamma^0(h) = (S_i(h), S_{-i}(h))$.

(Further rounds) For every $k \geq 2$, and every information set h, let $\Gamma^k(h)$ be the reduced decision problem which results if we eliminate from $\Gamma^{k-1}(h)$, for every player i, those strategies that are strictly dominated at some reduced decision problem $\Gamma^{k-1}(h')$ that weakly follows h and at which player i is active.

Strategy s_i survives the backward dominance procedure if s_i is in $\Gamma^k(\emptyset)$ for all k.

- Taken from Perea (2014).
- Perea (2014) has shown that it characterizes those strategies that can rationally be chosen under common belief in future rationality.

Definition (Backward dominance procedure)

Consider a finite dynamic game.

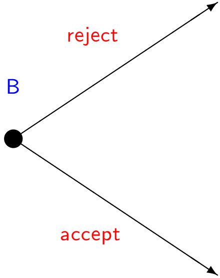
(Round 0) For every information set $h \in H$, create the full decision problem $\Gamma^0(h) = (S_i(h), S_{-i}(h))$.

(Further rounds) For every $k \geq 2$, and every information set h, let $\Gamma^k(h)$ be the reduced decision problem which results if we eliminate from $\Gamma^{k-1}(h)$, for every player i, those strategies that are strictly dominated at some reduced decision problem $\Gamma^{k-1}(h')$ that weakly follows h and at which player i is active.

Strategy s_i survives the backward dominance procedure if s_i is in $\Gamma^k(\emptyset)$ for all k.

- The algorithm always stops within finitely many steps.
- At every information set, it yields a nonempty set of strategies .
- The order in which we eliminate strategies – including the order in which we walk through the information sets – is not important for the final result.

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

350, 500

Round 1

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$

reject

B

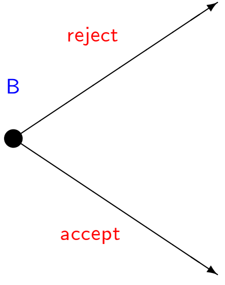
accept

$350, 500$

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Round 1

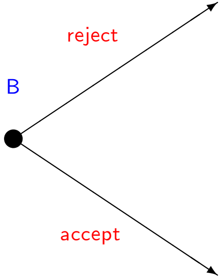
$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Round 1

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$

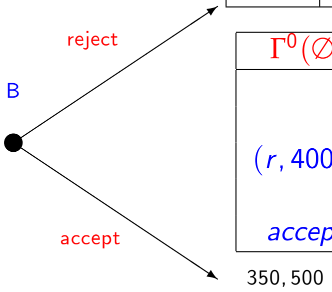


$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

$350, 500$

Round 1

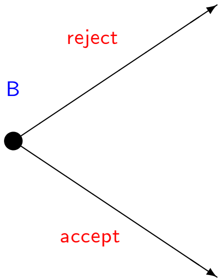
$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0



$\Gamma^0(\varnothing)$	200	300	400	500
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
accept	350, 500	350, 500	350, 500	350, 500

Round 1

$\Gamma^0(h_1)$	200	300	400	
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	

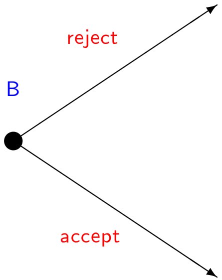


$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

$350, 500$

Round 1

$\Gamma^1(h_1)$	200	300	400	
$(r, 200)$	100, 100	200, 0	200, 0	
$(r, 300)$	0, 200	150, 150	300, 0	
$(r, 400)$	0, 200	0, 300	200, 200	

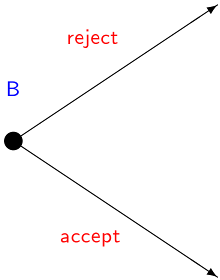


$\Gamma^1(\emptyset)$	200	300	400
$(r, 400)$	0, 200	0, 300	200, 200
accept	350, 500	350, 500	350, 500

