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EXPOSITA NOTE

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Revision of conjectures about the opponent's utilities in signaling games

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Abstract In this paper we apply the concept of preference conjecture equilibrium introduced in Perea (2005) to signaling games and show its relation to sequential equilibrium. We introduce the concept of minimum revision equilibrium and show how this can be interpreted as a refinement of sequential equilibrium.

Keywords Signaling games · Preference conjecture equilibrium · Utility revision

JEL Classification Numbers C72

1 Introduction

In this paper we deal with the question how a receiver in a signaling game should react if he observes an unexpected message. In the concept of sequential equilibrium (Kreps and Wilson 1982) this is dealt with by requiring that the receiver has beliefs on information sets that are not reached in equilibrium and that he decides optimally given these beliefs. However, in signaling games sequential equilibrium does not put any further restrictions on these beliefs. In order to make the concept more powerful, several refinements were introduced in literature, such as perfect sequential equilibrium (Grossman and Perry 1986), the intuitive criterion (Cho and Kreps 1987) and divine equilibrium (Banks and Sobel 1987). In all these refinements the idea is that player 2, upon observing an unexpected message, makes a distinction between "less plausible" and "more plausible" types, and attaches positive probability only to the more plausible types. Throughout this reasoning process the utility functions are assumed to be fixed, which implies that player 2

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does not revise his beliefs regarding player 1's payoffs but only determines the probabilities of the nodes in the current information set.

We will follow an alternative path, namely to insist on a player's belief of having a rational opponent. This leads us to assume that player 2 has a conjecture about his opponent's utility function, which he may revise after observing player 1's message. This revision should be such that the observed message becomes optimal for player 1. This way we define the new concept of preference conjecture equilibrium, first formalized in Perea (2005). We show that for signaling games the predictions made by this concept coincide with sequential equilibrium. Thus, our alternative path can be seen as an alternative foundation for sequential equilibrium.

Next we impose the condition that the revisions should be as limited as possible, which means that the revised conjectures should be as close as possible to the initial conjecture. This leads to the concept of *minimum revision equilibrium*, a refinement of preference conjecture equilibrium – and hence of sequential equilibrium – that imposes further restrictions on the belief revisions based on the *revision index*, a measure for the number of revisions required.

2 Preliminaries

For a finite set Q, $\Delta(Q)$ denotes the set of all probability distributions over Q, and $\Delta^0(Q)$ denotes the set of completely mixed probability distributions over Q.

Definition 2.1 *A signaling game is a tuple $\mathcal{S} = (T, M, A, p, u_1, u_2)$ where T, M and A are finite sets, p is an element of $\Delta^0(T)$, and u_1 and u_2 are functions from $T \times M \times A$ to $\mathbb{R}$.*

The game is played as follows. First Nature selects type $t \in T$ of player 1 with probability $p(t) > 0$. Next player 1, knowing his type, chooses a message $m \in M$. Then player 2, only knowing the probability distribution p, observes message m and subsequently chooses an action $a \in A$. Finally, player 1 receives payoff $u_1(t, m, a)$ and player 2 receives $u_2(t, m, a)$.

2.1 Sequential equilibrium

A pure strategy for player 1 is a map s_1 from T to M that specifies a message $s_1(t)$ for every type t. The set of pure strategies for player 1 is denoted by S_1. A mixed strategy for player 1 is an element σ_1 in $\Delta(S_1)$ and $\sigma_1(s_1)$ denotes the probability that pure strategy s_1 is played in σ_1. A pure strategy for player 2 is a map s_2 from M to A that specifies an action $s_2(m)$ for every message m. The set of pure strategies for player 2 is denoted by S_2, a mixed strategy for player 2 is an element σ_2 in $\Delta(S_2)$ and $\sigma_2(s_2)$ denotes the probability that s_2 is played in σ_2. A system of beliefs of player 2 is a vector $\beta := (\beta(m))_{m \in M}$. In this notation we have $\beta(m) := (\beta(t \mid m))_{t \in T}$ for each message m, and $\beta(t \mid m)$ is the probability player 2 attaches to player 1 being of type t given that player 2 received message m.

