Algorithms for Cautious Reasoning in Games*

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Abstract

We provide comparable algorithms for the Dekel-Fudenberg procedure, iterated admissibility and proper rationalizability by means of the concepts of preference restrictions and likelihood orderings. We apply the algorithms for comparing iterated admissibility and proper rationalizability, and provide a sufficient condition under which iterated admissibility does not rule out properly rationalizable strategies. Finally, we use the algorithms to examine an economically relevant strategic situation, namely a bilateral commitment bargaining game.

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1 Introduction

In non-cooperative game theory, a player is *cautious* if he takes into account all opponents' strategies, also strategies that seem very unlikely to be chosen by the opponent. Cautious reasoning of a player can be modeled by a *lexicographic belief* (Blume et al. (1991a)), which allows this player to deem some opponent's strategy a infinitely more likely than some other strategy b, while still taking b into account. What outcomes of a strategic game are consistent with common belief of the event that all players are rational and cautious?

Various concepts in the literature provide different answers to this question. Still, there is a common idea underlying each of these concepts, namely that a player should deem an opponent's strategy a infinitely more likely than b whenever he considers a a "better choice" for his opponent than b. The question then remains what we mean by a "better choice".

As an illustration, consider the following economic example. An entrant (firm 1) and an incumbent (firm 2) must decide which type of good to bring on the market: x, y or z. The entrant expects a revenue of 3 as long as it produces a good different from the incumbent, and a revenue of 2 if it produces the same good. Its production costs for each of the goods is 2. The incumbent expects, for every production choice, a revenue of 3. The only exception is when the goods x and z are both brought on the market. Since these goods are complementary, the incumbent expects a revenue of 6 in this case. The incumbent has produced good x in the past, which would therefore have the lowest costs (normalized to 0). Producing goods y and z would cost the incumbent 1 and 2, respectively, since good y is more similar to x than z is. The profits for both firms can be found in Figure 1.

		Incumbent		
		x_2	y_2	z_2
Entrant	x_1	$0,3$	$1,2$	$1,4$
	y_1	$1,3$	$0,2$	$1,1$
	z_1	$1,6$	$1,2$	$0,1$

Figure 1

Here, we denote the choices for firm i by x_i, y_i and z_i. Note that for firm 2, production choice y_2 can never be optimal, whereas x_2 and z_2 can be optimal for some belief about firm 1's choice. One could therefore argue that x_2 and z_2 are better choices for firm 2 than y_2, and hence firm 1 should deem x_2 and z_2 infinitely more likely than y_2. But then, if firm 1 takes all possible choices by firm 2 into account, its unique optimal choice would be to implement production plan y_1. The line of argument we have followed here is *iterated admissibility* (or, iterated elimination of weakly dominated strategies), for which an epistemic foundation has been provided in Brandenburger et al. (2008).

Iterated admissibility is not the only plausible concept for cautious reasoning, however. Consider again the example above. If firm 2 would indeed believe that firm 1 makes production choice y_1, which is what iterated admissibility requires, then choice y_2 would actually be better for firm 2 than choice z_2. So, *ex-post* one could argue that firm 1 should deem y_2 infinitely *more* likely than z_2, and not infinitely *less* likely, as iterated admissibility imposes. Hence, by applying the procedure of iterated admissibility one may *ex-ante* impose some conditions on lexicographic beliefs which *ex-post* need no longer be that convincing.

The concept of *proper rationalizability* (Schuhmacher (1999), Asheim (2001)) takes a different viewpoint. The key condition is that a player should deem an opponent's strategy a infinitely more likely than b whenever he believes that the opponent, after completing his reasoning process, prefers a to b. We say that the player *respects the opponent's preferences.* So, in a sense, the approach in proper rationalizability is entirely *ex-post,* since an opponent's strategy a is only deemed better than b if it is believed to be better *ex-post,* after the opponent has formed his final belief.

To see what difference this approach makes, let us return to the example. It is clear that for firm 2, choice x_2 is better than choice y_2, whereas z_2 need not be better than y_2. Proper rationalizability therefore only requires that firm 1 deems x_2 infinitely more likely than y_2, but does not require that it deems z_2 infinitely more likely than y_2. If firm 1 indeed holds such a belief, then it prefers y_1 to x_1, and hence firm 2 should deem y_1 infinitely more likely than x_1. But then, firm 2 will prefer x_2 to y_2, and y_2 to z_2. Hence, firm 1 should deem x_2 infinitely more likely than y_2, and y_2 infinitely more likely than z_2. As a consequence, firm 1 should choose production plan z_1, and not y_1, as iterated admissibility requires.

Both conceps, *iterated admissibility* and *proper rationalizability,* are reasonable concepts with their own intuitive appeal, but may lead to completely different choices as we have seen. It therefore seems worthwile to investigate their differences and similarities in some more detail, and this is exactly what this paper is trying to accomplish.

To make the picture complete, we will also investigate a third concept for cautious reasoning, namely the *Dekel-Fudenberg procedure* (Dekel and Fudenberg, 1990), where one round of elimination of *weakly* dominated strategies is followed by iterated elimination of *strictly* dominated strategies. This procedure has been given an epistemic foundation by Brandenburger (1992) and Börgers (1994), and it is weaker than both iterated admissibility and proper rationalizability. One could view the Dekel-Fudenberg procedure as a basic starting point for cautious reasoning, in the sense that the eliminated strategies are definitely incompatible with common belief of the event that all players are rational and cautious.

Both the Dekel-Fudenberg procedure and iterated admissibility are defined in terms of algorithms. The case of proper rationalizability is different. This concept was defined by Schuhmacher (1999) and Asheim (2001) by means of epistemic conditions. Schuhmacher (1999) provides an algorithm, *iteratively proper trembling,* which generates for a given $\varepsilon > 0$ the set of mixed strategy profiles that can be chosen under common belief of the ε-proper trembling condition. This procedure does not yield the set of properly rationalizable strategies directly, as we must still let ε go to zero, and see which strategies survive in the limit. Recently, Perea (2009) has provided an algorithm that *directly* computes the set of properly rationalizable strategies in every game.

The purpose of this paper is to closely investigate the differences and similarities between the Dekel-Fudenberg procedure, iterated admissibility and proper rationalizability. As to achieve this, we present algorithms for the Dekel-Fudenberg procedure and iterated admissibility that build on the key concepts introduced by Perea (2009), thereby making such established procedures comparable to the new algorithm for proper rationalizability. In Section 2, we introduce these key concepts: *preference restrictions* and *likelihood orderings*. In Section 3, we construct algorithms for the Dekel-Fudenberg procedure and iterated admissibility that are comparable with the one for proper rationalizability. In Section 4, we then put these algorithms to use. In particular, we offer examples illuminating the differences between iterated admissibility and proper rationalizability. Moreover, we provide a sufficient condition under which iterated admissibility does not rule out properly rationalizable strategies. Finally, we use the algorithms to examine an economically relevant strategic situation, namely a bilateral commitment bargaining game which has recently been analyzed by Ellingsen and Miettinen (2008). In Section 5 we offer concluding remarks, while an appendix contains all proofs.

2 Preference Restrictions and Likelihood Orderings

Consider a finite *strategic game* $G = (S_i, u_i)_{i \in I}$ with n players, where $I = \{1, 2, ..., n\}$ is the set of players, the finite set S_i denotes the set of strategies for player i and $u_i : S_1 \times ... \times S_n \to \mathbb{R}$ denotes player i's utility function. As usual, we extend u_i to subjective probability distributions $\lambda_i \in \Delta(S_{-i})$ over the opponents' strategy combinations, writing $u_i(s_i, \lambda_i)$ for the resulting subjective expected utility. Here, $S_{-i} := \times_{j \neq i} S_j$ denotes the set of opponents' strategy combinations.