350, 500

End of Round 1

$\Gamma^1(h_1)$	200	300	400	
$(r, 200)$	100, 100	200, 0	200, 0	
$(r, 300)$	0, 200	150, 150	300, 0	
$(r, 400)$	0, 200	0, 300	200, 200	

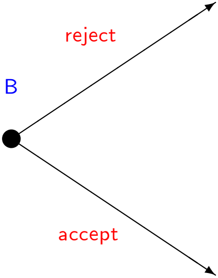


$\Gamma^1(\emptyset)$	200	300	400
accept	350, 500	350, 500	350, 500

350, 500

Round 2

$\Gamma^1(h_1)$	200	300	400	
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	

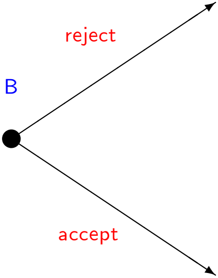


$\Gamma^1(\emptyset)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

$350, 500$

Round 2

$\Gamma^1(h_1)$	200	300		
$(r, 200)$	$100, 100$	$200, 0$		
$(r, 300)$	$0, 200$	$150, 150$		

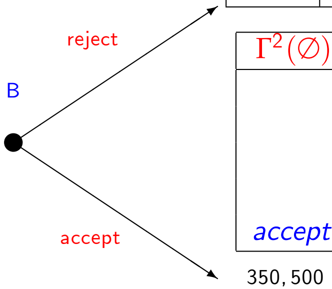


$\Gamma^1(\emptyset)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

$350, 500$

Round 2

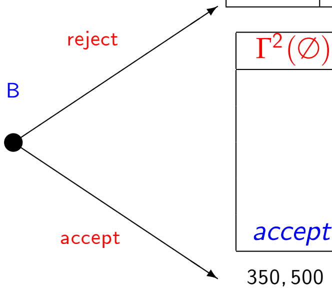
$\Gamma^2(h_1)$	200	300		
$(r, 200)$	$100, 100$	$200, 0$		
$(r, 300)$	$0, 200$	$150, 150$		



$\Gamma^2(\varnothing)$	200	300
accept	$350, 500$	$350, 500$

End of Round 2

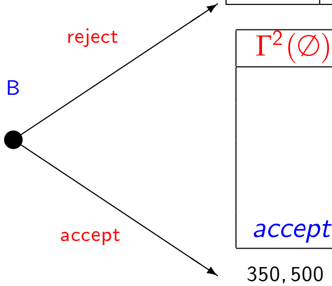
$\Gamma^2(h_1)$	200	300		
$(r, 200)$	$100, 100$	$200, 0$		



$\Gamma^2(\varnothing)$	200	300
accept	$350, 500$	$350, 500$

Round 3

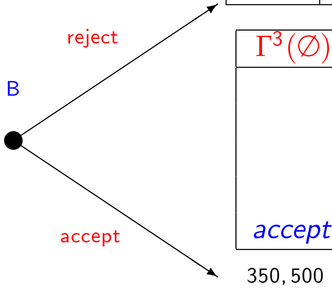
$\Gamma^2(h_1)$	200			
$(r, 200)$	$100, 100$			



$\Gamma^2(\emptyset)$	200	300
accept	$350, 500$	$350, 500$

Round 3

$\Gamma^3(h_1)$	200			
$(r, 200)$	$100, 100$			



$\Gamma^3(\emptyset)$	200
accept	$350, 500$

End of procedure

Backward induction

- For dynamic games with perfect information, the backward dominance procedure reduces to a very simple procedure called backward induction.

Definition (Game with perfect information)

A dynamic game is with perfect information if at every information set there is only one active player, and this player always knows exactly what choices have been made by his opponents in the past.

Theorem (Common belief in future rationality leads to backward induction)

Consider a dynamic game with perfect information.

Then, the strategies that can rationally be chosen under common belief in future rationality are exactly the backward induction strategies.

Theorem (Common belief in future rationality leads to backward induction)

Consider a dynamic game with perfect information.

