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Structure-preserving transformations of epistemic models

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Abstract

The prevailing approaches to modeling interactive uncertainty with epistemic models in economics are state-based and type-based. We explicitly formulate two general procedures that transform state models into type models and vice versa. Both transformation procedures preserve the belief hierarchies as well as the common prior assumption. By means of counterexamples it is shown that our procedures are not inverse to each other. However, if attention is restricted to maximally reduced epistemic models, then isomorphisms can be constructed and an inverse relationship emerges.

KEYWORDS

belief hierarchies, common prior assumption, epistemic game theory, epistemic models, games, interactive epistemology, isomorphism, maximal reduction, possible worlds, states, transformation procedures, types

JEL CLASSIFICATION

C70, C72, D80

1 | INTRODUCTION

In game theory it is fundamental to model interactive beliefs to capture the players' reasoning about each other. It is assumed that a player holds beliefs about his opponents' choices, about his opponents' beliefs about their opponents' choices, about his opponents' beliefs about their opponents' beliefs about their opponents' choices, etc. Such infinite doxastic sequences can be formally expressed by the notion of a belief hierarchy.

Initially proposed in the context of incomplete information by Harsanyi (1967–68), a belief hierarchy of a player—in the case of strategic uncertainty (e.g., Böge & Eisele, 1979; Brandenburger & Dekel, 1993; Mertens & Zamir, 1985)—specifies a belief about the basic space of uncertainty that is, the opponents' choice combinations (first-order belief), a belief about the opponents' choice combinations and the opponents' first-order beliefs (second-order belief), a belief about the opponents' choice combinations, the opponents' first-order beliefs, and the opponents' second-order beliefs (third-order belief), etc. Thus, a kth-order belief fixes a belief about the basic space of uncertainty and about each of the lower-order beliefs of the opponents. A player's belief hierarchy can be seen as the formalization of his entire interactive thinking about the game. Different patterns of reasoning (e.g., common belief in rationality) can then be modeled as conditions imposed on a player's belief hierarchy.

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Unfortunately, belief hierarchies are cumbersome objects due to their infinite nature. However, there exist finite encodings of belief hierarchies that render them more tractable. The standard way to represent belief hierarchies in a compact and convenient way is due to Harsanyi's (1967–68) seminal idea of types. Accordingly, a type induces a belief about the opponents' combinations of choices and types. Any belief of higher order can then be derived. An alternative implicit description of belief hierarchies is based on the idea of states or possible words due to Kripke (1963) and Aumann (1974). Any belief of higher order is inferable from a player's belief at a given possible world about the worlds in combination with the players' choices and beliefs at worlds. The relation between the so-called *type-based* and *state-based* approaches to modeling belief hierarchies have been investigated by Tan and Werlang (1992) as well as by Brandenburger and Dekel (1993). They essentially show that hypotheses involving common knowledge are preserved across these two epistemic frameworks.

We compare the type-based and state-based approaches to formalizing interactive thinking from a broader perspective and provide two general transformation procedures between type and state models. Belief hierarchies as well as the common prior assumption are preserved by these procedures. In this sense the two different epistemic approaches are equivalent. We then explore whether our procedures constitute operational inverses to each other by means of an isomorphism. It turns out that they do not do so unless attention is restricted to maximally reduced models which exclude the existence of "superfluous" worlds and types, respectively. This insight emphasizes that type and state models actually exhibit some foundational differences despite their equivalence in terms of preserving belief hierarchies and the common prior assumption. The underlying conceptual reason dwells in the distinct degrees of granularity: while the type-based approach only represents the players' interactive thinking the state-based approach additionally also fixes their choices.

We proceed as follows. Section 2 lays out the formal framework and notation. In particular, type-based and state-based approaches to interactive epistemology are presented. In Section 3 we provide a transformation procedure (Definition 5) to convert state models into type models. Belief hierarchies (Theorem 1) as well as the common prior assumption (Theorem 2) are preserved. Then, in Section 4 our point of departure are type models and we propose a second transformation procedure (Definition 6) to turn them into state models. Again, preservation holds with regard to belief hierarchies (Theorem 3) as well as the common prior assumption (Theorem 4). While the general conclusions of Theorems 1 and 3 about the structural conservation of belief hierarchies are likely to be implicitly known in the game theory community, our purpose is, first, to render these foundational insights explicit in an accessible way, and second, to provide concrete tools to switch back and forth between state and type models. Besides, in Section 5 an interplay of our transformation procedures and Theorem 2 with Hellman and Samet's (2012) results on the restrictiveness of the common prior assumption is explored. Section 6 then addresses structural identities within a given epistemic framework. It turns out that our two transformation procedures are not inverse to each other (Examples 5 and 6). By restricting to maximally reduced models inverse relationships between the two operations then ensue (Theorems 6 and 7). Finally, some conceptual matters are discussed and concluding remarks are offered in Section 7.

2 | PRELIMINARIES

A game is modeled as a tuple $\Gamma = \langle I, (C_i, U_i)_{i \in I} \rangle$, where I is a finite set of players, C_i denotes player i's finite choice set, and $U_i : \times_{j \in I} C_j \to \mathbb{R}$ constitutes player i's utility function, which assigns a real number $U_i(c) \in \mathbb{R}$ to every choice combination $c \in \times_{j \in I} C_j$. In terms of notation, given a collection $\{S_n : n \in N\}$ of sets and probability distribution $p_n \in \Delta(S_n)$ for all $n \in N$, the set S_{-n} refers to the product set $\times_{m \in N \setminus \{n\}} S_m$ and the probability distribution p_{-n} refers to the product distribution $\Pi_{m \in N \setminus \{n\}} p_m \in \Delta(S_{-n})$ on S_{-n}. Given a probability distribution $p \in \Delta_{n \in N}(\times S_n)$ on a product set, for the sake of simplicity any marginal is also denoted by p if the intended usage is clear from the context.

Belief hierarchies can be inductively formalized as sequences of probability distributions. In the context of games, construct for every player $i \in I$ a sequence $\left(X_i^n\right)_{n \in \mathbb{N}}$ of spaces, where

$$X_i^1 := C_{-i},$$

$$X_i^2 := X_i^1 \times \left(\times_{j \in I \setminus \{i\}} \Delta\left(X_j^1\right)\right),$$

$$\vdots$$

$$X_i^k := X_i^{k-1} \times \left(\times_{j \in I \setminus \{i\}} \Delta\left(X_j^{k-1}\right)\right),$$

$$\vdots$$

and a belief hierarchy of player i is then defined as a sequence $\eta_i := \left(\eta_i^n\right)_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} \left(\Delta\left(X_i^n\right)\right)$ of probability distributions. For every level $k \in \mathbb{N}$, the probability distribution $\eta_i^k \in \Delta\left(X_i^k\right)$ is called i's kth-order belief. Note that

$$X_i^k = C_{-i} \times \left(\times_{j \in I \setminus \{i\}} \Delta\left(X_j^1\right)\right) \times \left(\times_{j \in I \setminus \{i\}} \Delta\left(X_j^2\right)\right) \times \ldots \times \left(\times_{j \in I \setminus \{i\}} \Delta\left(X_j^{k-1}\right)\right)$$

holds for all $k \in \mathbb{N}$.

The standard implicit representation of belief hierarchies in terms of types is due to Harsanyi (1967–68). According to this epistemic approach the game-theoretic framework—given by Γ—is enriched by a type-based structure.

Definition 1. Let Γ be a game. A type model of Γ is a tuple $\mathcal{T}^\Gamma = \langle (T_i, b_i)_{i \in I} \rangle$, where for every player $i \in I$,

- *T_i is a finite set of types,*
- *$b_i : T_i \to \Delta(C_{-i} \times T_{-i})$ is i's belief function that assigns to every type $t_i \in T_i$ a probability distribution $b_i[t_i]$ on the set of opponents' choice type combinations.*

A type t_i of some player i naturally induces a belief hierarchy:

$$\eta_i^1[t_i](c_{-i}) := \sum_{t_{-i} \in T_{-i}} b_i[t_i](c_{-i}, t_{-i})$$

for all $c_{-i} \in X_i^1$, as well as

$$\eta_i^k[t_i]\left(c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \ldots, \eta_{-i}^{k-1}\right) := \sum_{t_{-i} \in T_{-i} : \eta_{-i}^l[t_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k-1} b_i[t_i](c_{-i}, t_{-i})$$

for all $\left(c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \ldots, \eta_{-i}^{k-1}\right) \in X_i^k$ and for all $k \geq 2$, where the sequence $\eta_i[t_i] := \left(\eta_i^n[t_i]\right)_{n \in \mathbb{N}}$ is called the *t_i-induced belief hierarchy of player i*. The set $H_i\left[\mathcal{T}^\Gamma\right] := \left\{\eta_i \in \times_{n \in \mathbb{N}} \left(\Delta\left(X_i^n\right)\right) : \text{there exists } t_i \in T_i \text{ such that } \eta_i[t_i] = \eta_i\right\}$ is called the *$\mathcal{T}^\Gamma$-induced set of belief hierarchies of player i*.

An alternative way to represent interactive thinking in games is based on the idea of possible worlds—sometimes also called states—due to Kripke (1963) and Aumann (1974). This epistemic approach employs a state-based structure as formal framework added to Γ.

Definition 2. Let Γ be a game. A state model of Γ is a tuple $\mathcal{S}^\Gamma = \langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$, where

- Ω is a finite set of all possible worlds,
  and for every player $i \in I$,
- $\mathcal{I}_i \subseteq 2^\Omega$ is a possibility partition of Ω,
- $\sigma_i : \Omega \to C_i$ is a $\mathcal{I}_i$-measurable choice function,
- $\pi_i \in \Delta(\Omega)$ is a subjective prior on Ω such that $\pi_i(\mathcal{I}_i(\omega)) > 0$ for every world $\omega \in \Omega$ with $\mathcal{I}_i(\omega)$ denoting the cell of $\mathcal{I}_i$ containing ω.

The requirement on the prior that $\pi_i(\mathcal{I}_i(\omega)) > 0$ for all $\omega \in \Omega$ and for all $i \in I$ is sometimes also called the non-null information condition.

Belief hierarchies also naturally emerge in state models. Given some player i, a possible world ω induces a belief hierarchy as follows:

$$\eta_i^1[\omega](c_{-i}) := \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = c_{-i}} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

for all $c_{-i} \in X_i^1$, as well as

$$\eta_i^k[\omega]\left(c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \ldots, \eta_{-i}^{k-1}\right) \coloneqq \sum_{\omega' \in \mathcal{I}_i(\omega): \sigma_{-i}(\omega') = c_{-i}, \eta_{-i}^l[\omega'] = \eta_{-i}^l \text{ for all } 1 \le l \le k-1} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

for all $\left(c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \ldots, \eta_{-i}^{k-1}\right) \in X_i^k$ and for all $k \geq 2$, where the sequence $\eta_i[\omega] \coloneqq \left(\eta_i^n[\omega]\right)_{n \in \mathbb{N}}$ is called the *ω-induced belief hierarchy of player i*. The set $H_i\left[\mathcal{S}^\Gamma\right] \coloneqq \left\{\eta_i \in \times_{n \in \mathbb{N}}\left(\Delta\left(X_i^n\right)\right) : \text{there exists } \omega \in \Omega \text{ such that } \eta_i[\omega] = \eta_i\right\}$ is called the *$\mathcal{S}^\Gamma$-induced set of belief hierarchies of player i*.

By the $\mathcal{I}_i$-measurability of σ_i the same choice for player i is assigned throughout an information cell, that is, $\sigma_i(\omega') = \sigma_i(\omega)$ for all $\omega' \in \mathcal{I}_i(\omega)$. Every information cell $P_i \in \mathcal{I}_i$ thus induces a choice $\sigma_i(P_i) \in C_i$, where $\sigma_i(P_i) \coloneqq \sigma_i(\omega)$ for all $\omega \in P_i$. Moreover, since the belief hierarchies are constructed on the basis of posterior beliefs, it follows that i's belief hierarchies are also constant throughout his information cells, that is, $\eta_i[\omega'] = \eta_i[\omega]$ for all $\omega' \in \mathcal{I}_i(\omega)$.

The common prior assumption constitutes a frequently used premise in game theory. Accordingly, all beliefs are derived from a single probability distribution. The common prior assumption formalizes the conceptual viewpoint that differences in beliefs are only due to differences in information.

Within the framework of type models the common prior assumption requires the probability distribution of every type induced by the belief function to be obtained via Bayesian conditionalization on some common probability distribution on all players' choice type combinations.

Definition 3. Let Γ be a game and $\mathcal{T}^\Gamma$ a type model of Γ. The type model $\mathcal{T}^\Gamma$ satisfies the common prior assumption, if there exists a probability distribution $\rho \in \Delta\left(\times_{j \in I}\left(C_j \times T_j\right)\right)$ such that for every player $i \in I$, and for every type $t_i \in T_i$ it is the case that $\rho(t_i) > 0$ and

$$b_i[t_i](c_{-i}, t_{-i}) = \frac{\rho(c_i, c_{-i}, t_i, t_{-i})}{\rho(c_i, t_i)}$$

for all $c_i \in C_i$ with $\rho(c_i, t_i) > 0$, and for all $(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}$. The probability distribution ρ is called common prior.

The preceding formalization of the common prior assumption is equivalent to the conjunction of Dekel and Siniscalchil's (2015) Definition 12.13 with their Definition 12.15 as well as to Bach and Perea's (2020) Definition 4.

In state models the common prior assumption simply postulates all subjective priors to coincide.

Definition 4. Let Γ be a game and $\mathcal{S}^\Gamma$ a state model of Γ. The state model $\mathcal{S}^\Gamma$ satisfies the common prior assumption, if there exists a probability distribution $\pi \in \Delta(\Omega)$ such that $\pi_i = \pi$ for every player $i \in I$. The probability distribution π is called common prior.

3 | TRANSFORMATION OF STATE MODELS INTO TYPE MODELS

The following transformation procedure converts state models into type models.

Definition 5. Let Γ be a game, and $\mathcal{S}^\Gamma$ a state model of Γ. The tuple

$$\langle (T_i, b_i)_{i \in I} \rangle$$

forms the $\mathcal{S}^\Gamma$-generated type model of Γ, where for every player $i \in I$,

- $T_i \coloneqq \left\{t_i^{P_i} : P_i \in \mathcal{I}_i\right\}$ is i's set of types,
- $b_i : T_i \to \Delta(C_{-i} \times T_{-i})$ is i's belief function with

$$b_i\left[t_i^{P_i}\right]\left(c_{-i}, t_{-i}^{P_{-i}}\right) := \sum_{\omega\in P_i:\sigma_{-i}(\omega)=c_{-i},\mathcal{I}_{-i}(\omega)=P_{-i}} \pi_i(\omega \mid P_i),$$

for all $\left(c_{-i}, t_{-i}^{P_{-i}}\right) \in C_{-i} \times T_{-i}$ and for all $t_i^{P_i} \in T_i$.

In a nutshell, information cells are transformed into types and the types' beliefs are then given by the subjective priors conditionalized on the corresponding information cells. Note that the type model generated by a given state model actually is unique.

The transformation procedure established by Definition 5 is now illustrated by means of an example.