A triple $(\sigma_1, \sigma_2, \beta)$ is called an assessment. Such an assessment is called a sequential equilibrium if the following three conditions are satisfied. The belief β

should be *Bayesian consistent* with σ_1,[1] meaning that for all t, and all m that are sent with positive probability under σ_1,

$$\beta(t \mid m) = \frac{p(t) \cdot \sigma_1(m \mid t)}{\sum_{t' \in T} p(t') \cdot \sigma_1(m \mid t')}.$$

Here $\sigma_1(m \mid t) := \sum_{s_1 \in S_1 : s_1(t) = m} \sigma_1(s_1)$ is the probability that player 1 plays m when he is of type t. Secondly, the expected utility

$$U_1(t, \sigma_1, \sigma_2) := \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdot u_1(t, s_1(t), s_2(s_1(t)))$$

for player 1 if his type is t, he plays σ_1 and player 2 plays σ_2 should be maximal. In other words, given the strategy σ_2 of player 2, it should hold that $U_1(t, \sigma_1, \sigma_2) \geq U_1(t, \tau_1, \sigma_2)$ for all mixed strategies τ_1 and all types t. In this case we say that σ_1 is optimal with respect to σ_2. Thirdly, the expected payoff

$$U_2(m, \sigma_2, \beta) := \sum_{s_2 \in S_2} \sum_{t \in T} \sigma_2(s_2) \cdot \beta(t \mid m) \cdot u_2(t, m, s_2(m))$$

for player 2 of playing σ_2, if he observes message m and has belief β, should be maximal. Again, in other words, $U_2(m, \sigma_2, \beta) \geq U_2(m, \tau_2, \beta)$ for all mixed strategies τ_2 and all messages m. We say that σ_2 is optimal with respect to β in this case.

Definition 2.2 *The assessment* $(\sigma_1, \sigma_2, \beta)$ *is a sequential equilibrium if* β *is Bayesian consistent with* σ_1, σ_1 *is optimal with respect to* σ_2 *and* σ_2 *is optimal with respect to* β.[2]

2.2 Preference conjecture equilibrium

The concept of *preference conjecture equilibrium* was introduced in Perea (2005) for games in extensive form. In the present paper we study this concept for signaling games.[3] For a detailed discussion of this concept we refer to Perea (2005).

Let $\mu_{12} \in \Delta(S_2)$ be the conjecture that player 1 has at the beginning of the game about player 2's choice of strategy and let $u_{12} : T \times M \times A \to \mathbb{R}$ be player 1's conjecture about player 2's utility function. Write $c_{12} = (\mu_{12}, u_{12})$ for the entire conjecture of player 1. Since player 1 moves first and only once, he does not need to revise his conjecture. Player 2 holds conjectures both at the start of the game as well as at every information set $m \in M$. Depending on the observed message, player 2 may revise the conjecture held at the beginning. Denote the start of the game by the symbol h_0. We write $S_1(h_0) := S_1$, and for $m \in M$ we write

$$S_1(m) := \{s_1 \in S_1 \mid \text{there exists } t \in T \text{ with } s_1(t) = m\}.$$

[1] In signaling games, Bayesian consistency is equivalent to consistency as defined in Kreps and Wilson (1982).

[2] In the literature this form of optimality is also called sequential rationality, see Kreps and Wilson (1982).

[3] In Perea (2005) the variant of preference conjecture equilibrium that we discuss here is called weak 'preference conjecture equilibrium'.

Take an $m \in M^* := M \cup \{h_0\}$. Let $\mu_{21}(m) \in \Delta(S_1(m))$ denote player 2's conjecture at m about player 1's strategy choice and let $u_{21}(m) : T \times M \times A \to \mathbb{R}$ be player 2's conjecture at m about player 1's utility function. We write $c_{21}(m) = (\mu_{21}(m), u_{21}(m))$.

Definition 2.3 *The combination $c := (c_{12}, (c_{21}(m))_{m \in M^*})$ is called a conjecture profile.*

We introduce the optimality conditions a conjecture profile has to satisfy in order to be a preference conjecture equilibrium. We start with the condition that whenever possible player 2 revises his conjecture according to Bayes' rule. A conjecture profile $c := (c_{12}, (c_{21}(m))_{m \in M^*})$ satisfies Bayesian updating if

$$\mu_{21}(m)(s_1) = \frac{\mu_{21}(h_0)(s_1)}{\sum_{s_1' \in S_1(m)} \mu_{21}(h_0)(s_1')}$$

for all $s_1 \in S_1(m)$, and all $m \in M$ for which the denominator is strictly positive.

In order to enable players to compute conjectured expected utilities for their opponent, we make the (informal) assumption that conjectures are common belief. In particular, we assume that player 2's (undefined) conjecture about player 1's conjecture about player 2's choice of strategy coincides with player 1's conjecture about player 2's choice of strategy, and vice versa.