Each player i's preferences over his own strategies are determined by u_i and a *lexicographic probability system* (LPS) (Blume et al., 1991a) with full support on S_{-i}. An LPS consists of a finite sequence of subjective probability distributions, $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^K)$, where for each $k \in \{1, \ldots, K\}$, $\lambda_i^k \in \Delta(S_{-i})$. Player i prefers $a_i \in S_i$ to $s_i \in S_i$ if there exists $k \in \{1, \ldots, K\}$ such that (i) $u_i(a_i, \lambda_i^k) > u_i(s_i, \lambda_i^k)$ and (ii) $u_i(a_i, \lambda_i^{k'}) = u_i(s_i, \lambda_i^{k'})$ for all $k' \in \{1, \ldots, k-1\}$. The LPS $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^K)$ has *full support* on S_{-i} if for every $s_{-i} \in S_{-i}$, there exists some $k \in \{1, \ldots, K\}$ such that $\lambda_i^k(s_{-i}) > 0$. Player i deems s_{-i} *infinitely more likely than* s'_{-i} (written $s_{-i} \gg_i s'_{-i}$) if there exists $k \in \{1, \ldots, K\}$ such that (i) $\lambda_i^k(s_{-i}) > 0$ and (ii) $\lambda_i^{k'}(s'_{-i}) = 0$ for all $k' \in \{1, \ldots, k\}$. It follows that $\gg_i$ is an asymmetric and transitive binary relation. For a given opponent $j \neq i$, we say that player i deems s_j infinitely more likely than s'_j (written $s_j \gg_i s'_j$) if there is some $s_{-ij} \in S_{-ij}$ such that $(s_j, s_{ij}) \gg_i (s'_j, s'_{-ij})$ for all $s'_{-ij} \in S_{-ij}$. Here, $S_{-ij} := \times_{\ell \neq i,j} S_\ell$ denotes the set of strategy combinations for players in $I \backslash \{i, j\}$.

The following two definitions, which are taken from Perea (2009), provide the key concepts for our algorithms.

Definition 1 (Preference restriction) A preference restriction on S_i is a pair (s_i, A_i), where $s_i \in S_i$ and A_i is a nonempty subset of S_i.

The interpretation of a preference restriction (s_i, A_i) is that player i prefers some strategy in A_i to s_i. Let $\mathcal{R}_i^*$ denote the collection of all sets of preference restrictions.

For any set R_i of preference restrictions, define the *choice set* $C_i(R_i)$ as follows:

$$C_i(R_i) := \{s_i \in S_i \mid \nexists A_i \subseteq S_i \text{ with } (s, A_i) \in R_i\}.$$

It follows that $C_i(R'_i) \cap C_i(R''_i) = C_i(R'_i \cup R''_i)$ for every R'_i, $R''_i \in \mathcal{R}_i^*$. In particular, $C_i(R'_i) \supseteq C_i(R''_i)$ whenever $R'_i \subseteq R''_i$.

Definition 2 (Likelihood ordering) A *likelihood ordering* on S_{-i} is an ordered partition $L_i = (L_i^1, L_i^2, \ldots, L_i^K)$ of S_{-i}.

A likelihood ordering $L_i = (L_i^1, L_i^2, \ldots, L_i^K)$ on S_{-i} determines the infinitely-more-likely relation on the set of opponents' strategy combinations: $s_{-i} \gg_i s'_{-i}$ if and only if $s_{-i} \in L_i^k$ and $s'_{-i} \in L_i^{k'}$ with $k < k'$. Similarly, $s_j \gg_i s'_j$ if there is some $s_{-ij} \in S_{-ij}$ such that $(s_j, s_{-ij}) \gg_i (s'_j, s'_{-ij})$ for all $s'_{-ij} \in S_{-ij}$. Let $\mathcal{L}_i^*$ denote the set of all likelihood orderings on S_{-i}.

For any subset $\mathcal{L}_i$ of likelihood orderings on S_{-i}, let $R_i(\mathcal{L}_i)$ denote the set of preference restrictions *derived* from $\mathcal{L}_i$ in the following manner:

$$
\begin{aligned}
R_i(\mathcal{L}_i) := \{(s_i, A_i) \in S_i \times 2^{S_i} \mid \forall L_i = (L_i^1, \ldots L_i^K) \in \mathcal{L}_i, \exists k \in \{1, \ldots, K\} \text{ and} \\
\mu_i \in \Delta(A_i) \text{ such that } s_i \text{ is weakly dominated by } \mu_i \text{ on } L_i^1 \cup \cdots \cup L_i^k\}\,.
\end{aligned}
$$

Here, we say that s_i is *weakly dominated* by μ_i on some subset $A_{-i} \subseteq S_{-i}$ if $u_i(s_i, s_{-i}) \leq u_i(\mu_i, s_{-i})$ for every $s_{-i} \in A_{-i}$, with strict inequality for some $s_{-i} \in A_{-i}$. It follows that $R_i(\mathcal{L}'_i) \cap R_i(\mathcal{L}''_i) = R_i(\mathcal{L}'_i \cup \mathcal{L}''_i)$ for every $\mathcal{L}'_i$, $\mathcal{L}''_i \in \mathcal{L}_i^*$. In particular, $R_i(\mathcal{L}'_i) \supseteq R_i(\mathcal{L}''_i)$ whenever $\mathcal{L}'_i \subseteq \mathcal{L}''_i$.

Likelihood-orderings can be related to the ordinary *belief* operator as well as the *assumption* operator, as proposed by Brandenburger et al. (2008) (and discussed by Asheim and Søvik, 2005, Section 6).

Definition 3 (Believing an event) For a given subset $A_{-i} \subseteq S_{-i}$, we say that the likelihood ordering L_i *believes* A_{-i} if, for every $s_{-i} \in S_{-i} \backslash A_{-i}$, $a_{-i} \gg_i s_{-i}$ for some $a_{-i} \in A_{-i}$.

Definition 4 (Assuming an event) For a given subset $A_{-i} \subseteq S_{-i}$, we say that the likelihood ordering L_i *assumes* A_{-i} if, for every $s_{-i} \in S_{-i} \backslash A_{-i}$, $a_{-i} \gg_i s_{-i}$ for every $a_{-i} \in A_{-i}$.

So, if L_i assumes a non-empty event A_{-i} it also believes the event A_{-i}, but not vice versa. Likelihood-orderings can also be related to *respect of preferences* as introduced by Blume et al. (1991b). In the following definition, $\mathcal{R}_{-i}^* := \cup_{j \neq i} \mathcal{R}_j^*$ denotes the set of preference restrictions for i's opponents.

Definition 5 (Respecting preferences) For a given subset $R_{-i} \subseteq \mathcal{R}_{-i}^*$ of preference restrictions, we say that the likelihood ordering L_i *respects* R_{-i} if, for every $(s_j, A_j) \in R_{-i}$, $a_j \gg_i s_j$ for some $a_j \in A_j$.

For a given set $R_{-i} \subseteq \mathcal{R}^*_{-i}$ of preference restrictions, we define $C_{-i}(R_{-i}) := \times_{j \neq i} C_j(R_j)$, where $R_j = R_{-i} \cap \mathcal{R}^*_j$ for every $j \neq i$. Hence, if L_i respects the set R_{-i} of preference restrictions, it also believes the event $C_{-i}(R_{-i})$, but not vice versa.

Let $\mathcal{L}^b_i(R_{-i})$ denote the set of likelihood orderings that *believe* the opponents' rationality when their preferences satisfy the set R_{-i} of preference restrictions:

$$\mathcal{L}^b_i(R_{-i}) := \{L_i \in \mathcal{L}^*_i \mid L_i \text{ believes } C_{-i}(R_{-i})\}\,.$$

Let $\mathcal{L}^a_i(R_{-i})$ denote the set of likelihood orderings that *assume* the opponents' rationality when their preferences satisfy the set R_{-i} of preference restrictions:

$$\mathcal{L}^a_i(R_{-i}) := \{L_i \in \mathcal{L}^*_i \mid L_i \text{ assumes } C_{-i}(R_{-i})\}\,.$$

Finally, let $\mathcal{L}^r_i(R_{-i})$ denote the set of likelihood orderings that *respect* the opponents' preferences when their preferences satisfy the set R_{-i} of preference restrictions:

$$\mathcal{L}^r_i(R_{-i}) := \{L_i \in \mathcal{L}^*_i \mid L_i \text{ respects } R_{-i}\}\,.$$