Example 1. Let Γ be a game with $I = \{Alice, Bob\}$, $C_{Alice} = \{a, b\}$ as well as $C_{Bob} = \{c, d\}$. Consider the state model $\mathcal{S}^\Gamma$ of Γ with

- $\Omega = \{\omega_1, \omega_2, \omega_3\}$,
- $\mathcal{I}_{Alice} = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$,
- $\mathcal{I}_{Bob} = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$,
- $\sigma_{Alice}(\omega_1) = \sigma_{Alice}(\omega_2) = a$ and $\sigma_{Alice}(\omega_3) = b$,
- $\sigma_{Bob}(\omega_1) = c$ and $\sigma_{Bob}(\omega_2) = \sigma_{Bob}(\omega_3) = d$,
- $\pi_{Alice}(\omega_1) = 0.2$, $\pi_{Alice}(\omega_2) = 0.5$, and $\pi_{Alice}(\omega_3) = 0.3$,
- $\pi_{Bob}(\omega_1) = 0.4$, $\pi_{Bob}(\omega_2) = 0$, and $\pi_{Bob}(\omega_3) = 0.6$.

Applying the transformation procedure of Definition 5 yields the following objects

- $T_{Alice} = \left\{t_{Alice}^{\{\omega_1,\omega_2\}}, t_{Alice}^{\{\omega_3\}}\right\}$,
- $T_{Bob} = \left\{t_{Alice}^{\{\omega_1\}}, t_{Alice}^{\{\omega_2,\omega_3\}}\right\}$,
- $b_{Alice}\left[t_{Alice}^{\{\omega_1,\omega_2\}}\right] = \frac{0.2}{0.2+0.5}\left(c, t_{Bob}^{\{\omega_1\}}\right) + \frac{0.5}{0.2+0.5}\left(d, t_{Bob}^{\{\omega_1,\omega_2\}}\right) = \frac{2}{7}\left(c, t_{Bob}^{\{\omega_1\}}\right) + \frac{5}{7}\left(d, t_{Bob}^{\{\omega_1,\omega_2\}}\right)$,
- $b_{Alice}\left[t_{Alice}^{\{\omega_3\}}\right] = \frac{0.3}{0.3}\left(d, t_{Bob}^{\{\omega_2,\omega_3\}}\right) = \left(d, t_{Bob}^{\{\omega_2,\omega_3\}}\right)$,
- $b_{Bob}\left[t_{Bob}^{\{\omega_1\}}\right] = \frac{0.4}{0.4}\left(a, t_{Alice}^{\{\omega_1,\omega_2\}}\right) = \left(a, t_{Alice}^{\{\omega_1,\omega_2\}}\right)$,
- $b_{Bob}\left[t_{Bob}^{\{\omega_2,\omega_3\}}\right] = \frac{0}{0+0.6}\left(a, t_{Alice}^{\{\omega_1,\omega_2\}}\right) + \frac{0.6}{0+0.6}\left(b, t_{Alice}^{\{\omega_3\}}\right) = \left(b, t_{Alice}^{\{\omega_3\}}\right)$.

The assemblage of these objects $\langle T_{Alice}, b_{Alice}, T_{Bob}, b_{Bob}\rangle$ then constitutes the $\mathcal{S}^\Gamma$-generated type model of Γ. ♣

It turns out that the transformation procedure laid out in Definition 5 preserves the induced belief hierarchies of state models.

Theorem 1. *Let Γ be a game, $\mathcal{S}^\Gamma$ a state model of Γ with $\mathcal{S}^\Gamma$-generated type model $\langle (T_i, b_i)_{i\in I}\rangle$ of Γ, $i \in I$ some player, and $\omega \in \Omega$ some world. Then,*

$$\eta_i[\omega] = \eta_i\left[t_i^{\mathcal{I}_i(\omega)}\right].$$

Proof. It is shown inductively that $\eta_i^k[\omega] = \eta_i^k\left[t_i^{\mathcal{I}_i(\omega)}\right]$ holds for all $k \geq 1$. It then directly follows that $\eta_i[\omega] = \left(\eta_i^n[\omega]\right)_{n\in\mathbb{N}} = \left(\eta_i^n\left[t_i^{\mathcal{I}_i(\omega)}\right]\right)_{n\in\mathbb{N}} = \eta_i\left[t_i^{\mathcal{I}_i(\omega)}\right]$.

First of all, observe that

$$\eta_i^1[\omega](c_{-i})$$

$$= \sum_{\omega' \in \mathcal{I}_i(\omega): \sigma_{-i}(\omega') = c_{-i}} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

$$= \sum_{t_{-i}^{P_{-i}} \in T_{-i}} \sum_{\omega' \in \mathcal{I}_i(\omega): \sigma_{-i}(\omega') = c_{-i}, \mathcal{I}_{-i}(\omega') = P_{-i}} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

$$= \sum_{t_{-i}^{P_{-i}} \in T_{-i}} b_i \left[t_i^{\mathcal{I}_i(\omega)} \right] \left(c_{-i}, t_{-i}^{P_{-i}} \right)$$

$$= \eta_i^1 \left[t_i^{\mathcal{I}_i(\omega)} \right] (c_{-i})$$

for all $c_{-i} \in C_{-i}$.

Now, suppose that $\eta_i^k[\omega] = \eta_i^k \left[t_i^{\mathcal{I}_i(\omega)} \right]$ holds up to some $k > 1$. It then follows that

$$\eta_i^{k+1}[\omega] \left(c_{-i}, \eta_{-i}^1, \ldots, \eta_i^k \right)$$

$$= \sum_{\omega' \in \mathcal{I}_i(\omega): \sigma_{-i}(\omega') = c_{-i}, \eta_{-i}^l[\omega'] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

$$= \sum_{t_{-i}^{P_{-i}} \in T_{-i}: \eta_{-i}^l \left[t_{-i}^{P_{-i}} \right] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \sum_{\omega' \in \mathcal{I}_i(\omega): \sigma_{-i}(\omega') = c_{-i}, \mathcal{I}_{-i}(\omega') = P_{-i}} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

$$= \sum_{t_{-i}^{P_{-i}} \in T_{-i}: \eta_{-i}^l[t_i] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} b_i \left[t_i^{\mathcal{I}_i(\omega)} \right] \left(c_{-i}, t_{-i}^{P_{-i}} \right)$$

$$= \eta_i^{k+1} \left[t_i^{\mathcal{I}_i(\omega)} \right] \left(c_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^k \right)$$

for all $\left(c_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^k \right) \in X_i^{k+1}$. ■

Also, the common prior assumption is maintained from state to type models.

Theorem 2. *Let Γ be a game, and $\mathcal{S}^\Gamma$ a state model of Γ satisfying the common prior assumption. Then, the $\mathcal{S}^\Gamma$-generated type model $\langle (T_i, b_i)_{i \in I} \rangle$ of Γ satisfies the common prior assumption.*

Proof. Define a probability distribution $\rho \in \Delta(\times_{i \in I}(C_i \times T_i))$ in the $\mathcal{S}^\Gamma$-generated type model $\langle (T_i, b_i)_{i \in I} \rangle$ such that for all $\left(c_i, t_i^{P_i} \right)_{i \in I} \in \times_{i \in I}(C_i \times T_i)$

$$\rho\left(\left(c_i, t_i^{P_i} \right)_{i \in I} \right) := \begin{cases} \pi(\cap_{i \in I} P_i), & \text{if } \sigma_i(P_i) = c_i \text{ for all } i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

First of all it is established that $\rho\left(t_i^{P_i} \right) > 0$ holds for all $t_i^{P_i} \in T_i$ and for all $i \in I$. Let $t_i^{P_i} \in T_i$ and observe that $\rho\left(t_i^{P_i} \right) = \sum_{t_{-i}^{P_{-i}} \in T_{-i}} \sum_{(c_j)_{j \in I} \in \times_{j \in I} C_j} \rho\left(\left(c_j, t_j^{P_j} \right)_{j \in I} \right) = \sum_{P_{-i} \in \mathcal{I}_{-i}} \pi\left(\cap_{j \in I} P_j \right) = \pi(P_i)$ and since $\pi(P_i) > 0$ it thus follows that $\rho\left(t_i^{P_i} \right) > 0$ holds.

Next it is shown that for all $i \in I$ and for all $t_i^{P_i} \in T_i$, the equation

$$b_i\left[t_i^{P_i}\right]\left(c_{-i}, t_{-i}^{P_{-i}}\right) = \frac{\rho\left(c_i, c_{-i}, t_i^{P_i}, t_{-i}^{P_{-i}}\right)}{\rho\left(c_i, t_i^{P_i}\right)}$$

holds for all $c_i \in C_i$ with $\rho\left(c_i, t_i^{P_i}\right) > 0$, and for all $\left(c_{-i}, t_{-i}^{P_{-i}}\right) \in C_{-i} \times T_{-i}$. Note that $\rho\left(c_i, t_i^{P_i}\right) = \sum_{t_{-i}^{P_{-i}} \in T_{-i}} \sum_{c_{-i} \in C_{-i}} \rho\left(\left(c_j, t_j^{P_j}\right)_{j\in I}\right) = \sum_{\omega \in \Omega: \sigma_i(\omega) = c_i, \mathcal{I}_i(\omega) = P_i} \pi\left(\cap_{j \in I} \mathcal{I}_j(\omega)\right) = \pi(P_i) > 0$ holds, if and only if, $\sigma_i(P_i) = c_i$. Thus, the following equation

$$b_i\left[t_i^{P_i}\right]\left(c_{-i}, t_{-i}^{P_i}\right) = \frac{\rho\left(\sigma_i(P_i), c_{-i}, t_i^{P_i}, t_{-i}^{P_i}\right)}{\rho\left(\sigma_i(P_i), t_i^{P_i}\right)}$$

has to be validated for all $\left(c_{-i}, t_{-i}^{P_{-i}}\right) \in C_{-i} \times T_{-i}$ and for all $t_i^{P_i} \in T_i$.

Consider some $P_i \in \mathcal{I}_i$ and distinguish two cases (I) and (II).

Case (I). Suppose that $P_i \cap (\cap_{j \in I \setminus \{i\}} P_j) \neq \emptyset$ and $c_j = \sigma_j(P_j)$ for all $j \in I \setminus \{i\}$. Then,

$$b_i\left[t_i^{P_i}\right]\left(c_{-i}, t_{-i}^{P_{-i}}\right) = b_i\left[t_i^{P_i}\right]\left(\sigma_{-i}(P_{-i}), t_{-i}^{P_{-i}}\right)$$

$$= \sum_{\omega \in P_i : \sigma_{-i}(\omega) = \sigma_{-i}(P_{-i}), \mathcal{I}_{-i}(\omega) = P_{-i}} \pi(\omega \mid P_i)$$

$$= \sum_{\omega \in P_i : \omega \in P_j \text{ for all } j \in I \setminus \{i\}} \pi(\omega \mid P_i)$$

$$= \frac{\pi(\cap_{k \in I} P_k)}{\pi(P_i)}$$

$$= \frac{\pi(\cap_{k \in I} P_k)}{\sum_{\hat{P}_j \in \mathcal{I}_j \text{ for all } j \in I \setminus \{i\}\}} \pi\left(P_i \cap \left(\cap_{j \in I \setminus \{i\}} \hat{P}_j\right)\right)}$$

$$= \frac{\rho\left(\sigma_i(P_i), t_i^{P_i}, \sigma_{-i}(P_{-i}), t_{-i}^{P_{-i}}\right)}{\sum_{\hat{P}_{-i} \in \mathcal{I}_{-i}} \rho\left(\sigma_i(P_i), t_i^{P_i}, \sigma_{-i}\left(\hat{P}_{-i}\right), t_{-i}^{\hat{P}_{-i}}\right)}$$

$$= \frac{\rho\left(\sigma_i(P_i), t_i^{P_i}, \sigma_{-i}(P_{-i}), t_{-i}^{P_{-i}}\right)}{\sum_{\left(c_{-i}, t_{-i}^{\hat{P}_{-i}}\right) \in C_{-i} \times T_{-i}} \rho\left(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{\hat{P}_{-i}}\right)}$$

$$= \frac{\rho\left(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{P_{-i}}\right)}{\rho\left(\sigma_i(P_i), t_i^{P_i}\right)}$$

for all $\left(c_{-i}, t_{-i}^{P_{-i}}\right) \in C_{-i} \times T_{-i}$.

Case (II). Suppose that $P_i \cap (\cap_{j \in I \setminus \{i\}} P_j) = \emptyset$ or $c_j \neq \sigma_j(P_j)$ for some $j \in I \setminus \{i\}$. Then, $\rho\left(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{P_{-i}}\right) = 0$ holds by definition of ρ as well as $b_i\left[t_i^{P_i}\right]\left(c_{-i}, t_{-i}^{P_{-i}}\right) = \sum_{\omega \in P_i : \sigma_{-i}(\omega) = c_{-i}, \mathcal{I}_{-i}(\omega) = P_{-i}} \frac{\pi(\{\omega\} \cap P_i)}{\pi_i(P_i)} = \sum_{\omega \in P_i : \sigma_{-i}(\omega) = c_{-i}, \mathcal{I}_{-i}(\omega) = P_{-i}} \pi(\omega \mid P_i) = 0$. It directly follows that

$$b_i\left[t_i^{P_i}\right]\left(c_{-i}, t_{-i}^{P_{-i}}\right) = \frac{\rho\left(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{P_{-i}}\right)}{\rho\left(\sigma_i(P_i), t_i^{P_i}\right)}$$

for all $\left(c_{-i}, t_{-i}^{P_{-i}}\right) \in C_{-i} \times T_{-i}$.

Therefore, the $\mathcal{S}^\Gamma$-generated type model $\langle (T_i, b_i)_{i \in I} \rangle$ satisfies the common prior assumption. ■

4 | TRANSFORMATION OF TYPE MODELS INTO STATE MODELS

Taking type models as input the following transformation procedure defines corresponding state models.

Definition 6. Let Γ be a game, and $\mathcal{T}^\Gamma$ be a type model of Γ. The tuple

$$\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$$

forms a $\mathcal{T}^\Gamma$-generated state model of Γ, where

- $\Omega := \left\{ \omega^{(c_i, t_i)_{i \in I}} : c_i \in C_i, t_i \in T_i \text{ for all } i \in I \right\}$ is the set of all possible worlds, and for every player $i \in I$,
- $\mathcal{I}_i \subseteq 2^\Omega$ is i's possibility partition with

$$\mathcal{I}_i\left(\omega^{\left(c_j, t_j\right)_{j \in I}}\right) := \left\{ \omega^{\left(c_i, t_i, c'_{-i}, t'_{-i}\right)} \in \Omega : c'_{-i} \in C_{-i}, t'_{-i} \in T_{-i} \right\}$$

for all $\omega^{\left(c_j, t_j\right)_{j \in I}} \in \Omega$,
- σ_i: $\Omega \to C_i$ is i's choice function with

$$\sigma_i\left(\omega^{\left(c_i, t_i, c_{-i}, t_{-i}\right)}\right) := c_i$$

for all $\omega^{\left(c_j, t_j\right)_{j \in I}} \in \Omega$,
- $\pi_i \in \Delta(\Omega)$ is i's subjective prior with

$$\pi_i\left(\omega^{\left(c_i, t_i, c_{-i}, t_{-i}\right)} \mid \mathcal{I}_i\left(\omega^{\left(c_i, t_i, c_{-i}, t_{-i}\right)}\right)\right) := b_i[t_i](c_{-i}, t_{-i})$$

for all $\omega^{\left(c_i, t_i, c_{-i}, t_{-i}\right)_{j \in I}} \in \Omega$.

Our transformation procedure generates a possible world for every combination of choices and types of the players. An information cell is associated with a choice type pair of a player and contains those worlds where the choices and types are varied for the opponents. The choice functions determine the players' decisions in line with the corresponding worlds. Finally, the subjective priors are picked such that their induced posteriors are in accordance with the types' beliefs. Note that, in general, many different subjective priors exist that meet this requirement. The belief of a given player i about a world conditional on his information is defined as the belief of the corresponding type about the opponents' choice type combinations attached to the world. Only varying i's choices thus results in the same belief. Observe that for every cell the conditional probability distributions on the set of possible worlds do indeed sum up to one and are well-defined.

