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Player splitting in extensive form games*

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Abstract. By a player splitting we mean a mechanism that distributes the information sets of a player among so-called agents. A player splitting is called independent if each path in the game tree contains at most one agent of every player. Following Mertens (1989), a solution is said to have the player splitting property if, roughly speaking, the solution of an extensive form game does not change by applying independent player splittings. We show that Nash equilibria, perfect equilibria, Kohlberg-Mertens stable sets and Mertens stable sets have the player splitting property. An example is given to show that the proper equilibrium concept does not satisfy the player splitting property. Next, we give a definition of invariance under (general) player splittings which is an extension of the player splitting property to the situation where we also allow for dependent player splittings. We come to the conclusion that, for any given dependent player splitting, each of the above solutions is not invariant under this player splitting. The results are used to give several characterizations of the class of independent player splittings and the class of single appearance structures by means of invariance of solution concepts under player splittings.

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1. Introduction

In an extensive form game, the moments in time at which players have to make decisions are modeled by so-called decision nodes. At a given decision

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node, the corresponding player can base his decision only on the information available to him in that node. The collection of decision nodes between which the player can not distinguish a-priori is called an information set. Hence, the player can deduce from his information that he is in one of the nodes in this set, but he can make no further distinction between these nodes.

In this paper we analyze the situation in which each player delegates the decision making at his information sets to a number of representatives which are called his *agents*. In effect, this means that the original extensive form game is transformed into a new game in which the player set equals the set of agents. Such a transformation is called a *player splitting*. As an extreme example, every player could decide to assign each of his information sets to a different agent, transforming the game into its agent normal form. However, we allow for a more general setting in which it is possible to assign several information sets to the same agent. The aim of the paper is to study different types of player splittings and check to what extent various solution concepts are robust against these player splittings.

In most literature, attention is restricted to *independent* player splittings in which in every play of the game at most one agent of every player appears. Intuitively, by applying an independent player splitting a player only loses the capability to coordinate his actions in different branches of the game tree. However, once the play has reached one branch of the game tree, what happens in other branches is irrelevant. For this reason, robustness against independent player splittings seems to be an appealing property for a solution concept, as Mertens (1989) already indicated. This type of robustness is formalized in the well-known *player splitting property* (Mertens, 1989). We verify this property for a number of solution concepts.

Next, we make the step to *dependent* player splittings in which in a certain play of the game two agents of the same player appear. It seems logical that under a dependent player splitting, the loss of capability to coordinate actions in the same branch of the game tree highly restricts the strategic abilities of a player. This intuition is supported by the fact that all solution concepts considered in this paper are sensitive to dependent player splittings.

The paper is organized as follows. In Section 2 and 3 we present some preliminaries and a formal definition of the player splitting property. Further, we discuss some useful technical results on independent player splittings.

In Section 4 we prove that Nash equilibria, perfect equilibria, Kohlberg-Mertens stable sets and Mertens stable sets satisfy the player splitting property. In an example, it is shown that the proper equilibrium concept does not satisfy the player splitting property. However, proper equilibria are shown to satisfy a weaker version of the player splitting property which is called the *weak player splitting property*.

Section 5 contains the definition of invariance under general player splittings, which is an extension of the player splitting property to the case where player splittings may be dependent. In Section 6 we show that, for every given dependent player splitting, each of the solution concepts mentioned above is not invariant under this player splitting. These results are in accordance with the intuition that dependent player splittings, in contrast to independent player splittings, really change the strategic abilities of the players.

This insight is used in Section 7 to give several characterizations of the class of independent player splittings. It turns out to be the largest class of player splittings under which Nash equilibria, perfect equilibria, Kohlberg-

Mertens stable sets and Mertens stable sets are invariant. Furthermore, it is the largest class of player splittings under which the proper equilibrium concept is weakly invariant. Finally in this section, we consider the class of *single appearance games* in which every player appears at most once in every play of the game. Using previous results, we give several characterizations of this class by means of the invariance of solutions under player splittings.

2. Preliminaries

In an extensive form game, the graphical structure of the game consisting of the game tree, the information sets, the actions, the chance moves and the player labels is called the *extensive form structure*. We denote the set of players by N. The information sets are denoted by h, whereas H is the collection of all information sets. At an information set h, $A(h)$ is the set of actions available at h. We assume that $|A(h)| \geq 2$ for all h. The set of terminal nodes is denoted by Z. The payoff for player i at a terminal node z is given by $u_i(z)$. An extensive form game with extensive form structure $\mathcal{S}$ and payoffs u is denoted as $\Gamma = \langle \mathcal{S}, u \rangle$. Often, we write game instead of extensive form game and structure instead of extensive form structure.