It follows from the observations that assumption implies belief, but not vice versa, and respect of preferences implies belief of rationality, but not vice versa. That is,

$$\mathcal{L}^b_i(R_{-i}) \supseteq \mathcal{L}^a_i(R_{-i}) \cup \mathcal{L}^r_i(R_{-i})$$

for every $R_{-i} \in \mathcal{R}^*_{-i}$ with $C_{-i}(R_{-i}) \neq \emptyset$. Since the belief operator satisfies conjunction and monotonicity, the properties of the choice correspondence $C_i(\cdot)$ imply that

$$\mathcal{L}^b_i(R'_{-i}) \cap \mathcal{L}^b_i(R''_{-i}) = \mathcal{L}^b_i(R'_{-i} \cup R''_{-i})\,.$$

for every R'_{-i}, $R''_{-i} \in \mathcal{R}^*_{-i}$. However, since the assumption operator satisfies conjunction but not monotonicity, it holds for every R'_{-i}, $R''_{-i} \in \mathcal{R}^*_{-i}$ that

$$\mathcal{L}^a_i(R'_{-i}) \cap \mathcal{L}^a_i(R''_{-i}) \subseteq \mathcal{L}^a_i(R'_j \cup R''_j)\,,$$

while the inverse inclusion need not hold. Finally, Definition 5 implies that

$$\mathcal{L}^r_i(R'_{-i}) \cap \mathcal{L}^r_i(R''_{-i}) = \mathcal{L}^r_i(R'_{-i} \cup R''_{-i})\,.$$

In particular, $\mathcal{L}^b_i(R'_{-i}) \supseteq \mathcal{L}^b_i(R''_{-i})$ and $\mathcal{L}^r_i(R'_{-i}) \supseteq \mathcal{L}^r_i(R''_{-i})$ whenever $R'_{-i} \subseteq R''_{-i}$. This conclusion need not hold for $\mathcal{L}^a_i(\cdot)$ since a likelihood ordering L_i may assume A'_{-i} but not A''_{-i} even though $A'_{-i} \subset A''_{-i}$. Hence, we may have $\mathcal{L}^a_i(R'_{-i}) \not\subseteq \mathcal{L}^a_i(R''_{-i})$ and $\mathcal{L}^a_i(R'_{-i}) \not\supseteq \mathcal{L}^a_i(R''_{-i})$ even though $R'_{-i} \subset R''_{-i}$.

3 Algorithms

In this section we provide comparable algorithms for the Dekel-Fudenberg procedure, iterated admissibility and proper rationalizability.

3.1 An algorithm for the Dekel-Fudenberg procedure

We first consider the *Dekel-Fudenberg procedure* (Dekel and Fudenberg, 1990), which is the procedure where one round of maximal elimination of weakly dominated strategies is followed by iterated maximal elimination of strictly dominated strategies. Following Brandenburger (1992), strategies surviving the Dekel-Fudenberg procedure are referred to as *permissible*.

Consider the following algorithm, which iteratedly increases the set of preference restrictions for both players:

Ini For all players i, let $R_i^0 = \emptyset$.

DF For every $n \geq 1$, and all players i, let $R_i^n = R_i(\mathcal{L}_i^b(R_{-i}^{n-1}))$.

Here, $R_{-i}^{n-1} := \cup_{j \neq i} R_j^{n-1}$. From the properties of $\mathcal{L}_i^b(\cdot)$ and $R_i(\cdot)$, it follows that **Ini** and **DF** determines, for each player, a non-decreasing sequence of sets of preference restrictions and a non-increasing sequence of sets of likelihood orderings (where non-decreasing and non-increasing are defined w.r.t. set inclusion). As a consequence, the sequence $C_i(R_i^n)$ of choice sets is non-increasing. Since the set of preference restrictions is finite, the algorithm converges after a finite number of rounds.

For both players i, let $R_i^\infty := \bigcup_{n=1}^\infty R_i^n$ be the limiting set of preference restrictions produced by the algorithm defined by **Ini** and **DF**.

Proposition 1 *Let G be a finite strategic game. Then, for all players i, a strategy s_i is permissible if and only if $s_i \in C_i(R_i^\infty)$.*

Proof. See the appendix. ■

3.2 An algorithm for iterated admissibility

Iterated admissibility is the procedure of iterated maximal elimination of weakly dominated strategies.

Consider the following algorithm:

Ini For all players i, let $R_i^0 = \emptyset$.

IA For every $n \geq 1$, and all players i, let

$$R_i^n = R_i\big(\mathcal{L}_i^a(R_{-i}^0) \cap \mathcal{L}_i^a(R_{-i}^1) \cap \cdots \cap \mathcal{L}_i^a(R_{-i}^{n-1})\big)\,.$$

From the properties of $R_i(\cdot)$, it follows that **Ini** and **IA** determines, for each player, a non-decreasing sequence of sets of preference restrictions and a non-increasing sequence $\mathcal{L}_i^a(R_{-i}^0) \cap \mathcal{L}_i^a(R_{-i}^1) \cap \cdots \cap \mathcal{L}_i^a(R_{-i}^n)$ of sets of likelihood orderings. As a consequence, the sequence $C_i(R_i^n)$ of choice sets is non-increasing. Since the set of preference restrictions is finite, the algorithm converges after a finite number of rounds.

For both players i, let $R_i^\infty := \bigcup_{n=1}^\infty R_i^n$ be the limiting set of preference restrictions produced by the algorithm defined by **Ini** and **IA**.

Proposition 2 *Let G be a finite strategic game. Then, for all players i, a strategy s_i survives iterated admissibility if and only if $s_i \in C_i(R_i^\infty)$.*

Proof. See the appendix. ■

Proposition 2 echoes Brandenburger et al.'s (2008, Theorem 9.1) epistemic characterization of iterated admissibility (see also the observation that Stahl, 1995, makes in his theorem), by pointing out that iterated admissibility corresponds to likelihood orderings where strategies eliminated in a later round are deemed infinitely more likely than strategies eliminated in an earlier round, and surviving strategies are deemed infinitely more likely than strategies eliminated in some round. Here we let these likelihood orderings interplay with sets of preference restrictions, thereby allowing comparison with the algorithm for proper rationalizability, presented next.

3.3 An algorithm for proper rationalizability

We finally consider *proper rationalizability*, a concept defined by Schuhmacher (1999) and characterized by Asheim (2001). We refer to these references for details.

Consider the following algorithm:

Ini For all players i, let $R_i^0 = \emptyset$.

PR For every $n \geq 1$, and all players i, let $R_i^n = R_i(\mathcal{L}_i^r(R_{-i}^{n-1}))$.

From the properties of $\mathcal{L}_i^r(\cdot)$ and $R_i(\cdot)$, it follows that **Ini** and **PR** determines, for each player, a non-decreasing sequence of sets of preference restrictions and a non-increasing sequence of sets of likelihood orderings. Since the set of preference restrictions is finite, the algorithm converges after a finite number of rounds.

For both players i, let $R_i^\infty := \bigcup_{n=1}^\infty R_i^n$ be the limiting set of preference restrictions produced by the algorithm defined by **Ini** and **PR**.

Proposition 3 *Let G be a finite strategic game. Then, for all players i, a strategy s_i is properly rationalizable if and only if $s_i \in C_i(R_i^\infty)$.*

Proof. Perea (2009). ■

4 Applying the algorithms

In this section we put the algorithms to work. In the first subsection we present three examples illustrating how the sequences of preference restrictions that the algorithms give rise to shed light on differences between iterated admissibility and proper rationalizability. In particular, in the first example, the set of strategies surviving iterated admissibility is a strict subset of the set of properly rationalizable strategies, while the sequences of preference restrictions for iterated admissibility and proper rationalizability coincide in the latter two examples.

In the second subsection we build on insights conveyed by the examples and provide through Proposition 4 a sufficient condition ensuring that any properly rationalizable strategy survives iterated admissibility. In particular, since proper equilibrium always exists and any strategy being used with positive probability in a proper equilibrium is properly rationalizable, we reach the following conclusion: If a game—for which iterated admissibility leads to a unique strategy for each player—satisfies the sufficient condition of Proposition 4, then the surviving strategies are the unique properly rationalizable strategies and the corresponding strategy profile is the unique proper equilibrium.

In the third subsection we consider a recent contribution on commitment bargaining (Ellingsen and Miettinen, 2008) and use the algorithm of Section 3.3 to show how proper rationalizability yields the results they seek, while other procedures do not.

4.1 Examples

We now illustrate our algorithms by means of three examples. Before doing so, we introduce the following piece of notation: For a given set R_i of preference restrictions on S_i, define the monotonic cover of R_i by

$$mcR_i := \{(s_i, A_i) \mid \exists \hat{A}_i \subseteq A_i \text{ with } (s_i, \hat{A}_i) \in R_i\} \,.$$

Every set R_i^n of preference restrictions produced by each algorithm on the way to R_i^∞ can clearly be written as the monotonic cover of some smaller set.

[Figure 1 about here.]