Given this assumption, consider an $m \in M^*$. The conjecture of player 2 at m about player 1's expected utility if player 1 has type t and plays pure strategy s_1 must now be equal to $U_{21}(m)(t, s_1, \mu_{12})$, where $U_{21}(m)$ is the expected payoff function given payoff function $u_{21}(m)$. A pure strategy s_1 of player 1 is optimal with respect to μ_{12} and $u_{21}(m)$ if for each type t and for all $s_1' \in S_1$ it holds that

$$U_{21}(m)(t, s_1, \mu_{12}) \geq U_{21}(m)(t, s_1', \mu_{12}).$$

The conjecture of player 1 regarding the expected utility of player 2 at information set $m \in M$ (so $m \neq h_0$) when he plays pure strategy s_2 is given by

$$U_{12}(m, \mu_{21}(m), s_2) := \sum_{t \in T} p(t) \sum_{s_1 : s_1(t) = m} \mu_{21}(m)(s_1) \cdot u_{12}(t, m, s_2(m))$$

because the assumption of common belief of conjectures requires $\mu_{21}(m)$ to be the conjecture of player 1 regarding the conjecture of player 2. The pure strategy s_2 is said to be optimal with respect to $\mu_{21}(m)$ and u_{12} if for all $s_2' \in S_2$ it holds that

$$U_{12}(m, \mu_{21}(m), s_2) \geq U_{12}(m, \mu_{21}(m), s_2').$$

Definition 2.4 *A conjecture profile c is a preference conjecture equilibrium if it satisfies Bayesian updating, and moreover*

1. *for every $m \in M^*$, $\mu_{21}(m)(s_1) > 0$ implies that s_1 is optimal w.r.t. μ_{12} and $u_{21}(m)$*
2. *for every $m \in M$, $\mu_{12}(s_2) > 0$ implies that s_2 is optimal w.r.t. $\mu_{21}(m)$ and u_{12}.*

Observe that in this definition optimality on the part of player 2 is not required at the start of the game. It is easy to verify that this is implied by the second part of the definition in combination with Bayesian updating.

3 Relation with sequential equilibrium

In this section we show that in a signaling game each preference conjecture equilibrium induces in a natural way a sequential equilibrium and that conversely each sequential equilibrium is induced this way by at least one preference conjecture equilibrium. In particular this shows that a preference conjecture equilibrium always exists. First we explain how a preference conjecture equilibrium induces a sequential equilibrium. To this end, take a conjecture profile

$$c = (\mu_{12}, u_{12}, (\mu_{21}(m), u_{21}(m))_{m \in M^*}).$$

The assessment $(\sigma_1, \sigma_2, \beta)$ is said to be induced by the conjecture profile c if

$$\sigma_1 = \mu_{21}(h_0) \quad \text{and} \quad \sigma_2 = \mu_{12} \tag{1}$$

and for all types t and messages m,

$$\beta(t \mid m) = \frac{\sum_{s_1 : s_1(t) = m} p(t) \cdot \mu_{21}(m)(s_1)}{\sum_{t' \in T} \sum_{s_1 : s_1(t') = m} p(t') \cdot \mu_{21}(m)(s_1)}. \tag{2}$$

The denominator in the definition of β is automatically positive. The definitions in (1) state that in the induced sequential equilibrium a player should actually play what his opponent conjectures him to play. Given these choices, equality (2) simply follows from Bayesian updating.

Theorem 3.1 *Let $\mathcal{S} = (T, M, A, p, u_1, u_2)$ be a signaling game and let c be a preference conjecture equilibrium with $u_{21}(h_0) = u_1$ and $u_{12} = u_2$. Then the assessment induced by c is a sequential equilibrium in $\mathcal{S}$.*

Proof We first show that σ_1 is optimal with respect to σ_2. Take a type $t \in T$, a pure strategy $s_1 \in S_1$ with $\sigma_1(s_1) > 0$ and an arbitrary strategy $s_1' \in S_1$. It suffices to show that

$$U_1(t, s_1, \sigma_2) \geq U_1(t, s_1', \sigma_2).$$

To this end, notice that also $\mu_{21}(h_0)(s_1) > 0$, because $\sigma_1 = \mu_{21}(h_0)$. So, because c is a preference conjecture equilibrium, it follows that s_1 is optimal with respect to μ_{12} and $u_{21}(h_0)$. In particular,

$$U_{21}(h_0)(t, s_1, \mu_{12}) \geq U_{21}(h_0)(t, s_1', \mu_{12}).$$

With $u_{21}(h_0) = u_1$ and $\sigma_2 = \mu_{12}$ this means exactly that $U_1(t, s_1, \sigma_2) \geq U_1(t, s_1', \sigma_2)$.