In an extensive form game, a pure strategy for player i is a function m_i assigning to each information set $h \in H_i$ an action $m_i(h) \in A(h)$. The set of player i pure strategies will be denoted by M_i. For a pure strategy profile $m = (m_1, \dots, m_n)$, $v_i(m)$ is the payoff to player i when m is played.

The normal form game $\Gamma_N = \langle M, v \rangle$, where $M := \prod_i M_i$ and $v := (v_1, \dots, v_n)$ is called the *normal form* of Γ.

A *mixed strategy* for player i is a probability distribution p_i on M_i. For a mixed strategy profile $p = (p_1, \dots, p_n)$, $\mathbb{P}_p$ denotes the probability distribution on the terminal nodes generated by p and $v_i(p)$ is the (expected) payoff to player i induced by p.

For a mixed strategy profile p and a mixed strategy p_i', the mixed strategy profile in which player i plays according to p_i' and all other players play according to p is denoted by $p \backslash p_i'$.

Since the set of mixed strategies profiles depends only on the extensive form structure $\mathcal{S}$, we sometimes talk about mixed strategies in $\mathcal{S}$ instead of mixed strategies in Γ.

Player splitting property

Consider an extensive form structure $\mathcal{S}$. A *player splitting* on $\mathcal{S}$ is a mechanism which divides the information sets of each player between so-called agents. Formally, a player splitting on $\mathcal{S}$ is a function π which defines for every player i a partition $\{H_{ij} \,|\, j \in J(i)\}$ of the collection H_i of information sets controlled by player i.

The player splitting π induces a new extensive form structure $\mathcal{S}^\pi$ in which the player set is given by $N' = \{ij | i \in N, j \in J(i)\}$ and every player ij controls the information sets in H_{ij}. The players ij are called *agents* of player i.

For every extensive form game Γ with structure $\mathcal{S}$, π induces a new game Γ^π with structure $\mathcal{S}^\pi$ and payoffs $u_{ij}(z)$ given by $u_{ij}(z) := u_i(z)$ for every terminal node z.

In $\mathscr{S}^\pi$, pure strategies are usually denoted by m_{ij} and the set of pure strategies of agent ij is given by M_{ij}. We use q_{ij} and q to represent mixed strategies and mixed strategy profiles in $\mathscr{S}^\pi$ repectively.

A player splitting on $\mathscr{S}$ is called *independent* if on every path in the game tree, at most one agent of every player is present.

As is argued by Mertens (1989), independent player splittings do not really change the strategic abilities of the players. Therefore, it is considered a desirable property for a solution to be robust against independent player splittings: a property which is known as the *player splitting property* and has been introduced by Mertens (1989). In order to recall the definition of the player splitting property, we need some more definitions.

Let π be an independent player splitting on $\mathscr{S}$. By f^π, we denote the function which assigns to every mixed strategy profile p in $\mathscr{S}$ the mixed strategy profile $q = (q_{ij})_{i \in N, j \in J(i)}$ in $\mathscr{S}^\pi$ given by

$$q_{ij}(m_{ij}) = \sum_{m_i : (m_i)_{ij} = m_{ij}} p_i(m_i)$$

for every $i \in N, j \in J(i)$ and every $m_{ij} \in M_{ij}$. Here, $(m_i)_{ij}$ denotes the restriction of m_i on the information sets in H_{ij}.

The function f^π transforms every mixed strategy p_i of $\mathscr{S}$ into an equivalent mixed strategy vector[1] $(q_{ij})_{j \in J(i)}$ for the agents of i in $\mathscr{S}^\pi$. The word equivalent means that in every mixed strategy profile q' of $\mathscr{S}^\pi$ in which player i plays $(q_{ij})_{j \in J(i)}$, we can replace $(q_{ij})_{j \in J(i)}$ by p_i without changing the probabilities on the terminal nodes. Throughout this paper, we say that *p generates q*.

Note that the function f^π also works for games without perfect recall. This is the reason that we do not require perfect recall when working with independent player splittings. Later on, when studying dependent player splittings, perfect recall is needed.