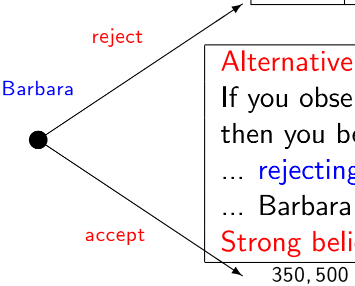
Then, the strategies that can rationally be chosen under common belief in future rationality are exactly the backward induction strategies.

- Hence, common belief in future rationality can be viewed as an epistemic foundation for backward induction.
- Other epistemic foundations for backward induction: Aumann (1995), Samet (1996), Stalnaker (1996, 1998), Balkenborg and Winter (1997), Asheim (2002), Quesada (2002, 2003), Clausing (2003, 2004), Feinberg (2005).
- See Perea (2007) for an overview.

Strong belief in the opponents' rationality

- So far, we have discussed the concept of common belief in future rationality.
- Main idea: Whatever you observe in the game, you always believe that your opponents will choose rationally from now on.
- It may not be the only plausible way of reasoning in a dynamic game.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Alternative way of reasoning:

If you observe that Barbara has rejected offer, then you believe that

... rejecting offer is part of a rational strategy,

... Barbara will choose price 400.

Strong belief in Barbara's rationality.

You will choose price 300.

- A strategy s_i is called rational for a type t_i, if at every information set $h \in H_i(s_i)$, the strategy s_i is optimal for the conditional belief $b_i(t_i, h)$.

- Idea of strong belief in the opponents' rationality:

- If at information set $h \in H_i$, it is possible for player i to believe that each of his opponents is implementing a rational strategy,

- then player i must believe so at h.

- Consider an epistemic model M and a type t_i within M.
- Type t_i strongly believes, at information set $h \in H_i$, in the opponents' rationality if:
- (richness condition) whenever h can be reached by opponents' strategies $(s_j)_{j \neq i}$ that are rational for some opponents' types in some epistemic model, the epistemic model must contain types for which these strategies s_j are rational, and
- (optimality condition) in this case, the conditional belief $b_i(t_i, h)$ assigns only positive probability to strategy-type pairs (s_j, t_j) where s_j is rational for t_j.
- Iterating this condition leads to common strong belief in rationality.
- Based on Battigalli and Siniscalchi (2002).
- There is no equilibrium analogue to common strong belief in rationality. See Perea (2017a).
- Details in Chapter 9 of the book.

- Common strong belief in rationality is a forward induction concept: Whenever possible, you try to explain the past choices made by your opponent.
- In contrast to common belief in future rationality, which is a backward induction concept: You ignore the opponent's past choices, and concentrate solely on the game that lies ahead.
- Battigalli and Siniscalchi (2002) show that common strong belief in rationality characterizes the concept of extensive-form rationalizability (Pearce (1984), Battigalli (1997)).

Definition (Iterated conditional dominance procedure)

Consider a finite dynamic game.

(Round 0) For every information set $h \in H$, create the full decision problem $\Gamma^0(h) = (S_i(h), S_{-i}(h))$.

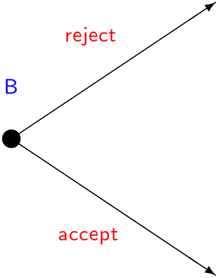
(Further rounds) For every $k \geq 2$, and every information set h, let $\Gamma^k(h)$ be the reduced decision problem which results if we eliminate from $\Gamma^{k-1}(h)$, for every player i, those strategies that are strictly dominated at some reduced decision problem $\Gamma^{k-1}(h')$ where i is active,

unless by doing so we would remove all remaining strategies for player i at h. In this case we remove nothing at h.

Strategy s_i survives the backward dominance procedure if s_i is in $\Gamma^k(\varnothing)$ for all k.

- Taken from Shimoji and Watson (1998).
- Characterizes the strategies that can rationally be chosen under common strong belief in rationality.