Next, we show that σ_2 is optimal with respect to β. Let m be a message, take a pure strategy s_2 with $\sigma_2(s_2) > 0$ and a pure strategy s_2'. Again it suffices to show that

$$U_2(m, s_2, \beta) \geq U_2(m, s_2', \beta).$$

Since $\mu_{12}(s_2) = \sigma_2(s_2) > 0$ and c is a preference conjecture equilibrium, we know that s_2 is optimal w.r.t. $\mu_{21}(m)$ and u_{12}. So, in particular

$$U_{12}(m, \mu_{21}(m), s_2) \geq U_{12}(m, \mu_{21}(m), s_2').$$

However, using the definition of $\beta(t \mid m)$, rearranging terms shows that

$$U_{12}(m, \mu_{21}(m), s_2) = \left[\sum_{t' \in T} \sum_{s_1 : s_1(t') = m} p(t') \cdot \mu_{21}(m)(s_1)\right] \cdot U_2(m, s_2, \beta)$$

$$U_{12}(m, \mu_{21}(m), s_2') = \left[\sum_{t' \in T} \sum_{s_1 : s_1(t') = m} p(t') \cdot \mu_{21}(m)(s_1)\right] \cdot U_2(m, s_2', \beta)$$

Hence, since the bracketed factor is always positive and the same for both s_2 and s_2', we see that $U_2(m, s_2, \beta) \geq U_2(m, s_2', \beta)$.

Finally, we show that β is Bayesian consistent with σ_1. Take a message m and a type t. Assume that $\sum_{t' \in T} p(t')\sigma_1(m \mid t') > 0$. We will show that

$$\beta(t \mid m) = \frac{p(t) \cdot \sigma_1(m \mid t)}{\sum\limits_{t' \in T} p(t') \cdot \sigma_1(m \mid t')}.$$

By the above assumption also $\sum_{s_1' \in S_1(m)} \mu_{21}(h_0)(s_1') = \sum_{s_1' \in S_1(m)} \sigma_1(s_1') > 0$. Hence, using the fact that c satisfies Bayesian updating to get the second equality,

$$\begin{aligned}
\beta(t \mid m) &= \frac{\sum_{s_1 : s_1(t) = m} p(t) \cdot \mu_{21}(m)(s_1)}{\sum_{t' \in T} \sum_{s_1 : s_1(t') = m} p(t') \cdot \mu_{21}(m)(s_1)} \\
&= \frac{\sum_{s_1 : s_1(t) = m} p(t) \cdot \mu_{21}(h_0)(s_1)}{\sum_{t' \in T} \sum_{s_1 : s_1(t) = m} p(t') \cdot \mu_{21}(h_0)(s_1)} \\
&= \frac{\sum_{s_1 : s_1(t) = m} p(t) \cdot \sigma_1(s_1)}{\sum_{t' \in T} \sum_{s_1 : s_1(t) = m} p(t') \cdot \sigma_1(s_1)} = \frac{p(t) \cdot \sigma_1(m \mid t)}{\sum_{t' \in T} p(t') \cdot \sigma_1(m \mid t')}
\end{aligned}$$

which shows that β is Bayesian consistent with σ_1. □

Conversely, for every sequential equilibrium there is a preference conjecture equilibrium that induces this equilibrium as the next theorem states. Since we prove a somewhat stronger result in the second part of Theorem 4.3, we postpone the proof to the next section.

Theorem 3.2 *Suppose that the assessment* $(\sigma_1, \sigma_2, \beta)$ *is a sequential equilibrium of a signaling game* $\mathcal{S} = (T, M, A, p, u_1, u_2)$. *Then there exists a preference conjecture equilibrium* $(\mu_{12}, u_{12}, (\mu_{21}(m), u_{21}(m))_{m \in M^*})$ *with* $u_{21}(h_0) = u_1$ *and* $u_{12} = u_2$ *that induces* $(\sigma_1, \sigma_2, \beta)$.

4 Minimum revision equilibrium

In a preference conjecture equilibrium player 2 can revise his conjecture about the utilities and strategies of player 1. Intuitively, one would like to keep these revisions as limited as possible. In this section we propose a way to measure this, and we study the resulting refinement of preference conjecture equilibrium and the associated refinement of sequential equilibrium.

A (weak) *ordering* on a finite set E is a complete and transitive binary relation on E. For an ordering R we denote by P and I its asymmetric and symmetric parts, respectively. For $x, y \in E$, the expressions xRy, xPy, and xIy, are interpreted as 'x is weakly preferred to y', 'x is strictly preferred to y', and 'x is equivalent to y'. Let R and R' be two orderings. Define

$$d(x, y) := \begin{cases} 1 & \text{if } xPy \text{ and not } xP'y \\ 1 & \text{if } xIy \text{ and not } xI'y \\ 1 & \text{if } yPx \text{ and not } yP'x \\ 0 & \text{otherwise.} \end{cases}$$