Two extensive form games $\Gamma = \langle \mathscr{S}, u \rangle$ and $\Gamma' = \langle \mathscr{S}, u' \rangle$ are called *equivalent* if $u_i(z) = u_i'(z)$ whenever player i appears on the path to z. Intuitively, this means that, whenever a player has to move, the payoffs for this player are the same in the remainder of the games Γ and Γ'.

Let φ be a solution assigning to every extensive form game a collection of sets of mixed strategy profiles. We say that the solution φ has the *player splitting property* if for every extensive form game Γ, every independent player splitting π on the structure of Γ and every game Γ' which is equivalent to Γ^π we have that

$$\varphi(\Gamma') = \{f^\pi(S) \mid S \in \varphi(\Gamma)\} \tag{2.1}$$

and for every $T \in \varphi(\Gamma')$ it holds that

$$\bigcup \{S \in \varphi(\Gamma) \mid f^\pi(S) = T\} = (f^\pi)^{-1}(T). \tag{2.2}$$

[1] In the sequel, when we write mixed strategy *vector*, we always mean a set of mixed strategies in the game Γ^π belonging to agents ij of the *same* player i. This is done in order to avoid confusion with mixed strategy *profiles* in Γ^π, which contain a mixed strategy for *each* agent.

Here, S and T denote sets of mixed strategy profiles. In words, condition (2.1) means that the solution sets in Γ' are exactly the images under f^π of the solution sets in Γ, whereas condition (2.2) states that the inverse image of a solution set T in Γ' is the union of solution sets in Γ which are mapped onto T.

In the case where φ is a point valued solution, the two conditions are equivalent to

$$\varphi(\Gamma) = (f^\pi)^{-1}(\varphi(\Gamma')).$$

We say that φ satisfies the *weak player splitting property* if it holds that

$$\{f^\pi(S) \mid S \in \varphi(\Gamma)\} \subset \varphi(\Gamma').$$

Intuitively, this means that f^π transforms solution sets in Γ into solution sets in Γ'.

The definition of the player splitting property is illustrated by the following diagram.

$$\begin{array}{ccccc} \mathscr{S} & \xrightarrow{u} & \Gamma & \xrightarrow{\varphi} & \varphi(\Gamma) \\ \pi\big\downarrow & & & & \big\downarrow f^\pi \\ \mathscr{S}^\pi & \xrightarrow[u']{} & \Gamma' & \xrightarrow[\varphi]{} & \varphi(\Gamma') \end{array}$$

3. Technical properties of independent player splittings

Independent player splittings have the special property that in the resulting game every path crosses at most one agent of every player. In this section, we show that this leads to some very special relationships between the expected payoffs of the original game and the expected payoffs of the new game. Since these results are used repeatedly in section 4, we dedicate a separate section to these relationships.

Throughout this section, let π be an independent player splitting on an extensive form structure $\mathscr{S}$, $\Gamma = \langle \mathscr{S}, u \rangle$ an extensive form game with structure $\mathscr{S}$ and $\Gamma' = \langle \mathscr{S}^\pi, u' \rangle$ a game which is equivalent to Γ^π. Let p be a mixed strategy profile in Γ and q a mixed strategy profile in Γ' generated by p. The expected payoffs in Γ and Γ' are denoted by $v_i(p), v_{ij}(q)$ and $v'_{ij}(q)$ respectively, whereas the payoffs at the terminal nodes are given by $u_i(z), u_{ij}(z)$ and $u'_{ij}(z)$ respectively.

For a collection H' of information sets, $Z(H')$ denotes the set of terminal nodes which follow H'. Since π is independent, $Z(H_i)$ is the disjoint union of the sets $Z(H_{ij})$. Hence, for every mixed strategy profile q in Γ' and every agent ij,

$$v'_{ij}(q) = \sum_{z \in Z} \mathbb{P}_q(z) u'_{ij}(z) = \sum_{z \in Z(H_{ij})} \mathbb{P}_q(z) u_i(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_q(z) u'_{ij}(z), \qquad (3.1)$$

where the second equality follows from the fact that $u'_{ij}(z) = u_i(z)$ for all $z \in Z(H_{ij})$. This relation plays an important role as well as the following lemma.