As already emphasized above, a state model constructed by the transformation procedure of Definition 6 on the basis of a type model is generally not unique, because the subjective priors can be varied. This property is in contrast to the transformation of state models into type models according to of Definition 5, where uniqueness obtains. The possible multiplicity of generated state models ensues because of their richer structure compared to type models and a degree of freedom in imposing the subjective priors. While type models only specify posterior beliefs, state models fix prior beliefs and choices on top of (implicit) posterior beliefs. In terms of interactive thinking this additional information is superfluous, as the posterior beliefs are the decision-relevant doxastic mental states of the agents. Nevertheless this additional information results in some ambiguity when deducing a state model from a type model as the latter is a sparser formal representation of reasoning. Only the engendered posterior beliefs are required to coincide with the beliefs of the types.

The following example illustrates how the transformation procedure of Definition 6 converts type models to state models.

Example 2. Let Γ be a game with $I = \{Alice, Bob\}$, $C_{Alice} = \{a, b\}$ as well as $C_{Bob} = \{c, d\}$. Consider the type model $\mathcal{T}^{\Gamma}$ of Γ with

- $T_{Alice} = \{t^1_{Alice}, t^2_{Alice}\}$,
- $T_{Bob} = \{t^1_{Bob}, t^2_{Bob}\}$,
- $b_{Alice}[t^1_{Alice}] = \frac{1}{3}(c, t^1_{Bob}) + \frac{1}{3}(d, t^1_{Bob}) + \frac{1}{3}(d, t^2_{Bob})$,
- $b_{Alice}[t^2_{Alice}] = \frac{1}{2}(d, t^1_{Bob}) + \frac{1}{2}(d, t^2_{Bob})$,
- $b_{Bob}[t^1_{Bob}] = \frac{2}{3}(a, t^1_{Alice}) + \frac{1}{3}(b, t^2_{Alice})$,
- $b_{Bob}[t^2_{Bob}] = (a, t^2_{Alice})$,

Applying the transformation procedure of Definition 6 yields the following objects

- Ω with elements
  - $\omega^{(a,t^1_{Alice},c,t^1_{Bob})}$
  - $\omega^{(b,t^1_{Alice},c,t^1_{Bob})}$
  - $\omega^{(a,t^2_{Alice},c,t^1_{Bob})}$
  - $\omega^{(b,t^2_{Alice},c,t^1_{Bob})}$
  - $\omega^{(a,t^1_{Alice},d,t^1_{Bob})}$
  - $\omega^{(b,t^1_{Alice},d,t^1_{Bob})}$
  - $\omega^{(a,t^2_{Alice},d,t^1_{Bob})}$
  - $\omega^{(b,t^2_{Alice},d,t^1_{Bob})}$
  - $\omega^{(a,t^1_{Alice},c,t^2_{Bob})}$
  - $\omega^{(b,t^1_{Alice},c,t^2_{Bob})}$
  - $\omega^{(a,t^2_{Alice},c,t^2_{Bob})}$
  - $\omega^{(b,t^2_{Alice},c,t^2_{Bob})}$
  - $\omega^{(a,t^1_{Alice},d,t^2_{Bob})}$
  - $\omega^{(b,t^1_{Alice},d,t^2_{Bob})}$
  - $\omega^{(a,t^2_{Alice},d,t^2_{Bob})}$
  - $\omega^{(b,t^2_{Alice},d,t^2_{Bob})}\}$,
- $\mathcal{I}_{Alice}$ with partition cells
  - $P^1_{Alice} = \left\{\omega^{(a,t^1_{Alice},c,t^1_{Bob})}, \omega^{(a,t^1_{Alice},d,t^1_{Bob})}, \omega^{(a,t^1_{Alice},c,t^2_{Bob})}, \omega^{(a,t^1_{Alice},d,t^2_{Bob})}\right\}$,
  - $P^2_{Alice} = \left\{\omega^{(b,t^1_{Alice},c,t^1_{Bob})}, \omega^{(b,t^1_{Alice},d,t^1_{Bob})}, \omega^{(b,t^1_{Alice},c,t^2_{Bob})}, \omega^{(b,t^1_{Alice},d,t^2_{Bob})}\right\}$,
  - $P^3_{Alice} = \left\{\omega^{(a,t^2_{Alice},c,t^1_{Bob})}, \omega^{(a,t^2_{Alice},d,t^1_{Bob})}, \omega^{(a,t^2_{Alice},c,t^2_{Bob})}, \omega^{(a,t^2_{Alice},d,t^2_{Bob})}\right\}$,
  - $P^4_{Alice} = \left\{\omega^{(b,t^2_{Alice},c,t^1_{Bob})}, \omega^{(b,t^2_{Alice},d,t^1_{Bob})}, \omega^{(b,t^2_{Alice},c,t^2_{Bob})}, \omega^{(b,t^2_{Alice},d,t^2_{Bob})}\right\}$,
- $\mathcal{I}_{Bob}$ with partition cells
  - $P^1_{Bob} = \left\{\omega^{(a,t^1_{Alice},c,t^1_{Bob})}, \omega^{(b,t^1_{Alice},c,t^1_{Bob})}, \omega^{(a,t^2_{Alice},c,t^1_{Bob})}, \omega^{(b,t^2_{Alice},c,t^1_{Bob})}\right\}$,
  - $P^2_{Bob} = \{\omega^{(a,t^1_{Alice},d,t^1_{Bob})}, \omega^{(b,t^1_{Alice},d,t^1_{Bob})}, \omega^{(a,t^2_{Alice},d,t^1_{Bob})}, \omega^{(b,t^2_{Alice},d,t^1_{Bob})}\}$,
  - $P^3_{Bob} = \left\{\omega^{(a,t^1_{Alice},c,t^2_{Bob})}, \omega^{(b,t^1_{Alice},c,t^2_{Bob})}, \omega^{(a,t^2_{Alice},c,t^2_{Bob})}, \omega^{(b,t^2_{Alice},c,t^2_{Bob})}\right\}$,
  - $P^4_{Bob} = \left\{\omega^{(a,t^1_{Alice},d,t^2_{Bob})}, \omega^{(b,t^1_{Alice},d,t^2_{Bob})}, \omega^{(a,t^2_{Alice},d,t^2_{Bob})}, \omega^{(b,t^2_{Alice},d,t^2_{Bob})}\right\}$,
- $\sigma_{Alice}(\omega) = a$ for all $\omega \in P^1_{Alice} \cup P^3_{Alice}$ and $\sigma_{Alice}(\omega) = b$ for all $\omega \in P^2_{Alice} \cup P^4_{Alice}$,
- $\sigma_{Bob}(\omega) = c$ for all $\omega \in P^1_{Bob} \cup P^3_{Bob}$ and $\sigma_{Bob}(\omega) = d$ for all $\omega \in P^2_{Bob} \cup P^4_{Bob}$,
- π_{Alice} with probabilities

- $\pi_{Alice}\left(\omega^{\left(a,t^1_{Alice},c,t^1_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Alice}\left(\omega^{\left(b,t^1_{Alice},c,t^1_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Alice}\left(\omega^{\left(a,t^2_{Alice},c,t^1_{Bob}\right)}\right) = 0$,
- $\pi_{Alice}\left(\omega^{\left(b,t^2_{Alice},c,t^1_{Bob}\right)}\right) = 0$,
- $\pi_{Alice}\left(\omega^{\left(a,t^1_{Alice},d,t^1_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Alice}\left(\omega^{\left(b,t^1_{Alice},d,t^1_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Alice}\left(\omega^{\left(a,t^2_{Alice},d,t^1_{Bob}\right)}\right) = \frac{1}{8}$,
- $\pi_{Alice}\left(\omega^{\left(b,t^2_{Alice},d,t^1_{Bob}\right)}\right) = \frac{1}{8}$,
- $\pi_{Alice}\left(\omega^{\left(a,t^1_{Alice},c,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Alice}\left(\omega^{\left(b,t^1_{Alice},c,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Alice}\left(\omega^{\left(a,t^2_{Alice},c,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Alice}\left(\omega^{\left(b,t^2_{Alice},c,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Alice}\left(\omega^{\left(a,t^1_{Alice},d,t^2_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Alice}\left(\omega^{\left(b,t^1_{Alice},d,t^2_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Alice}\left(\omega^{\left(a,t^2_{Alice},d,t^2_{Bob}\right)}\right) = \frac{1}{8}$,
- $\pi_{Alice}\left(\omega^{\left(b,t^2_{Alice},d,t^2_{Bob}\right)}\}\right) = \frac{1}{8}$,

– π_{Bob} with probabilities

- $\pi_{Bob}\left(\omega^{\left(a,t^1_{Alice},c,t^1_{Bob}\right)}\right) = \frac{2}{12}$,
- $\pi_{Bob}\left(\omega^{\left(b,t^1_{Alice},c,t^1_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(a,t^2_{Alice},c,t^1_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(b,t^2_{Alice},c,t^1_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Bob}\left(\omega^{\left(a,t^1_{Alice},d,t^1_{Bob}\right)}\right) = \frac{2}{12}$,
- $\pi_{Bob}\left(\omega^{\left(b,t^1_{Alice},d,t^1_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(a,t^2_{Alice},d,t^1_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(b,t^2_{Alice},d,t^1_{Bob}\right)}\right) = \frac{1}{12}$,
- $\pi_{Bob}\left(\omega^{\left(a,t^1_{Alice},c,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(b,t^1_{Alice},c,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(a,t^2_{Alice},c,t^2_{Bob}\right)}\right) = \frac{1}{4}$,
- $\pi_{Bob}\left(\omega^{\left(b,t^2_{Alice},c,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(a,t^1_{Alice},d,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(b,t^1_{Alice},d,t^2_{Bob}\right)}\right) = 0$,
- $\pi_{Bob}\left(\omega^{\left(a,t^2_{Alice},d,t^2_{Bob}\right)}\right) = \frac{1}{4}$,
- $\pi_{Bob}\left(\omega^{\left(b,t^2_{Alice},d,t^2_{Bob}\right)}\}\right) = 0$.

The assemblage of these objects $\langle \Omega, \mathcal{I}_{Alice}, \sigma_{Alice}, \pi_{Alice}, \mathcal{I}_{Bob}, \sigma_{Bob}, \pi_{Bob} \rangle$ then constitutes a $\mathcal{T}^{\Gamma}$-generated state model of Γ. ♣

The transformation procedure yields the same induced belief hierarchies in the type model of departure and its corresponding state models.

Theorem 3. *Let Γ be a game, and $\mathcal{T}^{\Gamma}$ a type model of Γ with some $\mathcal{T}^{\Gamma}$-generated state model $\mathcal{S}^{\Gamma}$ of Γ, $i \in I$ some player, and $t_i \in T_i$ some type of player i. Then,*

$$\eta_i[t_i] = \eta_i\left[\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right]$$

for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$.

Proof. It is shown inductively that $\eta_i^k[t_i] = \eta_i^k\left[\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right]$ holds for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$, and for all $k \geq 1$. It then directly follows that $\eta_i[t_i] = \left(\eta_i^n[t_i]\right)_{n \in \mathbb{N}} = \left(\eta_i^n\left[\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right]\right)_{n \in \mathbb{N}} = \eta_i\left[\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right]$ for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$.

First of all, let $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$ and observe that

$$\begin{aligned}
&\eta_i^1[t_i](c'_{-i}) \\
&= \sum_{t'_{-i} \in T_{-i}} b_i[t_i](c'_{-i}, t'_{-i}) \\
&= \sum_{t'_{-i} \in T_{-i}} \pi_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})}\right)\right) \\
&= \sum_{t'_{-i} \in T_{-i}} \pi_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i\left(\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right)\right) \\
&= \sum_{\omega \in \mathcal{I}_i\left(\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right) : \sigma_{-i}(\omega) = c'_{-i}} \pi_i\left(\omega \mid \mathcal{I}_i\left(\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right)\right) \\
&= \eta_i^1\left[\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right](c'_{-i})
\end{aligned}$$

holds for all $c'_{-i} \in C_{-i}$.

Now, suppose that $\eta_i^k[t_i] = \eta_i^k\left[\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right]$ holds for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$ up to some $k > 1$. Let $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$ and observe that

$$\begin{aligned}
&\eta_i^{k+1}[t_i](c'_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^k) \\
&= \sum_{t'_{-i} \in T_{-i} : \eta_{-i}^l[t'_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} b_i[t_i](c'_{-i}, t'_{-i}) \\
&= \sum_{t'_{-i} \in T_{-i} : \eta_{-i}^l[t'_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \pi_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})}\right)\right) \\
&= \sum_{t'_{-i} \in T_{-i} : \eta_{-i}^l[t'_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \pi_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i\left(\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right)\right)
\end{aligned}$$

for all $\left(c'_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^k\right) \in X_i^{k+1}$.

By the inductive assumption, it is the case $\eta_j^l[t_j] = \eta_j^l\left[\omega^{(c_j, t_j, c_{-j}, t_{-j})}\right]$ for all $j \in I \setminus \{i\}$, for all $t_j \in T_j$, for all $1 \leq l \leq k$, and for all $c_j, c_{-j}, t_{-j} \in C_j \times C_{-j} \times T_{-j}$. Therefore,

$$= \sum_{t'_{-i} \in T_{-i} : \eta^l_{-i}[t'_{-i}] = \eta^l_{-i} \text{ for all } 1 \leq l \leq k} \pi_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i\left(\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right)\right)$$

$$= \sum_{\omega' \in \mathcal{I}_i\left(\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right) : \eta^l_{-i}[\omega'] = \eta^l_{-i} \text{ for all } 1 \leq l \leq k, \sigma_{-i}(\omega') = c'_{-i}} \pi_i\left(\omega' \mid \mathcal{I}_i\left(\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right)\right)$$

$$= \eta^{k+1}_i\left[\omega^{(c_i, t_i, c_{-i}, t_{-i})}\right]\left(c'_{-i}, \eta^1_{-i}, \ldots, \eta^k_{-i}\right)$$

for all $\left(c'_{-i}, \eta^1_{-i}, \ldots, \eta^k_{-i}\right) \in X^{k+1}_i$. ■

The common prior assumption is preserved from type to state models, too.

Theorem 4. *Let Γ be a game, and $\mathcal{T}^\Gamma$ a type model of Γ satisfying the common prior assumption. Then, there exists a $\mathcal{T}^\Gamma$-generated state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ of Γ that satisfies the common prior assumption.*

Proof. Define a state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ of Γ with the objects $\Omega, (\mathcal{I}_i, \sigma_i)_{i \in I}$ as in Definition 6, as well as with a probability distribution $\pi \in \Delta(\Omega)$ such that $\pi\left(\omega^{(c_i, t_i)_{i \in I}}\right) := \rho\left((c_i, t_i)_{i \in I}\right)$ for all $\omega^{(c_i, t_i)_{i \in I}} \in \Omega$ and $\pi_i = \pi$ for all $i \in I$. By construction $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ thus satisfies the common prior assumption. Since

$$\pi_i\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i\left(\omega^{(c_j, t_j)_{j \in I}}\right)\right) = \frac{\pi\left(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})}\right)}{\pi\left(\mathcal{I}_i\left(\omega^{(c_j, t_j)_{j \in I}}\right)\right)} = \frac{\rho(c_i, t_i, c'_{-i}, t'_{-i})}{\rho(c_i, t_i)} = b_i[t_i]\left(c'_{-i}, t'_{-i}\right)$$

holds for all $\left(c'_{-i}, t'_{-i}\right) \in C_{-i} \times T_{-i}$, for all $\omega^{(c_j, t_j)_{j \in I}} \in \Omega$ and for all $i \in I$, the state model $\langle \left(\Omega, (\mathcal{I}_i, \pi_i, \sigma_i)_{i \in I}\right) \rangle$ also forms a $\mathcal{T}^\Gamma$-generated state model of Γ. ■

5 | RELATION TO HELLMAN AND SAMET (2012)

The common prior assumption is also addressed by Hellman and Samet (2012) who explore its restrictiveness using a state-based approach. More specifically, they analyze how the topological size of posterior belief profiles deducible from a common prior depends on properties of the underlying state model. In contrast we pursue a different question here by investigating whether the common prior assumption is preserved across epistemic models with Theorems 2 and 4. It is nonetheless insightful to relate our results to Hellman and Samet (2012). In particular, it seems intriguing to connect their Theorem 1 which makes explicit the restrictiveness of the common prior assumption to our Theorem 2 which establishes the preservation of the common prior assumption when moving from state models to type models.

