Epistemic Game Theory
Part 1: Standard Beliefs in Static Games

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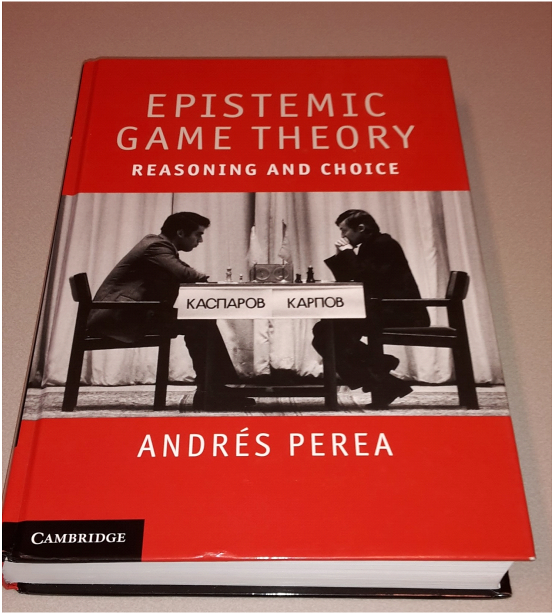
Introduction

- Game theory studies situations where you make a decision, but where the final outcome also depends on the choices of others.
- Before you make a choice, it is natural to reason about your opponents – about their choices but also about their beliefs.
- Oskar Morgenstern, in 1935, already stresses the importance of such reasoning for games.

- Classical game theory has focused mainly on the choices of the players.
- Epistemic game theory asks: Where do these choices come from?
- More precisely, it studies the beliefs that motivate these choices.
- Since the late 80's it has developed a broad spectrum of epistemic concepts for games.
- Some of these characterize existing concepts in classical game theory, others provide new ways of reasoning.


EPISTEMIC
GAME THEORY
REASONING AND CHOICE
КАСПАРОВ
КАРПОВ
ANDRÉS PEREA
CAMBRIDGE

Key properties of the book

- Takes seriously that game theory is about human beings.
- Zooms in on the reasoning of people before they make a decision in a game.
- One-person perspective.
- Examples from everyday life.
- Written for a broad audience.

EPICENTER Spring Course in
Epistemic Game Theory

June 22 – July 4, 2020
Maastricht University

Outline for the three days

- Part 1: Standard beliefs in static games
- Part 2: Lexicographic beliefs in static games
- Part 3: Conditional beliefs in dynamic games

Outline for today

- In the first part, we focus on standard beliefs in static games.
- We discuss, and formalize, the idea of common belief in rationality.
- We present a recursive procedure to compute the induced choices .
- We have a quick look at Nash equilibrium, and see that it requires more than just common belief in rationality.

Common belief in rationality

Idea

- If you are an expected utility maximizer, you form a belief about the opponents' choices, and make a choice that is optimal for this belief.
- That is, you choose rationally given your belief.
- It seems reasonable to believe that your opponents will choose rationally as well, ...
- and that your opponents believe that the others will choose rationally as well, and so on.
- Common belief in rationality.

Example: Going to a party

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

Story

- This evening, you are going to a party together with your friend Barbara.
- You must both decide which color to wear: blue, green, red or yellow.
- Your preferences for wearing these colors are as in the table. These numbers are called utilities.
- You dislike wearing the same color as Barbara: If you both would wear the same color, your utility would be 0.
- What color would you choose, and why?

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

- What choices are optimal for you for some belief?
- Choosing blue is optimal if you believe that Barbara chooses green.
- Choosing green is optimal if you believe that Barbara chooses blue.
- Choosing red is optimal if you believe that, with probability 0.6, Barbara chooses blue, and that with probability 0.4 she chooses green.
- Hence, blue, green and red are rational choices for you.

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

- Choosing yellow can never be optimal for you, even if you hold a probabilistic belief about Barbara's choice.
- If you assign probability less than 0.5 to Barbara's choice blue, then by choosing blue yourself, your expected utility will be at least $(0.5) \cdot 4 = 2$.
- If you assign probability at least 0.5 to Barbara's choice blue, then by choosing green yourself your expected utility will be at least $(0.5) \cdot 3 = 1.5$.
- So, yellow can never be optimal for you, and is therefore an irrational choice for you.

