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ORIGINAL ARTICLE

Proper belief revision and rationalizability in dynamic games

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Abstract In this paper we develop an epistemic model for dynamic games in which players may revise their beliefs about the opponents' utility functions as the game proceeds. Within this framework, we propose a rationalizability concept that is based upon the following three principles: (1) at every instance of the game, a player should believe that his opponents are carrying out optimal strategies, (2) a player, at information set h, should not change his belief about an opponent's relative ranking of two strategies s and s' if both s and s' could have led to h, and (3) the players' initial beliefs about the opponents' utility functions should agree on a given profile u of utility functions. Common belief in these events leads to the concept of *persistent rationalizability* for the profile u of utility functions. It is shown that for a given game tree with observable deviators and a given profile u of utility functions, every properly point-rationalizable strategy is a persistently rationalizable strategy for u. This result implies that persistently rationalizable strategies always exist for all game trees with observable deviators and all profiles of utility functions. We provide an algorithm that can be used to compute the set of persistently rationalizable strategies for a given profile u of utility functions. For generic games with perfect information, persistent rationalizability uniquely selects the backward induction strategy for every player.

Keywords Rationalizability · Dynamic games · Belief revision

JEL Classification Numbers C72

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1 Introduction

1.1 Persistent rationalizability

In most existing equilibrium and rationalizability concepts for dynamic games it is assumed that players do not revise their belief about the opponents' utility functions during the game. That is, the utilities we write at the terminal nodes are usually assumed to be "fixed", as players are supposed never to question these utilities even if this means that certain moves have to be interpreted as irrational moves. Consequently, common belief in rationality will in general not be possible in such models. Reny (1992a, 1993) has shown, for instance, that within the class of games with perfect information there are only very few games in which common belief in rationality is possible at all information sets, provided that players do not revise their belief about the opponents' utility functions.

In this paper we take an alternative approach: we allow players to revise their belief about the opponents' utility functions, but at the same time require players to interpret every opponent's move as a rational move. We call this belief in sequential rationality (BSR). The other key ingredient in our model, proper belief revision (PBR), states that players, when changing their belief about an opponent's utility function, should not carry out "unnecessary" changes. More precisely, if player i decides to change his belief about player j's utility function and/or player j's belief about the other players' strategy choices, then player i also changes his belief about player j's ranking of his strategies, since this ranking is induced by player j's utility function and belief about the other players' choices. Suppose that player i observes that his information set h_i has been reached, and in order to explain this event he decides to change his belief about player j's utility function and/or player j's belief about the other players' choices. Suppose also that s_j and s_j' are two strategies for player j that might have led to h_i. PBR states that in this case, player i should maintain his initial belief about player j's ranking of the two strategies s_j and s_j' [see also Perea (2006) for a formulation of this principle within an equilibrium framework]. The intuition behind this condition is one of minimal belief revision: the fact that h_i has been reached does not provide absolute evidence against player i's initial belief about player j's relative ranking of s_j and s_j', and therefore, within the spirit of minimal belief revision, player i should maintain his initial belief about this relative ranking.

The last condition we impose states that the players' *initial* beliefs about the opponents' utility functions should agree on some profile $u = (u_i)_{i \in I}$ of utility functions. We call this initial belief in u (IBu). This condition is not crucial conceptually, but helps us to compare our concept with existing rationality concepts in the literature that assume a "fixed" profile u of utilities, in the sense explained above. An important difference with these existing concepts is that our concept allows players to change their belief about the opponents' utilities as the game moves on.

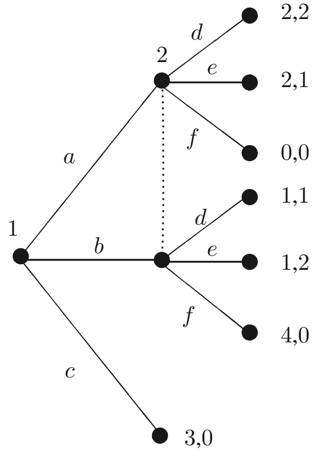


Fig. 1 Implications of persistent rationalizability

In order to formalize the three conditions above we develop an appropriate *epistemic model* for dynamic games. A *type* for player i has a utility function over the terminal nodes and holds at every information set h_i a conditional probabilistic belief about the possible opponents' strategy choices *and types*. Since different types may hold different utility functions, and since types may change their belief about the opponents' types, our model allows in particular for belief revision about the opponents' utility functions during the game, and is thus suited for our approach. A type t_i for player i is said to be *persistently rationalizable* for a given profile u of utility functions if t_i respects common belief, at every information set, in the events BSR, PBR and IBu. A strategy that is sequentially rational for such a type t_i is accordingly called *persistently rationalizable for u*.

In order to understand the implications of persistent rationalizability, consider the game in Fig. 1. Let $u = (u_1, u_2)$ be the profile of utility functions depicted at the terminal nodes. We show that d is player 2's only persistently rationalizable strategy for u. Namely, let t_2 be a persistently rationalizable type for player 2 for u. By common belief in BSR, t_2 initially believes that player 1 initially believes that player 2 chooses rationally at his information set. By common belief in IBu, t_2 initially believes that player 1 initially believes that player 2 has utility function u_2. As such, t_2 initially believes that player 1 initially believes that player 2 will not choose f. By IBu, t_2 initially believes that player 1 has utility function u_1. Combining this with the previous insight, we may conclude that t_2 initially believes that player 1 strictly prefers c to a, and strictly prefers a to b. By BSR, t_2 initially believes that player 1 chooses rationally, and hence t_2 initially believes that player 1 chooses c. Now, at player 2's information set type t_2 must conclude that player 1 did not choose c, and hence BSR forces t_2 to change his belief about player 1's utility function and/or player 1's belief about player 2's strategy choice. Since t_2 initially believes that player 1 ranks a strictly above b, and since a and b both lead to player 2's information set, PBR implies that t_2 should still believe at his information set that player 1 ranks a strictly above b. By BSR, t_2 should then believe at his information set that player 1 has chosen a. Since t_2 has utility function u_2, type t_2's unique sequentially rational strategy is d. Hence, d is player 2's only persistently rationalizable strategy for u.

Note, however, that t_2 *must* revise his belief about player 1's utility function when observing that player 1 has not chosen c. Namely, upon observing this event, t_2 must still believe that player 1 initially believes that player 2 has utility function u_2 and chooses rationally. As such, t_2 must still believe that player 1 initially believes that player 2 will not choose f. By BSR, t_2 must believe at his information set that player 1 has chosen rationally, and this can only be realized if t_2 changes his belief about player 1's utility function. For instance, t_2 could believe, upon observing that player 1 has not chosen c, that player 1's utility function is not u_1, but $(2,2,0,1,1,4,1)$, while maintaining his previous belief about player 1's beliefs. Here, the first utility in the vector corresponds to the highest terminal node, and the last utility to the lowest terminal node. This belief revision policy satisfies PBR, since player 2 will still believe at his information set that player 1 strictly prefers a to b.

1.2 Relation with proper rationalizability

It turns out that the concept of *proper rationalizability* (Schuhmacher 1999; Asheim 2001) also uniquely selects the strategy d for player 2 in Fig. 1. However, the line of reasoning that leads to this strategy choice is at some points crucially different from persistent rationalizability. The key idea in proper rationalizability, and also in Myerson's (1978) *proper equilibrium* and Schulte's (2003) *respect for public preferences*, is that a player, when choosing his strategy, should not exclude any of the opponents' strategies, yet should deem one opponent strategy "infinitely more likely" than another if he believes the opponent to prefer the former to the latter. Here, the notion of "infinitely more likely" can be made explicit by the use of lexicographic probability distributions, as has been done by Blume et al. (1991a, b) and Asheim (2001) in their characterizations of proper equilibrium and proper rationalizability, respectively. Moreover, proper rationalizability implicitly assumes that players never revise their beliefs about the opponents' utility functions during the game.

In the example of Fig. 1, the reasoning of proper rationalizability implies that, since player 2 should not exclude that player 1 may choose a or b, he should strictly prefer d and e to f. Player 1, knowing this, should then deem d and e infinitely more likely than f, and hence should strictly prefer c to a and strictly prefer a to b. Player 2, at the beginning of the game, should then deem c infinitely more likely than a, and deem a infinitely more likely than b. This implies that player 2, upon observing that player 1 has chosen a or b, should still deem a infinitely more likely than b, and hence player 2 should choose d at his information set. However, when player 2 observes that player 1 has not chosen c, he must conclude that player 1 has chosen irrationally, since proper rationalizability does not allow player 2 to change his belief about player 1's utility function during the game.

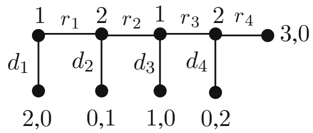
The crucial difference between persistent rationalizability and proper rationalizability in this example is thus the following: within the context of persistent rationalizability, player 2 believes at his information set that player 1 has rationally foregone the option of choosing c. To this purpose, player 2 changes his initial belief about player 1's utility function upon observing that player 1 has not chosen c. Within the context of proper rationalizability, on the other hand, player 2 believes at his information set that player 1 has chosen irrationally, while maintaining his initial belief about player 1's utility function. However, both concepts lead player 2 to believe that player 1 has chosen a upon observing that player 1 has not chosen c, and this eventually leads to the same strategy choice for player 2 in both concepts.

The first main result in this paper, Theorem 5.3, shows that the relationship in the example between properly rationalizable strategies and persistently rationalizable strategies is not a coincidence. Namely, we shall prove that some refinement of proper rationalizability, to which we refer as *proper point-rationalizability*, implies persistent rationalizability whenever the game tree satisfies the so-called *observable deviators* condition. Here, a game tree is said to be with *observable deviators* (see Battigalli 1996) if for every information set h the following holds: if every player chooses a strategy that *may possibly* lead to h, then the resulting profile *will* lead to h. Consequently, if player i believes at information set h_i that his information set h_i' will not be reached, but finds out later, by surprise, that his information set h_i' *has* been reached, then player i knows precisely at h_i' about which opponents he needs to revise his belief in order to make his new belief compatible with the event of reaching h_i'. This assumption appears to be crucial, since persistently rationalizable types and strategies need not exist in game trees that violate observable deviators (see Sect. 5.2). Formally, Theorem 5.3 states that for every game tree with observable deviators and every possible profile u of utility functions, every properly point-rationalizable strategy for u is persistently rationalizable for u. Since properly point-rationalizable strategies exist for every game tree and every u, this result implies the existence of persistently rationalizable strategies for every game tree with observable deviators and every profile u of utility functions.

1.3 Relation with backward induction

If the concept of persistent rationalizability is applied to generic games with perfect information, it uniquely selects the backward induction strategy for every player. To illustrate this relationship, consider the example in Fig. 2, which is taken from Reny (1992b). Let $u = (u_1, u_2)$ be the pair of utility functions

Fig. 2 Persistent rationalizability leads to backward induction



depicted at the terminal nodes, and let t_1 be a persistently rationalizable type for player 1 for u. By IBu, t_1 must initially believe that player 2, at his last information set, prefers strategy (r_2,d_4) to (r_2,r_4). Since both (r_2,d_4) and (r_2,r_4) lead to player 1's second information set, PBR implies that t_1 should believe, at his second information set, that player 2, at his second information set, prefers (r_2,d_4) to (r_2,r_4). By BSR, t_1 should believe at his second information set that player 2 chooses (r_2,d_4).

Now, let t_2 be a persistently rationalizable type for player 2 for u. Since t_2 initially believes that player 1 satisfies IBu, PBR and BSR, we know by the above argument that t_2 initially believes that player 1 believes at his second information set that player 2 chooses (r_2,d_4). Since t_2 initially believes that player 1 has utility function u_1, t_2 initially believes that player 1, at his second information set, prefers (r_1,d_3) to (r_1,r_3). As both (r_1,d_3) and (r_1,r_3) lead to player 2's first information set, PBR implies that t_2, at his first information set, should believe that player 1 prefers (r_1,d_3) to (r_1,r_3). By BSR, t_2 should believe at his first information set that player 1 chooses (r_1,d_3). Since t_2 has utility function u_2, we conclude that t_2 has a unique sequentially rational strategy, namely d_2, which is player 2's backward induction strategy with respect to u.