Hellman and Samet consider knowledge spaces which are state models without choice functions and subjective priors. They define a belief function for every player that assigns to every world a probability distribution on the set of all possible worlds. A profile of belief functions is then called *consistent*, whenever it is inferable from a common prior. Before Hellman and Samet's crucial ingredient of tightness can be explicated some standard notions of state models need to be introduced. Events $E \subseteq \Omega$ are sets of possibles worlds and correspond to some property shared by all the worlds they contain. For instance, the event of it raining in London contains all possible worlds in which it indeed rains in London. A definition of common knowledge of an event due to Aumann (1976) is based on the meet of the agents' possibility partitions. Given two possibility partitions $\mathcal{I}_1$ and $\mathcal{I}_2$, the partition $\mathcal{I}_1$ is called finer than the partition $\mathcal{I}_2$ (or $\mathcal{I}_2$ coarser than $\mathcal{I}_1$), if each cell of $\mathcal{I}_1$ is a subset of some cell of $\mathcal{I}_2$. The partition $\mathcal{I}_1$ is called strictly finer than the partition $\mathcal{I}_2$, if $\mathcal{I}_1$ is finer than $\mathcal{I}_2$ and there exists a cell of $\mathcal{I}_1$ which is a strict subset of some cell of $\mathcal{I}_2$. Given a possibility partition profile $(\mathcal{I}_i)_{i \in I}$, the finest partition that is coarser than all of them is called the meet and is denoted by $\bigwedge_{i \in I} \mathcal{I}_i$. Common knowledge of an event E is then defined as

$$CK(E) := \left\{\omega \in \Omega : \left(\bigwedge_{i \in I} \mathcal{I}_i\right)(\omega) \subseteq E\right\},$$

where $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega)$ denotes the cell of the meet that contains the world ω.[1] An event E is common knowledge at some world ω, whenever $\omega \in CK(E)$. We say that knowledge is tight at some world ω, whenever the following conditional holds: if the cell of some agent at ω can be strictly refined, then the meet cell of ω is thereby also being strictly refined. The possibility partition profile in a knowledge space is called *tight*, whenever there exists a world at which the event of knowledge being tight is common knowledge. Hellman and Samet (2012, Theorem 1.1) establish that if a possibility partition profile is tight, then every profile of belief functions (that can be constructed in the corresponding knowledge structure) is consistent. Moreover, according to Hellman and Samet (2012, Theorem 1.2), if a possibility partition profile is not tight, then the set of consistent profiles of belief functions is nowhere dense (i.e., small in a topological sense). We now relate these two results of Hellman and Samet by means of two examples to our Theorem 2.

In the following example a knowledge space with a tight possibility partition profile is equipped with a common prior and then converted via our transformation procedure from Definition 5 into its corresponding type model.

Example 3. Let Γ be a game with $I = \{Alice, Bob\}$, $C_{Alice} = \{a, b\}$ as well as $C_{Bob} = \{c, d\}$. Consider a knowledge space (with choice functions) given by

- $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$,
- $\mathcal{I}_{Alice} = \{\{\omega_1\}, \{\omega_2, \omega_4\}, \{\omega_3\}\}$,
- $\mathcal{I}_{Bob} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$,
- $\sigma_{Alice}(\omega_1) = \sigma_{Alice}(\omega_3) = a$ and $\sigma_{Alice}(\omega_2) = \sigma_{Alice}(\omega_4) = b$,
- $\sigma_{Bob}(\omega_1) = \sigma_{Bob}(\omega_2) = c$ and $\sigma_{Bob}(\omega_3) = \sigma_{Bob}(\omega_4) = d$.

We first show that the possibility partition profile in this knowledge space is tight. At ω_1 the cell of *Bob* can be strictly refined from $\mathcal{I}_{Bob}(\omega_1) = \{\omega_1, \omega_2\}$ to $\mathcal{I}'_{Bob}(\omega_1) = \{\omega_1\}$ which induces a strict refinement of the meet cell from $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega_1) = \Omega$ to $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega_1) = \{\omega_1\}$, while no strict refinement for *Alice* is possible at ω_1. At ω_2 the cell of *Alice* can be strictly refined from $\mathcal{I}_{Alice}(\omega_2) = \{\omega_2, \omega_4\}$ to $\mathcal{I}'_{Alice}(\omega_2) = \{\omega_2\}$ which induces a strict refinement of the meet cell from $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega_2) = \Omega$ to $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega_2) = \{\omega_1, \omega_2\}$, while the cell of *Bob* can be strictly refined from $\mathcal{I}_{Bob}(\omega_2) = \{\omega_1, \omega_2\}$ to $\mathcal{I}'_{Bob}(\omega_2) = \{\omega_2\}$ which induces a strict refinement of the meet cell from $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega_2) = \Omega$ to $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega_2) = \{\omega_2, \omega_3, \omega_4\}$. At ω_3 the cell of *Bob* can be strictly refined from $\mathcal{I}_{Bob}(\omega_3) = \{\omega_3, \omega_4\}$ to $\mathcal{I}'_{Bob}(\omega_3) = \{\omega_3\}$ which induces a strict refinement of the meet cell from $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega_3) = \Omega$ to $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega_3) = \{\omega_3\}$, while no strict refinement for *Alice* is possible at ω_3. At ω_4 the cell of *Alice* can be strictly refined from $\mathcal{I}_{Alice}(\omega_4) = \{\omega_2, \omega_4\}$ to $\mathcal{I}'_{Alice}(\omega_4) = \{\omega_4\}$ which induces a strict refinement of the meet cell from $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega_4) = \Omega$ to $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega_4) = \{\omega_3, \omega_4\}$, while the cell of *Bob* can be strictly refined from $\mathcal{I}_{Bob}(\omega_4) = \{\omega_3, \omega_4\}$ to $\mathcal{I}'_{Bob}(\omega_4) = \{\omega_4\}$ which induces a strict refinement of the meet cell from $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega_4) = \Omega$ to $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega_4) = \{\omega_1, \omega_2, \omega_4\}$. Since knowledge is tight at all four worlds in the knowledge space, it immediately follows that the event of knowledge being tight is common knowledge everywhere. Consequently, the possibilty partition profile of the knowledge space qualifies as tight. By Hellman and Samet (2012, Theorem 1.1), every profile of posterior beliefs thus has a common prior. To see this consider arbitrary posterior beliefs for the two agents, that is,

- $\pi_{Alice}(\omega_1 \mid \{\omega_1\}) = \pi_{Alice}(\omega_3 \mid \{\omega_3\}) = 1$,
- $\pi_{Alice}(\omega_2 \mid \{\omega_2, \omega_4\}) = \alpha$ *and* $\pi_{Alice}(\omega_4 \mid \{\omega_2, \omega_4\}) = 1 - \alpha$,
- $\pi_{Bob}(\omega_1 \mid \{\omega_1, \omega_2\}) = \beta$ *and* $\pi_{Bob}(\omega_2 \mid \{\omega_1, \omega_2\}) = 1 - \beta$,
- $\pi_{Bob}(\omega_3 \mid \{\omega_3, \omega_4\}) = \gamma$ *and* $\pi_{Bob}(\omega_4 \mid \{\omega_3, \omega_4\}) = 1 - \gamma$,

where $\alpha, \beta, \gamma \in [0, 1]$ and $\pi_{Alice}(\omega_1 \mid \{\omega_1\})$ and $\pi_{Alice}(\omega_3 \mid \{\omega_3\})$ cannot vary at all due to the given specific partitional structure. These posterior beliefs are consistent with a common prior $\pi \in \Delta(\Omega)$ that satisfies the following four conditions

- $\pi(\omega_1) = \frac{\alpha \cdot \beta \cdot (1-\gamma)}{\alpha \cdot (1-\gamma) + (1-\alpha) \cdot (1-\beta)}$,
- $\pi(\omega_2) = \frac{\alpha \cdot (1-\beta) \cdot (1-\gamma)}{\alpha \cdot (1-\gamma) + (1-\alpha) \cdot (1-\beta)}$,
- $\pi(\omega_3) = \frac{(1-\alpha) \cdot (1-\beta) \cdot \gamma}{\alpha \cdot (1-\gamma) + (1-\alpha) \cdot (1-\beta)}$,
- $\pi(\omega_4) = \frac{(1-\alpha) \cdot (1-\beta) \cdot (1-\gamma)}{\alpha \cdot (1-\gamma) + (1-\alpha) \cdot (1-\beta)}$.

For the conversion of the knowledge space into a type model with our transformation procedure suppose the common prior $\pi \in \Delta(\Omega)$, where $\pi(\omega_1) = \pi(\omega_2) = \pi(\omega_3) = \frac{1}{3}$ and $\pi(\omega_4) = 0$.[2] Note that this probability distribution satisfies the non-null information condition of Definition 2. The $\langle \Omega, (\mathcal{I}_i, \sigma_i)_{i \in I}, \pi \rangle$-induced type model of Γ is given by $\langle (T_i, b_i)_{i \in I} \rangle$ with

- $T_{Alice} = \left\{ t_{Alice}^{\{\omega_1\}}, t_{Alice}^{\{\omega_2,\omega_4\}}, t_{Alice}^{\{\omega_3\}} \right\}$ and $T_{Bob} = \left\{ t_{Bob}^{\{\omega_1,\omega_2\}}, t_{Bob}^{\{\omega_3,\omega_4\}} \right\}$,
- $b_{Alice}\left[t_{Alice}^{\{\omega_1\}} \right]\left(c, t_{Bob}^{\{\omega_1,\omega_2\}} \right) = 1$,
- $b_{Alice}\left[t_{Alice}^{\{\omega_2,\omega_4\}} \right]\left(c, t_{Bob}^{\{\omega_1,\omega_2\}} \right) = 1$,
- $b_{Alice}\left[t_{Alice}^{\{\omega_3\}} \right]\left(d, t_{Bob}^{\{\omega_3,\omega_4\}} \right) = 1$,
- $b_{Bob}\left[t_{Bob}^{\{\omega_1,\omega_2\}} \right]\left(a, t_{Alice}^{\{\omega_1\}} \right) = b_{Bob}\left[t_{Bob}^{\{\omega_1,\omega_2\}} \right]\left(b, t_{Alice}^{\{\omega_2,\omega_4\}} \right) = \frac{1}{2}$,
- $b_{Bob}\left[t_{Bob}^{\{\omega_3,\omega_4\}} \right]\left(a, t_{Alice}^{\{\omega_3\}} \right) = 1$.

By Theorem 2, the type model $\langle (T_i, b_i)_{i \in I} \rangle$ must satisfy the common prior assumption. Indeed, observe that the probability distribution $\rho \in \Delta(\times_{i \in I}(C_i \times T_i))$, where

- $\rho\left(\left(a, t_{Alice}^{\{\omega_1\}} \right), \left(c, t_{Bob}^{\{\omega_1,\omega_2\}} \right) \right) = \frac{1}{3}$,
- $\rho\left(\left(a, t_{Alice}^{\{\omega_3\}} \right), \left(d, t_{Bob}^{\{\omega_3,\omega_4\}} \right) \right) = \frac{1}{3}$,
- $\rho\left(\left(b, t_{Alice}^{\{\omega_2,\omega_4\}} \right), \left(c, t_{Bob}^{\{\omega_1,\omega_2\}} \right) \right) = \frac{1}{3}$,

constitutes a common prior for $\langle (T_i, b_i)_{i \in I} \rangle$ according to Definition 3. ♣

The preceding example illustrates Hellman and Samet (2012, Theorem 1.1), our Theorem 2, as well as to some extent the interplay of the two results. The point of departure is a tight knowledge space. Indeed, as predicted by Hellman and Samet (2012, Theorem 1.1), for every belief profile there exists a common prior that induces it. Hence, the posteriors can be chosen arbitrarily, yet they always satisfy the common prior assumption. Our transformation procedure from Definition 5 then yields a type model corresponding to the knowledge space. In line with our Theorem 2 this type model also satisfies the common prior asumption.

Next a knowledge space is provided with a non-tight possibility partition profile and a common prior. Its induced type model is then derived via our transformation procedure of Definition 5 and shown to preserve the common prior assumption.

Example 4. Let Γ be a game with $I = \{Alice, Bob\}$, $C_{Alice} = \{a, b\}$ as well as $C_{Bob} = \{c, d\}$. Consider a knowledge space (with choice functions) given by

- $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$,
- $\mathcal{I}_{Alice} = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$,
- $\mathcal{I}_{Bob} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$,
- $\sigma_{Alice}(\omega_1) = \sigma_{Alice}(\omega_3) = a$ and $\sigma_{Alice}(\omega_2) = \sigma_{Alice}(\omega_4) = b$,
- $\sigma_{Bob}(\omega_1) = \sigma_{Bob}(\omega_2) = c$ and $\sigma_{Bob}(\omega_3) = \sigma_{Bob}(\omega_4) = d$.

Observe that the possibility partition profile in this knowledge space is not tight, as knowledge is not tight at any world. Indeed, if the cell of *Alice* at $\omega \in \{\omega_1, \omega_3\}$ is strictly refined from $\mathcal{I}_{Alice}(\omega) = \{\omega_1, \omega_3\}$ to $\mathcal{I}'_{Alice}(\omega) = \{\omega\}$, then the meet cell is $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega) = \Omega$ and hence does not strictly refine $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega) = \Omega$. If the cell of *Alice* at $\omega \in \{\omega_2, \omega_4\}$ is strictly refined from $\mathcal{I}_{Alice}(\omega) = \{\omega_2, \omega_4\}$ to $\mathcal{I}'_{Alice}(\omega) = \{\omega\}$, then the meet cell is $(\bigwedge_{i \in I} \mathcal{I}_i)'(\omega) = \Omega$ and hence does not strictly refine $(\bigwedge_{i \in I} \mathcal{I}_i)(\omega) = \Omega$. Since the possibility partition profile is not tight, it follows from Hellman and Samet (2012, Theorem 1.2) that the set of posterior belief profiles stemming from a common prior is very small. Nonetheless, suppose that the knowledge space is equipped with a common prior $\pi \in \Delta(\Omega)$, where $\pi(\omega_1) = \pi(\omega_2) = \pi(\omega_3) = \frac{1}{3}$ and $\pi(\omega_4) = 0$, which is in line with the non-null information condition of Definition 2. The posterior beliefs for the two agents then obtain as

- $\pi_{Alice}(\omega_1 \mid \{\omega_1, \omega_3\}) = \pi_{Alice}(\omega_3 \mid \{\omega_1, \omega_3\}) = \frac{1}{2}$,
- $\pi_{Alice}(\omega_2 \mid \{\omega_2, \omega_4\}) = 1$ and $\pi_{Alice}(\omega_4 \mid \{\omega_2, \omega_4\}) = 0$,
- $\pi_{Bob}(\omega_1 \mid \{\omega_1, \omega_2\}) = \pi_{Bob}(\omega_2 \mid \{\omega_1, \omega_2\}) = \frac{1}{2}$,
- $\pi_{Bob}(\omega_3 \mid \{\omega_3, \omega_4\}) = 1$ and $\pi_{Bob}(\omega_4 \mid \{\omega_3, \omega_4\}) = 0$.