In G_1, illustrated in Figure 1 (and discussed by Asheim and Dufwenberg, 2003), iterated admissibility works by eliminating D, R, and M, leading to (U, L), while the concept of proper rationalizability rules out just D. In the first round, the only restriction imposed by both iterated admissibility and proper rationalizability is that U is preferred to D and thus, (s_1, A_1) is a preference restriction for 1 if and only if $s_1 = D$ and $A_1 \ni U$ (which in the notation just introduced is written $R_1^1 = mc\{(D, \{U\})\}$). In the algorithm of proper rationalizability, this means that the likelihood ordering over player 1's strategies must satisfy that U is infinitely more likely than D. Since this does not imply anything about the relative likelihood of M and D, which is what the preferences of player 2 depend on, no preference restriction is imposed on 2. Thus the algorithm converges after one round.

In contrast, since $C_1(mc\{(D, \{U\})\}) = \{U, M\}$, a likelihood ordering assumes $C_1(mc\{(D, \{U\})\})$ if *each* of U and M is infinitely more likely than D. This in turn means that L is preferred to R and U is preferred to M in the algorithm of preference restrictions that characterizes iterated admissibility (cf. Section 3.2), with $(\{U\}, \{M\}, \{D\})$ and $(\{L\}, \{R\})$ as the corresponding likelihood orderings. The likelihood ordering, $(\{L\}, \{R\})$, for player 2 entails that player 1 deems L infinitely more likely than R and therefore prefers D to M (and, of course, U to D since the former weakly dominates the latter). However, this means that the likelihood ordering $(\{U\}, \{M\}, \{D\})$ for player 1 determined by the algorithm characterizing iterated admissibility does not respect the preferences of player 1 that the same algorithm gives rise to.

[Figure 2 about here.]

Compare G_1 to G_2, which is the game illustrated in Figure 2. In G_2, the algorithms of iterated admissibility and proper rationalizability coincide in terms of the sets of preference restrictions. In the first round, the only restriction imposed by both iterated admissibility and proper rationalizability is that U is preferred to D; i.e., $R_1^1 = mc\{(D, \{U\})\}$. Even though the set of likelihood orderings that assumes $C_1(mc\{(D, \{U\})\})$ is a strict subset of the set of likelihood orderings that respects $mc\{(D, \{U\})\}$ (since only the former requires that M must be deemed infinitely more likely than D), every member of each set deems U infinitely more likely than D. This is sufficient to conclude L is preferred to R and U is preferred to M in the algorithms of iterated admissibility and proper rationalizability.

A key observation for game G_2 is that U weakly dominates D, and that L weakly dominates R on both $\{U\}$ (which is the strategy used to eliminate D in the first round of iterated admissibility) and $\{U, M\}$ (which is the set of strategies for player 1 surviving the first round of iterated admissibility). The same kind of observation can be made for the centipede game, which we turn to next.

[Figure 3 about here.]

In the four-legged centipede game illustrated in Figure 3 it is also the case that the algorithms of iterated admissibility and proper rationalizability coincide in terms of the sets of preference restrictions. In the first round, the only restriction imposed by both iterated admissibility and proper rationalizability is that *fd* is preferred to *ff*; i.e., $R_2^1 = mc\{(\mathit{ff}, \{\mathit{fd}\})\}$. Even though the set of likelihood orderings that assumes $C_2(mc\{(\mathit{ff}, \{\mathit{fd}\})\})$ is a strict subset of the set of likelihood orderings that respects $mc\{(\mathit{ff}, \{\mathit{fd}\})\}$ (since only the former requires that d must be deemed infinitely more likely than *ff*), every member of each set deems *fd* infinitely more likely than *ff*. This is sufficient to conclude *FD* is preferred to *FF*. Even though the set of likelihood orderings that assumes $C_1(mc\{(FF, \{FD\})\})$ is a strict subset of the set of likelihood orderings that respects $mc\{(FF, \{FD\})\}$ (since only the former requires that D must be deemed infinitely more likely than *FF*), every member of each set deems *FD* infinitely more likely than *FF*. This is sufficient to conclude d is preferred to *fd* and D is preferred to *FD*.

Note that in the second round, *FD* weakly dominates *FF* on both $\{\mathit{fd}\}$ (which is the strategy used to eliminate *ff* in the first round of iterated admissibility) and $\{d, \mathit{fd}\}$ (which is the set of strategies for player 2 surviving the first round of iterated admissibility). Likewise, in the third round, d weakly dominates *fd* and *ff* on both $\{FD\}$ (which is the strategy used to eliminate *FF* in the second round of iterated admissibility) and $\{D, FD\}$ (which is the set of strategies for player 1 surviving the second round of iterated admissibility). Similar conclusions hold for any centipede game independent of size and illustrates how both iterated admissibility and proper rationalizability correspond to the procedure of backward induction in such games.[1]

4.2 A sufficient condition

The following proposition presents a sufficient condition under which iterated admissibilty does not rule out properly rationalizable strategies.

Proposition 4 *Consider a finite strategic game G where the procedure of iterated admissibility leads to the sequence $(S_i^n)_{i\in I,n\in\mathbb{N}}$ of surviving strategy sets. Suppose that there exists a sequence $(A_i^n)_{i\in I,n\in\mathbb{N}}$ of strategy sets satisfying, for all players i, $A_i^0 = S_i$ and for each $n \in \mathbb{N}$,*

- *$A_i^n \subseteq S_i^n$,*
- *if $S_i^n \neq S_i^{n-1}$, then, for every $s_i \in S_i \backslash S_i^n$, s_i is weakly dominated by every $a_i \in A_i^n$ on either (A_{-i}^{n-1} and S_{-i}^{n-1}) or S_{-i},*
- *if $S_i^n = S_i^{n-1}$, then $A_i^n = A_i^{n-1}$.*

Then, for both players i, if s_i is properly rationalizable, then $s_i \in \bigcap_{n=1}^{\infty} S_i^n$.

Proof. See the appendix. ■

Both G_2 of Figure 2 and G_3 of Figure 3 can be used to illustrate Proposition 4. In G_2, the procedure of iterated admissibility yields the following sequence of strategy sets: $S_1^1 = S_1^2 = \{U, M\}$ and $S_1^n = \{U\}$ for $n \geq 3$, and $S_2^1 = \{L, R\}$ and $S_2^n = \{L\}$ for $n \geq 2$. Choose $A_1^n = \{U\}$ for $n \geq 1$, and $A_2^1 = \{L, R\}$ and $A_2^n = \{L\}$ for $n \geq 2$. It is straightforward to check that the conditions of Proposition 4 are satisfied; in particular, L weakly dominates R on both $A_1^1 = \{U\}$ and $S_1^1 = \{U, M\}$, and U weakly dominates M on $A_2^2 = S_2^2 = \{L\}$, and weakly dominates D on S_2.

In G_3, the procedure of iterated admissibility yields the following sequence of strategy sets: $S_1^1 = \{D, FD, FF\}$, $S_1^2 = S_1^3 = \{D, FD\}$ and $S_1^n = \{D\}$ for $n \geq 4$,

[1] For finite perfect information games without relevant payoff ties, proper rationalizability leads to the unique profile of backward induction *strategies* (Schuhmacher, 1999; Asheim, 2001), and iterated admissibility leads to the backward induction *outcome* (see Battigalli, 1997, pp. 52–53, for relevant references). While the algorithms of Sections 3.2 and 3.3 correspond to the backward induction *procedure* in the subclass of centipede games, this does not hold for the whole class of finite perfect information games without relevant payoff ties.

and $S_2^1 = S_2^2 = \{d, fd\}$ and $S_1^n = \{d\}$ for $n \geq 3$. Choose $A_1^1 = \{D, FD, FF\}$, $A_1^2 = A_1^3 = \{FD\}$ and $A_1^n = \{D\}$ for $n \geq 4$, and $A_2^1 = A_2^2 = \{fd\}$ and $A_2^n = \{d\}$ for $n \geq 3$. Again, it is straightforward to check that the conditions of Proposition 4 are satisfied; in particular, FD weakly dominates FF on both $A_2^1 = \{fd\}$ and $S_2^1 = \{d, fd\}$, d weakly dominates both fd and ff on both $A_1^2 = \{FD\}$ and $S_1^2 = \{D, FD\}$, and D weakly dominates both FD and FF on $A_2^3 = S_2^3 = \{d\}$.