Summarizing, player 2 has a unique persistently rationalizable strategy for u, namely his backward induction strategy for u. Theorem 7.1 shows that this result holds in general: for a given game tree with perfect information and generic profile u of utility functions, every player has a unique persistently rationalizable strategy for u, namely his backward induction strategy for u.

1.4 Relation with other rationality concepts

The concept of *common certainty of rationality at the beginning of the game* (Ben-Porath 1997), also called *weak sequential rationalizability*, requires common belief at the beginning of the game that players choose rationally at each of their information sets. The only restriction on the players' belief revision policies, however, is that players should not change their belief about the opponents' utility functions. In particular, players may believe that an opponent has chosen irrationally if the initial belief about this opponent's strategy choice has been contradicted by the play of the game. In the game of Fig. 1, common certainty of rationality at the beginning implies that player 1 should believe that player 2 will not choose f, and that player 2 should believe initially that player 1 chooses c. However, if player 2 is led to revise his belief about player 1 upon observing that player 1 has not chosen c, he may believe that player 1 has chosen a or b, and player 2 may choose both d and e. Hence, in this example, every persistently rationalizable strategy for u is also weakly sequentially rationalizable, but not vice versa. In Sect. 7 we show that this relationship holds in general: for every game tree and every profile u of utility functions, every strategy that is persistently rationalizable for u is also weakly sequentially rationalizable for u.

The concept of *extensive form rationalizability* (Pearce 1984; Battigalli 1997), on the other hand, places important restrictions on player 2's belief revision procedure in Fig. 1, and eventually singles out the choice *e* for player 2. In words, the concept requires a player, at each of his information sets, to maintain his original belief about the opponents' utility functions and to look for the "highest possible degree of interactive belief in rationality"[1] that rationalizes the event of reaching this information set. The player should then form his current and future beliefs on the basis of this degree until this degree will be contradicted by some other event in the future. In the game of Fig. 1, this means that player 2, upon observing that player 1 has chosen *a* or *b*, should still believe that player 1 has utility function u_1, and should attempt to explain this event by a theory in which player 1 is believed to choose rationally. If this is possible, then player 2 should try to find a "more sophisticated" theory explaining this event in which player 1 is not only believed to choose rationally, but is also believed to believe that player 2 will choose rationally at his information set. If this is not possible, then player 2 should stick to his first theory. If the more sophisticated theory *is* possible, then player 2 should attempt to find a theory with an even higher degree of interactive belief in rationality, and so on. According to this line of reasoning, player 2's "most sophisticated" theory that explains the event of player 1 choosing *a* or *b*, without changing his belief about player 1's utility function, is the following: player 1 is believed to rationally choose *b*, and player 1 is believed to believe with high probability that player 2 will irrationally respond with *f*. Consequently, player 2 should choose *e*. Since we have seen that player 2's unique persistently rationalizable strategy for *u* is *d*, we conclude that there is no general logical relationship between persistent rationalizability and extensive form rationalizability.

The outline of this paper is as follows. In Sect. 2 we first present some preliminary definitions and notation in extensive form games. Section 3 lays out the epistemic model we use. The concept of persistent rationalizability is introduced in Sect. 4. In Sect. 5 we prove our result concerning the relationship between proper and persistent rationalizability. In Sect. 6 we present an algorithm that can be used to compute the set of persistently rationalizable strategies for a given extensive form game. In Sect. 7 we use this algorithm to study the relationships with backward induction and weak sequential rationalizability. All proofs are collected in the appendix.

2 Extensive form structures

In this section we present the notation and some basic definitions. The rules of the game are represented by an *extensive form structure* $\mathcal{S}$ consisting of a finite game tree, a finite set of players I, a finite collection H_i of information sets for each player i and at each information set $h_i \in H_i$ a finite collection $A(h_i)$ of actions for the player. The set of terminal nodes in $\mathcal{S}$ is denoted by Z, whereas $H = \cup_{i \in I} H_i$ denotes the collection of all information sets. By h_0 we denote the beginning of the game, and we use the notation $H_i^* = H_i \cup \{h_0\}$ for

[1] Battigalli and Siniscalchi (2002) call it "highest possible degree of strategic sophistication".

every player i. We assume throughout that the extensive form structure satisfies perfect recall and that no chance moves occur.

The concept of strategy we use is different from the usual one since it does not require a player to specify actions at information sets that are avoided by the same strategy. It thus coincides with the concept of *plan of action* in Rubinstein (1991). The use of this alternative definition is not really relevant for the analysis, but rather avoids the inclusion of redundant information in the definition of a strategy. Formally, let $\tilde{H}_i \subseteq H_i$ be some collection of information sets for player i, not necessarily containing *all* player i's information sets, and let s_i be a mapping that assigns to every $h_i \in \tilde{H}_i$ some available action $s_i(h_i) \in A(h_i)$. We say that some information set $h^* \in H$ is *avoided* by the mapping s_i if for every profile of actions $(a(h))_{h\in H}$ with $a(h) \in A(h)$ for all h and $a(h_i) = s_i(h_i)$ for all $h_i \in \tilde{H}_i$, it holds that $(a(h))_{h\in H}$ avoids the information set h^*. We say that s_i is a *strategy* if its domain $\tilde{H}_i$ is equal to the collection of player i's information sets that are not avoided by s_i. Obviously, every strategy s_i can be obtained by first prescribing some action at all player i's information sets (that is, defining a strategy in the usual sense) and then deleting those player i's information sets that are avoided by it. Let S_i denote the set of player i's strategies, and let $S = \times_{i\in I} S_i$ be the set of all strategy profiles.

For a given information set h, let $S(h)$ be the set of strategy profiles that reach h. For a given player i, not necessarily the player who moves at h, let $S_i(h)$ be the set of strategies $s_i \in S_i$ for which there is some opponents' strategy profile $s_{-i} \in S_{-i} := \times_{j\neq i} S_j$ such that (s_i, s_{-i}) reaches h. We say that $\mathcal{S}$ is with *observable deviators* (see Battigalli 1996, among others) if $S(h) = \times_{i\in I} S_i(h)$ for every information set h. That is, if every player i chooses a strategy s_i that *may possibly reach* h, then the strategy profile $(s_i)_{i\in I}$ *will* reach h. For two-player games, the condition is implied by perfect recall. This is not true for more than two players.

3 Epistemic framework

In this section we formally model the players in an extensive form structure as decision makers under uncertainty. As already outlined in the introduction, such model should allow players to have uncertainty about the opponents' utilities, and to revise their beliefs about the opponents' utilities as the game proceeds. In addition, the model should provide a language in which beliefs about beliefs about …about beliefs of arbitrary length can be formalized. That is, it should allow for statements of the form "player i believes with probability α_i at information set h_i that player j believes with probability α_j at information set h_j that player k chooses strategy s_k" or "player i believes with probability α_i at information set h_i that player j believes with probability α_j at information set h_j that player k has utility function u_k". By applying techniques similar to Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Battigalli and Siniscalchi (1999), this can be achieved by constructing for each player i a set T_i of *types* such that every type $t_i \in T_i$ can be identified with a profile

$$(u_i(t_i), \mu_i(t_i, h_i)_{h_i \in H_i^*}),$$

where $u_i(t_i) : Z \to [-M, M]$ represents t_i's von Neumann–Morgenstern utility function from the set Z of terminal nodes to the interval $[-M, M]$, and $\mu_i(t_i, h_i) \in \Delta(S_{-i}(h_i) \times T_{-i})$ is t_i's probabilistic belief at h_i about the opponents' strategy choices and types. Here, M is some large, fixed, positive number. By $\Delta(X)$, we denote the set of probability distributions on a set X, whereas $S_{-i}(h_i)$ and T_{-i} are short ways to write $\times_{j \neq i} S_j(h_i)$ and $\times_{j \neq i} T_j$, respectively. Recall that $H_i^* = H_i \cup \{h_0\}$. Hence, every type is assumed to have a conditional belief at the beginning of the game (his initial belief) and at each of his information sets. For a formal construction of these type spaces T_i, the reader is referred to a previous version of this paper (Perea 2003). The main idea in the construction is to recursively define, for every player i, a set of kth order conditional beliefs, consisting of conditional beliefs about the opponents' possible strategies and $(k-1)$th order conditional beliefs. An important technical feature is that for every k, the set of kth order conditional beliefs is compact with respect to the weak topology on probability measures. Together with a coherence condition, this property implies that every infinite hierarchy of conditional beliefs, consisting of kth order conditional beliefs for every k, can be identified with a type as described above. This eventually leads to *complete* type spaces T_i for every player i, which are uncountably infinite compact metric spaces, where "compact" is defined with respect to the weak topology on probability measures. Moreover, for every player i there is a homeomorphism

$$f_i : T_i \to U \times (\times_{h_i \in H_i^*} \Delta(S_{-i}(h_i) \times T_{-i})),$$

where U is the set of utility functions from Z to $[-M, M]$. The function f_i thus identifies every type $t_i \in T_i$ with the profile $f_i(t_i) = (u_i(t_i), \mu_i(t_i, h_i)_{h_i \in H_i^*})$ as described above.

We now formalize what it means that a type respects *common belief* in the event that types have certain properties. Let $E_i \subseteq T_i$ be a subset of player i's types for every i, and let $E = \times_{i \in I} E_i$. We say that type t_i *believes* in E if $\text{supp}\mu_i(t_i, h_i) \subseteq S_{-i}(h_i) \times E_{-i}$ for all $h_i \in H_i^*$, where $E_{-i} = \times_{j \neq i} E_j$. Hence, at every instance of the game type t_i assigns probability 1 to the event that all opponents' types belong to E. We recursively define

$$B_i^1(E) = \{t_i \in E_i | t_i \text{ believes in } E\}$$

for all i, and

$$B_i^k(E) = \{t_i \in B_i^{k-1}(E) | t_i \text{ believes in } \times_{j \in I} B_j^{k-1}(E)\}$$

for all i and all $k \geq 2$. By $B_i^\infty(E) = \cap_{k \in \mathbb{N}} B_i^k(E)$ we denote the set of player i's types that *respect common belief in E*. That is, a type $t_i \in B_i^\infty(E)$ belongs to E, believes throughout the game that all opponents' types belong to E, believes throughout the game that all opponents' types believe throughout the game that all the other players' types belong to E, and so forth.

4 Persistent rationalizability

In the concept of persistent rationalizability we impose two conditions on types, to which we refer as *PBR* and *BSR*. In the previous section, we have seen that every type $t_i \in T_i$ corresponds to a vector $(u_i(t_i), \mu_i(t_i, h_i)_{h_i \in H_i^*})$, where $u_i(t_i)$ is a utility function and $\mu_i(t_i, h_i)$ is a probability measure on $S_{-i}(h_i) \times T_{-i}$. *PBR* states that, whenever player i initially believes that player j strictly prefers some strategy s_j to some strategy s_j', then player i should continue to believe so at every information set that can both be reached by s_j and s_j'. Consequently, player i should attach probability zero to s_j' at every such information set if he believes player j to choose rationally. More precisely, let j be an opponent for player i, let s_j, s_j' be two strategies for player j, and let h_j be some information set in $H_j^*(s_j) \cap H_j^*(s_j')$. Here, by $H_j^*(s_j)$ we denote the collection of information sets $h_j \in H_j^*$ that are reachable by s_j. Similarly for $H_j^*(s_j')$. We say that a type t_j *strictly prefers s_j to s_j' at h_j* if

$$u_j(t_j)(s_j, \mu_j(t_j, h_j)) > u_j(t_j)(s_j', \mu_j(t_j, h_j)),$$

where $u_j(t_j)(s_j, \mu_j(t_j, h_j))$ denotes the expected utility induced by the utility function $u_j(t_j)$, the strategy s_j and the belief $\mu_j(t_j, h_j)$ at h_j over the opponents' strategy-type pairs. Similarly for $u_j(t_j)(s_j', \mu_j(t_j, h_j))$. For a given type t_i, we say that t_i initially believes that player j at h_j strictly prefers s_j to s_j', if t_i at h_0 attaches probability 1 to the set of player j's types that strictly prefer s_j to s_j' at h_j.