As $\langle \Omega, (\mathcal{I}_i, \sigma_i)_{i \in I}, \pi \rangle$-induced type model of Γ our transformation procedure of Definition 5 yields $\langle (T_i, b_i)_{i \in I} \rangle$ with

- $T_{Alice} = \left\{ t_{Alice}^{\{\omega_1,\omega_3\}}, t_{Alice}^{\{\omega_2,\omega_4\}} \right\}$ *and* $T_{Bob} = \left\{ t_{Bob}^{\{\omega_1,\omega_2\}}, t_{Bob}^{\{\omega_3,\omega_4\}} \right\}$,
- $b_{Alice}\left[t_{Alice}^{\{\omega_1,\omega_3\}}\right]\left(c, t_{Bob}^{\{\omega_1,\omega_2\}}\right) = b_{Alice}\left[t_{Alice}^{\{\omega_1,\omega_3\}}\right]\left(d, t_{Bob}^{\{\omega_3,\omega_4\}}\right) = \frac{1}{2}$,
- $b_{Alice}\left[t_{Alice}^{\{\omega_2,\omega_4\}}\right]\left(c, t_{Bob}^{\{\omega_1,\omega_2\}}\right) = 1$,
- $b_{Bob}\left[t_{Bob}^{\{\omega_1,\omega_2\}}\right]\left(a, t_{Alice}^{\{\omega_1,\omega_3\}}\right) = b_{Bob}\left[t_{Bob}^{\{\omega_1,\omega_2\}}\right]\left(b, t_{Alice}^{\{\omega_2,\omega_4\}}\right) = \frac{1}{2}$,
- $b_{Bob}\left[t_{Bob}^{\{\omega_3,\omega_4\}}\right]\left(a, t_{Alice}^{\{\omega_1,\omega_3\}}\right) = 1$.

By Theorem 2, the type model $\langle (T_i, b_i)_{i \in I} \rangle$ must satisfy the common prior assumption. Indeed, observe that the probability distribution $\rho \in \Delta(\times_{i \in I}(C_i \times T_i))$, where

- $\rho\left(\left(a, t_{Alice}^{\{\omega_1,\omega_3\}}\right), \left(c, t_{Bob}^{\{\omega_1,\omega_2\}}\right)\right) = \frac{1}{3}$,
- $\rho\left(\left(a, t_{Alice}^{\{\omega_1,\omega_3\}}\right), \left(d, t_{Bob}^{\{\omega_3,\omega_4\}}\right)\right) = \frac{1}{3}$,
- $\rho\left(\left(b, t_{Alice}^{\{\omega_2,\omega_4\}}\right), \left(c, t_{Bob}^{\{\omega_1,\omega_2\}}\right)\right) = \frac{1}{3}$,

constitutes a common prior for $\langle (T_i, b_i)_{i \in I} \rangle$ according to Definition 3. ♣

In the preceding example Hellman and Samet (2012, Theorem 1.2) is illustrated together with our Theorem 2. A non-tight knowledge space is equipped with a common prior. According to Hellman and Samet (2012, Theorem 1.2) there are only very few belief functions consistent with a common prior. The posterior beliefs of the agents must therefore be chosen carefully. However, the transformation procedure of Definition 5 is still guaranteed by our Theorem 2 to generate an induced type model that also satisfies the common prior assumption.

Hellman and Samet's notion of tightness can also be formally connected to our transformation procedures. Indeed, it actually turns out that Definition 6 always generates non-tight state models.

Theorem 5. *Let Γ be a game such that $|I| \geq 2$ as well as $|C_i| \geq 2$ for all $i \in I$, and $\mathcal{T}^\Gamma$ a type model of Γ. Every $\mathcal{T}^\Gamma$-generated state model of Γ is not tight.*

Proof. Let $\mathcal{S}^\Gamma$ be a $\mathcal{T}^\Gamma$-generated state model of Γ. By Definition 6, the set of all possible worlds is given by $\Omega = \left\{ \omega^{(c_i,t_i)_{i \in I}} : c_i \in C_i, t_i \in T_i \text{ for all } i \in I \right\}$.

First of all, we show that $(\bigwedge_{i \in I} \mathcal{I}_i) = \{\Omega\}$. Toward a contradiction, suppose that there exists two non-empty cells C and C' such that $C, C' \in (\bigwedge_{i \in I} \mathcal{I}_i)$ and $C \cap C' = \emptyset$. Let $\omega^{(c_i,t_i)_{i \in I}} \in C$ and $\omega^{(c'_i,t'_i)_{i \in I}} \in C'$. Consider some player $j \in I$. It follows by the construction of $\mathcal{I}_j$ in Definition 6 that $\omega^{(c_j,t_j,c'_{-i},t'_{-i})} \in \mathcal{I}_j\left(\omega^{(c_i,t_i)_{i \in I}}\right)$. Let $k \in I \setminus \{j\}$ be some other player. Then, by the construction of $\mathcal{I}_k$, it holds that $\omega^{(c'_j,t'_j)_{j \in I}} \in \mathcal{I}_k\left(\omega^{(c_j,t_j,c'_{-i},t'_{-i})}\right)$. Consequently, $\omega^{(c_j,t_j,c'_{-i},t'_{-i})} \in \mathcal{I}_j\left(\omega^{(c_i,t_i)_{i \in I}}\right) \cap \mathcal{I}_k\left(\omega^{(c_j,t_j,c'_{-i},t'_{-i})}\right)$. As $\mathcal{I}_j\left(\omega^{(c_i,t_i)_{i \in I}}\right) \subseteq C$ and $\mathcal{I}_k\left(\omega^{(c_j,t_j,c'_{-i},t'_{-i})}\right) \subseteq C'$, it follows that $C \cap C' \neq \emptyset$, a contradiction.

Now, take some world $\omega^{(c_i,t_i)_{i \in I}}$ and some player $j \in I$. Let $\mathcal{I}_j\left(\omega^{(c_i,t_i)_{i \in I}}\right)$ be strictly refined into $\mathcal{I}'_j\left(\omega^{(c_i,t_i)_{i \in I}}\right)$. Note that a strict refinement is always possible, since Γ contains at least two players with at least two choices per player. Thus, there exists some player $k \in I \backslash \{j\}$ with $|C_k| \geq 2$, which ensures that $\mathcal{I}_j\left(\omega^{(c_i,t_i)_{i\in I}}\right)$ has at least two worlds and can therefore be strictly refined. Consider some world $\omega^{\left(c_i',t_i'\right)_{i\in I}} \in \Omega$ and distinguish two cases (I) and (II).

Case (I). Suppose that $\left(c_j', t_j'\right) \neq (c_j, t_j)$. Take some player $k \in I \backslash \{j\}$. Then, by Definition 6, it follows that $\omega^{\left(c_k,t_k,c_{-k}',t_{-k}'\right)} \in \mathcal{I}_k\left(\omega^{(c_i,t_i)_{i\in I}}\right)$ and it is also the case that $\omega^{\left(c_i',t_i'\right)_{i\in I}} \in \mathcal{I}_j\left(\omega^{\left(c_k,t_k,c_{-k}',t_{-k}'\right)}\right)$. As only player j's cell containing $\omega^{(c_i,t_i)_{i\in I}}$ has been strictly refined, $\mathcal{I}_j'\left(\omega^{\left(c_k,t_k,c_{-k}',t_{-k}'\right)}\right) = \mathcal{I}_j\left(\omega^{\left(c_k,t_k,c_{-k}',t_{-k}'\right)}\right)$. Hence, $\omega^{\left(c_i',t_i'\right)_{i\in I}} \in \mathcal{I}_j'\left(\omega^{\left(c_k,t_k,c_{-k}',t_{-k}'\right)}\right)$ holds, too. Consequently, $\omega^{\left(c_i',t_i'\right)_{i\in I}} \in \left(\bigwedge_{i\in I\backslash\{j\}}\mathcal{I}_i \wedge \mathcal{I}_j'\right)\left(\omega^{(c_i,t_i)_{i\in I}}\right)$. Since $\omega^{\left(c_i',t_i'\right)_{i\in I}} \in \Omega$ has been arbitrarily chosen, $\left(\bigwedge_{i\in I\backslash\{j\}}\mathcal{I}_i \wedge \mathcal{I}_j'\right)$ $\left(\omega^{(c_i,t_i)_{i\in I}}\right) = \Omega$ obtains.

Case (II). Suppose that $\left(c_j', t_j'\right) = (c_j, t_j)$. Consider some pair $\left(c_j'', t_j''\right) \in C_j \times T_j$ such that $\left(c_j'', t_j''\right) \neq (c_j, t_j)$. Then, by Case ($I$), $\omega^{\left(c_j'',t_j'',c_{-j}',t_{-j}'\right)} \in \left(\bigwedge_{i\in I\backslash\{j\}}\mathcal{I}_i \wedge \mathcal{I}_j'\right)\left(\omega^{(c_i,t_i)_{i\in I}}\right)$. Take some player $k \in I \backslash \{j\}$. Definition 6 ensures that $\omega^{\left(c_j',t_j',c_{-j}',t_{-j}'\right)} \in \mathcal{I}_k\left(\omega^{\left(c_j'',t_j'',c_{-j}',t_{-j}'\right)}\right)$. As $\omega^{\left(c_j'',t_j'',c_{-j}',t_{-j}'\right)} \in \left(\bigwedge_{i\in I\backslash\{j\}}\mathcal{I}_i \wedge \mathcal{I}_j'\right)\left(\omega^{(c_i,t_i)_{i\in I}}\right)$ and $\mathcal{I}_k\left(\omega^{\left(c_j'',t_j'',c_{-j}',t_{-j}'\right)}\right) \subseteq \left(\bigwedge_{i\in I\backslash\{j\}}\mathcal{I}_i \wedge \mathcal{I}_j'\right)$ $\left(\omega^{(c_i,t_i)_{i\in I}}\right)$, it follows that $\omega^{\left(c_j',t_j',c_{-j}',t_{-j}'\right)} \in \left(\bigwedge_{i\in I\backslash\{j\}}\mathcal{I}_i \wedge \mathcal{I}_j'\right)\left(\omega^{(c_i,t_i)_{i\in I}}\right)$. Since $\omega^{\left(c_i',t_i'\right)_{i\in I}} \in \Omega$ has been arbitrarily chosen, $\left(\bigwedge_{i\in I\backslash\{j\}}\mathcal{I}_i \wedge \mathcal{I}_j'\right)\left(\omega^{(c_i,t_i)_{i\in I}}\right) = \Omega$ obtains.

Because the world $\omega^{(c_i,t_i)_{i\in I}} \in \Omega$ and the player $j \in I$ have been arbitrarily chosen, knowledge is not tight at any world. A fortiori the event of knowledge being tight cannot be common knowledge. Therefore, the possibility partition profile of $\mathcal{S}^\Gamma$ is not tight. ■

Independent from the properties of the type model of departure, Definition 6 thus constantly brings about a state model i.e., non-tight. This is essentially due to the way the possibility partitions are generated from type models without any exogenous restrictions on the beliefs. By construction, type models induce rectangular possibility partitions for the induced state models in the sense that every information cell of any player contains, for a fixed choice type pair of that player, all possible choice type combinations of his opponents. In other words, every information cell can be written as the product of a fixed choice type pair of the corresponding player and all choice type combinations of his opponents. Such rectangular possibility structures are always non-tight, as refining any cell still leaves all the worlds in the new finer cells connected to all other worlds, which in turn precludes the meet from being refinable.

Besides, note that the assumption requiring at least two choices per player in Theorem 5 makes the tightness condition non-trivial, as our transformation procedure then implies that every information cell in the induced state model contains at least two worlds and can thus be strictly refined. In the case of a cell containing a single world, knowledge being tight at that world would be trivially satisfied due to a false antecedent in its definition.

A direct consequence of Theorem 5 is that our two transformation procedures map tight state models into non-tight state models via applying Definition 5 followed by Definition 6.

Corollary 1. *Let Γ be a game such that $|I| \geq 2$ as well as $|C_i| \geq 2$ for all $i \in I$, and $\mathcal{S}^\Gamma$ a tight state model of Γ with its $\mathcal{S}^\Gamma$-generated type model $\mathcal{T}^\Gamma$ of Γ. Then, every $\mathcal{T}^\Gamma$-generated state model of Γ is not tight.*

By Theorem 5 it is ensured that every $\mathcal{T}^\Gamma$-generated state model of Γ is non-tight. Conceptually, Corollary 1 reflects an information loss from the input state model to the output state model. The doxastic structure embodied by possibility partitions in state models is qualitative and vanishes when constructing the induced type models, which only preserve the quantitative doxastic structure that is, the beliefs given by the probabilities. This qualitative doxastic structure of the possibility partitions imposes exogenous restrictions on the agents' reasoning that are absent in type models. With regard to Corollary 1, the lack of exogenous restrictions in the $\mathcal{S}^\Gamma$-generated type model $\mathcal{T}^\Gamma$ thus morphs into a $\mathcal{T}^\Gamma$-generated state model that does not exhibit any exogenous restrictions which in turn makes it non-tight. In contrast, $\mathcal{S}^\Gamma$ as the state model of departure may well contain some exogeneous restrictions.

Besides, Theorem 5 in connection with Hellman and Samet (2012, Theorem 1) sheds some light on the possibility of preserving common priors in line with our transformation procedures if slight belief perturbations are admitted. Let the point of departure be a tight state model. By Hellman and Samet (2012, Theorem 1.1) the state model's beliefs are derivable from a common prior and its induced type model of Definition 5 also satisfies the common prior assumption by Theorem 3. Before transforming the induced type model back into a state model via Definition 6 let its beliefs be slightly perturbed. By Theorem 5 this new state model violates tightness. It then follows from Hellman and Samet (2012, Theorem 1.2) that the likelihood of it violating the common prior assumption is vast.

6 | ISOMORPHISM

The transformation procedure in Definition 5 converts state models into type models, while the one in Definition 6 moulds state models from type models. In terms of structural equivalence of epistemic models the question whether these two transformation procedures are inverse to each other naturally emerges. We explore the relationship between our two transformation procedures by means of isomorphism. Intuitively, two epistemic models are isomorphic if they formalize the same interactive thinking. In the context of our transformation procedures two issues need to be addressed. Firstly, it has to be determined whether a type model is isomorphic to the type model generated via Definition 5 by the state model which itself is generated via Definition 6 by the type model of departure. Secondly, it needs to be established whether a state model is isomorphic to the state model generated via Definition 6 by the type model which itself is generated via Definition 5 by the state model of departure.

For the epistemic framework of type models the notion of isomorphism can be spelled out as follows.

Definition 7. Let Γ be a game, and $\langle (T_i, b_i)_{i\in I}\rangle$ as well as $\langle \left(\tilde{T}_i, \tilde{b}_i\right)_{i\in I}\rangle$ be type models of Γ. The type models $\langle (T_i, b_i)_{i\in I}\rangle$ and $\langle \left(\tilde{T}_i, \tilde{b}_i\right)_{i\in I}\rangle$ are isomorphic, if for all $i \in I$ there exists a bijection $f_i : T_i \to \tilde{T}_i$ such that

$$\tilde{b}_i[f_i(t_i)](c_{-i}, f_{-i}(t_{-i})) = b_i[t_i](c_{-i}, t_{-i})$$

for all $(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}$ and for all $t_i \in T_i$.

Intuitively, in two isomorphic type models the same belief hierarchies are present—in fact only their labels differ—and thus the described interactive thinking is alike. The bijection in Definition 7 is essentially equivalent to the notion of type isomorphism due to Heifetz and Samet (1998, Definition 3.2).