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

- What if you also believe that Barbara chooses rationally?
- If Barbara chooses rationally, she would never choose green.
- Hence, if you believe that Barbara chooses rationally, you must believe that Barbara will not choose green.
- Then, green will always be better for you than red.
- Conclusion: If you choose rationally, and believe that Barbara chooses rationally, you will not choose yellow or red.

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

- What if you also believe that Barbara believes in your rationality?
- If Barbara believes in your rationality, she will believe that you do not choose yellow.
- Then, yellow will be better for Barbara than blue.
- Hence, if you believe that Barbara chooses rationally, and that Barbara believes in your rationality, then you will believe that Barbara will not choose blue or green.
- Your unique best choice will be blue.
- Conclusion: If you choose rationally, believe that Barbara chooses rationally, and believe that Barbara believes that you choose rationally, then you must go for blue.

Belief Hierarchies

- To formalize the idea of common belief in rationality, we need to specify
- your belief about the opponents' choices (first-order belief),
- your belief about what your opponents believe about their opponents' choices (second-order belief),
- a belief about what the opponents believe that their opponents believe about the other players' choices (third-order belief),
- and so on, ad infinitum.
- Writing down a belief hierarchy explicitly is impossible.
- Is there an easy way to encode a belief hierarchy?

Types

- In a belief hierarchy, you hold a belief about
- the opponents' choices,
- the opponents' first-order beliefs,
- the opponents' second-order beliefs,
- and so on.

- Hence, in a belief hierarchy you hold a belief about
- the opponents' choices, and the opponents' belief hierarchies.

- Following Harsanyi (1967–1968), call a belief hierarchy a type.
- Then, a type holds a belief about the opponents' choices and the opponents' types.

- Let $I = \{1, ..., n\}$ be the set of players.
- For every player i, let C_i be the finite set of choices.

Definition (Epistemic model)

A finite epistemic model specifies for every player i a finite set T_i of possible types.

Moreover, for every type t_i it specifies a probabilistic belief $b_i(t_i)$ over the set $C_{-i} \times T_{-i}$ of opponents' choice-type combinations.

- Implicit epistemic model: For every type, we can derive the belief hierarchy induced by it.
- This is the model as used by Tan and Werlang (1988).
- Builds upon work by Harsanyi (1967–1968), Armbruster and Böge (1979), Böge and Eisele (1979), and Bernheim (1984).

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

$$
\begin{aligned}
b_1(t_1^{blue}) &= (red,\ t_2^{red}) \\
b_1(t_1^{green}) &= (blue,\ t_2^{blue}) \\
b_1(t_1^{red}) &= (0.6)\cdot(blue,\ t_2^{blue}) + (0.4)\cdot(green,\ t_2^{green}) \\
b_1(t_1^{yellow}) &= (yellow,\ t_2^{yellow})
\end{aligned}
$$

$$
\begin{aligned}
b_2(t_2^{blue}) &= (0.6)\cdot(red,\ t_1^{red}) + (0.4)\cdot(yellow,\ t_1^{yellow}) \\
b_2(t_2^{green}) &= (green,\ t_1^{green}) \\
b_2(t_2^{red}) &= (blue,\ t_1^{blue}) \\
b_2(t_2^{yellow}) &= (red,\ t_1^{red})
\end{aligned}
$$

- Remember: A type t_i holds a probabilistic belief $b_i(t_i)$ over the set $C_{-i} \times T_{-i}$ of opponents' choice-type combinations.

- For a choice c_i, let

$$u_i(c_i, t_i) := \sum_{(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}} b_i(t_i)(c_{-i}, t_{-i}) \cdot u_i(c_i, c_{-i})$$

  be the expected utility that type t_i obtains by choosing c_i.

- Choice c_i is optimal for type t_i if

$$u_i(c_i, t_i) \geq u_i(c_i', t_i) \text{ for all } c_i' \in C_i.$$

Definition (Belief in the opponents' rationality)

Type t_i believes in the opponents' rationality if his belief $b_i(t_i)$ only assigns positive probability to opponents' choice-type pairs (c_j, t_j) where choice c_j is optimal for type t_j.

Definition (Common belief in rationality)

(Induction start) Type t_i expresses 1-fold belief in rationality if t_i believes in the opponents' rationality.

(Inductive step) For every $k \geq 2$, type t_i expresses k-fold belief in rationality if t_i only assigns positive probability to opponents' types that express $(k-1)$-fold belief in rationality.

Type t_i expresses common belief in rationality if t_i expresses k-fold belief in rationality for all k.