Definition 4.1 *A type t_i is said to satisfy proper belief revision* (**PBR**) *if for every opponent j, every two strategies s_j and s_j' for player j, and every information set $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$ the following holds: if t_i initially believes that player j at h_j strictly prefers s_j to s_j', then t_i assigns probability zero to s_j' at every information set $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$.*

Here, $H_i^*(s_j)$ denotes the collection of player i information sets $h_i \in H_i^*$ that are reachable by s_j. We next define BSR. A strategy s_i is *sequentially rational* for type t_i if at every information set $h_i \in H_i^*(s_i)$ there is no strategy $s_i' \in S_i(h_i)$ that is strictly preferred to s_i by t_i at h_i. Let $(S_i \times T_i)^{\text{sr}}$ be the set of strategy-type pairs for player i at which s_i is sequentially rational for t_i.

Definition 4.2 *A type t_i believes in sequential rationality (BSR) if* $\text{supp}\,\mu_i(t_i, h_i) \subseteq \times_{j \neq i}(S_j \times T_j)^{\text{sr}}$ *for every $h_i \in H_i^*$.*

We are now ready to formalize our concept of persistent rationalizability.

Definition 4.3 *A type t_i is persistently rationalizable if it respects common belief in the event that types satisfy PBR and BSR.*

By our definition of "common belief", the condition that a type respects common belief in BSR implies in particular that the type itself satisfies BSR. Similarly, saying that the type respects common belief in PBR implies that the type itself satisfies PBR.

Finally, let $u = (u_i)_{i \in I}$ be some profile of utility functions at the terminal nodes. We say that a type t_i *initially believes in* u (IBu) if for every opponent j the initial belief $\mu_i(t_i, h_0)$ assigns probability 1 to the set of player j's types with utility function u_j.

Definition 4.4 *A type t_i is persistently rationalizable for $(\mathcal{S}, u)$ if (1) t_i is persistently rationalizable, (2) $u_i(t_i) = u_i$, and (3) t_i respects common belief in the event IBu. A strategy $s_i \in S_i$ is persistently rationalizable for $(\mathcal{S}, u)$ if there is some persistently rationalizable type t_i for $(\mathcal{S}, u)$ such that s_i is sequentially rational for t_i.*

5 Relation to proper rationalizability

5.1 Proper rationalizability

Schuhmacher (1999) introduced the concept of *proper rationalizability* as a rationalizability-type analogue to proper equilibrium, and showed that it uniquely selects the backward induction strategies in generic games with perfect information. Asheim (2001) provided a characterization of proper rationalizability in terms of lexicographic beliefs for the case of two players. In this section, we introduce a refinement of properly rationalizable strategies to which we refer as "properly point-rationalizable strategies". For the definition of properly point-rationalizable strategies, we use Asheim's characterization of proper rationalizability and extend it to games with more than two players.

Consider some type space R_i[2] for every player i with the property that every type $r_i \in R_i$ can be identified with some pair $(u_i(r_i), \lambda_i(r_i))$, where $u_i(r_i)$ is a von Neumann–Morgenstern utility function, and $\lambda_i(r_i)$ is a *cautious lexicographic probability distribution on* $S_{-i} \times R_{-i}$. By a lexicographic probability distribution we mean a vector $\lambda_i(r_i) = (\lambda_i^1(r_i), \lambda_i^2(r_i), \ldots, \lambda_i^K(r_i))$ of probability distributions on $S_{-i} \times R_{-i}$, and we call $\lambda_i^k(r_i)$ the kth order belief in $\lambda_i(r_i)$. The interpretation is that player i assigns "infinitely more importance" to his kth order beliefs than to his $(k+1)$th order beliefs, without completely discarding the latter beliefs.

For every opponent j, let $R_j(r_i)$ be the set of player j's types that r_i deems possible, that is, $r_j \in R_j(r_i)$ if there is some k such that $\lambda_i^k(r_i)$ assigns positive probability to some (s_{-i}, r_{-i}) in which r_j is present. We say that $\lambda_i(r_i)$ is *cautious* if for every opponent j, every type $r_j \in R_j(r_i)$ and every strategy $s_j \in S_j$, there is some k for which $\lambda_i^k(r_i)$ assigns positive probability to some $(s_{-i}, r_{-i}) \in S_{-i} \times R_{-i}$ in which (s_j, r_j) is present. That is, no opponent's strategy is excluded for any type $r_j \in R_j(r_i)$.

[2] Here, we use different symbols for types as to distinguish them from the types introduced in Sect. 3.

For every strategy s_i and every order k, let

$$u_i^k(r_i)(s_i) = \sum_{(s_{-i},r_{-i})\in S_{-i}\times R_{-i}} \lambda_i^k(r_i)(s_{-i},r_{-i})u_i(r_i)(s_i,s_{-i})$$

be the kth order expected utility of strategy s_i, where $u_i(r_i)(s_i,s_{-i})$ is the utility at the terminal node reached by (s_i,s_{-i}). Type r_i *strictly prefers* strategy s_i to strategy s_i' if there is some k with $u_i^k(r_i)(s_i) > u_i^k(r_i)(s_i')$ and $u_i^l(r_i)(s_i) = u_i^l(r_i)(s_i')$ for all $l < k$. For every player j strategy-type pair (s_j,r_j) with $r_j \in R_j(r_i)$, let $k(r_i)(s_j,r_j)$ be the first k such that $\lambda_i^k(r_i)$ assigns positive probability to some (s_{-i},r_{-i}) in which (s_j,r_j) is present. Type r_i deems (s_j,r_j) *infinitely more likely* than (s_j',r_j) if $k(r_i)(s_j,r_j) < k(r_i)(s_j',r_j)$.

Definition 5.1 *Type r_i respects the opponents' preferences if for every opponent j, every type $r_j \in R_j(r_i)$ and all strategies s_j,s_j' such that r_j strictly prefers s_j to s_j', it holds that r_i deems (s_j,r_j) infinitely more likely than (s_j',r_j).*

Hence, player i should deem superior strategies infinitely more likely than inferior strategies.

So far, we have followed Asheim's model. We now impose an additional condition on types. Type r_i has *point-beliefs on types* if $R_j(r_i)$ only contains one type for every opponent j. Hence, r_i only deems possible one type for every opponent.

Let $E_i \subseteq R_i$ be a subset of player i's types for every i, and let $E = \times_{i\in I}E_i$. We say that r_i *believes in E* if for every k and every opponent j, $\lambda_i^k(r_i)$ only assigns positive probability to player j's types in E_j. We recursively define

$$B_i^1(E) = \{r_i \in E_i | r_i \text{ believes in } E\}$$

for every i, and

$$B_i^k(E) = \left\{r_i \in B_i^{k-1}(E) | r_i \text{ believes in } \times_{j\in I} B_j^{k-1}(E)\right\}$$

for every i and every $k \geq 2$. By $B_i^\infty(E) = \cap_{k\in\mathbb{N}} B_i^k(E)$ we denote the set of player i's types that *respect common belief in E*.

Definition 5.2 *Let $\mathcal{S}$ be an extensive form structure and u a profile of utility functions. A type $r_i \in R_i$ is properly point-rationalizable for $(\mathcal{S},u)$ if r_i respects common belief in the events that types (1) have utility functions as specified by u, (2) respect the opponents' preferences, and (3) have point-beliefs on types. A strategy s_i is properly point-rationalizable for $(\mathcal{S},u)$ if there is a properly point-rationalizable type r_i for $(\mathcal{S},u)$ such that s_i is optimal for r_i.*

The difference between proper point-rationalizability, as we define it, and proper rationalizability, as characterized by Asheim (2001), lies in the condition of point-beliefs on types. Asheim's characterization of properly rationalizable types and strategies, namely, is obtained by imposing common belief in the events (1) and (2) only. Following Asheim (2001), it can be shown that every strategy s_i that is assigned positive probability in some mixed strategy proper equilibrium for $(\mathcal{S}, u)$ is properly point-rationalizable for $(\mathcal{S}, u)$. Therefore, we may conclude that properly point-rationalizable strategies always exist for every $(\mathcal{S}, u)$.

5.2 Relation between persistent and proper point-rationalizability

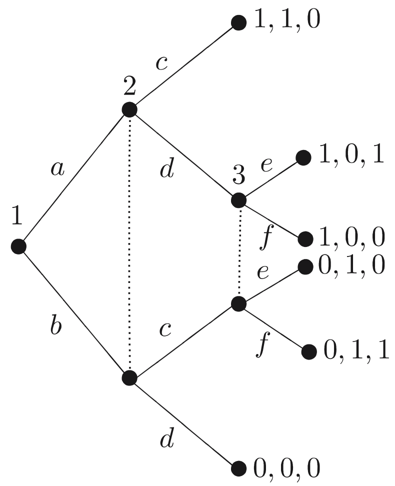
We now prove that in every game tree with observable deviators, every properly point-rationalizable strategy for $(\mathcal{S}, u)$ is persistently rationalizable for $(\mathcal{S}, u)$. This implies that persistently rationalizable strategies always exist for games with observable deviators.

Theorem 5.3 *Let $\mathcal{S}$ be an extensive form structure with observable deviators and $u = (u_i)_{i \in I}$ a profile of utility functions. Then, every properly point-rationalizable strategy for $(\mathcal{S}, u)$ is persistently rationalizable for $(\mathcal{S}, u)$.*

It is easily seen that the converse of this theorem is not true in general: in a simultaneous move game, the concept of persistent rationalizability coincides with "ordinary" rationalizability, as defined by Bernheim (1984) and Pearce (1984). Since it is well-known that not every rationalizable strategy is properly rationalizable (and hence not properly point-rationalizable), this implies that persistently rationalizable strategies need not be properly (point-) rationalizable.

For the proof of the theorem, the assumption of "observable deviators" is crucial. It can even be shown that without this assumption, persistently rationalizable types and strategies may fail to exist for a given game $(\mathcal{S}, u)$. As to illustrate this fact, consider the game in Fig. 3. Let h be the information set controlled by player 3. By definition,

Fig. 3 Importance of observable deviators



$$S(h) = \{(a,d,e),(a,d,f),(b,c,e),(b,c,f)\},$$
$$S_1(h) = \{a,b\}, \quad S_2(h) = \{c,d\} \quad \text{and} \quad S_3(h) = \{e,f\}$$

which implies that $S(h) \neq S_1(h) \times S_2(h) \times S_3(h)$, and hence the game has no observable deviators. Let u_1, u_2 and u_3 be the utility functions depicted at the terminal nodes. We show that there is no persistently rationalizable type, and hence no persistently rationalizable strategy, for player 3 in $(\mathcal{S}, u)$. Suppose, on the contrary, that t_3 is a persistently rationalizable type for player 3 in $(\mathcal{S}, u)$. By IBu, player 3 initially believes that player 1 strictly prefers a to b, and that player 2 strictly prefers c to d. Since $S_1(h) = \{a,b\}$ and $S_2(h) = \{c,d\}$, PBR would require that player 3, at h, assigns probability zero to b and d. However, this is incompatible with the event that h has been reached. We conclude that there is no persistently rationalizable type for player 3 in $(\mathcal{S}, u)$.

The problem with the game in Fig. 3 is that the violation of observable deviators at information set h generates a conflict between BSR and PBR. Namely, persistent rationalizability for u implies that player 3 should initially believe that player 1 ranks a above b and that player 2 ranks c above d. When player 3 finds himself at his information set, he must conclude that either player 1 has not chosen a or player 2 has not chosen c, and hence BSR forces him to give up either his initial belief about player 1's ranking or his initial belief about player 2's ranking. PBR, on the other hand, tells player 3 not to change his belief about opponent j's ranking of strategies unless the observed play of the game provides absolute evidence against this belief. Since the structure of information set h violates observable deviators, the event of reaching player 3's information set does not provide absolute evidence against player 3's initial belief about player 1's ranking of strategies, nor does it provide absolute evidence against player 3's initial belief about player 2's ranking. PBR therefore tells player 3 to maintain his original belief about player 1's ranking and about player 2's ranking, which contradicts BSR at player 3's information set.