4.3 Commitment bargaining

The algorithms of Section 3 can be applied for the purpose of analyzing economically significant models, independently of whether the sufficient condition of Proposition 4 is satisfied. In particular, they can be used for comparing iterated admissibility to proper rationalizability in specific strategic situations. In this subsection we consider a model of bilateral commitment bargaining due to Ellingsen and Miettinen (2008, Section I).

Ellingsen and Miettinen (2008) reexamine the problem of observable commitments in bargaining, first studied by Schelling (1956) and later formalized by Crawford (1982). Ellingsen and Miettinen (2008) extends Crawford's (1982) analysis by considering iterated admissibility and refinements of Nash equilibrium. Here we show how some of the results of Ellingsen and Miettinen (2008), in particular Lemma 2 and Proposition 2, can be obtained by using proper rationalizability instead of iterated admissibility. We also believe there is a mistake in their Lemma 2, but we will come back to this later.

In order to turn their strategic situation where two players bargain over real numbered fractions of a surplus of size 1 into a *finite* one-stage game with simultaneous moves, we introduce a smallest money unit g. We measure all variables in terms of numbers of the smallest money unit, and assume that k units of the smallest money unit equals the total surplus (i.e., $k \cdot g = 1$). Hence, players 1 and 2 bargain over a surplus of size k.

Each player i chooses, simultaneously with the other, either to commit to some demand $s_i \in \{0, 1, \ldots, k\}$ or to wait and remain uncommitted. Let w denote the waiting strategy. Hence the strategy set of each player i is $S_i = \{0, 1, \ldots, k\} \cup \{w\}$. If both players choose w, then each player i receives $\beta_i > 0$, where $\beta_1 + \beta_2 = k$.

In the case with certain commitments and no commitment costs (Ellingsen and Miettinen, 2008, Section I) the payoffs are as follows: If only one player i makes a commitment s_i, then i receives s_i and the other player receives $k - s_i$. If both players make commitments, then each player i receives $x_i(s_i, s_j) \in \{s_i, s_i + 1, \ldots, k - s_j\}$, with $x_1(s_1, s_2) + x_2(s_1, s_2) \leq k$, if $s_1 + s_2 \leq k$ and nothing otherwise.

The payoff function $u_i(s_i, s_j)$ of each player i can be summarized as follows:

$$u_i(s_i, s_j) = \begin{cases} x_i(s_i, s_j) & \text{if } s_i + s_j \leq k\,, \\ 0 & \text{if } s_i + s_j > k\,, \\ s_i & \text{if } s_i \neq w \text{ and } s_j = w\,, \\ k - s_j & \text{if } s_i = w \text{ and } s_j \neq w\,, \\ \beta_i & \text{if } s_i = w = s_j\,. \end{cases}$$

Ellingsen and Miettinen (2008) claim through their Lemma 2 that, for each player i, iterated admissibility leads to the elimination of $0, 1, \ldots, \beta_i$ in the first round, and $\beta_i + 1, \beta_i + 2, \ldots, k - 1$ in the second round, leaving k and w as the surviving strategies. Actually, with only k and w as the surviving strategies, w is eliminated in the third round, since choosing k yields player i a payoff of 0 if the opponent also chooses k and k if the opponent chooses w, while choosing w yields player i a payoff of 0 if the opponent chooses k and β_i ($< k$) if the opponent also chooses w. Hence, the correct statement of Ellingsen and Miettinen's (2008) Lemma 2 is that only k is iteratively weakly undominated.

Ellingsen and Miettinen (2008) use Lemma 2 in their subsequent Proposition 2 to focus on Nash equilibria involving only the strategies k and w (including asymmetric equilibria where one commits to the entire surplus and the other waits), as opposed to the plethora of unrefined Nash equilibria that this game gives rise to (cf. Crawford, 1982). Their Proposition 2 states that only the two asymmetric equilibria along with the symmetric equilibrium where both claim the entire surplus are consistent with two rounds of elimination of weakly dominated strategies. This statement is correct, but it begs the question: *why stop with two rounds of weak elimination?* As the following proposition shows, proper rationalizability provides a reason for considering only the strategies k and w.

Proposition 5 *Consider the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost. The properly rationalizable strategies for each player is to commit to the whole surplus, i.e., to choose the strategy k, or to wait, i.e., to choose the strategy w.*

Proof. See the appendix. ■

The proof of Proposition 5 consists of two parts. The one part uses the algorithm of Section 3.3 to show that no strategy but k and w can be properly rationalizable. Since w weakly dominates $0, 1, \ldots, \beta_j$ for player j, respect of j's preferences forces player i to deem w infinitely more likely than each of $0, 1, \ldots, \beta_j$. This in turn implies that k weakly dominates $\beta_i + 1, \beta_i + 2, \ldots, k - 1$ for player i. Hence, only k and w can be best responses when players are cautious.

The other part uses the result of Asheim (2001, Proposition 2) — that any strategy being used with positive probability in a proper equilibrium is properly rationalizable — to show that k and w are properly rationalizable. In particular, the asymmetric equilibria where one player commits to the entire surplus and the other waits are proper. In addition, there is a proper equilibrium where both players choose k with probability 1.[2] It can be shown that in any proper equilibrium, at least one player chooses k with probability 1, at most one player chooses w with positive probability, and no other strategy is assigned positive probability. Thus, the concept of proper equilibrium focuses precisely on the equilibria highlighted in Ellingsen and Miettinen's (2008) Proposition 2.[3]

Ellingsen and Miettinen (2008, Section II) also consider a variant of Crawford's (1982) bilateral commitment bargaining game where commitments are uncertain. In their Proposition 4 they show that only k survives iterated admissibility if commitments are uncertain. Actually, the iterations involve one round of weak elimination, followed by two rounds of strict elimination. Hence, only k survives the Dekel-Fudenberg procedure, and it follows from the algorithms of Sections 3.1 and 3.3 that only k is properly rationalizable (and thus, (k, k) is the only proper equilibrium). In their Propositions 1 and 3 they consider costly commitments. In this case, it can be shown that every strategy surviving iterated elimination of *strictly* dominated strategies is properly rationalizable. Hence in all variants considered by Ellingsen and Miettinen (2008), proper rationalizability and proper equilibrium yield the results they seek, while other concepts do not.

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[2] This equilibrium involves likelihood orderings where $k - 1$ and w are at the second level. See the Claim of the Appendix.

[3] For each player i and any strategy $\ell \in \{\beta_i+1, \beta_i+2, \ldots, k-1\}$, there exists a perfect equilibrium in which player i assigns positive probability to ℓ, provided that this player also assigns sufficient positive probability to w, so that k is the unique best response for the other player. See the Claim of the Appendix. Hence, the concept of perfect equilibrium can not be used to rule out all equilibria but the ones highlighted in Ellingsen and Miettinen's (2008) Proposition 2.

5 Concluding remarks

In our opinion, proper rationalizability is an attractive concept which is based on appealing epistemic conditions. However, up to now, its applicability has been hampered by the lack of an algorithm leading directly to the properly rationalizable strategies. With Perea's (2009) algorithm, this roadblock has been removed.

In this paper we have compared proper rationalizability to the Dekel-Fudenberg procedure and iterated admissibility by presenting comparable algorithms for the two latter concepts. Through the example in the introduction, and the bilateral commitment bargaining game due to Crawford (1982) and Ellingsen and Miettinen (2008), we have illustrated the usefulness of proper rationalizability in economic applications.

A Proofs

In order to prove Proposition 1, we need the following lemma.

Lemma 1 *Let $s_i \in S_i$, $D_i \subseteq S_i$ and $D_{-i} \subseteq S_{-i}$. Then, s_i is strictly dominated by some $\mu_i \in \Delta(D_i)$ on D_{-i} if and only for every $(\emptyset \neq)$ $E_{-i} \subseteq D_{-i}$ strategy s_i is weakly dominated by some $\tilde{\mu}_i \in \Delta(D_i)$ on E_{-i}.*

Proof. *Only if.* If there exists $\mu_i \in \Delta(D_i)$ such that μ_i strictly dominates s_i on D_{-i}, then, for every $(\emptyset \neq)$ $E_{-i} \subseteq D_{-i}$, $\mu_i \in \Delta(D_i)$ weakly dominates s_i on E_{-i}.