Lemma 3.1. (a) *If*

$$v'_{ij}(q\backslash m_{ij}) < v'_{ij}(q\backslash l_{ij}),$$

then for all pure strategies m_i *and* l_i *of player* i *with* $(m_i)_{ij} = m_{ij}$, $(l_i)_{ij} = l_{ij}$ *and* $m_i(h) = l_i(h)$ *for all* $h \in H_i\backslash H_{ij}$ *it holds that*

$$v_i(p\backslash m_i) < v_i(p\backslash l_i).$$

(b) *If*

$$v_i(p\backslash m_i) < v_i(p\backslash l_i)$$

then there is an agent ij *with pure strategies* $m_{ij} = (m_i)_{ij}$ *and* $l_{ij} = (l_i)_{ij}$ *such that*

$$v'_{ij}(q\backslash m_{ij}) < v'_{ij}(q\backslash l_{ij}).$$

Proof: (a) Suppose that $v'_{ij}(q\backslash m_{ij}) < v'_{ij}(q\backslash l_{ij})$. Let m_i and l_i be pure strategies of player i with $(m_i)_{ij} = m_{ij}$, $(l_i)_{ij} = l_{ij}$ and $m_i(h) = l_i(h)$ if $h \in H_i\backslash H_{ij}$. Then, we have

$$\begin{aligned} v_i(p\backslash m_i) = v_{ij}(q\backslash m_i) &= \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z)u_i(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z)u_i(z) \\ &= \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash m_{ij}}(z)u'_{ij}(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z)u_i(z) \\ &= v'_{ij}(q\backslash m_{ij}) - \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q\backslash m_{ij}}(z)u'_{ij}(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z)u_i(z) \\ &< v'_{ij}(q\backslash l_{ij}) - \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q\backslash l_{ij}}(z)u'_{ij}(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q\backslash l_i}(z)u_i(z) \\ &= v_i(p\backslash l_i). \end{aligned}$$

Here, the inequality follows from the fact that (1) $v'_{ij}(q\backslash m_{ij}) < v'_{ij}(q\backslash l_{ij})$ and (2) $\mathbb{P}_{q\backslash m_{ij}}(z) = \mathbb{P}_{q\backslash l_{ij}}(z)$ and $\mathbb{P}_{q\backslash m_i}(z) = \mathbb{P}_{q\backslash l_i}(z)$ for $z \notin Z(H_{ij})$. The last equality is obtained if we substitute m_i by l_i in the first three equations.

(b) Suppose that $v_i(p\backslash m_i) < v_i(p\backslash l_i)$. For a terminal node $z \notin Z(H_i)$, $\mathbb{P}_q(z)$ does not depend on player i's strategy. Hence, for such a z and any pure strategy r_i of player i it holds that $\mathbb{P}_{q\backslash r_i}(z) = \mathbb{P}_q(z)$ leading to

$$v_i(p\backslash r_i) = \sum_{j \in J(i)} \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash r_i}(z)u_i(z) + \sum_{z \notin Z(H_i)} \mathbb{P}_q(z)u_i(z).$$

Hence, the inequality $v_i(p\backslash m_i) < v_i(p\backslash l_i)$ implies that

$$\sum_{j \in J(i)} \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z)u_i(z) < \sum_{j \in J(i)} \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash l_i}(z)u_i(z).$$

So we can find an agent ij such that

$$\sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z) u_i(z) < \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash l_i}(z) u_i(z).$$

By defining $m_{ij} := (m_i)_{ij}$ and $l_{ij} := (l_i)_{ij}$ we obtain

$$\begin{aligned} v'_{ij}(q\backslash m_{ij}) &= \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash m_{ij}}(z) u_i(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z) u'_{ij}(z) \\ &= \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash m_i}(z) u_i(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q}(z) u'_{ij}(z) \\ &< \sum_{z \in Z(H_{ij})} \mathbb{P}_{q\backslash l_i}(z) u_i(z) + \sum_{z \notin Z(H_{ij})} \mathbb{P}_{q}(z) u'_{ij}(z) \\ &= v'_{ij}(q\backslash l_{ij}). \end{aligned}$$

□

4. Behavior of solutions under independent player splittings

In this section, we investigate how Nash equilibria, perfect equilibria, proper equilibria, Kohlberg-Mertens stable sets and Mertens stable sets behave under independent player splittings. It turns out that all solution concepts listed above, except the proper equilibria, satisfy the player splitting property.