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250

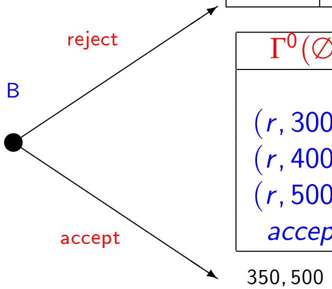


$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

350, 500

Round 1

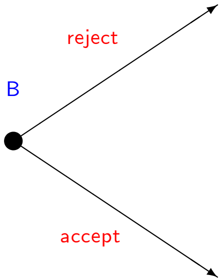
$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Round 1

$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250

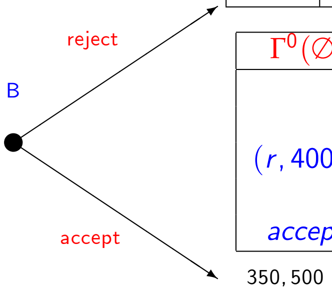


$\Gamma^0(\varnothing)$	200	300	400	500
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

350, 500

Round 1

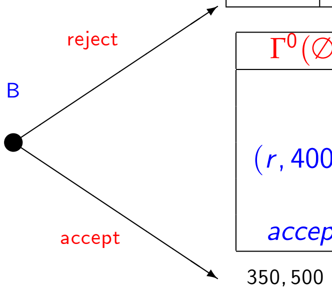
$\Gamma^0(h_1)$	200	300	400	500
$(r, 200)$	$100, 100$	$200, 0$	$200, 0$	$200, 0$
$(r, 300)$	$0, 200$	$150, 150$	$300, 0$	$300, 0$
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
$(r, 500)$	$0, 200$	$0, 300$	$0, 400$	$250, 250$



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Round 1

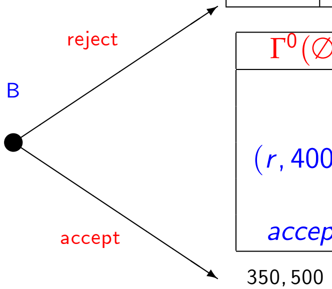
$\Gamma^0(h_1)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Round 1

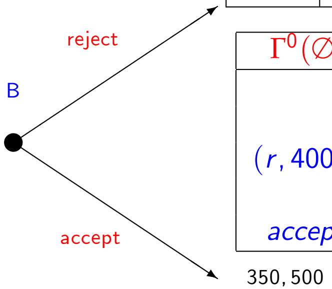
$\Gamma^0(h_1)$	200	300	400	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	



$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	$400, 0$
accept	$350, 500$	$350, 500$	$350, 500$	$350, 500$

Round 1

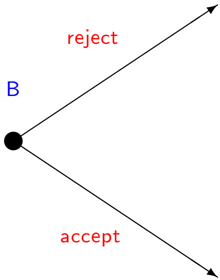
$\Gamma^0(h_1)$	200	300	400	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	



$\Gamma^0(\varnothing)$	200	300	400
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$
accept	$350, 500$	$350, 500$	$350, 500$

Round 1

$\Gamma^0(h_1)$	200	300	400	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	

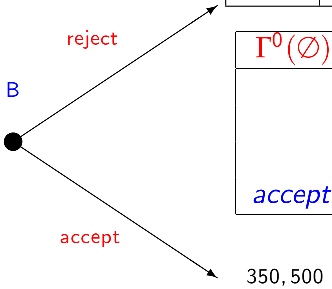


$\Gamma^0(\emptyset)$	200	300	400
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$
accept	$350, 500$	$350, 500$	$350, 500$