- Based on Spohn (1982) and Tan and Werlang (1988) .

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

$$
\begin{aligned}
b_1(t_1^{blue}) &= (red,\ t_2^{red}) \\
b_1(t_1^{green}) &= (blue,\ t_2^{blue}) \\
b_1(t_1^{red}) &= (0.6)\cdot(blue,\ t_2^{blue}) + (0.4)\cdot(green,\ t_2^{green}) \\
b_1(t_1^{yellow}) &= (yellow,\ t_2^{yellow})
\end{aligned}
$$

$$
\begin{aligned}
b_2(t_2^{blue}) &= (0.6)\cdot(red,\ t_1^{red}) + (0.4)\cdot(yellow,\ t_1^{yellow}) \\
b_2(t_2^{green}) &= (green,\ t_1^{green}) \\
b_2(t_2^{red}) &= (blue,\ t_1^{blue}) \\
b_2(t_2^{yellow}) &= (red,\ t_1^{red})
\end{aligned}
$$

Only the types t_1^{blue} and t_2^{red} express common belief in rationality.

Recursive Procedure

- Suppose we wish to find those choices you can rationally make under common belief in rationality.
- Is there a recursive procedure that helps us find these choices?
- Based on following result:

Lemma (Pearce (1984))

A choice c_i is optimal for some probabilistic belief about the opponents' choices, if and only if, c_i is not strictly dominated by any randomized choice.

- Here, a randomized choice r_i for player i is a probability distribution on i's choices.
- Choice c_i is strictly dominated by the randomized choice r_i if

$$u_i(c_i, c_{-i}) < u_i(r_i, c_{-i})$$

for every opponents' choice-combination $c_{-i} \in C_{-i}$.

Definition (Iterated elimination of strictly dominated choices)

Consider a finite static game Γ.

(Round 0) Let $\Gamma^0 := \Gamma$ be the original game.

(Further rounds) For every $k \geq 1$, let Γ^k be the game which results if we eliminate from Γ^{k-1} all choices that are strictly dominated within Γ^{k-1}.

- This procedure terminates within finitely many steps. That is, there is some K with $\Gamma^{K+1} = \Gamma^K$.
- The choices in Γ^K are said to survive iterated elimination of strictly dominated choices.
- It always yields a nonempty set of choices for all players.
- The final output does not depend on the order by which we eliminate choices.

Definition (Iterated elimination of strictly dominated choices)

Consider a finite static game Γ.

(Round 0) Let $\Gamma^0 := \Gamma$ be the original game.

(Further rounds) For every $k \geq 1$, let Γ^k be the game which results if we eliminate from Γ^{k-1} all choices that are strictly dominated within Γ^{k-1}.

- In two-player games, it yields exactly the rationalizable choices, as defined by Bernheim (1984) and Pearce (1984).
- For games with more than two players, rationalizability requires player i's belief about player j's choice to be stochastically independent from his belief about player k's choice.
- The procedure does not impose this independence condition.
- For games with more than two players, this procedure yields correlated rationalizability (Brandenburger and Dekel (1987)).

Theorem (Tan and Werlang (1988))

(1) For every $k \geq 1$, the choices that are optimal for a type that expresses up to k-fold belief in rationality are exactly those choices that survive $(k+1)$-fold elimination of strictly dominated choices.

(2) The choices that are optimal for a type that expresses common belief in rationality are exactly those choices that survive iterated elimination of strictly dominated choices.

Corollary (Common belief in rationality is always possible)

We can always construct an epistemic model in which all types express common belief in rationality.

Example: Going to a party

Barbara

You		blue	green	red	yellow
	blue	0, 0	4, 1	4, 4	4, 3
You	green	3, 2	0, 0	3, 4	3, 3
	red	2, 2	2, 1	0, 0	2, 3
	yellow	1, 2	1, 1	1, 4	0, 0

- **Round 1.** Your choice yellow is strictly dominated by randomized choice $(0.5) \cdot \textit{blue} + (0.5) \cdot \textit{green}$.
- Barbara's choice green is strictly dominated by randomized choice $(0.5) \cdot \textit{red} + (0.5) \cdot \textit{yellow}$.
- Eliminate your choice yellow and Barbara's choice green.

Example: Going to a party

Barbara

You		blue	red	yellow
	blue	0, 0	4, 4	4, 3
	green	3, 2	3, 4	3, 3
	red	2, 2	0, 0	2, 3

- **Round 2.** Your choice red is strictly dominated by green.
- Barbara's choice blue is strictly dominated by yellow.
- Eliminate your choice red and Barbara's choice blue.