4.1. Nash equilibria

The correspondence which assigns to every extensive form game Γ the set of Nash equilibria (Nash, 1950) of the normal form is denoted by NE.

Theorem 4.1. *The Nash equilibrium concept satisfies the player splitting property.*

Proof: Let Γ be an extensive form game, π an independent player splitting on the structure of Γ and Γ' a game which is equivalent to Γ^π. The expected payoffs in Γ, Γ^π and Γ' are denoted by $v_i(p), v_{ij}(q)$ and $v'_{ij}(q)$ respectively. We show that

$$(f^\pi)^{-1}(NE(\Gamma')) = NE(\Gamma).$$

(a) First we prove that $NE(\Gamma) \subset (f^\pi)^{-1}(NE(\Gamma'))$.

Let $p \in NE(\Gamma)$ and let $q = f^\pi(p)$. We prove that $q \in NE(\Gamma')$.

Let $m_{ij}, l_{ij} \in H_{ij}$ and $q_{ij}(m_{ij}) > 0$. We can choose m_i, l_i with $(m_i)_{ij} = m_{ij}$, $(l_i)_{ij} = l_{ij}$ and $m_i(h) = l_i(h)$ for all $h \in H_i \backslash H_{ij}$. Since p is a Nash equilibrium and $p_i(m_i) > 0$ we have $v_i(p\backslash m_i) \geq v_i(p\backslash l_i)$. By Lemma 3.1 (a), it follows that $v'_{ij}(q\backslash m_{ij}) \geq v'_{ij}(q\backslash l_{ij})$. Since this holds for all such m_{ij}, l_{ij} we may conclude that q is a Nash equilibrium in Γ'.

(b) Now, we show that $(f^{\pi})^{-1}(NE(\Gamma')) \subset NE(\Gamma)$.

Let $p \in (f^{\pi})^{-1}(NE(\Gamma'))$ and $q = f^{\pi}(p)$. So, by construction, $q \in NE(\Gamma')$. We prove that $p \in NE(\Gamma)$. Let $m_i, l_i \in M_i$ be such that $v_i(p \backslash m_i) < v_i(p \backslash l_i)$. Then, by Lemma 3.1 (b), there is an agent ij with pure strategies $m_{ij} = (m_i)_{ij}$ and $l_{ij} = (l_i)_{ij}$ such that $v'_{ij}(q \backslash m_{ij}) < v'_{ij}(q \backslash l_{ij})$. Since q is a Nash equilibrium in Γ', it follows that $q_{ij}(m_{ij}) = 0$. Therefore,

$$0 = q_{ij}(m_{ij}) = \sum_{r_i:(r_i)_{ij}=m_{ij}} p_i(r_i) \geq p_i(m_i),$$

which implies that $p_i(m_i) = 0$. Since this holds for every such m_i, it follows that p is a Nash equilibrium in Γ. □

4.2. Perfect equilibria

In this paper, we exploit the following characterization of perfect equilibria (Selten, 1975) for normal form games.

A mixed strategy profile p is a perfect equilibrium in a normal form game if and only if there is a sequence $(p^k)_{k \in \mathbb{N}}$ of completely mixed mixed strategy profiles converging to p such that $v_i(p^k \backslash m_i) < v_i(p^k \backslash l_i)$ for some k implies $p_i(m_i) = 0$. By completely mixed, we mean that every pure strategy is played with strictly positive probability. By PE, we denote the correspondence which assigns to every extensive form game the set of perfect equilibria of the *normal form*. This correspondence is therefore different from the original definition of perfect equilibria for extensive form games, which makes use of the *agent normal form* and is given in terms of behavior strategy profiles. Whenever we speak about a perfect equilibrium of the extensive form game Γ, we mean a perfect equilibrium of the normal form of Γ.

Before we come to the main result we need two technical lemmas. The first lemma can be found in Cook *et al.* (1986).

Lemma 4.2. *Let A be a real $m \times n$ matrix, $B := \{b | Ax \leq b \text{ is solvable}\}$ and $\psi(b) := \{x | Ax \leq b\}$ for every $b \in B$. Then, there is an $L > 0$ such that*

$$d_H(\psi(b), \psi(b')) \leq L \cdot \|b - b'\|$$

for every $b, b' \in B$.

Here, d_H denotes the Hausdorff-distance and $\| \cdot \|$ represents the maximum norm.