On the other hand, this conflict between BSR and PBR cannot occur if the game has observable deviators. Suppose that player i is surprised by the fact that his information set h_i has been reached, since it contradicts his initial belief about the opponents' conditional rankings over strategies. By the observable deviators condition at h_i, player i knows exactly for which opponents he needs to revise his belief about their ranking, and for which opponents he needs not, and hence the conflict as it appears in Fig. 3 cannot arise.

6 Algorithmic characterization of persistently rationalizable strategies

6.1 The algorithm

In this subsection we provide an algorithm that can be used to compute the set of persistently rationalizable strategies for a given extensive form game $(\mathcal{S}, u)$. In order to formally state the algorithm, we need the following definitions.

A *conditional belief vector* for player i is a vector $b_i = (b_i(h_i))_{h_i \in H_i^*}$ that to every $h_i \in H_i^*$ assigns a conditional probability distribution $b_i(h_i) \in \Delta(S_{-i}(h_i))$ on the feasible opponents' strategies. We say that strategy s_i is *sequentially rational* with respect to b_i if at every $h_i \in H_i^*(s_i)$ it holds that $u_i(s_i, b_i(h_i)) = \max_{s_i' \in S_i(h_i)} u_i(s_i', b_i(h_i))$. Here, $u_i(s_i, b_i(h_i))$ is the expected utility induced by the strategy s_i, the conditional belief $b_i(h_i)$ and the utility function u_i. For a given set B_i of conditional belief vectors for player i, and a strategy s_i, let $B_i(s_i)$ be the set of those belief vectors in B_i for which s_i is sequentially rational. For any two strategies s_i, s_i' and information set $h_i \in H_i^*(s_i) \cap H_i^*(s_i')$, say that s_i *is strictly preferred to* s_i' *at* h_i *with respect to* b_i if $u_i(s_i, b_i(h_i)) > u_i(s_i', b_i(h_i))$.

In our algorithm, we recursively define for every player i and every $k \in \mathbb{N}$ a set B_i^k of conditional belief vectors as follows. For $k = 0$, let B_i^0 be the set of all possible conditional belief vectors for every player i. Now, suppose that B_j^{k-1} has been defined for all players j. Then, B_i^k is defined to be the set of those conditional belief vectors b_i in B_i^{k-1} with the following two properties:

(A.1) $b_i(h_0)$ only assigns positive probability to player j's strategies s_j for which $B_j^{k-1}(s_j)$ is nonempty;

(A.2) if there are strategies s_j and s_j' for player j and an information set $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$ such that for every s_j'' assigned positive probability by $b_i(h_0)$ and every $b_j \in B_j^{k-1}(s_j'')$, strategy s_j is strictly preferred to s_j' at h_j with respect to b_j, then $b_i(h_i)$ assigns probability zero to s_j' at every $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$.

For every player i, let $B_i^\infty = \cap_{k \in \mathbb{N}} B_i^k$.

Theorem 6.1 *Let* $(\mathcal{S}, u)$ *be an extensive form game. Then,* s_i *is persistently rationalizable for* $(\mathcal{S}, u)$ *if and only if there is some* $b_i \in B_i^\infty$ *such that* s_i *is sequentially rational with respect to* b_i.

6.2 An illustration

We illustrate the algorithm by means of the example of Fig. 1. Let h_0 and h_1 denote the beginning of the game and player 2's information set, respectively. For every round k, the sets of conditional belief vectors $B_1^k, B_2^k, B_1^k(s_1)$ and $B_2^k(s_2)$, with $s_1 \in S_1$ and $s_2 \in S_2$, are given by Table 1. Here, $b_1(h_0)(f)$ denotes the probability that b_1 assigns at h_0 to f. Similarly for the other expressions. By (c, a) we denote the conditional belief vector for player 2 that initially assigns probability 1 to c, and at h_1 assigns probability 1 to a. The crucial step in the algorithm is to conclude that $B_2^2 = \{(c, a)\}$. In round 2 player 2 initially believes that player 1 chooses c, since $B_1^1(a)$ and $B_1^1(b)$ are empty. Hence, player 2 initially believes that player 1's conditional belief vector is in $B_1^1(c) = B_1^1$. Since for every $b_1 \in B_1^1$ it holds that a is preferred to b, player 2 should assign probability zero to b at h_1. Consequently, player 2's unique

Table 1 Illustration of the algorithm

$$B_1^0(a)=\emptyset,\quad B_1^0(b)=\left\{b_1|b_1(h_0)(f)\geq\tfrac{2}{3}\right\},\quad B_1^0(c)=\left\{b_1|b_1(h_0)(f)\leq\tfrac{2}{3}\right\}$$

$$B_2^0(d)=\left\{b_2|\begin{array}{c}b_2(h_0)(a)\geq b_2(h_0)(b)\\ b_2(h_1)(a)\geq\tfrac{1}{2}\end{array}\right\},\quad B_2^0(e)=\left\{b_2|\begin{array}{c}b_2(h_0)(a)\leq b_2(h_0)(b)\\ b_2(h_1)(a)\leq\tfrac{1}{2}\end{array}\right\},\quad B_2^0(f)=\emptyset$$

$$B_1^1=\{b_1|b_1(h_0)(f)=0\},\quad B_2^1=\{b_2|b_2(h_0)(a)=0\}$$

$$B_1^1(a)=\emptyset,\quad B_1^1(b)=\emptyset,\quad B_1^1(c)=B_1^1$$

$$B_2^1(d)=\left\{b_2|\begin{array}{c}b_2(h_0)=c\\ b_2(h_1)(a)\geq\tfrac{1}{2}\end{array}\right\},\quad B_2^1(e)=\left\{b_2|\begin{array}{c}b_2(h_0)(a)=0\\ b_2(h_1)(a)\leq\tfrac{1}{2}\end{array}\right\},\quad B_2^1(f)=\emptyset$$

$$B_1^2=B_1^1,\quad B_2^2=\{(c,a)\}$$

$$B_1^2(a)=\emptyset,\quad B_1^2(b)=\emptyset,\quad B_1^2(c)=B_1^1$$

$$B_2^2(d)=\{(c,a)\},\quad B_2^2(e)=\emptyset,\quad B_2^2(f)=\emptyset$$

$$B_1^3=B_1^\infty=\{d\},\quad B_2^3=B_2^\infty=\{(c,a)\}$$

conditional belief vector in B_2^2 is (c,a). The algorithm stops after three rounds, and selects a unique conditional belief vector for both players: d for player 1, and (c,a) for player 2. By Theorem 6.1, the unique persistently rationalizable strategies for the players are thus c and d.

6.3 Comparison with Schulte's algorithm

We proceed by comparing our algorithm above with Schulte's "iterated backward inference algorithm" (Schulte 2003), as both procedures are similar in spirit, although different on a more detailed level. While our procedure iteratively eliminates conditional belief vectors, Schulte's procedure iteratively eliminates, for every information set h, strategies that lead to h. That is, for every $k\in\mathbb{N}$, every information set h and every player i, Schulte iteratively defines monotonically decreasing sets $S_i^k(h)$ of strategies in $S_i(h)$. Intuitively, $S_i^k(h)$ represents the set of player i's strategies that one may attach positive probability to in round k of the procedure, conditional on the event that h has been reached. Hence, these sets $S_i^k(h)$ naturally induce, for every k and every player i, a set $\tilde{B}_i^k$ of "admissible" conditional belief vectors, where $b_i\in\tilde{B}_i^k$ if and only if b_i assigns at every $h_i\in H_i^*$ positive probability only to player j's strategies in $S_j^k(h_i)$. In order to compare Schulte's procedure with ours, it is convenient to provide an algorithmic characterization of Schulte's induced sets of admissible conditional belief vectors $\tilde{B}_i^k$, and compare them with our sets B_i^k of admissible conditional belief vectors as defined in our algorithm.

In Schulte's procedure, the induced sets $\tilde{B}_i^k$ of conditional belief vectors can be generated as follows: for $k=0$, let $\tilde{B}_i^0$ be the set of all conditional belief vectors for player i. For $k = 1$, let $\tilde{B}_i^1$ be the set of conditional belief vectors $b_i \in \tilde{B}_i^0$ with the following property:

(D.1) for every two strategies s_j and s_j' and every $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$ such that s_j is weakly preferred to s_j' at h_j with respect to every $b_j \in \tilde{B}_j^0$ and s_j is strictly preferred to s_j' at h_j with respect to some $b_j \in \tilde{B}_j^0$, it holds that $b_i(h_i)(s_j') = 0$ for all $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$.

For $k \geq 2$, let $\tilde{B}_i^k$ be the set of $b_i \in \tilde{B}_i^{k-1}$ with the following property:

(D.2) for every two strategies s_j and s_j' and every $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$ such that s_j is strictly preferred to s_j' at h_j with respect to every $b_j \in \tilde{B}_j^{k-1}$, it holds that $b_i(h_i)(s_j') = 0$ for all $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$.

As the reader may verify, the conditions (D.1) and (D.2) are very similar to the condition (A.2) in our algorithm. However, they differ slightly on a more detailed level. Condition (D.1), for instance, requires that conditional belief vectors should assign, at every information set h, probability zero to an opponent strategy s_j that is weakly dominated by some other strategy leading to h. Our condition (A.2) does not necessarily rule out such strategies. On the other hand, condition (D.2) is logically weaker than our condition (A.2): while condition (A.2) states that $b_i(h_i)(s_j') = 0$ whenever there are some strategy s_j and information set $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$ such that both s_j and s_j' lead to h_i and s_j is strictly preferred to s_j' at h_j with respect to every $b_j \in \cup_{s_j'' \in \mathrm{supp}(b_i(h_0))} B_j^{k-1}(s_j'') \subseteq B_j^{k-1}$, condition (D.2) only requires this whenever s_j is strictly preferred to s_j' at h_j with respect to every $b_j \in \tilde{B}_j^{k-1}$. Hence, both algorithms differ on a technical level, although they are similar in spirit.

6.4 Alternative characterization of persistently rationalizable strategies

Our algorithm presented above only uses belief revisions about strategy choices, and not belief revisions about opponents' utility functions. The concept of persistent rationalizability, on the other hand, has been defined assuming that types may, and sometimes must, revise their beliefs about the opponents' utility functions. The question thus arises whether one can construct an alternative epistemic model, leading to the same set of persistently rationalizable strategies, in which types do not revise their beliefs about the opponents' utility functions. The answer is "yes", as can be seen from the following lemma.

Lemma 6.2 *Let (S, u) be an extensive form game. Then, s_i is a persistently rationalizable strategy for (S, u) if and only if it is sequentially rational for some type t_i that has utility function u_i and respects common belief in the events that types (1) believe in u at all information sets, (2) satisfy PBR, and (3) initially believe in sequential rationality.*

Similarly to the proof of Theorem 6.1, it can be shown that the algorithm selects exactly those strategies that are sequentially rational for types satisfying the properties in Lemma 6.2. As such, those strategies coincide with the set of persistently rationalizable strategies for $(\mathcal{S}, u)$. Since the proof of this result is basically a copy of the proof of Theorem 6.1, we omit it.

The crucial difference with the original definition of persistently rationalizable strategies is that the alternative definition insists on players maintaining their original belief in the opponents' utility functions, while allowing them to drop their original belief in the opponents' rationality, whereas the original definition insists on players maintaining their original belief in the opponents' rationality, while allowing them to drop their original belief in the opponents' utility functions. In the game of Fig. 1, for instance, persistent rationalizability implies that player 2 initially believes player 1 to choose c, while he revises this belief to a upon observing that he has not chosen c. Within our original definition, player 2 interprets the strategy choice a as a *rational* choice for player 1 since he believes, upon observing that c is not chosen, that player 1's utility from (a, d) is higher than his utility from c. According to the alternative definition, player 2 interprets the strategy choice a as a *suboptimal* choice for player 1, since he maintains his original belief in player 1's utility function, and therefore still believes that c is better than a for player 1.

7 Relation to other concepts

In this section we use the algorithm above to compare persistent rationalizability with the concepts of backward induction, weak sequential rationalizability (Ben-Porath 1997) and extensive form rationalizability (Pearce 1984; Battigalli 1997).