If. Suppose there does not exist $\mu_i \in \Delta(D_i)$ such that μ_i strictly dominates s_i on D_{-i}. Hence, by Pearce (1984, Lemma 3), there exists $\lambda_i \in \Delta(D_{-i})$ such that $u_i(s_i, \lambda_i) \geq u_i(s_i', \lambda_i)$ for all $s_i' \in D_i$. Then, by Pearce (1984, Lemma 4), there does not exist $\tilde{\mu}_i \in \Delta(D_i)$ such that $\tilde{\mu}_i$ weakly dominates s_i on $E_{-i} := \text{supp}\lambda_i \subseteq D_{-i}$. ■

Proof of Proposition 1. The Dekel-Fudenberg procedure is given by the following sequence of strategy subsets:

(i) For each player i, let $S_i^0 = S_i$.

(ii) For each player i, let $S_i^1 = \{s_i \in S_i \mid s_i \text{ not weakly dominated on } S_{-i}\}$.

(iii) For every $n \geq 2$, and each player i, let

$$S_i^n = \{s_i \in S_i^{n-1} \mid s_i \text{ not strictly dominated on } S_{-i}^{n-1}\}.$$

Here, $S_{-i}^{n-1} := \times_{j \neq i} S_j^{n-1}$. We show, by induction on n, that $C_i(R_i^n) = S_i^n$ for each player i and all n.

Part (i). For $n = 0$, we have that

$$C_i(R_i^0) = C_i(\emptyset) = S_i = S_i^0$$

for each player i.

Part (ii). For $n = 1$, we have that $R_i^1 = R_i(\mathcal{L}_i^b(R_{-i}^0)) = R_i(\mathcal{L}_i^b(\emptyset))$. By definition,

$$\begin{aligned} \mathcal{L}_i^b(\emptyset) &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } C_{-i}(\emptyset)\} \\ &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } S_{-i}\} = \mathcal{L}_i^*. \end{aligned}$$

Hence,

$$R_i^1 = \{(s_i, A_i) \mid \exists \mu_i \in \Delta(A_i) \text{ that weakly dominates } s_i \text{ on } S_{-i}\}.$$

Therefore,

$$C_i(R_i^1) = \{s_i \in S_i \mid s_i \text{ not weakly dominated on } S_{-i}\} = S_i^1.$$

Part (iii). Now, let $n \geq 2$, and assume that for each player i, $C_i(R_i^{n-1}) = S_i^{n-1}$. We show that, for each player i, $C_i(R_i^n) = S_i^n$.

Fix a player i. By definition, $R_i^n = R_i(\mathcal{L}_i^b(R_{-i}^{n-1}))$. We have that

$$\begin{aligned} \mathcal{L}_i^b(R_{-i}^{n-1}) &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } C_{-i}(R_{-i}^{n-1})\} \\ &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ believes } S_{-i}^{n-1}\} \\ &= \{L_i \in \mathcal{L}_i^* \mid L_i^1 \subseteq S_{-i}^{n-1}\}, \end{aligned}$$

by our induction assumption. But then,

$$\begin{aligned} R_i^n = \{(s_i, A_i) \mid & \text{ for every } L_i^1 \subseteq S_{-i}^{n-1} \text{ there is } \mu_i \in \Delta(A_i) \text{ that} \\ & \text{weakly dominates } s_i \text{ on } L_i^1 \text{ or on } S_{-i}\}. \end{aligned}$$

Consider the strategies s_i where for every $L_i^1 \subseteq S_{-i}^{n-1}$ there is $\mu_i \in \Delta(A_i)$ such that μ_i weakly dominates s_i on L_i^1. By Lemma 1, we know that these are exactly the strategies s_i that are strictly dominated by some $\mu_i \in \Delta(A_i)$ on S_{-i}^{n-1}.

Hence, we may conclude that

$$\begin{aligned} R_i^n = \{(s_i, A_i) \mid & \text{ there is } \mu_i \in \Delta(A_i) \text{ that} \\ & \text{strictly dominates } s_i \text{ on } S_{-i}^{n-1}, \text{ or weakly dominates } s_i \text{ on } S_{-i}\}. \end{aligned}$$

Hence,

$$C_i(R_i^n) = \{s_i \in S_i \mid s_i \text{ not strictly dominated on } S_{-i}^{n-1},$$
$$\text{nor weakly dominated on } S_{-i}\} = S_i^n \,,$$

which completes the proof. ■

Proof of Proposition 2. Iterated admissibility is given by the following sequence of strategy subsets:

(i) For each player i, let $S_i^0 = S_i$.

(ii) For every $n \geq 1$, and each player i, let

$$S_i^n = \{s_i \in S_i^{n-1} \mid s_i \text{ not weakly dominated on } S_{-i}^{n-1}\}.$$

We show, by induction on n, that $C_i(R_i^n) = S_i^n$ for each player i and all n.

Part (i). For $n = 0$, we have that

$$C_i(R_i^0) = C_i(\emptyset) = S_i = S_i^0$$

for each player i.

Part (ii). Let $n \geq 1$, and assume that, for each player i, $C_i(R_i^k) = S_i^k$ for all $k < n$. We show that, for each player i, $C_i(R_i^n) = S_i^n$.

Fix a player i. By definition, we have that

$$R_i^n = R_i\big(\mathcal{L}_i^a(R_{-i}^0) \cap \mathcal{L}_i^a(R_{-i}^1) \cap \cdots \cap \mathcal{L}_i^a(R_{-i}^{n-1})\big)\,.$$

By the induction assumption, we know that $C_{-i}(R_{-i}^k) = S_{-i}^k$ for all $k < n$, and hence

$$\begin{aligned} \mathcal{L}_i^a(R_{-i}^k) &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ assumes } C_{-i}(R_{-i}^k)\} \\ &= \{L_i \in \mathcal{L}_i^* \mid L_i \text{ assumes } S_{-i}^k\} \end{aligned}$$

for all $k < n$. This implies that $\mathcal{L}_i^a(R_{-i}^0) \cap \mathcal{L}_i^a(R_{-i}^1) \cap \cdots \cap \mathcal{L}_i^a(R_{-i}^{n-1})$ is equal to

$$\{L_i \in \mathcal{L}_i^* \mid L_i \text{ assumes } S_{-i}^k \text{ for all } k < n\}\,.$$

Since $R_i^n = R_i\big(\mathcal{L}_i^a(R_{-i}^0) \cap \mathcal{L}_i^a(R_{-i}^1) \cap \cdots \cap \mathcal{L}_i^a(R_{-i}^{n-1})\big)$, it follows that R_i^n contains exactly those preference restrictions (s_i, A_i) such that s_i is weakly dominated by some $\mu_i \in \Delta(A_i)$ on some S_{-i}^k with $k < n$. Hence,

$$C_i(R_i^n) = \{s_i \in S_i \mid s_i \text{ not weakly dominated on any } S_{-i}^k \text{ with } k < n\} = S_i^n \,,$$

which completes the proof. ■

Proof of Proposition 4. Let $(R_i^n)_{i\in I,n\in\mathbb{N}}$ be the sequence of preference restrictions according to the algorithm of proper rationalizability (cf. Section 3.3). It is sufficient to show, under the assumptions of the proposition, that for all player i and each n, it holds that, for every $s_i \in S_i\backslash S_i^n$, $(s_i, \{a_i\}) \in R_i^n$ for every $a_i \in A_i^n$. In this case, namely, every properly rationalizable strategy must be in $\bigcap_{n=1}^{\infty} S_i^n$. We show by induction that the statement above is true.

Part (i). For $n = 0$, we have that $S_i^0 = S_i$, so that there is no $s_i \in S_i\backslash S_i^n$ and the statement is trivially true.

Part (ii). Let $n \geq 1$, and assume that, for each player i and each $m \in \{1, \ldots, n-1\}$, it holds that, for every $s_i \in S_i\backslash S_i^m$, $(s_i, \{a_i\}) \in R_i^m$ for every $a_i \in A_i^m$.

Fix a player i. We first make the observation that, for each $m \in \{1, \ldots, n-1\}$, every $L_i = (L_i^1, \ldots, L_i^K) \in \mathcal{L}_i^r(R_{-i}^m)$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $A_j^m \subseteq L_j^1 \cup \cdots \cup L_j^k \subseteq S_j^m$. This is true by the full support assumption if $S_j^m = S_j$ (and thus $A_j^m = S_j$). Assume now that $S_j^m \neq S_j$. If L_j respects R_j^m, then for every $s_j \in S_j\backslash S_j^m$, $a_j \gg_i s_j$ for every $a_j \in A_j^m$, and the observation follows also in this case.

If $S_i^n = S_i$, then the statement is trivially true also for n.