Lemma 4.3. *Let $\mathcal{S}$ be an extensive form structure and π an independent player splitting on $\mathcal{S}$. Moreover, let p be a mixed strategy profile in $\mathcal{S}$, $q = f^{\pi}(p)$ and q^k a sequence of completely mixed mixed strategy profiles in $\mathcal{S}^{\pi}$ converging to q. Then, there is a sequence p^k of completely mixed mixed strategy profiles in $\mathcal{S}$ converging to p with $f^{\pi}(p^k) = q^k$ for every k.*

Proof: Let $\mathcal{M}(\mathcal{S})$ and $\mathcal{M}(\mathcal{S}^{\pi})$ be the sets of mixed strategy profiles in $\mathcal{S}$ and $\mathcal{S}^{\pi}$ respectively. Since the function $f^{\pi} : \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{S}^{\pi})$ is linear and surjective and the set $\mathscr{M}(\mathscr{S})$ is determined by linear equalities and inequalities, there is a matrix A and an affine function b on $\mathscr{M}(\mathscr{S}^\pi)$ such that

$$(f^\pi)^{-1}(q) = \{p | Ap \leq b(q)\}.$$

So, by Lemma 4.2, we can find a constant $L > 0$ such that

$$d_H((f^\pi)^{-1}(q), (f^\pi)^{-1}(q')) \leq L \cdot \|b(q) - b(q')\|$$

for all q, q'. Therefore, since $b(q)$ is affine, there is an $L' > 0$ with

$$d_H((f^\pi)^{-1}(q), (f^\pi)^{-1}(q')) \leq L' \cdot \|q - q'\|$$

for all q, q'. Now, let p be a mixed strategy profile in $\mathscr{S}$, $q = f^\pi(p)$ and q^k a sequence of completely mixed mixed strategy profiles in $\mathscr{S}^\pi$ converging to q. By the inequality above we can find a sequence $\bar{p}^k \in (f^\pi)^{-1}(q^k)$ converging to p.

For every k, let $\hat{p}^k$ be the mixed strategy profile in $\mathscr{S}$ given by

$$\hat{p}_i^k(m_i) = \prod_{ij \in J(i)} q_{ij}^k((m_i)_{ij})$$

for all $i \in N$ and $m_i \in M_i$. Obviously, $\hat{p}^k$ is completely mixed for every k. For each k we define p^k by

$$p^k = \left(1 - \frac{1}{k}\right)\bar{p}^k + \frac{1}{k}\hat{p}^k.$$

Since $f^\pi(\hat{p}^k) = q^k$ and $f^\pi(\bar{p}^k) = q^k$, it follows by linearity of f^π that $f^\pi(p^k) = q^k$. The observation that p^k converges to p and p^k is completely mixed completes the proof. □

Theorem 4.4. *The perfect equilibrium concept satisfies the player splitting property.*

Proof: Let Γ be an extensive form game, π an independent player splitting on the structure of Γ and Γ' a game which is equivalent to Γ^π. The expected payoffs in Γ, Γ^π and Γ' are denoted by $v_i(p), v_{ij}(q)$ and $v'_{ij}(q)$ respectively.

(a) First we show that $(f^\pi)^{-1}(PE(\Gamma')) \subset PE(\Gamma)$.

Suppose that $p \in (f^\pi)^{-1}(PE(\Gamma'))$, which means that $q = f^\pi(p)$ is a perfect equilibrium in Γ'. Then there is a sequence q^k of completely mixed mixed strategy profiles converging to q such that $q_{ij}(m_{ij}) = 0$ if $v'_{ij}(q^k \backslash m_{ij}) < v'_{ij}(q^k \backslash l_{ij})$. By Lemma 4.3 there is a sequence p^k of completely mixed mixed strategy profiles in Γ converging to p with $f^\pi(p^k) = q^k$.

In order to show that p is perfect, we suppose that $v_i(p^k \backslash m_i) < v_i(p^k \backslash l_i)$ for pure strategies m_i and l_i of player i. By Lemma 3.1 (b) there is an agent ij with pure strategies $m_{ij} = (m_i)_{ij}$ and $l_{ij} = (l_i)_{ij}$ such that $v'_{ij}(q^k \backslash m_{ij}) <$

$v'_{ij}(q^k \backslash l_{ij})$. This implies that $q_{ij}(m_{ij}) = 0$. Since

$$q_{ij}(m_{ij}) = \sum_{r_i:(r_i)_{ij}=m_{ij}} p_i(r_i),$$

it follows that $p_i(r_i) = 0$ for all r_i with $(r_i)_{ij} = m_{ij}$. In particular, $p_i(m_i) = 0$, which implies that p is a perfect equilibrium for the game Γ.