7.1 Backward induction

We show that in generic games with perfect information, every player has a unique persistently rationalizable strategy, namely his backward induction strategy. A game with perfect information $(\mathcal{S}, u)$ is *in generic position* if for every player i and every pair z_1, z_2 of different terminal nodes, $u_i(z_1) \neq u_i(z_2)$. For such a game, let $a^*(h_i) \in A(h_i)$ denote the unique backward induction action at information set h_i. For every player i, there is a unique strategy s_i^* with $s_i^*(h_i) = a^*(h_i)$ for every $h_i \in H_i(s_i^*)$, to which we shall refer as the *backward induction strategy* in $(\mathcal{S}, u)$.

Theorem 7.1 *Let $(\mathcal{S}, u)$ be a game with perfect information in generic position. Then, every player has a unique persistently rationalizable strategy for $(\mathcal{S}, u)$, namely his backward induction strategy in $(\mathcal{S}, u)$.*

In view of Theorem 7.1, the concept of persistent rationalizability may be employed as an alternative epistemic foundation for backward induction in games with perfect information. There is an important difference with other foundations proposed in the literature, such as Aumann (1995), Samet (1996), Balkenborg and Winter (1997), Stalnaker (1998) and Asheim (2002), as persistent rationalizability allows players to revise their conjectures about the opponents' utility functions during the game, whereas the latter foundations do not. In turn, persistent rationalizability requires players to interpret "unexpected moves" (in this case, moves that deviate from the backward induction play) always as being in accordance with common belief in rationality.

7.2 Weak sequential rationalizability

Formally speaking, the concept of weak sequential rationalizability as we use it, is an extension of the notion of *common certainty of rationality at the beginning of the game*, defined in Ben-Porath (1997) for the class of games with perfect information, to the general class of extensive form games. For a given extensive form game $(\mathcal{S}, u)$, say that strategy s_i is *weakly sequentially rationalizable* for $(\mathcal{S}, u)$ if it is sequentially rational for a type t_i with the following properties: (1) t_i has utility function u_i, (2) t_i respects common belief in the event that types *throughout the game* believe in u, and (3) t_i respects common belief in the event that types *initially* believe in sequential rationality. In particular, after observing an unexpected move by player j, type t_i need no longer believe that player j acts rationally. On the other hand, t_i is assumed not to revise his belief about the opponents' utility functions as the game proceeds.

It is well-known that the set of weakly sequentially rationalizable strategies for $(\mathcal{S}, u)$ can be obtained by the following algorithm: first, eliminate strategies that are never sequentially rational for any conditional belief vector. Next, eliminate strategies that are never sequentially rational for any conditional belief vector that initially assigns probability zero to opponent strategies eliminated in the first round. Then, eliminate strategies that are never sequentially rational for any conditional belief vector that initially assigns probability zero to opponent strategies eliminated in the first and second round, and so on.[3] However, it is not hard to verify that this is exactly the procedure that is obtained by deleting the requirement (A.2) in our algorithm of Sect. 6, thus yielding a "weaker" algorithm. Together with Theorem 6.1, this leads to the observation that every persistently rationalizable strategy for $(\mathcal{S}, u)$ is also weakly sequentially rationalizable for $(\mathcal{S}, u)$. The other direction is not true, as we have seen in the introduction.

7.3 Extensive form rationalizability

We have seen that persistent rationalizability and extensive form rationalizability lead to disjoint sets of strategies for player 2 in the game of Fig. 1. However,

[3] For games with perfect information in generic position, this procedure coincides with the Dekel-Fudenberg procedure (Dekel and Fudenberg 1990), that is, one round of elimination of weakly dominated strategies followed by iterative elimination of strongly dominated strategies. See Ben-Porath (1997) for a proof of this result.

both concepts lead to the same outcome in this game, namely the terminal node following c. The question remains whether both concepts may also differ outcome-wise. To this purpose, consider a small variation of the game in Fig. 1 in which the utility-pair $(4,0)$ is replaced by $(4,4)$. Extensive form rationalizability then uniquely selects the strategies b and f, leading to the outcome (b,f). Namely, if player 2 observes that c is not chosen, then, according to extensive form rationalizability, he must conclude that player 1 has chosen b, since a is dominated by c, whereas b is not. Hence, player 2 must respond with f, and player 1 must choose b.

Using the algorithm in Sect. 6, it can be shown that

$$B_1^\infty = \{b_1 | b_1(h_0)(e) = 0\} \quad \text{and} \quad B_2^\infty = \{b_2 | b_2(h_0)(a) = 0\}.$$

Hence, $\{b,c\}$ and $\{d,f\}$ are the sets of persistently rationalizable strategies for the two players. In particular, the outcome c can be reached by persistent rationalizability, but not by extensive form rationalizability. So far, I did not manage to find an example in which the two concepts lead to *disjoint* sets of outcomes.

8 Appendix

In order to prove Theorem 5.3 we need the following two technical lemmas. The first lemma provides a useful technical property of extensive form structures with observable deviators. We need some additional notation. Let i and j be different players, $h_i \in H_i^*$ and $h_j \in H_j$. If h_j precedes h_i, let $A(h_j,h_i)$ be the set of actions at h_j which lead to the information set h_i, that is, $a \in A(h_j,h_i)$ if and only if there is some path from the root to h_i at which a is chosen at h_j. If h_j does not precede h_i, then define $A(h_j,h_i) = A(h_j)$. Recall that $S_j(h_i)$ is the set of player j's strategies that do not avoid h_i. Let $Z_j(h_i)$ be the set of terminal nodes that can be reached by choosing a strategy in $S_j(h_i)$. Let $H_j(s_j)$ be the collection of player j information sets in H_j that are not avoided by strategy s_j.

Lemma 8.1 *Let $\mathcal{S}$ be an extensive form structure with observable deviators. Let i and j be different players and let $h_i \in H_i^*$. Then, the following holds:*
(a) $s_j \in S_j(h_i)$ if and only if $s_j(h_j) \in A(h_j,h_i)$ for every $h_j \in H_j(s_j)$,
(b) $z \in Z_j(h_i)$ if and only if for every information set $h_j \in H_j$ on the path to z, the unique action at h_j leading to z belongs to $A(h_j,h_i)$.

Proof (a) Let $s_j \in S_j(h_i)$. Suppose that there is some $h_j \in H_j(s_j)$ with $s_j(h_j) \notin A(h_j,h_i)$. Then, necessarily, h_j precedes h_i. Hence, by the definition of $A(h_j,h_i)$, the action $s_j(h_j)$ avoids h_i. On the other hand, since h_j precedes h_i, there is some node $x \in h_j$ which leads to h_i. By perfect recall, there is some strategy profile s_{-j} such that (s_j,s_{-j}) reaches x. Hence, there is some strategy profile $(\tilde{s}_j,\tilde{s}_{-j})$ such that $(\tilde{s}_j,\tilde{s}_{-j})$ reaches x and h_i. Since $(\tilde{s}_j,\tilde{s}_{-j}) \in S(h_i)$ and since, by the observable deviators condition, $S(h_i) = \times_{k\in I} S_k(h_i)$, it follows that $\tilde{s}_{-j} \in \times_{k\neq j} S_k(h_i)$. Since $(\tilde{s}_j,\tilde{s}_{-j})$ reaches $x \in h_j$, we know, by perfect recall, that $\tilde{s}_j$ coincides with s_j on the player j information sets preceding h_j. Hence, $(s_j,\tilde{s}_{-j})$ reaches h_j. Since $s_j(h_j)$

avoids h_i, we have that $(s_j, \tilde{s}_{-j})$ does not reach h_i, and hence $(s_j, \tilde{s}_{-j}) \notin S(h_i)$. Since, by the observable deviators condition, $S(h_i) = \times_{k \in I} S_k(h_i)$ and $\tilde{s}_{-j} \in \times_{k \neq j} S_k(h_i)$, it thus follows that $s_j \notin S_j(h_i)$, which is a contradiction. We may thus conclude that $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$.

Now, let s_j be such that $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$. We prove that $s_j \in S_j(h_i)$. We distinguish two cases. Suppose first that there is no player j information set preceding h_i. Then, obviously, $s_j \in S_j(h_i)$. Suppose now that there is some player j information set preceding h_i. Let $h_j \in H_j(s_j)$ be a player j information set preceding h_i such that there is no other player j information set in $H_j(s_j)$ between h_j and h_i. By assumption, $s_j(h_j) \in A(h_j, h_i)$, hence there exists a node $x \in h_j$ such that h_i can be reached through x via action $s_j(h_j)$. By perfect recall, there is some strategy profile $\tilde{s}_{-j}$ for the opponents such that $(s_j, \tilde{s}_{-j})$ reaches x. Since there is no $h'_j \in H_j(s_j)$ between h_j and h_i, and since h_i can be reached through x via $s_j(h_j)$, we can choose $\tilde{s}_{-j}$ such that $(s_j, \tilde{s}_{-j})$ reaches h_i. But then, by definition, $s_j \in S_j(h_i)$. This completes the proof of part (a).

(b) Suppose that $z \in Z_j(h_i)$ and that $h_j \in H_j$ is a player j information set on the path to z. Then, obviously, the unique action at h_j leading to z belongs to $A(h_j, h_i)$. Suppose, on the other hand, that the terminal node z is such that for every player j information set h_j on the path to z, the unique action at h_j leading to z belongs to $A(h_j, h_i)$. Let s_j be a strategy such that at every information set $h_j \in H_j(s_j)$ on the path to z, the strategy s_j chooses the unique action at h_j leading to z, and at every other information set $h_j \in H_j(s_j)$ the strategy s_j chooses some action in $A(h_j, h_i)$. Then, $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$, and hence, by part (a), $s_j \in S_j(h_i)$. Since z can be reached by strategy s_j, it follows that $z \in Z_j(h_i)$. This completes the proof. □

The second lemma deals with the problem of transforming a type from the "proper rationalizability type space" into a type from the "persistent rationalizability type space" while preserving its "relevant properties". Such transformations are relevant for the problem at hand since, in order to prove that properly point-rationalizable strategies are persistently rationalizable, we shall show that every properly point-rationalizable *type* can be transformed into a persistently rationalizable type, while preserving its "relevant properties". Before stating the lemma, we need some additional definitions. For every two players i and j and information set $h_i \in H_i^*$, recall that $Z_j(h_i)$ denotes the set of terminal nodes that can be reached by some strategy in $S_j(h_i)$. Let the utility functions $(u_i)_{i \in I}$ be given. Define for every player i, every $h_i \in H_i^*$ and every opponent j the player j utility function $\tilde{u}_j(h_i) : Z \to \mathbb{R}$ by

$$\tilde{u}_j(h_i)(z) = \begin{cases} u_j(z), & \text{if } z \in Z_j(h_i), \\ u_j(z) - K_j(h_i), & \text{if } z \notin Z_j(h_i), \end{cases} \tag{8.1}$$

where the constant $K_j(h_i) > 0$ is chosen such that $u_j(z_1) > u_j(z_2) - K_j(h_i)$ for all $z_1 \in Z_j(h_i)$ and all $z_2 \notin Z_j(h_i)$. In the proof of Theorem 5.3, $\tilde{u}_j(h_i)$ shall represent player i's belief about player j's utility function once information set h_i has been reached.

Let R_i and T_i denote the set of player i's types in the proper rationalizability model and persistent rationalizability model, respectively. Let R_i^* be the set of types in R_i that respect common belief in the event that types have point-beliefs on types. For every type $r_i \in R_i^*$ and opponent j, let $r_j(r_i)$ be the unique player j type that r_i deems possible. For every type $r_i \in R_i^*$ and every information set $h_i \in H_i^*$, let $\lambda_i(r_i, h_i)$ be the marginal of the lexicographic probability distribution $\lambda_i(r_i)$ on $S_{-i}(h_i)$, and let $\mu_i(r_i, h_i)$ be the first-order probability distribution (or first-order belief) of $\lambda_i(r_i, h_i)$ on $S_{-i}(h_i)$.