Define the *distance* between R and R' as

$$d(R, R') := \frac{1}{2} \sum_{x \in E} \sum_{y \in E} d(x, y).$$

Now consider a conjecture profile $c = (\mu_{12}, u_{12}, (\mu_{21}(m), u_{21}(m))_{m \in M^*})$. In this profile the conjecture of player 2 in information set m regarding the expected utility function of player 1's type t is given by

$$U_{21}(m)(t, m', \mu_{12}) := \sum_{s_2 \in S_2} \mu_{12}(s_2) \cdot u_{21}(m)(t, m', s_2(m')).$$

The preference ordering R_m^t on M for type t is defined by, for all $k, l \in M$

$$kR_m^t l \quad \text{if and only if} \quad U_{21}(m)(t, k, \mu_{12}) \geq U_{21}(m)(t, l, \mu_{12}).$$

Now, the distance $d(R_{h_0}^t, R_m^t)$ between the orderings R_m^t and $R_{h_0}^t$ counts the number of 'utility changes' player 2 makes for type t of player 1 if he observes message m. Based on this, we define the *revision index* of the conjecture profile c by

$$d(c) := \sum_{m \in M} \sum_{t \in T} d(R_{h_0}^t, R_m^t).$$

By requiring that the revision index be as small as possible, we obtain the announced refinement of preference conjecture equilibrium.

Definition 4.1 *Let c be a preference conjecture equilibrium with $u_{12} = u_2$ and $u_{21}(h_0) = u_1$. Then c is a minimum revision equilibrium if $d(c) \leq d(c')$ for all preference conjecture equilibria c' with $u'_{12} = u_2$ and $u'_{21}(h_0) = u_1$.*

Since a preference conjecture equilibrium always exists by Theorem 3.2, it follows that a minimum revision equilibrium always exists.

4.1 Relation with sequential equilibrium

Since any preference conjecture equilibrium induces a sequential equilibrium by Theorem 3.1, the concept of minimum revision equilibrium can be used to obtain a refinement of sequential equilibrium. This works as follows. The expected utility for player 1 if his type is t, he chooses message m, and player 2 plays σ_2 is given by

$$U_1(t, m, \sigma_2) := \sum_{s_2 \in S_2} \sigma_2(s_2) \cdot u_1(t, m, s_2(m)).$$

Now, for a sequential equilibrium $(\sigma_1, \sigma_2, \beta)$ of $\mathcal{S}$, let $R(\sigma_1, \sigma_2, \beta)$ denote the set of triples (t, m, m') in $T \times M \times M$ for which $\beta(t \mid m) > 0$ and $U_1(t, m, \sigma_2) < U_1(t, m', \sigma_2)$.

Definition 4.2 *The revision index of $(\sigma_1, \sigma_2, \beta)$ is defined as the number $|R(\sigma_1, \sigma_2, \beta)|$ of elements of the set $R(\sigma_1, \sigma_2, \beta)$. It is denoted by $r(\sigma_1, \sigma_2, \beta)$.*

If (t, m, m') is in $R(\sigma_1, \sigma_2, \beta)$, player 2 believes that type t has positive probability although the observed message m is inferior to m' for t. If player 2 believes that player 1 is rational he should make some 'revision' in order to rationalize this. This explains the term 'revision index' in Definition 4.2. The following theorem justifies the use of this particular expression.

Theorem 4.3 *Let $\mathcal{S} = (T, M, A, p, u_1, u_2)$ be a signaling game and let $(\sigma_1, \sigma_2, \beta)$ be a sequential equilibrium in $\mathcal{S}$. If a preference conjecture equilibrium c with $u_{12} = u_2$ and $u_{21}(h_0) = u_1$ induces $(\sigma_1, \sigma_2, \beta)$, then $d(c) \geq r(\sigma_1, \sigma_2, \beta)$. Moreover, there is a preference conjecture equilibrium c with $u_{12} = u_2$, $u_{21}(h_0) = u_1$ and $d(c) = r(\sigma_1, \sigma_2, \beta)$ that induces $(\sigma_1, \sigma_2, \beta)$.*

Proof (a) Let c be a preference conjecture equilibrium with $u_{12} = u_2$ and $u_{21}(h_0) = u_1$ which induces $(\sigma_1, \sigma_2, \beta)$. Take an element (t, m, m') in $R(\sigma_1, \sigma_2, \beta)$. It suffices to show that $m R^t_m m'$ and $m' P^t_{h_0} m$ because this implies that

$$d(c) = \sum_{m \in M} \sum_{t \in T} d(R^t_{h_0}, R^t_m) \geq |R(\sigma_1, \sigma_2, \beta)| = r(\sigma_1, \sigma_2, \beta).$$

In order to show that $m R^t_m m'$, note that $\beta(t \mid m) > 0$ because (t, m, m') is an element of $R(\sigma_1, \sigma_2, \beta)$. So, by (2), there exists an $s_1 \in S_1$ such that $s_1(t) = m$ and $\mu_{21}(m)(s_1) > 0$. Since c is a preference conjecture equilibrium, this implies that

$$U_{21}(m)(t, m, \mu_{12}) \geq U_{21}(m)(t, m', \mu_{12})$$

and hence R^t_m orders m weakly above m'. On the other hand, $U_1(t, m, \sigma_2) < U_1(t, m', \sigma_2)$ since (t, m, m') is an element of $R(\sigma_1, \sigma_2, \beta)$. This is equivalent to

$$U_{21}(h_0)(t, m, \mu_{12}) < U_{21}(h_0)(t, m', \mu_{12})$$

and hence $R^t_{h_0}$ orders m strictly below m'.