$350, 500$

End of Round 1

$\Gamma^0(h_1)$	200	300	400	
$(r, 400)$	$0, 200$	$0, 300$	$200, 200$	



$\Gamma^0(\varnothing)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

Round 2

$\Gamma^0(h_1)$		300	400	
$(r, 400)$		$0, 300$	$200, 200$	

reject

B

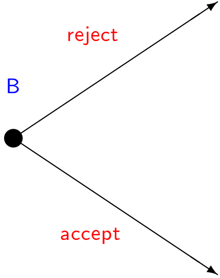
accept

$350, 500$

$\Gamma^0(\emptyset)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

Round 2

$\Gamma^0(h_1)$		300		
$(r, 400)$		$0, 300$		

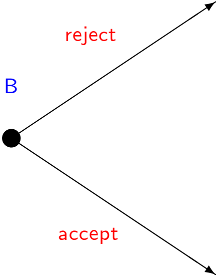


$\Gamma^0(\emptyset)$	200	300	400
accept	$350, 500$	$350, 500$	$350, 500$

$350, 500$

Round 2

$\Gamma^0(h_1)$		300		
$(r, 400)$		$0, 300$		



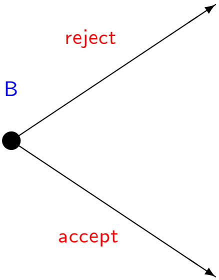
$350, 500$

$\Gamma^0(\varnothing)$	300
accept	$350, 500$

Round 2

$\Gamma^0(h_1)$		300		
$(r, 400)$		$0, 300$		

$\Gamma^0(\emptyset)$	300
accept	$350, 500$



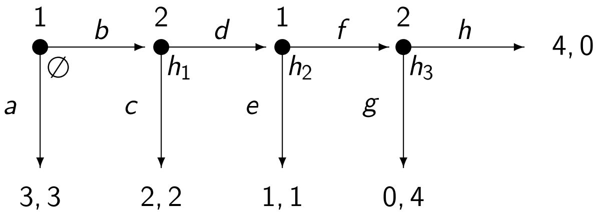
$350, 500$

End of procedure

Comparison with common belief in future rationality

- Common strong belief in rationality and common belief in future rationality represent completely different lines of reasoning.
- The example "Painting Chris' house" has shown that in terms of strategies selected, there is no logical relationship between the two concepts. Both concepts lead to a unique, yet different, strategy choice for you.
- However, both concepts lead to the same outcome in that example, namely that Barbara accepts the colleague's offer at the beginning.
- What about dynamic games with perfect information?

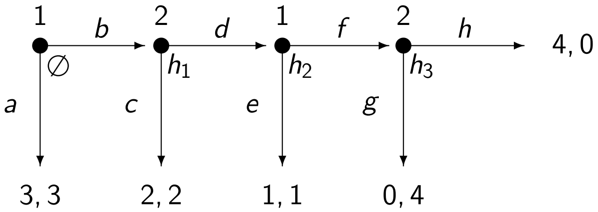
Example: Centipede game.



Common belief in future rationality: Do backward induction.

- At h_3, player 2's backward induction choice is g.
- At h_2, player 1's backward induction choice is e.
- At h_1, player 2's backward induction choice is c.
- At $\emptyset$, player 1's backward induction choice is a.

- Hence, common belief in future rationality uniquely selects strategy c for player 2.
- Induced outcome is a .



- Common strong belief in rationality:
- At h_1, player 2 must believe that player 1 is choosing a rational strategy.
- Hence, at h_1 player 2 must believe that player 1 is implementing the strategy (b, f).
- But then, the unique optimal strategy for player 2 is (d, g).
- Hence, common strong belief in rationality uniquely selects the strategy (d, g) for player 2.
- Induced outcome is a .

Theorem (Outcomes under common strong belief in rationality and common belief in future rationality)

Every outcome that is possible under common strong belief in rationality, is also possible under common belief in future rationality.

- A proof can be found in Perea (2017b).
- This result does not hold for strategies.

Theorem (Outcomes under common strong belief in rationality and common belief in future rationality)

Every outcome that is possible under common strong belief in rationality, is also possible under common belief in future rationality.

- Remember that in games with perfect information, common belief in future rationality leads to the backward induction strategies, and hence to the backward induction outcomes.
- In generic games with perfect information, the backward induction outcome is unique.

Corollary (Battigalli's Theorem)

Consider a generic dynamic game with perfect information. Then, the only outcome that is possible under common strong belief in rationality is the backward induction outcome.

- Result does not hold for strategies.

Corollary (Battigalli's Theorem)

Consider a generic dynamic game with perfect information. Then, the only outcome that is possible under common strong belief in rationality is the backward induction outcome.

- This result was first shown by Battigalli (1997).
- Other proofs can be found in Chen and Micali (2013), Heifetz and Perea (2015), Catonini (2017) and Perea (2018).

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