Take some type model $\mathcal{T}^\Gamma = \langle (T_i, b_i)_{i\in I}\rangle$ as input and construct a type model $\hat{\mathcal{T}}^\Gamma = \langle \left(\hat{T}_i, \hat{b}_i\right)_{i\in I}\rangle$ as output by first applying Definition 6 to $\mathcal{T}^\Gamma$ and then Definition 5 to the $\mathcal{T}^\Gamma$-generated state model. It turns out that the isomorphic relationship does actually not always hold between such input and output type models. To see this consider the following example.

Example 5. Let Γ be a game with $I = \{Alice, Bob\}$, $C_{Alice} = \{a\}$ as well as $C_{Bob} = \{b, c\}$, and $\mathcal{T}^\Gamma$ a type model of Γ with

- $T_{Alice} = \{t_{Alice}\}$,
- $T_{Bob} = \{t_{Bob}\}$,
- $b_{Alice}[t_{Alice}](b, t_{Bob}) = \frac{1}{2}$ and $b_{Alice}[t_{Alice}](c, t_{Bob}) = \frac{1}{2}$,
- $b_{Bob}[t_{Bob}](a, t_{Alice}) = 1$.

Then, $\langle \Omega, \mathcal{I}_{Alice}, \sigma_{Alice}, \pi_{Alice}, \mathcal{I}_{Bob}, \sigma_{Bob}, \pi_{Bob}\rangle$ with

- $\Omega = \left\{\omega^{(a,t_{Alice},b,t_{Bob})}, \omega^{(a,t_{Alice},c,t_{Bob})}\right\}$,
- $\mathcal{I}_{Alice} = \{\Omega\}$,
- $\mathcal{I}_{Bob} = \left\{\left\{\omega^{(a,t_{Alice},b,t_{Bob})}\right\}, \left\{\omega^{(a,t_{Alice},c,t_{Bob})}\right\}\right\}$,
- $\sigma_{Alice}\left(\omega^{(a,t_{Alice},b,t_{Bob})}\right) = \sigma_{Alice}\left(\omega^{(a,t_{Alice},c,t_{Bob})}\right) = a$,
- $\sigma_{Bob}\left(\omega^{(a,t_{Alice},b,t_{Bob})}\right) = b$ as well as $\sigma_{Bob}\left(\omega^{(a,t_{Alice},c,t_{Bob})}\right) = c$,
- and $\pi_{Alice}(\omega) = \pi_{Bob}(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$

forms a $\mathcal{T}^\Gamma$-induced state model of Γ. The $\langle \Omega, \mathcal{I}_{Alice}, \sigma_{Alice}, \pi_{Alice}, \mathcal{I}_{Bob}, \sigma_{Bob}, \pi_{Bob}\rangle$-induced type model of Γ is given by $\hat{\mathcal{T}}^\Gamma = \langle \hat{T}_{Alice}, \hat{b}_{Alice}, \hat{T}_{Bob}, \hat{b}_{Bob}\rangle$ with

- $\hat{T}_{Alice} = \{t^{\Omega}_{Alice}\}$ and $T_{Bob} = \left\{t_{Bob}^{\left\{\omega^{(a,t_{Alice},b,t_{Bob})}\right\}}, t_{Bob}^{\left\{\omega^{(a,t_{Alice},c,t_{Bob})}\right\}}\right\}$,

– $\hat{b}_{Alice}\left[t^{\Omega}_{Alice}\right]\left(b, t_{Bob}^{\left\{\omega^{(a,t_{Alice},b,t_{Bob})}\right\}}\right)$

$$= \sum_{\omega\in\Omega:\sigma_{Bob}(\omega)=b,\mathcal{I}_{Bob}(\omega)=\left\{\omega^{(a,t_{Alice},b,t_{Bob})}\right\}} \pi_{Alice}(\omega \mid \{\Omega\}) = \frac{1}{2}$$

as well as $\hat{b}_{Alice}\left[t^{\Omega}_{Alice}\right]\left(c, t_{Bob}^{\left\{\omega^{(a,t_{Alice},c,t_{Bob})}\right\}}\right)$

$$= \sum_{\omega\in\Omega:\sigma_{Bob}(\omega)=c,\mathcal{I}_{Bob}(\omega)=\left\{\omega^{(a,t_{Alice},c,t_{Bob})}\right\}} \pi_{Alice}(\omega \mid \{\Omega\}) = \frac{1}{2},$$

– $\hat{b}_{Bob}\left[t_{Bob}^{\left\{\omega^{(a,t_{Alice},b,t_{Bob})}\right\}}\right]\left(a, t^{\Omega}_{Alice}\right)$

$$= \sum_{\omega\in\Omega:\sigma_{Alice}(\omega)=a,\mathcal{I}_{Alice}=\{\Omega\}} \pi_{Bob}\left(\omega \mid \left\{\omega^{(a,t_{Alice},b,t_{Bob})}\right\}\right) = 1,$$

– $\hat{b}_{Bob}\left[t_{Bob}^{\left\{\omega^{(a,t_{Alice},c,t_{Bob})}\right\}}\right]\left(a, t^{\Omega}_{Alice}\right)$

$$= \sum_{\omega\in\Omega:\sigma_{Alice}(\omega)=a,\mathcal{I}_{Alice}=\{\Omega\}} \pi_{Bob}\left(\omega \mid \left\{\omega^{(a,t_{Alice},c,t_{Bob})}\right\}\right) = 1.$$

Since $|T_{Bob}| < |\hat{T}_{Bob}|$, there does not exist a bijection $f_{Bob} : T_{Bob} \to \hat{T}_{Bob}$ and consequently $\mathcal{T}^{\Gamma}$ and $\hat{\mathcal{T}}^{\Gamma}$ are not isomorphic. ♣

In the preceding example the input type model only contains one type for *Bob*, yet there are two cells for him in the generated state model, which in turn imply two corresponding types in its induced type model. It thus becomes impossible to construct a bijection between the two type models. However, one of *Bob*'s two types in the output type model is superfluous in the sense of interactive thinking, as it encodes precisely the same belief hierarchy as the other type.

To remove any superflous ingredients from type models we now introduce the idea of reduction.

Definition 8. Let Γ be a game, and $\langle(T_i, b_i)_{i\in I}\rangle$ as well as $\langle(\tilde{T}_i, \tilde{b}_i)_{i\in I}\rangle$ type models of Γ.

(a) The type model is $\langle(\tilde{T}_i, \tilde{b}_i)_{i\in I}\rangle$ is a reduction of the type model $\langle(T_i, b_i)_{i\in I}\rangle$, if for every player $i \in I$ there exists a reduction function $r_i : T_i \to \tilde{T}_i$ such that r_i is surjective and

$$\tilde{b}_i(r_i(t_i))\left(\left(c_j, \tilde{t}_j\right)_{j\in I\{i\}}\right) = b_i(t_i)\left(\left(\left\{c_j\right\} \times r_j^{-1}\left(\tilde{t}_j\right)\right)_{j\in I\setminus\{i\}}\right) \quad (1)$$

for all $\left(c_j, \tilde{t}_j\right)_{j\in I\setminus\{i\}} \in \times_{j\in I\setminus\{i\}}\left(C_j, \tilde{T}_j\right)$ and for all $t_i \in T_i$.

(b) The type model $\langle(\tilde{T}_i, \tilde{b}_i)_{i\in I}\rangle$ is a strict reduction of the type model $\langle(T_i, b_i)_{i\in I}\rangle$, if $\langle(\tilde{T}_i, \tilde{b}_i)_{i\in I}\rangle$ is a reduction of $\langle(T_i, b_i)_{i\in I}\rangle$ and $|\tilde{T}_j| < |T_j|$ for some $j \in I$.
(c) The type model $\langle(\tilde{T}_i, \tilde{b}_i)_{i\in I}\rangle$ is a maximal reduction of the type model $\langle(T_i, b_i)_{i\in I}\rangle$, if $\langle(\tilde{T}_i, \tilde{b}_i)_{i\in I}\rangle$ is a reduction of $\langle(T_i, b_i)_{i\in I}\rangle$ and there exists no strict reduction of $\langle(\tilde{T}_i, \tilde{b}_i)_{i\in I}\rangle$.

Note that the reduction functions r_i for all $i \in I$ correspond to surjective type morphisms of Heifetz and Samet (1998, Definition 3.2).

A couple of preparatory results about reduced type models are established next.

An importantly property of reductions is that they actually preserve belief hierarchies. This can be formalized as follows.

Lemma 1. *Let Γ be a game, $\mathcal{T}^\Gamma$ a type model of Γ, $\tilde{\mathcal{T}}^\Gamma$ a reduction of $\mathcal{T}^\Gamma$ with reduction function $r_j : T_j \to \tilde{T}_j$ for every player $j \in I$, and $i \in I$ some player. Then, $\eta_i[t_i] = \eta_i[r_i(t_i)]$ for all $t_i \in T_i$.*

Proof. It is shown inductively that $\eta_i[t_i]^k = \eta_i[r_i(t_i)]^k$ holds for all $t_i \in T_i$, for all $i \in I$, and for all $k \geq 1$. It then directly follows that $\eta_i[t_i] = \left(\eta_i^n[t_i]\right)_{n\in\mathbb{N}} = \left(\eta_i^n[r_i(t_i)]\right)_{n\in\mathbb{N}}\eta_i[r_i(t_i)]$ for all $t_i \in T_i$ and for all $i \in I$.

Let $k = 1$ and consider some player $i \in I$, some type $t_i \in T_i$ of player i, as well as some opponents' choice combination $c_{-i} \in C_{-i}$. By definition,

$$\eta_i^1[t_i](c_{-i}) = \sum_{t_{-i}\in T_{-i}} b_i[t_i](c_{-i}, t_{-i}).$$

Moreover, as

$$\tilde{b}_i[r_i(t_i)](c_{-i}, \tilde{t}_{-i}) = \sum_{t_{-i}\in T_{-i}:r_{-i}(t_{-i})=\tilde{t}_{-i}} b_i[t_i](c_{-i}, t_{-i}),$$

it follows that

$$\eta_i^1[r_i(t_i)](c_{-i}) = \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}} \tilde{b}_i[r_i(t_i)](c_{-i}, \tilde{t}_{-i})$$

$$= \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}} \sum_{t_{-i}\in T_{-i}:r_{-i}(t_{-i})=\tilde{t}_{-i}} b_i[t_i](c_{-i}, t_{-i}) = \sum_{t_{-i}\in T_{-i}} b_i[t_i](c_{-i}, t_{-i}) = \eta_i^1[t_i](c_{-i}).$$

Let $k \geq 2$ and assume that $\eta_i[t_i]^l = \eta_i[r_i(t_i)]^l$ holds for all $t_i \in T_i$, for all $i \in I$, and for all $l \leq k-1$. Consider some player $i \in I$, some type $t_i \in T_i$ of player i, and some tuple $\left(c_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^{k-1}\right) \in X_{-i}^k$. By definition,

$$\eta_i^k[t_i]\left(c_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^{k-1}\right) = \sum_{t_{-i}\in T_{-i}:\eta_{-i}^l[t_{-i}]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[t_i](c_{-i}, t_{-i}).$$

Consequently,

$$\eta_i^k[r_i(t_i)] = \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}:\eta_{-i}^l\left[\tilde{t}_{-i}\right]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[r_i(t_i)](c_{-i}, \tilde{t}_{-i})$$

$$= \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}:\eta_{-i}^l\left[\tilde{t}_{-i}\right]=\eta_{-i}^l \text{ for all } l\leq k-1} \sum_{t_{-i}\in T_{-i}:r_{-i}(t_{-i})=\tilde{t}_{-i}} b_i[t_i](c_{-i}, t_{-i})$$

$$= \sum_{t_{-i}\in T_{-i}:\eta_{-i}^l[r_{-i}(t_{-i})]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[t_i](c_{-i}, t_{-i})$$

$$= \sum_{t_{-i}\in T_{-i}:\eta_{-i}^l[t_{-i}]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[t_i](c_{-i}, t_{-i})$$

$$= \eta_i^k[t_i]\left(c_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^{k-1}\right),$$

where the fourth equality follows from the inductive hypothesis. ■

Thus, type models are structurally equivalent to their reduced counterparts. No essential information is lost and the same interactive reasoning is represented. Note that Lemma 1 follows from Heifetz and Samet (1998, Proposition 5.1). However, since their formal framework is slightly different and to keep our paper self-contained, we still provide a direct proof.

A further significant feature of reductions is that any two maximally reduced type models only contain distinct belief hierarchies. This is substantiated by the following result.

Lemma 2. *Let Γ be a game, $\mathcal{T}^{\Gamma}$ a type model of Γ such that there exists no strict reduction of $\mathcal{T}^{\Gamma}$, and $i \in I$ some player. Then, $\eta_i[t_i'] \neq \eta_i[t_i'']$ for all $t_i', t_i'' \in T_i$ such that $t_i' \neq t_i''$.*

Proof. By contraposition, suppose that there exist $t_i', t_i'' \in T_i$ such that $t_i' \neq t_i''$ and $\eta_i[t_i'] = \eta_i[t_i'']$. For every player $j \in I$ recall the set $H_j[\mathcal{T}^{\Gamma}] := \left\{\eta_j \in \times_{n \in \mathbb{N}} \Delta\left(X_j^n\right) : \text{There exists } t_j \in T_j \text{ such that } \eta_j[t_j] = \eta_j\right\}$ of induced belief hierarchies in the type model $\mathcal{T}^{\Gamma}$. Construct a type model $\tilde{\mathcal{T}}^{\Gamma} = \langle(\tilde{T}_i, \tilde{b}_i)_{j \in I}\rangle$ where $\tilde{T}_j := H_j[\mathcal{T}^{\Gamma}]$ for every player $j \in I$ and

$$\tilde{b}_j[h_j](c_{-j}, h_{-j}) := \sum_{t_{-j} \in T_{-j} : \eta_{-j}[t_{-j}] = h_{-j}} b_j[t_j](c_{-j}, t_{-j}) \tag{2}$$

such that $\eta_j[t_j] = h_j$, for all $(c_{-j}, h_{-j}) \in C_{-j} \times \tilde{T}_{-j}$, for all $h_j \in \tilde{T}_j$, and for all $j \in I$. Observe that the belief functions are well-defined, since every two types $t_j, t_j' \in T_j$ such that $\eta_j[t_j] = \eta_j[t_j']$ satisfy

$$\sum_{t_{-j} \in T_{-j} : \eta_{-j}[t_{-j}] = h_{-j}} b_j[t_j](c_{-j}, t_{-j}) = \sum_{t_{-j} \in T_{-j} : \eta_{-j}[t_{-j}] = h_{-j}} b_j[t_j'](c_{-j}, t_{-j})$$

for all $(c_{-j}, t_{-j}) \in C_{-j} \times T_{-j}$ and for all $h_{-j} \in \tilde{T}_{-j}$.

For every player $j \in I$ define a surjection $r_j : T_j \to \tilde{T}_j$ such that

$$r_j(t_j) := \eta_j[t_j] \tag{3}$$

for all $t_j \in T_j$. By Equations (2) and (3) it follows that

$$\tilde{b}_j[r_j(t_j)](c_{-j}, \tilde{t}_{-j}) = b_j[t_j]\left(\{c_{-j}\} \times r_{-j}^{-1}(\tilde{t}_{-j})\right)$$

for all $(c_{-j}, \tilde{t}_{-j}) \in C_{-j} \times \tilde{T}_{-j}$, for all $t_j \in T_j$, and for all $j \in I$. Consequently, $\tilde{\mathcal{T}}^{\Gamma}$ constitutes a reduction of $\mathcal{T}^{\Gamma}$. Since $\eta_i[t_i'] = \eta_i[t_i'']$, it is the case that $|\tilde{T}_i| = |H_i[\mathcal{T}^{\Gamma}]| < |T_i|$ for player i. Therefore, $\tilde{\mathcal{T}}^{\Gamma}$ actually is a strict reduction of $\mathcal{T}^{\Gamma}$. ■

Accordingly, any two different types in an epistemic model without strict reduction possibilities induce distinct belief hierarchies. In this sense, maximally reduced type models do not carry any superfluous ingredients.