If $S_i^n \neq S_i$, let $(1 \leq)\ n' \leq n$ satisfy $S_i^n = S_i^{n'} \neq S_i^{n'-1}$. By a premise of the proposition, for every $s_i \in S_i\backslash S_i^{n'}$, s_i is weakly dominated by every $a_i \in A_i^{n'}$ on either ($A_j^{n'-1}$ and $S_j^{n'-1}$) or S_j. If s_i is weakly dominated by a_i on $A_j^{n'-1}$ and $S_j^{n'-1}$, then s_i is weakly dominated by a_i on each strategy set D_j satisfying $A_j^{n'-1} \subseteq D_j \subseteq S_j^{n'-1}$. By the observation that every $L_j = (L_j^1, \ldots, L_j^K) \in \mathcal{L}_j^r(R_j^{n'-1})$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $A_j^{n'-1} \subseteq L_j^1 \cup \cdots \cup L_j^k \subseteq S_j^{n'-1}$ it follows that $(s_i, \{a_i\}) \in R_i^{n'} = R_i(\mathcal{L}_j^r(R_j^{n'-1}))$. If s_i is weakly dominated by a_i on S_j, then by the full support assumption, $(s_i, \{a_i\}) \in R_i^1 = R_i(\mathcal{L}_j^*)$. Hence, since the sequence of sets of preference restrictions is non-decreasing, for every $s_i \in S_i\backslash S_i^n$, $(s_i, \{a_i\}) \in R_i^n$ for every $a_i \in A_i^n$. ■

Proof of Proposition 5. The proof is divided into two parts. In part (i) we show that the strategies in $S_i\backslash(\{k\} \cup \{w\})$ are *not* properly rationalizable. In part (ii) we show that k and w *are* properly rationalizable.

Part (i). Let $\langle R_1^n, R_2^n\rangle_{n=1}^{\infty}$ be the sequence of preference restrictions for the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost, according to the algorithm of proper rationalizability (cf. Section 3.3). In order to show that the strategies in $S_i \backslash (\{k\} \cup \{w\}) = \{0, 1, \ldots, k-1\}$ are not properly rationalizable, it is sufficient to show that for each player i, it holds that (a) for every $s_i \in \{0, 1, \ldots, \beta_i\}$, $(s_i, \{w\}) \in R_i^1$, and (b) for every $s_i \in \{\beta_i+1, \beta_i+2, \ldots, k-1\}$, $(s_i, \{k\}) \in R_i^2$, keeping in mind that the sequence of sets of preference restrictions is non-decreasing.

Result (a) follows from the fact that, for each player i and for every $s_i \in \{0, 1, \ldots, \beta_i\}$, w weakly dominates s_i on S_j. Hence, for each player i and for every $s_i \in \{0, 1, \ldots, \beta_i\}$, $(s_i, \{w\}) \in R_i^1 = R_i(\mathcal{L}_i^*)$. This result implies that, for each player i, every $L_i = (L_i^1, \ldots, L_i^K) \in \mathcal{L}_i^r(R_j^1)$ satisfies that there exists $k \in \{1, \ldots, K\}$ such that $\{w\} \subseteq L_i^1 \cup \cdots \cup L_i^k \subseteq \{\beta_j+1, \beta_j+2, \ldots, k\} \cup \{w\}$. Result (b) follows from the fact that, for each player i and for every $s_i \in \{\beta_i+1, \beta_i+2, \ldots, k-1\}$, k weakly dominates s_i on each strategy set D_j satisfying $\{w\} \subseteq D_j \subseteq \{\beta_j+1, \beta_j+2, \ldots, k\} \cup \{w\}$. Hence, for each player i and for every $s_i \in \{\beta_i+1, \beta_i+2, \ldots, k-1\}$, $(s_i, \{k\}) \in R_i^2 = R_i(\mathcal{L}_i^r(R_j^1))$.

Part (ii). We establish that k and w *are* properly rationalizable in the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost, by showing that both k and w can be used with positive probability in a proper equilibrium; thus, they are properly rationalizable (Asheim, 2001, Proposition 2). To prove this claim, consider the likelihood orderings $L_1 = \{\{k\}, \{k-1\}, \ldots, \{\beta_1+1\}, \{w\}, \{\beta_1\}, \{\beta_1-1\}, \ldots, \{1\}, \{0\}\}$ and $L_2 = \{\{w\}, \{1\}, \{2\}, \ldots, \{\beta_2-1\}, \{k\}, \{k-1\}, \ldots, \{\beta_2+1\}, \{\beta_2\}\}$. Since each element in either of these partitions contains only one strategy, they determine a pair of LPSs. It is straightforward to check that this pair of LPSs is a proper equilibrium, according to Blume et al.'s (1991b, Proposition 5) characterization, where player 1 chooses k with probability 1 and player 2 chooses w with probability 1. ■

Claim *Consider the finite version of Ellingsen and Miettinen's (2008, Section I) bilateral commitment bargaining game with zero commitment cost.*
(i) There exists a proper equilibrium where both players assign probability 1 to k.
(ii) For both players i, there exists a perfect equilibrium where player i assigns positive probability to $\beta_i+1, \beta_i+2, ..., k-1$ and player j assigns probability 1 to k.

Proof. *Part (i).* Consider the LPSs

$$\lambda_1 = \{\lambda_1^1, \ldots, \lambda_1^{k+1}\}$$
$$\lambda_2 = \{\lambda_2^1, \ldots, \lambda_2^{k+1}\},$$

where for both players i and each $\ell \in \{1, \ldots, k+1\}$, the support of λ_i^ℓ is included in $\{w, k+1-\ell\}$ for $\ell \in \{1, \ldots, \beta_i + 1\}$, $\{w, 1\}$ for $\ell = \beta_i + 2$, $\{w, k+2-\ell\}$ for $\ell \in \{\beta_i + 3, \ldots, k\}$, and $\{w, 0\}$ for $\ell = k+1$. Let, for each $\ell \in \{1, \ldots, k+1\}$, λ_i^ℓ be determined by $u_i(w, \lambda_i^\ell) = u_i(k-1, \lambda_i^\ell)$. This means that $\lambda_i^1(w) = 0$, $\lambda_i^{\beta_i+2}(w) = 0$ and $\lambda_i^{k+1}(w) \in [0, 1)$, and $\lambda_i^\ell(w) \in (0, 1)$ otherwise with $\lambda_i^2(w) = 1/\beta_j$, $\lambda_i^3(w) = 2/(\beta_j + 1)$, etc.

The LPSs λ_1 and λ_2 determine the following likelihood orderings:

$$L_1 = \{\{k\}, \{w, k-1\}, \{k-2\}, \ldots, \{\beta_1 + 1\}, \{\beta_1\}, \{1\}, \{\beta_1 - 1\}, \ldots, \{2\}, \{0\}\}$$
$$L_2 = \{\{k\}, \{w, k-1\}, \{k-2\}, \ldots, \{\beta_2 + 1\}, \{\beta_2\}, \{1\}, \{\beta_2 - 1\}, \ldots, \{2\}, \{0\}\}\,.$$

It can be checked that L_1 respects the set of preference restrictions that u_2 and λ_2 give rise to, and that L_2 respects the set of preference restrictions that u_1 and λ_1 give rise to. It follows from Blume et al. (1991b, Proposition 5) that $(\lambda_1^1, \lambda_2^1)$ is a proper equilibrium. Note that, for both players i, $\lambda_i^1(k) = 1$.

Part (ii). Let $(\lambda_1^1, \lambda_2^1)$ satisfy that (1) $\lambda_1^1(k) = 1$, and (2) $\lambda_2^1(w) = \beta_1/k$ and $\lambda_2^1(\ell) = 1 - \beta_1/k$ for some $\ell \in \beta_2 + 1, \beta_2 + 2, \ldots, k-1$ (so that $\lambda_2^1(s_2) = 0$ otherwise). Then λ_1^1 is not weakly dominated and the unique best response for 1 to λ_2^1, and λ_2^1 is not weakly dominated[4] and a best response for 2 to λ_1^1. The result that $(\lambda_1^1, \lambda_2^1)$ is perfect follows from the fact that, in two player games, any Nash-equilibrium in strategies that are not weakly dominated is perfect.

Likewise for $(\lambda_1^1, \lambda_2^1)$ with (1) $\lambda_1^1(w) = \beta_2/k$ and $\lambda_1^1(\ell) = 1 - \beta_2/k$ for some $\ell \in \beta_1 + 1, \beta_1 + 2, \ldots, k-1$ (so that $\lambda_1^1(s_1) = 0$ otherwise), and (2) $\lambda_2^1(k) = 1$. ■

References

Asheim, G.B. (2001). 'Proper rationalizability in lexicographic beliefs.' *International Journal of Game Theory* **30**, 453–478.