Lemma 8.2 *There is a transformation mapping t^* which to every type $r_i \in R_i^*$ and every opponent's information set h_l assigns some type $t^*(r_i, h_l)$ in T_i such that for all r_i, h_l and h_i:*
(a) $t^*(r_i, h_l)$ has utility function $\tilde{u}_i(h_l)$ for all r_i and h_l,
(b) type $t^*(r_i, h_l)$ assigns at h_i probability one to type $t^*(r_j(r_i), h_i) \in T_j$ for every opponent j,
(c) the probability that type $t^*(r_i, h_l)$ assigns at h_i to s_{-i} is equal to the probability that $\mu_i(r_i, h_i)$ assigns to s_{-i}, for all $s_{-i} \in S_{-i}(h_i)$.

The lengthy proof of this result can be found in a previous version (Perea 2003) of this paper, and is omitted here for the sake of brevity.

Proof of Theorem 5.3 Lemma 8.2 guarantees that there is some transformation mapping t^* which to every type $r_i \in R_i^*$ and information set h_l assigns some type $t^*(r_i, h_l) \in T_i$ satisfying the properties (a), (b) and (c) as stated in that lemma. As a preliminary step we first show the following claim, stating that these types $t^*(r_i, h_l)$ have properties that can be used later to show that every properly point-rationalizable strategy for $(\mathcal{S}, u)$ is persistently rationalizable for $(\mathcal{S}, u)$.
Claim 1 For every player i, every properly point-rationalizable type $r_i^* \in R_i^*$, every player $l \neq i$ and every $h_l \in H_l^*$, the type $t^*(r_i^*, h_l)$ satisfies IBu, PBR and BSR.
Proof of Claim 1 Fix a type $t_i^* = t^*(r_i^*, h_l)$, induced by a properly point-rationalizable type r_i^*.

1. *Initial belief in u.* By Lemma 8.2 (b), we know that $\mu_i(t_i^*, h_0)$ assigns probability 1 to type $t^*(r_j(r_i^*), h_0)$ for every opponent j. By Lemma 8.2 (a), such type $t^*(r_j(r_i^*), h_0)$ has utility function $\tilde{u}_j(h_0)$. Since $Z_j(h_0) = Z$, it follows from (8.1) that $\tilde{u}_j(h_0) = u_j$. Hence, t_i^* believes at h_0 that all opponents j hold utility function u_j, and hence t_i^* satisfies IBu.

2. *Proper belief revision.* Suppose that t_i^* initially believes that player j, at information set h_j, strictly prefers strategy s_j to strategy s_j', where $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$. By Lemma 8.2 (b), we know that $\mu_i(t_i^*, h_0)$ assigns probability 1 to player j type $t^*(r_j(r_i^*), h_0)$. Therefore, type $t^*(r_j(r_i^*), h_0)$ strictly prefers s_j to s_j' at h_j. Since we have seen above that $t^*(r_j(r_i^*), h_0)$ has utility function u_j, and since $t^*(r_j(r_i^*), h_0)$ satisfies property (c) above, it follows that

$$u_j(s_j, \mu_j(r_j(r_i^*), h_j)) > u_j(s_j', \mu_j(r_j(r_i^*), h_j)). \tag{8.2}$$

Now, let $\tilde{s}_j$ be the unique strategy in $S_j(h_j)$ that coincides with s_j at every information set $h'_j \in H_j(\tilde{s}_j)$ following h_j, and coincides with s'_j at all information sets $h'_j \in H_j(\tilde{s}_j)$ not following h_j. Then, it follows immediately from (8.2) that

$$u_j(\tilde{s}_j, \mu_j(r_j(r_i^*), h_j)) = u_j(s_j, \mu_j(r_j(r_i^*), h_j)) > u_j(s'_j, \mu_j(r_j(r_i^*), h_j)). \tag{8.3}$$

We shall use (8.3) to prove that $r_j(r_i^*)$ strictly prefers $\tilde{s}_j$ to s'_j.

Recall that $r_j(r_i^*)$ holds a lexicographic probability distribution $\lambda_j(r_j(r_i^*))$ on $S_{-j} \times R_{-j}$, and that $\lambda_j(r_j(r_i^*), h_j)$ is the marginal of $\lambda_j(r_j(r_i^*))$ on $S_{-j}(h_j)$. By $\mu_j(r_j(r_i^*), h_j)$ we have denoted the first-order probability distribution on $S_{-j}(h_j)$ induced by $\lambda_j(r_j(r_i^*), h_j)$. Suppose that $\lambda_j(r_j(r_i^*)) = (\lambda_j^1, \ldots, \lambda_j^L)$, and let l^* be the first order for which $\lambda_j^{l^*}(S_{-j}(h_j) \times R_{-j}) > 0$.

Claim 1.1 For all $l < l^*$, we have that

$$u_j(\tilde{s}_j, \lambda_j^l) = u_j(s'_j, \lambda_j^l).$$

Proof of Claim 1.1 Since, by the observable deviators condition, $S(h_j) = S_j(h_j) \times S_{-j}(h_j)$, and since $\lambda_j^l(S_{-j}(h_j) \times R_{-j}) = 0$ for all $l < l^*$, it follows that both $(\tilde{s}_j, \lambda_j^l)$ and (s'_j, λ_j^l) reach h_j with probability zero for all $l < l^*$. By construction, $\tilde{s}_j$ and s'_j only differ at information sets following h_j, and hence $u_j(\tilde{s}_j, \lambda_j^l) = u_j(s'_j, \lambda_j^l)$ for all $l < l^*$. This completes the proof of Claim 1.1.

Claim 1.2 $u_j(\tilde{s}_j, \lambda_j^{l^*}) < u_j(s'_j, \lambda_j^{l^*})$.

Proof of Claim 1.2 Let $Z(h_j)$ be the set of terminal nodes that follow h_j. Then, we have

$$\begin{aligned} u_j(\tilde{s}_j, \lambda_j^{l^*}) &= \lambda_j^{l^*}(S_{-j}(h_j) \times R_{-j})\, u_j(\tilde{s}_j, \mu_j(r_j(r_i^*), h_j)) + \sum_{z \notin Z(h_j)} \mathbb{P}_{(\tilde{s}_j, \lambda_j^{l^*})}(z) u_j(z) \\ &> \lambda_j^{l^*}(S_{-j}(h_j) \times R_{-j}) u_j(s'_j, \mu_j(r_j(r_i^*), h_j)) + \sum_{z \notin Z(h_j)} \mathbb{P}_{(\tilde{s}_j, \lambda_j^{l^*})}(z) u_j(z) \\ &= \lambda_j^{l^*}(S_{-j}(h_j) \times R_{-j}) u_j(s'_j, \mu_j(r_j(r_i^*), h_j)) + \sum_{z \notin Z(h_j)} \mathbb{P}_{(s'_j, \lambda_j^{l^*})}(z) u_j(z) \\ &= u_j(s'_j, \lambda_j^{l^*}). \end{aligned}$$

Here, $\mathbb{P}_{(\tilde{s}_j, \lambda_j^{l^*})}(z)$ denotes the probability of reaching terminal node z under $(\tilde{s}_j, \lambda_j^{l^*})$. The first equality follows from the observation that (1) $(\tilde{s}_j, s_{-j})$ leads to a terminal node in $Z(h_j)$ if and only if $s_{-j} \in S_{-j}(h_j)$, and (2) $\mu_j(r_j(r_i^*), h_j)$ is the conditional distribution of $\lambda_j^{l^*}$ on $S_{-j}(h_j)$. The inequality follows from (8.3) and the assumption that $\lambda_j^{l^*}(S_{-j}(h_j) \times R_{-j}) > 0$. The second equality follows from the fact that $\tilde{s}_j$ and s'_j only differ at information sets following h_j, and hence

$\mathbb{P}_{(\tilde{s}_j,\lambda_j^{l*})}(z) = \mathbb{P}_{(s_j',\lambda_j^{l*})}(z)$ for all $z \notin Z(h_j)$. The last equality follows from the same argument as used for the first equality.

By Claims 1.1 and 1.2, we may conclude that type $r_j(r_i^*)$ strictly prefers strategy $\tilde{s}_j$ to s_j'. In order to prove that t_i^* satisfies PBR, we must show that $t_i^* = t^*(r_i^*, h_l)$ assigns at every information set $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$ probability zero to s_j'. Let h_i be an information set in $H_i^*(s_j) \cap H_i^*(s_j')$, that is, s_j and s_j' are both in $S_j(h_i)$. Hence, by Lemma 8.1 (a), at every information set $h_j \in H_j$ preceding h_i, both s_j and s_j' choose an action in $A(h_j, h_i)$. Since the actions chosen by $\tilde{s}_j$ coincide either with s_j or s_j', it follows that at every information set $h_j \in H_j$ preceding h_i, also $\tilde{s}_j$ chooses an action in $A(h_j, h_i)$. Therefore, by Lemma 8.1 (a), both $\tilde{s}_j$ and s_j' are in $S_j(h_i)$. Since r_i^* is properly point-rationalizable, and $r_j(r_i^*)$ strictly prefers strategy $\tilde{s}_j$ to s_j', it follows that r_i^* deems $\tilde{s}_j$ infinitely more likely than s_j'. In particular, since $\tilde{s}_j, s_j' \in S_j(h_i)$, it follows that $\mu_i(r_i^*, h_i)$ assigns probability zero to s_j'. By Lemma 8.2 (c) we may then conclude that $t_i^* = t^*(r_i^*, h_l)$ assigns at h_i probability zero to s_j', which was to show. Hence, t_i^* satisfies PBR.

3. *Belief in sequential rationality.* We finally show that $t_i^* = t^*(r_i^*, h_l)$ satisfies BSR. Hence, we must prove that $\mu_i(t_i^*, h_i)$ assigns probability one to the set of sequentially rational strategy-type pairs (s_j, t_j) for all players j and at all information sets $h_i \in H_i^*$. Fix an information set h_i^* and an opponent j. Then, by Lemma 8.2 we know that $\mu_i(t_i^*, h_i^*)$ assigns probability one to type $t_j = t^*(r_j(r_i^*), h_i^*)$, with utility function $u_j(t_j) = \tilde{u}_j(h_i^*)$. Suppose that s_j is a strategy in $S_j(h_i^*)$ that is not sequentially rational for t_j. We prove that $\mu_i(t_i^*, h_i^*)$ puts probability zero on s_j.

Since $u_j(t_j) = \tilde{u}_j(h_i^*)$, and s_j is not sequentially rational for t_j, there exists some information set $h_j^* \in H_j^*(s_j)$ such that s_j is not optimal given the probability distribution $\mu_j(t_j, h_j^*)$ on $S_{-j}(h_j^*) \times T_{-j}$ and the utility function $\tilde{u}_j(h_i^*)$. Since s_j is not optimal at h_j^*, there is some other strategy $s_j^1 \in S_j(h_j^*)$ such that

$$\tilde{u}_j(h_i^*)(s_j, \mu_j(t_j, h_j^*)) < \tilde{u}_j(h_i^*)(s_j^1, \mu_j(t_j, h_j^*)), \tag{8.4}$$

where $\tilde{u}_j(h_i^*)(s_j, \mu_j(t_j, h_j^*))$ is the expected utility induced by the utility function $\tilde{u}_j(h_i^*)$, the strategy s_j and the belief $\mu_j(t_j, h_j^*)$ at h_j^* about the opponents' strategy-type pairs. In order to prove that $\mu_i(t_i^*, h_i^*)$ puts probability zero on s_j, we shall show the following claim.

Claim 1.3 There exists some $s_j' \in S_j(h_i^*)$ such that $r_j(r_i^*)$ strictly prefers s_j' to s_j.

Suppose, namely, that this claim would be true. Then, since r_i^* is properly rationalizable and hence respects the opponents' preferences, it would follow that r_i^* deems s_j infinitely less likely than s_j'. As both s_j and s_j' belong to $S_j(h_i^*)$, this would imply that $\mu_i(r_i^*, h_i^*)$ assigns probability zero to s_j. But then, part (c) in Lemma 8.2 would guarantee that $t_i^* = t^*(r_i^*, h_l)$, at information set h_i^*, attaches probability zero to s_j, which was to show. It thus suffices to prove Claim 1.3 in order to prove BSR of t_i^*.