If the input type model and output type model of the successive application of the two transformation procedures are considered in their maximally reduced form, then an isomorphism does emerge between the input and output type models.

Theorem 6. *Let Γ be a game, $\mathcal{T}^{\Gamma}$ a type model of Γ, and $\hat{\mathcal{T}}^{\Gamma}$ the type model of Γ generated by a $\mathcal{T}^{\Gamma}$-generated state model. Then, every maximal reduction of $\mathcal{T}^{\Gamma}$ is isomorphic to every maximal reduction of $\hat{\mathcal{T}}^{\Gamma}$.*

Proof. Let $i \in I$ be a player and note that the set $\hat{T}_i$ from $\hat{\mathcal{T}}^{\Gamma}$ can be expressed as $\left\{\hat{t}_i^{P_i^{(c_i, t_i)}} : t_i \in T_i, c_i \in C_i\right\}$, where T_i belongs to $\mathcal{T}^{\Gamma}$. Construct a correspondence $e_i : T_i \to \hat{T}_i$ such that

$$e_i(t_i) := \left\{ \hat{t}_i^{P_i^{(c_i,t_i)}} : c_i \in C_i \right\}$$

for all $t_i \in T_i$. Thus, e_i maps i's types from the initial input model to the respective types in the output model. Hence, by construction, $\hat{T}_i = \cup_{t_i \in T_i} e_i(t_i)$. By Theorems 1 and 3 it follows that every $t_i \in T_i$ and every $\hat{t}_i \in \hat{T}_i$ such that $\hat{t}_i \in e_i(t_i)$ induce the same belief hierarchy, that is, $\eta_i[t_i] = \eta_i[\hat{t}_i]$. Consequently, for every player $i \in I$ there exists a collection of belief hierarchies $H_i \subseteq \times_{n \in \mathbb{N}} \left(\Delta\left(X_i^n\right)\right)$ such that

$$H_i\left[\mathcal{T}^\Gamma\right] = H_i\left[\hat{\mathcal{T}^\Gamma}\right] = H_i. \tag{4}$$

Let $\mathcal{T}_\downarrow^\Gamma = \langle (T_{\downarrow i}, b_{\downarrow i})_{i \in I} \rangle$ be a maximal reduction of $\mathcal{T}^\Gamma$ and $\hat{\mathcal{T}}_\downarrow^\Gamma = \langle \left(\hat{T}_{\downarrow i}, \hat{b}_{\downarrow i}\right)_{i \in I} \rangle$ a maximal reduction of $\hat{\mathcal{T}^\Gamma}$. By Lemma 1 and Equation (4) it follows that $H_i\left[\mathcal{T}_\downarrow^\Gamma\right] = H_i\left[\mathcal{T}^\Gamma\right] = H_i$ as well as $H_i\left[\hat{\mathcal{T}_\downarrow^\Gamma}\right] = H_i\left[\hat{\mathcal{T}^\Gamma}\right] = H_i$ for all $i \in i$. Moreover, Lemma 2 implies that two distinct types in $\mathcal{T}_\downarrow^\Gamma$ induce different belief hierarchies. The same holds for $\hat{\mathcal{T}_\downarrow^\Gamma}$. Consequently, for every player $i \in I$ and for every belief hierarchy $h_i \in H_i$ there exists a unique type $t_i \in T_{\downarrow i} \in T_{\downarrow i}$ and a unique type $\hat{t}_i \in \hat{T}_{\downarrow i}$ such that $\eta_i[t_i] = \eta_i\left[\hat{t}_i\right] = h_i$.

It follows that for every player $i \in I$ a bijection $f_i : T_{\downarrow i} \to \hat{T}_{\downarrow i}$ can be defined such that

$$\eta_i[t_i] = \eta_i[f_i(t_i)] \tag{5}$$

for all $t_i \in T_{\downarrow i}$. Besides, Equation (5) implies that $\hat{b}_{\downarrow i}[f_i(t_i)](c_{-i}, f(t_{-i})) = b_{\downarrow i}[t_i](c_{-i}, t_{-i})$ for all $(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}$ and for all $t_i \in T_{\downarrow i}$. Therefore, $T_{\downarrow i}$ and $\hat{T}_{\downarrow i}$ are isomorphic. ■

A type model can thus be said to be structurally equivalent to its two-fold transformed counterpart modulo superfluous ingredients.

An notion of isomorphism can also be laid out for the epistemic framework of state models.

Definition 9. Let Γ be a game, and $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ as well as $\langle \tilde{\Omega}, \left(\widetilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i \in I} \rangle$ state models of Γ. The state models $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ and $\langle \tilde{\Omega}, \left(\widetilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i \in I} \rangle$ are isomorphic, if there exists a bijection $f : \Omega \to \tilde{\Omega}$ such that for all $\omega \in \Omega$ and for all $i \in I$ it is the case that

$$\widetilde{\mathcal{I}}_i(f(\omega)) = \{f(\omega') : \omega' \in \mathcal{I}_i(\omega)\}, \tag{6}$$

$$\tilde{\pi}_i\big(f(\omega) \mid \hat{\mathcal{I}}_i(f(\omega))\big) = \pi_i(\omega \mid \mathcal{I}_i(\omega)), \tag{7}$$

$$\tilde{\sigma}_i(f(\omega)) = \sigma_i(\omega). \tag{8}$$

In two isomorphic state models the corresponding worlds induce the same information, posterior beliefs, and choices for all players. The subjective priors can be distinct yet the models qualify as isomorphic, because the players' belief hierarchies that is, their full interactive thinking are fixed by the posterior beliefs. Indeed, a difference in priors is not a relevant issue, as the posterior beliefs are the relevant doxastic mental configurations upon which the agents act. In that sense subjective prior beliefs could be viewed as artifacts of the state-based approach.

Take some state model $\mathcal{S}^\Gamma = \langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ as input and construct a state model $\hat{\mathcal{S}}^\Gamma = \langle \hat{\Omega}, (\hat{\mathcal{I}}_i, \hat{\sigma}_i, \hat{\pi}_i)_{i \in I} \rangle$ as output by first applying Definition 5 to $\mathcal{S}^\Gamma$ and then Definition 6 to the $\mathcal{S}^\Gamma$-generated type model. By counterexample it is now illustrated that such input and output state models are not necessarily isomorphic.

Example 6. Let Γ be a game with $I = \{Alice, Bob\}$, $C_{Alice} = \{a\}$ as well as $C_{Bob} = \{b\}$. Consider the state model $\mathcal{S}^\Gamma$ of Γ with

- $\Omega = \{\omega_1, \omega_2\}$,
- $\mathcal{I}_{Alice} = \mathcal{I}_{Bob} = \{\Omega\}$,
- $\sigma_{Alice}(\omega) = a$ and $\sigma_{Bob}(\omega) = b$ for all $\omega \in \Omega$,
- $\pi_{Alice}(\omega) = \pi_{Bob}(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$.

Then, $\mathcal{T}^\Gamma = \langle T_{Alice}, b_{Alice}, T_{Bob}, b_{Bob}\rangle$ with

- $T_{Alice} = \{t^\Omega_{Alice}\}$ and $T_{Bob} = \{t^\Omega_{Bob}\}$,
- $b_{Alice}[t^\Omega_{Alice}](b, t^\Omega_{Bob}) = \sum_{\omega\in\Omega:\sigma_{Bob}(\omega)=b,\mathcal{I}_{Bob}(\omega)=\{\Omega\}} \pi_{Alice}(\omega \mid \{\Omega\}) = 1$,
- $b_{Bob}[t^\Omega_{Bob}](a, t^\Omega_{Alice}) = \sum_{\omega\in\Omega:\sigma_{Alice}(\omega)=a,\mathcal{I}_{Alice}(\omega)=\{\Omega\}} \pi_{Bob}(\omega \mid \{\Omega\}) = 1$,

constitutes the $\mathcal{S}^\Gamma$-generated type model of Γ. However, it directly follows that $\hat{\mathcal{S}}^\Gamma$ with $\hat{\Omega} = \left\{\omega^{(a,t^\Omega_{Alice},b,t^\Omega_{Bob})}\right\}$ as the set of all possible worlds forms the unique $\mathcal{T}^\Gamma$-generated state model of Γ. Consequently, there exists no bijection $f : \Omega \to \hat{\Omega}$. The state models $\mathcal{S}^\Gamma$ and $\hat{\mathcal{S}}^\Gamma$ are therefore not isomorphic. ♣

The possible worlds ω_1 and ω_2 in the input state model $\mathcal{S}^\Gamma$ of the preceding example induce the same choices and beliefs for both players. With regards to interactive thinking one of them is thus superfluous. These kind of redundancies prevent the isomorphic relationship between input and output state models to hold in general.

We call a state model $\mathcal{S}^\Gamma$ of Γ *non-redundant*, if for all $\omega, \omega' \in \Omega$ such that $\omega \neq \omega'$ it is the case that $\mathcal{I}_i(\omega) \neq \mathcal{I}_i(\omega')$ or $\sigma_i(\omega) \neq \sigma_i(\omega')$ for some $i \in I$. Intuitively, any two distinct worlds in the structure carry some difference for at least one of the players. Observe that non-redundancy implies that $\cap_{i\in I}\mathcal{I}_i(\omega) = \{\omega\}$ for all $\omega \in \Omega$. Essentially, the latter says that if the players' information is pooled, then all uncertainty is resolved.

To get rid of any superfluous ingredients we also need a notion of reduction for state models in addition to non-redundancy.

Definition 10. Let Γ be a game, and $\langle\Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$ as well as $\langle\tilde{\Omega}, \left(\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i\in I}\rangle$ be state models of Γ.

(a) The state model $\langle\tilde{\Omega}, \left(\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i\in I}\rangle$ is a reduction of the state model $\langle\Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$, if there exists a reduction function $r : \Omega \to \tilde{\Omega}$ such that r is surjective and for all $i \in i$

$$\tilde{\mathcal{I}}_i(r(\omega)) = \{r(\omega') : \omega' \in \mathcal{I}_i(\omega)\} \text{ for all } \omega \in \Omega, \tag{9}$$

$$\tilde{\sigma}_i(r(\omega)) = \sigma_i(\omega) \text{ for all } \omega \in \Omega \text{ such that } \pi_j(\omega \mid \mathcal{I}_j(\omega)) > 0 \text{ for some } j \in I \backslash \{i\}, \tag{10}$$

$$\tilde{\pi}_i\left(\tilde{\omega} \mid \tilde{\mathcal{I}}_i(r(\omega))\right) = \pi_i\left(r^{-1}(\tilde{\omega}) \mid \mathcal{I}_i(\omega)\right) \text{ for all } \omega \in \Omega \text{ and for all } \tilde{\omega} \in \tilde{\Omega}. \tag{11}$$

(b) The state model $\langle\tilde{\Omega}, \left(\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i\in I}\rangle$ is a strict reduction of the state model $\langle\Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$, if $\langle\tilde{\Omega}, \left(\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i\in I}\rangle$ is a reduction of $\langle\Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$ and $|\tilde{\Omega}| < |\Omega|$.
(c) The state model $\langle\tilde{\Omega}, \left(\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i\in I}\rangle$ is a maximal reduction of the state model $\langle\Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$, if $\langle\tilde{\Omega}, \left(\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i\in I}\rangle$ is a reduction of $\langle\Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$ and there exists no strict reduction of $\langle\tilde{\Omega}, \left(\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i\right)_{i\in I}\rangle$.

An auxiliary result about reductions of state models is collected before the issue of isomorphic relationship between state models and their two-fold transformed counterparts is addressed.

Lemma 3. *Let Γ be a game, and $\mathcal{S}^\Gamma$ a state model of Γ. If there exists no strict reduction of $\mathcal{S}^\Gamma$, then $\mathcal{S}^\Gamma$ is non-redundant.*

Proof. We proceed by contraposition. Suppose that $\mathcal{S}^\Gamma$ is redundant. Then there exist distinct worlds $\omega', \omega'' \in \Omega$ such that $\mathcal{I}_i(\omega') = \mathcal{I}_i(\omega'')$ as well as $\sigma_i(\omega') = \sigma_i(\omega'')$ for every player $i \in I$. Construct a state model $\tilde{\mathcal{S}}^\Gamma$ of Γ as follows:

- $\tilde{\Omega} := \Omega \backslash \{\omega', \omega''\} \cup \{\omega^*\}$

and for every player $j \in I$,

- $\tilde{\mathcal{I}}_j(\omega^*) := \mathcal{I}_j(\omega')\backslash\{\omega', \omega''\} \cup \{\omega^*\}$,
- $\tilde{\mathcal{I}}_j(\omega) := \begin{cases} \mathcal{I}_j(\omega), & \text{if } \omega', \omega'' \notin \mathcal{I}_j(\omega), \\ \tilde{\mathcal{I}}_j(\omega^*), & \text{otherwise,} \end{cases}$ for all $\omega \in \tilde{\Omega}\backslash\{\omega^*\}$,
- $\tilde{\sigma}_j(\omega^*) = \sigma_j(\omega')$,
- $\tilde{\sigma}_j(\omega) = \sigma_j(\omega)$ for all $\omega \in \tilde{\Omega}\backslash\{\omega^*\}$,
- $\tilde{\pi}_j(\omega^*) = \pi_j(\omega') + \pi_j(\omega'')$,
- and $\tilde{\pi}_j(\omega) = \pi_j(\omega)$ for all $\omega \in \tilde{\Omega}\backslash\{\omega^*\}$.

Define a function $r : \Omega \to \tilde{\Omega}$ by $r(\omega') = r(\omega'') = \omega^*$ and $r(\omega) = \omega$ for all $\omega \in \Omega \backslash\{\omega', \omega''\}$. Observe that r is surjective and also satisfies conditions (9), (10), and (11). As $|\tilde{\Omega}| = |\Omega| - 1 < |\Omega|$, the state model $\tilde{\mathcal{S}}^\Gamma$ constitutes a strict reduction of $\mathcal{S}^\Gamma$. ■

Accordingly, maximal reduction in the sense of the impossibility of strict reduction implies non-redundancy.

By considering maximally reduced models, the existence of superfluous worlds such as in Example 6 is blocked and an isomorphic relationship between input and output state models ensues.