(b) Let $(\sigma_1, \sigma_2, \beta)$ be a sequential equilibrium. We define the conjecture profile

$$c^* = (\mu^*_{12}, u^*_{12}, (\mu^*_{21}(m), u^*_{21}(m))_{m \in M^*})$$

as follows. For player 1 we define $\mu_{12}^* := \sigma_2$ and $u_{12}^* := u_2$. For player 2, we take $\mu_{21}^*(h_0) := \sigma_1$ and $u_{21}^*(h_0) := u_1$ at the start of the game. For any other $m \in M$ we define $u_{21}^*(m)$ by

$$u_{21}^*(m)(t,l,a) := \begin{cases} \max_{m' \in M} u_1(t,m',a) + 1 & \text{if l = m and moreover there exists an} \\ & m' \in M \text{ with } (t,m,m') \in R(\sigma_1,\sigma_2,\beta) \\ u_1(t,l,a) & \text{else.} \end{cases}$$

In the definition of $\mu_{21}^*(m)$ we distinguish two cases.

Case 1 Suppose there exists an $s_1 \in S_1(m)$ such that $\sigma_1(s_1) > 0$. Then we define $\mu_{21}^*(m)$ for each $s_1 \in S_1(m)$ by

$$\mu_{21}^*(m)(s_1) := \frac{\sigma_1(s_1)}{\sum_{s_1' \in S_1(m)} \sigma_1(s_1')}.$$

Case 2 Suppose that $\sigma_1(s_1) = 0$ for all $s_1 \in S_1(m)$. Take a pure strategy s_1^* with $\sigma_1(s_1^*) > 0$. Then $s_1^*(t) \neq m$ for each t. Define for type $t \in T$ the pure strategy $s_1^{t,m}$ by

$$s_1^{t,m}(t') := \begin{cases} m & \text{if } t' = t \\ s_1^*(t') & \text{if } t' \neq t. \end{cases}$$

Clearly, all strategies $s_1^{t,m}$ are elements of $S_1(m)$ and $s_1^{t,m} \neq s_1^{t',m}$ whenever $t \neq t'$. Thus we can define for each pure strategy $s_1 \in S_1(m)$

$$\mu_{21}(m)(s_1) := \begin{cases} \frac{\beta(t|m)}{p(t)} & \text{if } s_1 = s_1^{t,m} \\ 0 & \text{if } s_1 \neq s_1^{t,m}. \end{cases}$$

Next, define $\mu_{21}^*(m)$ by

$$\mu_{21}^*(m)(s_1) := \frac{\mu_{21}(m)(s_1)}{\sum_{s_1' \in S_1(m)} \mu_{21}(m)(s_1')}$$

for all pure strategies $s_1 \in S_1(m)$. The denominator is not equal to zero, because there is at least one type t with $\beta(t \mid m) > 0$, and for this type $\mu_{21}(m)(s_1^{t,m}) > 0$ while indeed $s^{t,m} \in S_1(m)$.

We show that c^* is a preference conjecture equilibrium with $d(c^*) = r(\sigma_1,\sigma_2,\beta)$ that induces $(\sigma_1,\sigma_2,\beta)$. First we show that c^* induces $(\sigma_1,\sigma_2,\beta)$. We only have to prove (2). Let $t \in T$ and $m \in M$. In Case 1,

$$\beta(t \mid m) = \frac{p(t)\cdot\sigma_1(m \mid t)}{\sum_{t' \in T} p(t')\cdot\sigma_1(m \mid t')} = \frac{\sum_{s_1 : s_1(t)=m} p(t)\cdot\mu_{21}^*(m)(s_1)}{\sum_{t' \in T}\sum_{s_1 : s_1(t')=m} p(t')\cdot\mu_{21}^*(m)(s_1)},$$

where the first equality follows from Bayesian consistency of β with σ_1 and the second one from the definitions of $\sigma_1(m \mid t)$ and $\mu_{21}^*(m)$. In Case 2,

$$\beta(t \mid m) = \frac{\beta(t \mid m)}{\sum_{t' \in T} \beta(t' \mid m)} = \frac{\sum_{s_1 : s_1(t) = m} p(t) \cdot \mu_{21}(m)(s_1)}{\sum_{t' \in T} \sum_{s_1 : s_1(t') = m} p(t') \cdot \mu_{21}(m)(s_1)} = \frac{\sum_{s_1 : s_1(t) = m} p(t) \cdot \mu^*_{21}(m)(s_1)}{\sum_{t' \in T} \sum_{s_1 : s_1(t') = m} p(t') \cdot \mu^*_{21}(m)(s_1)},$$

so also in this case (2) holds. Hence, c induces $(\sigma_1, \sigma_2, \beta)$.