(b) Next, we show that $PE(\Gamma) \subset (f^\pi)^{-1}(PE(\Gamma'))$.

Let $p \in PE(\Gamma)$. So there is a sequence p^k of completely mixed mixed strategy profiles converging to p such that $p_i(m_i) = 0$ if $v_i(p^k \backslash m_i) < v_i(p^k \backslash l_i)$. We prove that $q = f^\pi(p)$ is a perfect equilibrium in Γ'. If q^k is the mixed strategy profile in Γ' generated by p^k, then q^k is completely mixed and the sequence q^k converges to q.

In order to show that q is perfect, we suppose that $v'_{ij}(q^k \backslash m_{ij}) < v'_{ij}(q^k \backslash l_{ij})$. Let m_i, l_i be pure strategies with $(m_i)_{ij} = m_{ij}$, $(l_i)_{ij} = l_{ij}$ and $m_i(h) = l_i(h)$ for all $h \in H_i \backslash H_{ij}$. Then, by Lemma 3.1 (a), $v_i(p^k \backslash m_i) < v_i(p^k \backslash l_i)$ which implies that $p_i(m_i) = 0$. Since this holds for every pure strategy m_i with $(m_i)_{ij} = m_{ij}$ it follows that

$$q_{ij}(m_{ij}) = \sum_{m_i:(m_i)_{ij}=m_{ij}} p_i(m_i) = 0.$$

Hence q is a perfect equilibrium for the game Γ'. □

4.3. Proper equilibria

A mixed strategy profile p is called ε-proper in a normal form game for some $\varepsilon > 0$ if it is completely mixed and $v_i(p \backslash m_i) < v_i(p \backslash l_i)$ implies $p_i(m_i) \leq \varepsilon p_i(l_i)$. We call p a proper equilibrium (Myerson, 1978) if there is a sequence $(\varepsilon^k)_{k \in \mathbb{N}}$ of strictly positive numbers converging to zero and a sequence $(p^k)_{k \in \mathbb{N}}$ of ε^k-proper mixed strategy profiles converging to p. The correspondence which assigns to every extensive form game the set of proper equilibria of the normal form is called PR.

Remark 1: The proper equilibrium concept does not satisfy the player splitting property.

Proof: Consider the signaling game Γ below which is taken from Cho and Kreps (1987), pp. 200–201. Let π be the independent player splitting which divides the information sets of player 1 among the agents 1a and 1b respectively.

First we prove that $q = (a; c; e)$ is a proper equilibrium in Γ^π. For $\varepsilon > 0$ sufficiently small

$$q^\varepsilon = ((1 - \varepsilon^2)a + \varepsilon^2 b; (1 - \varepsilon^2)c + \varepsilon^2 d; (1 - \varepsilon^2)e + \varepsilon^2 f)$$

is ε-proper for the game Γ^π. Since $\lim_{\varepsilon \downarrow 0} q^\varepsilon = q$ it follows that q is a proper equilibrium in Γ^π.

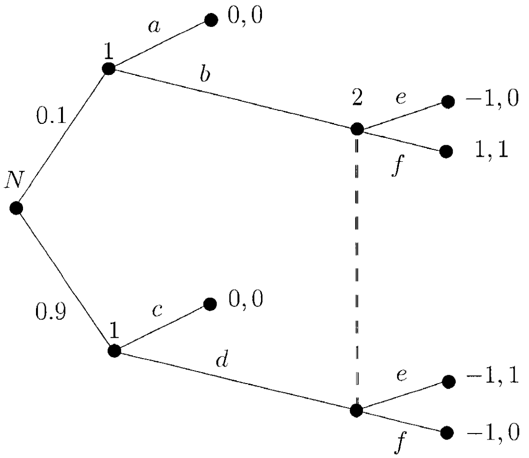

Fig. 1.