Proof of Claim 1.3 We shall prove Claim 1.3 through a series of smaller claims. Recall that, by (8.4), there is some strategy $s_j^1 \in S_j(h_j^*)$ such that

$$\tilde{u}_j(h_i^*)(s_j, \mu_j(t_j, h_j^*)) < \tilde{u}_j(h_i^*)(s_j^1, \mu_j(t_j, h_j^*)).$$

Claim 1.3.1 There is a strategy s_j^2, differing from s_j only at information sets following h_j^*, such that

$$\tilde{u}_j(h_i^*)(s_j, \mu_j(t_j, h_j^*)) < \tilde{u}_j(h_i^*)(s_j^2, \mu_j(t_j, h_j^*)).$$

Proof of Claim 1.3.1 Let s_j^2 be the unique strategy in $S_j(h_j^*)$ that coincides with s_j^1 on all information sets $h_j \in H_j(s_j^2)$ following h_j^*, and coincides with s_j on all information sets $h_j \in H_j(s_j^2)$ not following h_j^*. Then, s_j^2 only differs from s_j at information sets following h_j^*, and

$$\tilde{u}_j(h_i^*)(s_j^2, \mu_j(t_j, h_j^*)) = \tilde{u}_j(h_i^*)(s_j^1, \mu_j(t_j, h_j^*)),$$

which together with (8.4) completes the proof of Claim 1.3.1.
Claim 1.3.2 $\tilde{u}_j(h_i^*)(s_j, \mu_j(t_j, h_j^*)) = u_j(s_j, \mu_j(t_j, h_j^*))$.

Here, $u_j(s_j, \mu_j(t_j, h_j^*))$ denotes the expected utility induced by the utility function u_j.
Proof of Claim 1.3.2 Since $s_j \in S_j(h_i^*)$, we know that s_j can only lead to terminal nodes in $Z_j(h_i^*)$, and hence $(s_j, \mu_j(t_j, h_j^*))$ induces a probability distribution on $Z_j(h_i^*)$. By (8.1), $\tilde{u}_j(h_i^*)$ coincides with u_j on $Z_j(h_i^*)$, and hence the claim follows.

Note that s_j^2 is not necessarily a strategy in $S_j(h_i^*)$. However, we can prove the following.
Claim 1.3.3 There is a strategy s_j^3 in $S_j(h_i^*)$ such that

$$\tilde{u}_j(h_i^*)(s_j^2, \mu_j(t_j, h_j^*)) \leq \tilde{u}_j(h_i^*)(s_j^3, \mu_j(t_j, h_j^*)).$$

Proof of Claim 1.3.3 Let

$$\hat{H}_j = \{h_j \in H_j(s_j^2) | s_j^2(h_j) \notin A(h_j, h_i^*)\}.$$

By definition of $A(h_j, h_i^*)$, we have that $a \in A(h_j) \backslash A(h_j, h_i^*)$ if and only if h_j precedes h_i^* and a avoids h_i^*. Hence, if $a \in A(h_j) \backslash A(h_j, h_i^*)$ and $\tilde{h}_j$ follows h_j and a, then $\tilde{h}_j$ cannot precede h_i^*, and hence $A(\tilde{h}_j, h_i^*) = A(\tilde{h}_j)$. Consequently, if h_j and $\tilde{h}_j$ are both in $\hat{H}_j$, then h_j cannot precede nor follow $\tilde{h}_j$. Note that every h_j in $\hat{H}_j$ follows h_j^*. Namely, we have seen that s_j and s_j^2 can only differ at information sets following h_j^*. Since $s_j \in S_j(h_i^*)$, we have, by Lemma 8.1 (a), that $s_j(h_j) \in A(h_j, h_i^*)$ for all h_j. In particular, $s_j(h_j) \in A(h_j, h_i^*)$ at all information sets h_j not following h_j^*. Since s_j^2 coincides with s_j on these information sets, it follows that $s_j^2(h_j) \in A(h_j, h_i^*)$ at all information sets h_j not following h_j^*. Hence, $\hat{H}_j$ can only contain information sets following h_j^*.

Let s_j^3 be some strategy which coincides with s_j^2 on all information sets in $(H_j(s_j^3) \cap H_j(s_j^2)) \backslash \hat{H}_j$, and chooses some action in $A(h_j, h_i^*)$ at all other information sets in $H_j(s_j^3)$. Then, by construction, $s_j^3(h_j) \in A(h_j, h_i^*)$ at all information sets $h_j \in H_j(s_j^3)$. By Lemma 8.1 (a), it then follows that $s_j^3 \in S_j(h_i^*)$. By construction, s_j^3 only differs from s_j^2 at information sets h_j that either belong to $H_j(s_j^3) \cap H_j(s_j^2) \cap \hat{H}_j$, or that follow an information set in $H_j(s_j^3) \cap H_j(s_j^2) \cap \hat{H}_j$. At every information set $h_j \in H_j(s_j^3) \cap H_j(s_j^2) \cap \hat{H}_j$, the strategy s_j^3 chooses some $a \in A(h_j, h_i^*)$, which eventually leads to $Z_j(h_i^*)$. At such information sets h_j, the strategy s_j^2 chooses some action $a \notin A(h_j, h_i^*)$, eventually leading to $Z \backslash Z_j(h_i^*)$. The latter follows from Lemma 8.1 (b). By (8.1), we know that

$$\tilde{u}_j(h_i^*)(z_1) > \tilde{u}_j(h_i^*)(z_2)$$

for all $z_1 \in Z_j(h_i^*)$ and all $z_2 \in Z \backslash Z_j(h_i^*)$, which, together with the observations above, implies that

$$\tilde{u}_j(h_i^*)(s_j^2, \mu_j(t_j, h_j^*)) \leq \tilde{u}_j(h_i^*)(s_j^3, \mu_j(t_j, h_j^*)).$$

This completes the proof of Claim 1.3.3.
Claim 1.3.4 $\tilde{u}_j(h_i^*)(s_j^3, \mu_j(t_j, h_j^*)) = u_j(s_j^3, \mu_j(t_j, h_j^*))$.
Proof of Claim 1.3.4 The proof is identical to the proof of Claim 1.3.2, since $s_j^3 \in S_j(h_i^*)$.

By combining the Claims 1.3.1 until 1.3.4, we obtain that

$$u_j(s_j, \mu_j(t_j, h_j^*)) < u_j(s_j^3, \mu_j(t_j, h_j^*)), \tag{8.5}$$

where both s_j and s_j^3 belong to $S_j(h_j^*) \cap S_j(h_i^*)$, and s_j^3 and s_j only differ at information sets following h_j^*. We have seen namely, that s_j^2 only differs from s_j at information sets following h_j^*, while s_j^3 only differs from s_j^2 at information sets in, or following, $\hat{H}_j$. Since $\hat{H}_j$ only contains information sets following h_j^*, it follows that s_j^3 and s_j only differ at information sets following h_j^*. Summarizing, we thus know that (1) $t_j = t^*(r_j(r_i^*), h_i^*)$, (2) $u_j(s_j, \mu_j(t_j, h_j^*)) < u_j(s_j^3, \mu_j(t_j, h_j^*))$, and (3) s_j and s_j^3 only differ at information sets following h_j^*. By using the same techniques as in the proof of PBR above, one can now show that $r_j(r_i^*)$ strictly prefers s_j^3 to s_j. Since $s_j^3 \in S_j(h_i^*)$, Claim 1.3 follows. As we have seen above, this implies that t_i^* satisfies BSR.

This therefore completes the proof of Claim 1. We thus have shown that for every properly point-rationalizable type r_i for $(\mathcal{S}, u)$ and every information set h_l, the induced type $t^*(r_i, h_l)$ satisfies IBu, PBR and BSR.

Now, let T^* be the set of all types t in $\cup_{i\in I}T_i$ that can be written as $t = t^*(r_i, h_l)$ for some properly point-rationalizable type r_i for $(\mathcal{S}, u)$, and some information set h_l. For a given properly point-rationalizable type r_i for $(\mathcal{S}, u)$, it holds, by definition of proper point-rationalizability, that $r_j(r_i)$ is properly point-rationalizable for every opponent j. Together with property (b) in Lemma 8.2, it follows that every type t in T^* assigns, at every information set, probability 1 to opponents' types in T^*. Since we have seen that every type in T^* satisfies IBu, PBR and BSR, it follows that every type $t^*(r_i, h_l)$ in T^* respects common belief in the events IBu, PBR and BSR. However, this implies that every type $t^*(r_i, h_l)$ induced by a properly point-rationalizable type r_i for $(\mathcal{S}, u)$, is persistently rationalizable and respects common belief in the event IBu.

Now, let s_i^* be a properly point-rationalizable strategy for $(\mathcal{S}, u)$. Then, there is some properly point-rationalizable type r_i^* for $(\mathcal{S}, u)$ such that s_i^* is optimal for r_i^*. Let $t_i^* = t^*(r_i^*, h_0)$. Then, by property (a) in Lemma 8.2, t_i^* holds utility function $\tilde{u}_i(h_0) = u_i$. Since we have seen above that t_i^* is persistently rationalizable and respects common belief in the event IBu, it follows that t_i^* is persistently rationalizable for $(\mathcal{S}, u)$.

Since s_i^* is optimal for r_i^*, and since the lexicographic probability distribution $\lambda_i(r_i^*)$ is cautious, it follows that s_i^* is optimal with respect to $\mu_i(r_i^*, h_i)$ at every information set $h_i \in H_i^*(s_i^*)$. By property (c) in Lemma 8.2, we then know that s_i^* is optimal with respect to $\mu_i(t_i^*, h_i)$ for all $h_i \in H_i^*(s_i^*)$. This implies that s_i^* is sequentially rational for t_i^*, and hence s_i^* is persistently rationalizable for $(\mathcal{S}, u)$. We thus have shown that every properly point-rationalizable strategy for $(\mathcal{S}, u)$ is persistently rationalizable for $(\mathcal{S}, u)$. This completes the proof of this theorem. □

Proof of Theorem 6.1 For every player i, let T_i^{pr} be the set of persistently rationalizable types for $(\mathcal{S}, u)$. For a given type $t_i \in T_i^{\mathrm{pr}}$, let $b_i(t_i)$ be the induced conditional belief vector, and let $B_i^{\mathrm{pr}} = \{b_i(t_i) \mid t_i \in T_i^{\mathrm{pr}}\}$.
Claim 1 For every player i we have $B_i^{\mathrm{pr}} \subseteq B_i^{\infty}$.
Proof of Claim 1 We show by induction on k that $B_i^{\mathrm{pr}} \subseteq B_i^k$ for all k. By definition, it holds that $B_i^{\mathrm{pr}} \subseteq B_i^0$ for all players i. Now, take a player i, and suppose that $B_j^{\mathrm{pr}} \subseteq B_j^{k-1}$ for all players j. We show that $B_i^{\mathrm{pr}} \subseteq B_i^k$. Take some $b_i \in B_i^{\mathrm{pr}}$. Hence, there is some persistently rationalizable type t_i for $(\mathcal{S}, u)$ such that $b_i = b_i(t_i)$. Assume that $b_i(t_i)(h_0)$ assigns positive probability to some s_j. Hence, there is some $t_j \in T_j$ such that $\mu_i(t_i, h_0)$ assigns positive probability to (s_j, t_j). As t_i is persistently rationalizable for $(\mathcal{S}, u)$, it follows that t_j must be persistently rationalizable for $(\mathcal{S}, u)$ and that s_j must be sequentially rational for t_j. But then, s_j is sequentially rational with respect to $b_j(t_j) \in B_j^{\mathrm{pr}}$. Since, by induction assumption, $B_j^{\mathrm{pr}} \subseteq B_j^{k-1}$, it follows that $b_j(t_j) \in B_j^{k-1}(s_j)$, and hence $B_j^{k-1}(s_j)$ is nonempty. Hence, $b_i(t_i)$ satisfies (A.1) above.