Next we show that $d(c^*) = r(\sigma_1, \sigma_2, \beta)$. Let $t \in T$ and $m \in M$. If there is no m' with $(t, m, m') \in R(\sigma_1, \sigma_2, \beta)$, then $u^*_{21}(m) = u_1 = u^*_{21}(h_0)$ and therefore $d(R^t_{h_0}, R^t_m) = 0$. Otherwise, consider an $m' \in M$ with $(t, m, m') \in R(\sigma_1, \sigma_2, \beta)$. Then, by definition of $R(\sigma_1, \sigma_2, \beta)$, we have $\beta(t \mid m) > 0$ and $U_1(t, m, \sigma_2) < U_1(t, m', \sigma_2)$. Hence

$$\sum_{s_2 \in S_2} \mu^*_{12}(s_2) u^*_{21}(h_0)(t, m, s_2(m)) < \sum_{s_2 \in S_2} \mu^*_{12}(s_2) u^*_{21}(h_0)(t, m', s_2(m'))$$

and $R^t_{h_0}$ orders m strictly below m'. However, notice that in this case

$$u^*_{21}(m)(t, m, a) := \max_{m'' \in M} u_1(t, m'', a) + 1$$

and $u^*_{21}(m)(t, m', a) = u_1(t, m', a)$ for all $a \in A$. Hence, R^t_m orders m strictly above m' and it orders any two messages different from m in the same way as $R^t_{h_0}$. Thus, since $\beta(t \mid m) > 0$ in this case, we obtain:

$$d(R^t_{h_0}, R^t_m) = \left| \left\{ m' \in M \mid U_1(t, m, \sigma_2) < U_1(t, m', \sigma_2) \right\} \right|.$$

Summing up over all messages m and all types t, and using the fact that $d(R^t_{h_0}, R^t_m) = 0$ whenever $\beta(t \mid m) = 0$ to get the last equality, yields

$$r(\sigma_1, \sigma_2, \beta) = \sum_{(t,m) : \beta(t|m) > 0} \left| \{ m' \in M \mid U_1(t, m, \sigma_2) < U_1(t, m', \sigma_2) \} \right| = \sum_{(t,m) : \beta(t|m) > 0} d(R^t_{h_0}, R^t_m) = d(c).$$

Finally we show that c^* is a preference conjecture equilibrium. First note that $\mu^*_{21}(h_0) = \sigma_1$. So, for $m \in M$ such that $\sum_{s'_1 \in S_1(m)} \mu^*_{21}(h_0)(s'_1) > 0$, the conjecture $\mu^*_{21}(m)$ is defined by Case 1, and hence c^* satisfies Bayesian updating.

Secondly, we show part 2 of Definition 2.4. Take an $m \in M$ and an $s_2 \in S_2$ with $\mu^*_{12}(s_2) > 0$. Since $\sigma_2(s_2) = \mu^*_{12}(s_2) > 0$, optimality of σ_2 with respect to β implies that $U_2(m, s_2, \beta) \geq U_2(m, s'_2, \beta)$ for all $s'_2 \in S_2$. Hence, using the equality

$$U_2(m, s_2, \beta) = \sum_{t \in T} \beta(t \mid m) \cdot u_2(t, m, s_2(m)),$$

together with (2) and $u^*_{12} = u_2$, we see for all $s'_2 \in S_2$ that

$$U^*_{12}(m, \mu^*_{21}(m), s_2) = \sum_{t \in T} p(t) \sum_{s_1 : s_1(t) = m} \mu^*_{21}(m)(s_1) \cdot u^*_{12}(t, m, s_2(m)) \geq \sum_{t \in T} p(t) \sum_{s_1 : s_1(t) = m} \mu^*_{21}(m)(s_1) \cdot u^*_{12}(t, m, s'_2(m)) = U^*_{12}(m, \mu^*_{21}(m), s'_2).$$

Hence, s_2 is optimal for player 2 with respect to $\mu^*_{21}(m)$ and u^*_{12}.