Asheim, G.B., Dufwenberg, M., 2003. 'Admissibility and common belief.' *Games and Economic Behavavior* **42**, 208–234.

[4] Suppose that there exists λ_2' such that λ_2' weakly dominates λ_2^1. Since w weakly dominates all other strategies on $\{\beta_1, \beta_1 + 1, \ldots, k-1\}$, $\lambda_2'(w) \geq \lambda_2^1(w)$. Since $u_2(w, k-\ell) = u_2(\ell, k-\ell) = \ell > u_2(s_2, k-\ell)$ for $s_2 \neq w$, ℓ, it follows that $1 \geq \lambda_2'(w) + \lambda_2'(\ell) \geq \lambda_2^1(w) + \lambda_2^1(\ell) = 1$. Hence, the support of λ_2' is also $\{w, \ell\}$. However, if $\lambda_2'(w) > \lambda_2^1(w)$ and $\lambda_2'(\ell) < \lambda_2^1(\ell)$, then $\lambda_2'(w)u_2(w, w) + \lambda_2'(\ell)u_2(\ell, w) < \lambda_2^1(w)u_2(w, w) + \lambda_2^1(\ell)u_2(\ell, w)$, contradicting that λ_2' weakly dominates λ_2^1. Therefore, $\lambda_2' = \lambda_2^1$, showing that λ_2^1 is not weakly dominated.

Asheim, G.B., Søvik, Y., 2005. 'Preference-based belief operators.' *Mathematical Social Sciences* **50**, 61–82.

Battigalli, P. (1997). 'On rationalizability in extensive games.' *Journal of Economic Theory* **74**, 40–61.

Blume, L., Brandenburger, A., and Dekel, E. (1991a). 'Lexicographic probabilities and choice under uncertainty.' *Econometrica* **59**, 61–79.

Blume, L., Brandenburger, A., and Dekel, E. (1991b). 'Lexicographic probabilities and equilibrium refinements.' *Econometrica* **59**, 81–98.

Börgers, T. (1994). 'Weak dominance and approximate common knowledge.' *Journal of Economic Theory* **64**, 265–276.

Brandenburger, A. (1992). 'Lexicographic probabilities and iterated admissibility.' In *Economic Analysis of Markets and Games* (eds. P. Dasgupta, D. Gale, O. Hart, E. Maskin), pp. 282–290. Cambridge, MA: MIT Press.

Brandenburger, A., Friedenberg, A., and Keisler, H.J. (2008). 'Admissibility in games.' *Econometrica* **76**, 307–352.

Crawford, Vincent P. 1982. 'A theory of disagreement in bargaining.' *Econometrica* **50**, 607–37.

Dekel, E. and Fudenberg, D. (1990). 'Rational behavior with payoff uncertainty.' *Journal of Economic Theory* **52**, 243–67.

Ellingsen, T. and Miettinen, T. (2008). 'Commitment and conflict in bilateral bargaining.' *American Economic Review* **98**, 1629–1635.

Pearce, D.G. (1984), 'Rationalizable strategic behavior and the problem of perfection, *Econometrica* **52**, 1029–1050.

Perea, A. (2009). 'An algorithm for proper rationalizability.' Mimeo, Maastricht University.

Schelling, T.C. (1956). 'An essay on bargaining.' *American Economic Review* **46**, 281–306.

Schuhmacher, F. (1999). 'Proper rationalizability and backward induction.' *International Journal of Game Theory* **28**, 599–615.

Stahl, D.. (1995). 'Lexicographic rationality, common knowledge, and iterated admissibility.' *Economics Letters* **47**, 155–159.

$$
\begin{array}{c|c|c|}
 & L & R \\
\hline
U & 1,\ 1 & 1,\ 1 \\
\hline
M & 0,\ 1 & 2,\ 0 \\
\hline
D & 1,\ 0 & 0,\ 1 \\
\hline
\end{array}
$$

$$
\begin{array}{ll}
Dekel - Fudenberg and Proper rationalizability & \\
R_1^0 = \emptyset & R_2^0 = \emptyset \\
R_1^1 = mc\{(D, \{U\})\} & R_2^1 = \emptyset \\
\ldots & \ldots \\
R_1^\infty = mc\{(D, \{U\})\} & R_2^\infty = \emptyset \\
Iterated admissibility & \\
R_1^0 = \emptyset & R_2^0 = \emptyset \\
R_1^1 = mc\{(D, \{U\})\} & R_2^1 = \emptyset \\
R_1^2 = mc\{(D, \{U\})\} & R_2^2 = mc\{(R, \{L\})\} \\
R_1^3 = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\} & R_2^3 = mc\{(R, \{L\})\} \\
\ldots & \ldots \\
R_1^\infty = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\} & R_2^\infty = mc\{(R, \{L\})\}
\end{array}
$$

Figure 1: Iterated admissibility rules out properly rationalizable strategies (G_1).

	L	R
U	1, 1	1, 0
M	0, 1	2, 1
D	1, 0	0, 1

$Dekel - Fudenberg$

$R_1^0 = \emptyset$ $\qquad$ $R_2^0 = \emptyset$

$R_1^1 = mc\{(D, \{U\})\}$ $\qquad$ $R_2^1 = \emptyset$

$\dots$ $\qquad$ $\dots$

$R_1^\infty = mc\{(D, \{U\})\}$ $\qquad$ $R_2^\infty = \emptyset$

$Iterated admissibility and Proper rationalizability$

$R_1^0 = \emptyset$ $\qquad$ $R_2^0 = \emptyset$

$R_1^1 = mc\{(D, \{U\})\}$ $\qquad$ $R_2^1 = \emptyset$

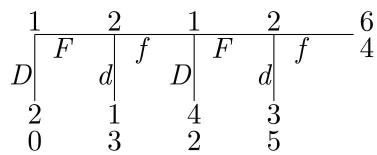
$R_1^2 = mc\{(D, \{U\})\}$ $\qquad$ $R_2^2 = mc\{(R, \{L\})\}$

$R_1^3 = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\}$ $\qquad$ $R_2^3 = mc\{(R, \{L\})\}$

$\dots$ $\qquad$ $\dots$

$R_1^\infty = mc\{(M, \{U\}), (M, \{D\}), (D, \{U\})\}$ $\qquad$ $R_2^\infty = mc\{(R, \{L\})\}$

Figure 2: Iterated admissibility coincides with proper rationalizability (G_2).



	d	fd	ff
D	2, 0	2, 0	2, 0
FD	1, 3	4, 2	4, 2
FF	1, 3	3, 5	6, 4

$Dekel - Fudenberg$

$$
\begin{array}{ll}
R_1^0 = \emptyset & R_2^0 = \emptyset \\
R_1^1 = \emptyset & R_2^1 = mc\{(ff, \{fd\})\} \\
R_1^2 = mc\{(FF, \{D, FD\})\} & R_2^2 = mc\{(ff, \{fd\})\} \\
\quad \dots & \quad \dots \\
R_1^\infty = mc\{(FF, \{D, FD\})\} & R_2^\infty = mc\{(ff, \{fd\})\}
\end{array}
$$

$Iterated admissibility and Proper rationalizability$

$$
\begin{array}{ll}
R_1^0 = \emptyset & R_2^0 = \emptyset \\
R_1^1 = \emptyset & R_2^1 = mc\{(ff, \{fd\})\} \\
R_1^2 = mc\{(FF, \{FD\})\} & R_2^2 = mc\{(ff, \{fd\})\} \\
R_1^3 = mc\{(FF, \{FD\})\} & R_2^3 = mc\{(fd, \{d\}), (ff, \{d\}), (ff, \{fd\})\} \\
R_1^4 = mc\{(FD, \{D\}), (FF, \{D\}), (FF, \{FD\})\} & R_2^4 = mc\{(fd, \{d\}), (ff, \{d\}), (ff, \{fd\})\} \\
\quad \dots & \quad \dots \\
R_1^\infty = mc\{(FD, \{D\}), (FF, \{D\}), (FF, \{FD\})\} & R_2^\infty = mc\{(fd, \{d\}), (ff, \{d\}), (ff, \{fd\})\}
\end{array}
$$

Figure 3: A four-legged centipede game (G_3).