Example: Going to a party

Barbara

		red	yellow
You	blue	4, 4	4, 3
	green	3, 4	3, 3

- **Round 3.** Your choice green is strictly dominated by blue.
- Barbara's choice yellow is strictly dominated by red.
- Eliminate your choice green and Barbara's choice yellow.

Example: Going to a party

		Barbara
		red
You	blue	4, 4

- Procedure stops.
- Under common belief in rationality, you can only rationally wear blue, and Barbara can only rationally wear red.

Nash equilibrium

- Nash equilibrium has dominated game theory for many years.
- But until the rise of Epistemic Game Theory it remained unclear what Nash equilibrium assumes about the reasoning of the players.
- Nash equilibrium requires more than just common belief in rationality.
- Nash equilibrium can be epistemically characterized by

  common belief in rationality + simple belief hierarchy.

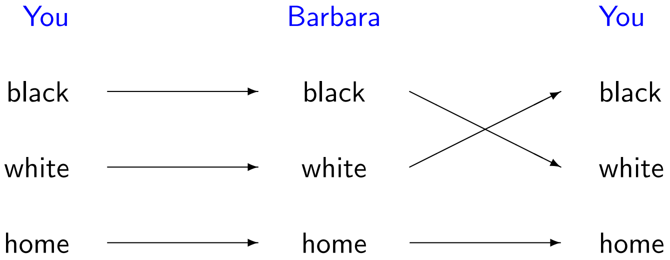
- However, the condition of a simple belief hierarchy is quite unnatural, and overly restrictive.

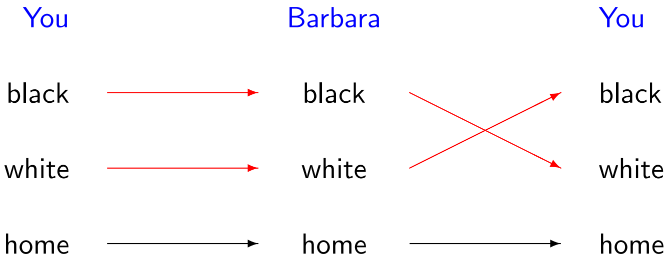
Example: Black or white?

Story

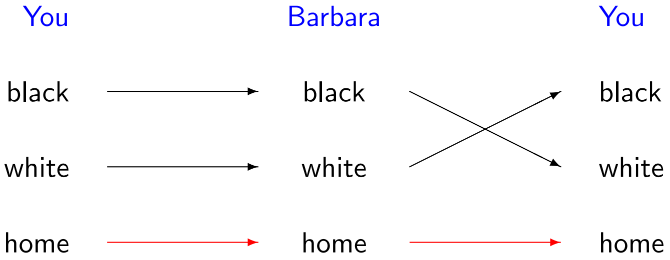
- You and Barbara are again invited for a party.
- You can only wear black or white, but you can also stay at home.
- Staying at home gives a utility of 2.
- Going to the party, seeing Barbara, and wearing the same color, gives you a utility of 3.
- Otherwise, your utility will be 0.
- Same for Barbara, except that she prefers to wear a different color than you.

		Barbara		
		black	white	home
You	black	3, 0	0, 3	0, 2
	white	0, 3	3, 0	0, 2
	home	2, 0	2, 0	2, 2





- All belief hierarchies express common belief in rationality.
- Under common belief in rationality, you can rationally make any choice.
- In your belief hierarchy that starts at your choice black, you believe that Barbara is wrong about your belief.
- This belief hierarchy is not simple.
- Same for your belief hierarchy that starts at your choice white.



- In your belief hierarchy that starts at your choice home, you believe that Barbara is correct about your belief.

- The whole belief hierarchy is generated by the beliefs $\sigma_1 =$ home and $\sigma_2 =$ home.

- This belief hierarchy is simple.

- It corresponds to the Nash equilibrium ($\sigma_1 =$ home, $\sigma_2 =$ home).

- In general, it can be shown that Nash equilibrium corresponds exactly to belief hierarchies that
- express common belief in rationality, and
- are simple.
- Details can be found in Chapter 4 of the book.
- In particular, Nash equilibrium assumes that a player believes that his opponents are correct about his beliefs.
- This is a strong, and somewhat unreasonable, assumption.

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