Suppose now that there are some s_j, s_j' and $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$ such that for all s_j'' assigned positive probability by $b_i(t_i)(h_0)$ and all $b_j \in B_j^{k-1}(s_j'')$, strategy s_j is strictly preferred to s_j' at h_j with respect to b_j. Let t_i initially assign positive probability to some strategy-type pair (s_j'', t_j). Since t_i is persistently rationalizable for $(\mathcal{S}, u)$, it follows that $t_j \in T_j^{\text{pr}}$. By induction assumption we have that $B_j^{\text{pr}} \subseteq B_j^{k-1}$, and hence $b_j(t_j) \in B_j^{\text{pr}} \subseteq B_j^{k-1}$. Since t_i satisfies BSR, it must be the case that s_j'' is sequentially rational for t_j, and hence s_j'' is sequentially rational with respect to $b_j(t_j)$. Combined with the fact that $b_j(t_j) \in B_j^{k-1}$, it follows that $b_j(t_j) \in B_j^{k-1}(s_j'')$. By our assumption above, we then know that s_j is strictly preferred to s_j' at h_j with respect to $b_j(t_j)$. We thus have shown that every strategy-type pair (s_j'', t_j) to which t_i initially assigns positive probability has the property that s_j is strictly preferred to s_j' at h_j with respect to $b_j(t_j)$. Hence, t_i initially believes with probability 1 that player j, at h_j, strictly prefers s_j to s_j'. By PBR of t_i, we may then conclude that t_i, at every $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$, assigns probability zero to s_j'. As such, $b_i(t_i)$ assigns probability zero to s_j' at every $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$. This implies that $b_i(t_i)$ satisfies (A.2) above, and hence $b_i(t_i) \in B_i^k$. We have thus shown that $B_i^{\text{pr}} \subseteq B_i^k$. By induction, it follows that $B_i^{\text{pr}} \subseteq B_i^\infty$ for all players i, which completes the proof of this claim.

Now, suppose that s_i is persistently rationalizable for $(\mathcal{S}, u)$. Then, there is some type $t_i \in T_i^{\text{pr}}$ such that s_i is sequentially rational for t_i, implying that s_i is sequentially rational with respect to $b_i(t_i) \in B_i^{\text{pr}}$. Since $B_i^{\text{pr}} \subseteq B_i^\infty$, it follows that s_i is sequentially rational with respect to some $b_i \in B_i^\infty$, thus establishing the 'only-if' part of the theorem.

In order to prove the 'if' part, choose for every player i a *finite* subset $\hat{B}_i^\infty \subseteq B_i^\infty$ such that for every $b_i \in B_i^\infty$ there is some $\hat{b}_i \in \hat{B}_i^\infty$ with the following property: for any two strategies s_i, s_i' and every information set $h_i \in H_i^*(s_i) \cap H_i^*(s_i')$, strategy s_i is strictly preferred to s_i' at h_i with respect to b_i if and only if s_i is strictly preferred to s_i' at h_i with respect to $\hat{b}_i$.[4] Recall by our notation introduced above, that $\hat{B}_i^\infty(s_i)$ denotes the set of those conditional belief vectors in $\hat{B}_i^\infty$ for which strategy s_i is sequentially rational. By construction of the sets B_i^∞ and $\hat{B}_i^\infty$, we have that every $b_i \in \hat{B}_i^\infty$ satisfies the following two properties:

(B.1) $b_i(h_0)$ only assigns positive probability to player j's strategies s_j for which $\hat{B}_j^\infty(s_j)$ is nonempty;

(B.2) if there are some strategies s_j and s_j' and an information set $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$ such that for all s_j'' assigned positive probability by $b_i(h_0)$ and all $b_j \in \hat{B}_j^\infty(s_j'')$, strategy s_j is strictly preferred to s_j' at h_j with respect to b_j, then $b_i(h_i)$ assigns probability zero to s_j' at all $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$.

For every strategy s_i with nonempty $\hat{B}_i^\infty(s_i)$, define $B_i^*(s_i) := \hat{B}_i^\infty(s_i)$. For every strategy s_i with empty $\hat{B}_i^\infty(s_i)$, let $B_i^*(s_i)$ be some arbitrary subset of $\hat{B}_i^\infty$. Then, we may define for every strategy s_i and every conditional belief vector

[4] Finding such finite subsets $\hat{B}_i^\infty$ is always possible since there are only finitely many information sets $h_i \in H_i^*$, and for every information set h_i there are only finitely many preference relations over strategies in $S_i(h_i)$.

$b_i \in B_i^*(s_i)$ a type $t_i(s_i, b_i)$, with a utility function that may differ from u_i, with the following properties:

(C.1) s_i is sequentially rational for $t_i(s_i, b_i)$;
(C.2) $t_i(s_i, b_i)$ has utility function u_i whenever $\hat{B}_i^\infty(s_i)$ is nonempty;
(C.3) the probability that $t_i(s_i, b_i)$ assigns at $h_i \in H_i^*$ to a strategy-type pair (s_j, t_j) is equal to

$$\begin{cases} b_i(h_i)(s_j)/|B_j^*(s_j)|, & \text{if } t_j = t_j(s_j, b_j) \text{ for some } b_j \in B_j^*(s_j), \\ 0, & \text{otherwise,} \end{cases}$$

where $b_i(h_i)(s_j)$ is the probability that b_i assigns at h_i to s_j.

For every player i, let T_i^* be the set of types $\{t_i(s_i, b_i) \mid s_i \in S_i \text{ and } b_i \in B_i^*(s_i)\}$ obtained in this way.

Claim 2 Every type $t_i \in T_i^*$ is persistently rationalizable, and respects common belief in the event IBu.

Proof of Claim 2 Since, by (C.3), every type $t_i \in T_i^*$ assigns at every $h_i \in H_i^*$ only positive probability to player j's types in T_j^*, it suffices to show that every type $t_i \in T_i^*$ satisfies IBu, BSR and PBR. Take some type $t_i \in T_i^*$, with $t_i = t_i(s_i, b_i)$ for some s_i and some $b_i \in B_i^*(s_i)$.

Initial belief in u. Suppose that $t_i(s_i, b_i)$ initially assigns positive probability to some player j type t_j. By (C.3), there must be some strategy s_j and $b_j \in B_j^*(s_j)$ such that $t_j = t_j(s_j, b_j)$ and $b_i(h_0)(s_j) > 0$. Since $b_i \in \hat{B}_i^\infty$, it follows by (B.1) that $\hat{B}_j^\infty(s_j)$ is nonempty. By (C.2), we may then conclude that $t_j = t_j(s_j, b_j)$ has utility function u_j. Hence, $t_i(s_i, b_i)$ satisfies IBu.

Belief in sequential rationality. By (C.3), type $t_i(s_i, b_i)$ only assigns positive probability to strategy-type pairs (s_j, t_j) where $t_j = t_j(s_j, b_j)$ for some $b_j \in B_j^*(s_j)$. Since, by (C.1), s_j is sequentially rational for $t_j(s_j, b_j)$, BSR follows.

Proper belief revision. Suppose that $t_i(s_i, b_i)$ initially believes with probability 1 that player j, at $h_j \in H_j^*(s_j) \cap H_j^*(s_j')$, strictly prefers s_j to s_j'. Recall that $t_i(s_i, b_i)$ initially believes that player j has utility function u_j. Hence, by (C.3) it follows that for every s_j'' with $b_i(h_0)(s_j'') > 0$, and for every $b_j \in B_j^*(s_j'')$, it holds that s_j is strictly preferred to s_j' at h_j with respect to b_j. Since $b_i \in \hat{B}_i^\infty$, it follows by (B.1) that $\hat{B}_j^\infty(s_j'')$ is nonempty for every s_j'' with $b_i(h_0)(s_j'') > 0$. But then, $B_j^*(s_j'') = \hat{B}_j^\infty(s_j'')$ for all s_j'' with $b_i(h_0)(s_j'') > 0$. We may thus conclude that for every s_j'' with $b_i(h_0)(s_j'') > 0$, and for every $b_j \in \hat{B}_j^\infty(s_j'')$, it holds that s_j is strictly preferred to s_j' at h_j with respect to b_j. By (B.2), we may then conclude that $b_i(h_i)(s_j') = 0$ for all $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$. Together with (C.3), it follows that $t_i(s_i, b_i)$ assigns at every $h_i \in H_i^*(s_j) \cap H_i^*(s_j')$ probability zero to s_j'. Hence, $t_i(s_i, b_i)$ satisfies PBR, which completes the proof of Claim 2.

Suppose, finally, that s_i is sequentially rational with respect to some $b_i \in B_i^\infty$. Then, s_i is sequentially rational with respect to some $b_i \in \hat{B}_i^\infty(s_i)$, and hence s_i is sequentially rational for the type $t_i = t_i(s_i, b_i) \in T_i^*$. In particular, $\hat{B}_i^\infty(s_i)$

is nonempty, which, by (C.2), implies that $t_i(s_i, b_i)$ has utility function u_i. Since $t_i(s_i, b_i)$ is persistently rationalizable, respects common belief in the event IBu, and since $t_i(s_i, b_i)$ has utility function u_i, it follows that $t_i(s_i, b_i)$ is persistently rationalizable for $(\mathcal{S}, u)$. This implies that s_i is persistently rationalizable for $(\mathcal{S}, u)$, which completes the 'if' part of this theorem. □

Proof of Theorem 7.1 Let $(\mathcal{S}, u)$ be a game with perfect information in generic position. For a given information set $h_i \in H_i^*$ and opponent j, let $S_j^*(h_i)$ denote the set of strategies $s_j \in S_j(h_i)$ such that at every information set $h_j \in H_j(s_j)$ following h_i, the strategy s_j prescribes the backward induction action $a^*(h_j)$. Say that h_i is followed by at most k information sets if every path from h_i to a terminal node passes through at most k information sets. Let H_i^k be the set of information sets $h_i \in H_i^*$ followed by at most k information sets. For every player i, let $B_i^1, B_i^2, \ldots$ be the sets of conditional belief vectors as specified by the algorithm in Sect. 6. We prove the following claim.

Claim For every k, every $b_i \in B_i^k$ and every $h_i \in H_i^k$, the conditional belief $b_i(h_i)$ assigns positive probability only to player j's strategies in $S_j^*(h_i)$.

Proof of Claim By induction on k. If $k = 0$, $b_i \in B_i^0$ and $h_i \in H_i^0$, then h_i is not followed by any information set. Hence, $S_j^*(h_i) = S_j(h_i)$, and the statement holds trivially. Now, assume that the statement holds for $k - 1$ and every player i. We prove that the statement holds for k and every player i. Choose a player i, a conditional belief vector $b_i \in B_i^k$ and an information set $h_i \in H_i^k$. Suppose that $s_j \in S_j(h_i)\backslash S_j^*(h_i)$. We show that $b_i(h_i)$ assigns probability zero to s_j. As $s_j \in S_j(h_i)\backslash S_j^*(h_i)$, there is some $h_j \in H_j(s_j)$ following h_i such that $s_j(h_j) \neq a^*(h_j)$. Take some $s_j^* \in S_j^*(h_i)$. Since h_j follows h_i and $h_i \in H_i^k$, we have that $h_j \in H_j^{k-1}$. Hence, we know by the induction assumption that for every $b_j \in B_j^{k-1}$, the conditional belief $b_j(h_j)$ assigns only positive probability to player k strategies in $S_k^*(h_j)$. This implies that for every $b_j \in B_j^{k-1}$, strategy s_j^* is strictly preferred to s_j at h_j. Since $s_j, s_j^* \in S_j(h_i)$, we have that $h_i \in H_i^*(s_j) \cap H_i^*(s_j^*)$. As $b_i \in B_i^k$, we may therefore conclude by (A.2) of the algorithm that $b_i(h_i)$ assigns probability zero to s_j, which was to show. By induction on k, the claim follows.

Now, choose a strategy s_i that is persistently rationalizable for $(\mathcal{S}, u)$. By Theorem 6.1, we know that s_i is sequentially rational for some conditional belief vector $b_i \in B_i^\infty$. Since $B_i^\infty = \cap_{k \in \mathbb{N}} B_i^k$, we know by the claim that for every information set $h_i \in H_i^*$, the conditional belief $b_i(h_i)$ assigns positive probability only to player j's strategies in $S_j^*(h_i)$. But then, the unique strategy that is sequentially rational with respect to b_i is the backward induction strategy in $(\mathcal{S}, u)$. Hence, s_i must be equal to the backward induction strategy in $(\mathcal{S}, u)$. This completes the proof of the theorem. □

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