Thirdly we show part 1 of Definition 2.4. Let $s_1 \in S_1$. If $\mu^*_{21}(h_0)(s_1) > 0$ then $\sigma_1(s_1) > 0$, so optimality of σ_1 with respect to σ_2 yields that s_1 is optimal with respect to σ_2 in the game $\mathcal{S}$. Hence s_1 is optimal with respect to $\mu^*_{12} = \sigma_2$ and $u^*_{21}(h_0) = u_1$.

Take an $m \in M$ and assume that $\mu^*_{21}(m)(s_1) > 0$. We will show that s_1 is optimal for player 1 with respect to μ^*_{12} and $u^*_{21}(m)$.

First consider Case 1 in the definition of $\mu^*_{21}(m)$. Clearly $\sigma_1(s_1) > 0$ in this case. So, s_1 is optimal with respect to σ_2 and u_1. Since $\mu^*_{12} := \sigma_2$, it remains to show that $u^*_{21}(m) = u_1$. Since we are in Case 1, there is an $s'_1 \in S_1(m)$ with $\sigma_1(s'_1) > 0$. Thus, m is played with positive probability under σ_1. Then the optimality of σ_1 with respect to σ_2 implies that

$$\{(t, m') \in T \times M \mid \beta(t \mid m) > 0 \text{ and } U_1(t, m, \sigma_2) < U_1(t, m', \sigma_2)\}$$

is empty. So, there is no (t, m') such that $(t, m, m') \in R(\sigma_1, \sigma_2, \beta)$. Hence, $u^*_{21}(m) = u_1$.

Next consider Case 2. So, $\sigma_1(s'_1) = 0$ for all $s'_1 \in S_1(m)$. Since $\mu^*_{21}(m)(s_1) > 0$, in this case $s_1 = s_1^{t,m}$ for a unique $t \in T$. So, $\beta(t \mid m) > 0$ for this t, by definition of $\mu^*_{21}(m)$, $\mu_{21}(m)$ and $s_1^{t,m}$. We distinguish two subcases.

First, suppose that message m is not optimal for t with respect to σ_2 and u_1. Then there exists an $m' \in M$ such that $U_1(t, m, \sigma_2) < U_1(t, m', \sigma_2)$, and therefore (t, m, m') is an element of $R(\sigma_1, \sigma_2, \beta)$. So, by definition,

$$u^*_{21}(m)(t, m, a) = \max_{m'' \in M} u_1(t, m'', a) + 1 > u_1(t, m', a) = u^*_{21}(m)(t, m', a)$$

for all a and $m' \neq m$. Thus, $s_1(t) = s_1^{t,m}(t) = m$ is optimal for type t w.r.t $u^*_{21}(m)$, even regardless of what μ^*_{12} is. Next, consider any other $t' \neq t$. By the choice of $s^*_1(t')$ in the definition of $s_1^{t,m}$, message $s_1(t') = s_1^{t,m}(t') = s^*_1(t')$ is optimal for t' w.r.t. σ_2 and u_1. Since $t' \neq t$, we have $u^*_{21}(m) = u_1$, and $s_1(t')$ is also optimal w.r.t. μ^*_{12} and $u^*_{21}(m)$.

Secondly, suppose that m is optimal for t with respect to σ_2 and u_1. Then there exists no m' such that $(t, m, m') \in R(\sigma_1, \sigma_2, \beta)$. Hence, $u^*_{21}(m) = u_1$, and $s_1(t) = s_1^{t,m}(t) = m$ is optimal w.r.t. μ^*_{12} and $u^*_{21}(m)$. The argument for types $t' \neq t$ is identical to the previous case. □

We conclude with a few direct consequences of Theorem 4.3. A sequential equilibrium $(\sigma_1, \sigma_2, \beta)$ in a signaling game $\mathcal{S} = (T, M, A, p, u_1, u_2)$ is called a minimum revision sequential equilibrium if it minimizes the revision index among all sequential equilibria of the game.

Corollary 4.4 *If c is a minimum revision equilibrium in $\mathcal{S}$, then the induced sequential equilibrium is a minimum revision sequential equilibrium. Conversely, for every minimum revision sequential equilibrium $(\sigma_1, \sigma_2, \beta)$ there exists a minimum revision equilibrium c with $u_{21}(h_0) = u_1$ and $u_{12} = u_2$ that induces $(\sigma_1, \sigma_2, \beta)$.*

Further, when under σ_1 each information set of player 2 is reached with positive probability, $R(\sigma_1, \sigma_2, \beta)$ will be empty and $(\sigma_1, \sigma_2, \beta)$ is a minimum revision sequential equilibrium with revision index 0. Finally, it can be shown in examples that minimum revision sequential equilibrium has no logical relation to the intuitive criterion.

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