Theorem 7. *Let Γ be a game, $\mathcal{S}^\Gamma$ a state model of Γ, and $\hat{\mathcal{S}}^\Gamma$ a state model of Γ generated by the $\mathcal{S}^\Gamma$-generated type model. Then, every maximal reduction of $\mathcal{S}^\Gamma$ is isomorphic to every maximal reduction of $\hat{\mathcal{S}}^\Gamma$.*

Proof. Consider a maximal reduction $\mathcal{S}^\Gamma_\downarrow$ of $\mathcal{S}^\Gamma$ and a maximal reduction $\hat{\mathcal{S}}^\Gamma_\downarrow$ of $\hat{\mathcal{S}}^\Gamma$. The set $\hat{\Omega}$ from $\hat{\mathcal{S}}^\Gamma_\downarrow$ is a subset of $\left\{\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} : c_i \in C_i \text{ for all } i \in I, \omega \in \Omega\right\}$, which is from $\hat{\mathcal{S}}^\Gamma$, and where Ω and $\mathcal{I}_i$ for all $i \in I$ belong to $\mathcal{S}^\Gamma_\downarrow$. It is first shown that for every world $\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} \in \hat{\Omega}$, it is the case that $c_i = \sigma_i(\omega)$, where σ_i belongs to $\mathcal{S}^\Gamma_\downarrow$, for all $i \in I$. Toward a contradiction suppose that there exists a world $\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} \in \hat{\Omega}$ such that $c_j \neq \sigma_j(\omega)$ for some player $j \in I$. By definition of the two transformation procedures,

$$\hat{\pi}_k\left(\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} \mid \hat{\mathcal{I}}_k\left(\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}}\right)\right)$$

$$= b_k\left[t_k^{\mathcal{I}_k(\omega)}\right]\left(c_{-k}, t_{-k}^{\mathcal{I}_{-k}(\omega)}\right)$$

$$= \sum_{\omega' \in \mathcal{I}_k(\omega) : \sigma_{-k}(\omega') = c_{-k}, \mathcal{I}_{-k}(\omega') = \mathcal{I}_{-k}(\omega)} \pi_k(\omega' \mid \mathcal{I}_k(\omega))$$

for all $k \in I \backslash\{j\}$.Since $c_j \neq \sigma_j(\omega)$ the $\mathcal{I}_j$-measurability of σ_j implies that $\sigma_j(\omega'') \neq c_j$ for all $\omega'' \in \mathcal{I}_j(\omega)$. Consequently, there exists no world $\omega' \in \mathcal{I}_k(\omega)$ such that $\sigma_j(\omega') = c_j$ and $\mathcal{I}_j(\omega') = \mathcal{I}_j(\omega)$. It follows that $\pi_k(\omega' \mid \mathcal{I}_k(\omega)) = 0$ for all $\omega' \in \mathcal{I}_k(\omega)$ such that $\sigma_{-k}(\omega') = c_{-k}$ and $\mathcal{I}_{-k}(\omega') = \mathcal{I}_{-k}(\omega)$. Thus, $\hat{\pi}_k\left(\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} \mid \hat{\mathcal{I}}_k\left(\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}}\right)\right) = 0$ for all $k \in I \backslash\{j\}$. Next define a state model $\tilde{S}^\Gamma$ based on $\tilde{\Omega} := \left\{\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} : \omega \in \Omega\right\}$ as set of all possible worlds and a surjection $r : \hat{\Omega} \to \tilde{\Omega}$ with $r\left(\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}}\right) = \hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}}$ for all $\hat{\omega}^{\left(c_i, t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} \in \hat{\Omega}$ such that for all $i \in I$:

- $\widetilde{\mathcal{I}}_i(r(\hat{\omega})) := \left\{r(\hat{\omega}') : \hat{\omega}' \in \hat{\mathcal{I}}_i(\hat{\omega})\right\}$ for all $r(\hat{\omega}) \in \tilde{\Omega}$,
- $\tilde{\sigma}_i\left(\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}}\right) := \sigma_i(\omega)$ for all $\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} \in \tilde{\Omega}$,
- $\tilde{\pi}_i\left(\tilde{\omega} \mid \widetilde{\mathcal{I}}_i(r(\hat{\omega}))\right) := \hat{\pi}_i\left(r^{-1}(\tilde{\omega}) \mid \hat{\mathcal{I}}_i(\hat{\omega})\right)$ for all $\tilde{\omega} \in \tilde{\Omega}$ and for all $\hat{\omega} \in \hat{\Omega}$.Note that whenever $\hat{\pi}_j\left(\hat{\omega} \mid \hat{\mathcal{I}}_j(\hat{\omega})\right) > 0$ for some $j \in I \backslash\{i\}$, it is the case that $\hat{\omega} \in \tilde{\Omega}$ hence $\tilde{\sigma}_i(r(\hat{\omega})) = \hat{\sigma}_i(\hat{\omega}) = \sigma_i(\omega)$, and thus Equation (10) is satisfied. As $|\tilde{\Omega}| < |\hat{\Omega}|$, the state model $\tilde{S}^\Gamma$ forms a strict reduction of $\hat{\mathcal{S}}^\Gamma_\downarrow$, a contradiction.

Construct a function $f : \Omega \to \hat{\Omega}$ such that $f(\omega) := \hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}}$ for all $\omega \in \Omega$. The function f is surjective, as for every world $\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}} \in \hat{\Omega}$ the pre-image $f^{-1}\left(\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i\in I}}\right) \supseteq \{\omega\}$ contains $\{\omega\}$ and is thus non-empty by the successive application of the two transformation procedures, that is, by first applying Definition 5 to $\mathcal{S}^\Gamma_\downarrow$ and then Definition 6 to the $\mathcal{S}^\Gamma_\downarrow$-generated type model to induce $\hat{\mathcal{S}}^\Gamma_\downarrow$. Suppose that $f(\omega') = f(\omega'')$, that is, $\hat{\omega}^{\left(\sigma_i(\omega'),t_i^{\mathcal{I}_i(\omega')}\right)_{i\in I}} = \hat{\omega}^{\left(\sigma_i(\omega''),t_i^{\mathcal{I}_i(\omega'')}\right)_{i\in I}}$, for some worlds $\omega', \omega'' \in \Omega$. Then, $\sigma_i(\omega') = \sigma_i(\omega'')$ as well as $\mathcal{I}_i(\omega') = \mathcal{I}_i(\omega'')$ for all $i \in I$. As $\mathcal{S}^\Gamma_\downarrow$ is non-redundant by Lemma 3, it follows that $\omega' = \omega''$. Hence, f is injective too and thus bijective.

We now show that the bijection f satisfies Equations (6)–(8) of Definition 10. First, observe that

$$\hat{\mathcal{I}}_i(f(\omega)) = \hat{\mathcal{I}}_i\left(\hat{\omega}^{\left(\sigma_j(\omega),t_j^{\mathcal{I}_j(\omega)}\right)_{j\in I}}\right)$$

$$= \left\{\hat{\omega}^{\left(\sigma_j(\omega'),t_j^{\mathcal{I}_j(\omega')}\right)_{j\in I}} \in \hat{\Omega} : \sigma_i(\omega') = \sigma_i(\omega), \mathcal{I}_i(\omega') = \mathcal{I}_i(\omega)\right\}$$

$$= \{f(\omega') : \sigma_i(\omega') = \sigma_i(\omega), \mathcal{I}_i(\omega') = \mathcal{I}_i(\omega)\} = \{f(\omega') : \omega' \in \mathcal{I}_i(\omega)\}$$

for all $\omega \in \Omega$ and for all $i \in I$. Therefore, f satisfies Equation (6). Second, $\hat{\sigma}_i(f(\omega)) = \hat{\sigma}_i\left(\hat{\omega}^{\left(\sigma_j(\omega),t_j^{\mathcal{I}_j(\omega)}\right)_{j\in I}}\right) = \sigma_i(\omega)$ for all $\omega \in \Omega$ and for all $i \in I$. Hence, f satisfies Equation (7). Third,

$$\hat{\pi}_i\big(f(\omega) \mid \hat{\mathcal{I}}_i(f(\omega))\big) = \hat{\pi}_i\left(\hat{\omega}^{\left(\sigma_j(\omega),t_j^{\mathcal{I}_j(\omega)}\right)_{j\in I}} \mid \hat{\mathcal{I}}_i(f(\omega))\right)$$

$$\hat{\pi}_i\left(\hat{\omega}^{\left(\sigma_j(\omega),t_j^{\mathcal{I}_j(\omega)}\right)_{j\in I}} \mid \hat{\mathcal{I}}_i\left(\hat{\omega}^{\left(\sigma_j(\omega),t_j^{\mathcal{I}_j(\omega)}\right)_{j\in I}}\right)\right)$$

$$= b_i\left[t_i^{\mathcal{I}_i(\omega)}\right]\left(\sigma_{-i}(\omega), t_{-i}^{\mathcal{I}_i(\omega)}\right)$$

$$= \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = \sigma_{-i}(\omega), \mathcal{I}_{-i}(\omega') = \mathcal{I}_{-i}(\omega)} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

$$= \sum_{\omega' \in \Omega : \mathcal{I}_i(\omega') = \mathcal{I}_i(\omega), \sigma_i(\omega') = \sigma_i(\omega), \sigma_{-i}(\omega') = \sigma_{-i}(\omega), \mathcal{I}_{-i}(\omega') = \mathcal{I}_{-i}(\omega)} \pi_i(\omega' \mid \mathcal{I}_i(\omega))$$

$$= \pi_i\left(f^{-1}\left(\hat{\omega}^{\left(\sigma_j(\omega),t_j^{\mathcal{I}_j(\omega)}\right)_{j\in I}}\right) \mid \mathcal{I}_i(\omega)\right) = \pi_i(\omega) \mid \mathcal{I}_i(\omega))$$

for all $\omega \in \Omega$, and for all $i \in I$. Thus, f satisfies Equation (8).Consequently, $\mathcal{S}^\Gamma_\downarrow$ and $\hat{\mathcal{S}}^\Gamma_\downarrow$ are isomorphic. ■

Hence, a state model is structurally equivalent to its two-fold transformed counterpart modulo superfluous ingredients.

A notable consequence obtains from Theorem 7 in conjunction with Theorem 5.

Corollary 2. *Let Γ be a game such that $|I| \geq 2$ as well as $|C_i| \geq 2$ for all $i \in I$. Consider a tight and maximally reduced state model $\mathcal{S}^\Gamma$ of Γ with its $\mathcal{S}^\Gamma$-generated type model $\mathcal{T}^\Gamma$ of Γ. Then, every maximally reduced $\mathcal{T}^\Gamma$-generated state model of Γ is isomorphic to $\mathcal{S}^\Gamma$ and not tight.*

Thus, a tight state model can be isomorphic to a non-tight state model. This may seem somewhat surprising at first sight. However, isomorphisms and tightness concern distinct kinds of doxastic attitudes. While two models are isomorphic whenever they represent the same actual doxastic attitudes, tightness imposes conditions on hypothetical doxastic attitudes. In a way, isomorphism and tightness thus refer to orthogonal properties of state models.

7 | DISCUSSION

Belief hierarchies as well as the common prior assumption are structurally preserved across the two epistemic frameworks by our transformation procedures. With regards to modeling interactive thinking in games the state-based and type-based approaches can thus be viewed as equivalent. None of the two models contains any *relevant* structure that the respective other lacks. Our two transformation procedures can be viewed as practical tools to switch back and forth between state-based and type-based interactive epistemology.

A somewhat more subtle difference between the two epistemic approaches surfaces, as the transformation procedures fail to constitute inverses. The underlying reason is attributable to the richer structure of state models compared to type models. While the latter only specify interactive thinking the former also fixes the players' choices. Once superfluous ingredients are wiped out—technically, by restricting attention to maximally reduced models—our two transformation procedures turn out to be inverse to each other. Besides, the notions of maximal reduction for state and type models can also serve to simplify a given epistemic structure while retaining the same interactive thinking in applications.

While this "isomorphic" disparity between the state and type models does not make a difference with respect to interactive thinking at all, the particular usage could determine which epistemic apparatus is more appropriate. The possible benefits of each of the two modeling approaches to interactive thinking can be illustrated with a simple game-theoretic example. Consider the two player game depicted in Figure 1 with players *Alice* and *Bob*, where *Alice* chooses a "row" (a or b) and *Bob* picks a "column" (c or d).

The solution concept of Nash equilibrium specifies for every player a probability distribution on his own choice set such that only best responses receive positive probability against the product of the opponents' probability distributions. The probability distribution tuple $\sigma = (\sigma_{Alice}, \sigma_{Bob})$, where $\sigma_{Alice}(b) = 1$ and $\sigma_{Bob}(c) = \sigma_{Bob}(d) = \frac{1}{2}$, forms a Nash equilibrium of the game in Figure 1, as b is a best response against σ_{Bob} and c as well as d are best responses against σ_{Alice}. In terms of reasoning this Nash equilibrium corresponds to the two belief hierarchies generated entirely by σ, that is,

- *Alice* believes with probability $\frac{1}{2}$ that *Bob* chooses c and with probability $\frac{1}{2}$ that *Bob* chooses d,
- *Alice* believes that *Bob* believes that she chooses b,
- *Alice* believes that *Bob* believes that she believes with probability $\frac{1}{2}$ that *Bob* chooses c and with probability $\frac{1}{2}$ that *Bob* chooses d,
- etc.

and

- *Bob* believes that *Alice* chooses b,
- *Bob* believes that *Alice* believes with probability $\frac{1}{2}$ that *Bob* chooses c and with probability $\frac{1}{2}$ that *Bob* chooses d,
- *Bob* believes that *Alice* believes that he believes that she chooses b,
- etc.

In the epistemic game-theoretic literature such belief hierarchies are called simple (Perea, 2012). These reasoning patterns underlying the Nash equilibrium σ can be modeled by either of the two epistemic approaches.

The type model $\langle (T_i, b_i)_{i\in I}\rangle$, where

	c	d
a	$2,0$	$0,1$
b	$1,0$	$1,0$

FIGURE 1 A two player game.

- $T_{Alice} = \{t_{Alice}\}$ and $T_{Bob} = \{t_{Bob}\}$,
- $b_{Alice}[t_{Alice}](c, t_{Bob}) = b_{Alice}[t_{Alice}](d, t_{Bob}) = \frac{1}{2}$,
- $b_{Bob}[t_{Bob}](b, t_{Alice}) = 1$,

formalizes the above two belief hierarchies. The structure is parsimonious, as it merely contains the absolute essentials to describe the belief hierarchies. Such an approach seems appropriate, if the only interest lies in the agent's interactive reasoning.

The state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ where

- $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$,
- $\mathcal{I}_{Alice} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$,
- $\mathcal{I}_{Bob} = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$,
- $\sigma_{Alice}(\omega_1) = \sigma_{Alice}(\omega_2) = b$ and $\sigma_{Alice}(\omega_3) = \sigma_{Alice}(\omega_4) = a$,
- $\sigma_{Bob}(\omega_1) = \sigma_{Bob}(\omega_3) = c$ and $\sigma_{Bob}(\omega_2) = \sigma_{Bob}(\omega_4) = d$,
- $\pi_{Alice}(\omega_1) = \pi_{Alice}(\omega_2) = \pi_{Alice}(\omega_3) = \pi_{Alice}(\omega_4) = \frac{1}{4}$,
- $\pi_{Bob}(\omega_1) = \pi_{Bob}(\omega_2) = \frac{1}{2}$ and $\pi_{Bob}(\omega_3) = \pi_{Bob}(\omega_4) = 0$,

also formalizes the above two belief hierarchies. While the state model is somewhat more involved than the type model, it allows for analyzing situations in the sense of interactive thinking *and* choice. In particular, the state model embeds two situations—the two distinct cells for *Alice*—with the same belief hierarchy for *Alice* yet distinct (and optimal) choices for her. A state-based approach seems appropriate, if the emphasis lies not only on the agents' interactive reasoning but also on their actual choices.

To conclude, if the focus is put on reasoning in games before decisions are made or the perspective of a particular player is considered, then type models may be more suitable. In contrast for analyses that are conducted from the perspective of a modeler the state-based framework could be preferable. After all there remains a degree of subjectivity whether the specification of beliefs only *or* beliefs and behavior is desired in an epistemic framework for games.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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ENDNOTES

[1] Intuitively, an event E is common knowledge, whenever all agents know E, all agents know that all agents know E, all agents know that all agents know that all agents know E, etc. The so-called iterative definition of common knowledge captures this intuition formally. It can be shown that common knowledge defined in terms of the meet is equivalent to the iterative definition (e.g., Aumann, 1976; Bach & Cabessa, 2017, Lemma 1; Tóbiás, 2021, Proposition 2).

[2] Of course this is not the only prior in line with our transformation procedure. In fact, any probability distribution on Ω also satisfying the four conditions would also work.

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