Next, we show that the unique mixed strategy profile $p = (ac; e)$ in Γ which induces q is not a proper equilibrium in Γ. The normal form of Γ is given by

	e	f
ac	$0, 0$	$0, 0$
ad	$-0.9, 0.9$	$-0.9, 0$
bc	$-0.1, 0$	$0.1, 0.1$
bd	$-1, 0.9$	$-0.8, 0.1$

Assume that p is a proper equilibrium for the game Γ. Then, for $\varepsilon > 0$ small enough there is an ε-proper profile $p^\varepsilon = (p_1^\varepsilon, p_2^\varepsilon)$ in Γ with $\lim_{\varepsilon\downarrow 0} p^\varepsilon = p$. If p_2^ε is close to e then ad and bd are both worse responses than bc. Hence, ad and bd are played with a very small probability compared to bc. But then, e is a worse response than f implying that e should be played with probability zero in p. This is a contradiction. Therefore, p is not a proper equilibrium in Γ implying that the proper equilibrium correspondence does not satisfy the player splitting property. □

However, we can show that the proper equilibria correspondence satisfies the *weak* player splitting property. In order to prove this, we need the following lemma. In this lemma, let Γ be an extensive form game, π an independent player splitting on the structure of Γ and Γ' a game which is equivalent to Γ^π.

Lemma 4.5. *If p is ε-proper in Γ, then $f^\pi(p)$ is ε-proper in Γ'.*

Proof: Let $\varepsilon > 0$ and p an ε-proper mixed strategy profile for the game Γ. Then, obviously, the mixed strategy profile $q = f^\pi(p)$ is completely mixed since p is completely mixed. In order to show that q is ε-proper for the game Γ', we prove that $q_{ij}(m_{ij}) \leq \varepsilon q_{ij}(l_{ij})$ if $v'_{ij}(q\backslash m_{ij}) < v'_{ij}(q\backslash l_{ij})$.

So suppose that $v'_{ij}(q\backslash m_{ij}) < v'_{ij}(q\backslash l_{ij})$. Then by Lemma 3.1 (a), $v_i(p\backslash m_i) < v_i(p\backslash l_i)$ for all pure strategies m_i and l_i with $(m_i)_{ij} = m_{ij}$, $(l_i)_{ij} = l_{ij}$ and $m_i(h) = l_i(h)$ for all $h \in H_i\backslash H_{ij}$. Since p is an ε-proper mixed strategy profile for the game Γ, $p_i(m_i) \leq \varepsilon p_i(l_i)$ for all such pairs m_i, l_i.

For every pure strategy m_i with $(m_i)_{ij} = m_{ij}$ we can find a corresponding pure strategy $m_i\backslash l_{ij}$ which prescribes l_{ij} at H_{ij} and coincides with m_i at $H_i\backslash H_{ij}$. Since $p_i(m_i) \leq \varepsilon p_i(m_i\backslash l_{ij})$ for such m_i we have

$$q_{ij}(m_{ij}) = \sum_{m_i:(m_i)_{ij}=m_{ij}} p_i(m_i) \leq \sum_{m_i:(m_i)_{ij}=m_{ij}} \varepsilon p_i(m_i\backslash l_{ij})$$

$$= \sum_{l_i:(l_i)_{ij}=l_{ij}} \varepsilon p_i(l_i) = \varepsilon q_{ij}(l_{ij}).$$

Since this holds for every m_{ij} and l_{ij} with $v'_{ij}(q\backslash m_{ij}) < v'_{ij}(q\backslash l_{ij})$, it follows that q is ε-proper in the game Γ'. □

This result immediately implies the following theorem.

Theorem 4.6. *The proper equilibria concept satisfies the weak player splitting property.*

4.4. Stable sets

Stable sets

In a normal form game Γ, a set S of mixed strategy profiles is called a Kohlberg-Mertens stable set (KM-stable set) (Kohlberg and Mertens, 1986) if it is minimal with respect to the following property: S is closed and for every open set $\mathcal{O}$ containing S there is an $\varepsilon > 0$ such that for every strictly positive mistake vector $\eta = (\eta_j(m_j))_{j\in N, m_j \in Mj} \leq \varepsilon$ we have

$$NE(\Gamma^\eta) \cap \mathcal{O} \neq \emptyset.$$

Here, Γ^η is the restriction of the game Γ to mixed strategies p_j with $p_j(m_j) \geq \eta_j(m_j)$ for all j and all $m_j \in M_j$. The game Γ^η is called a perturbed game. Let $\mathcal{KM}$ be the correspondence which assigns to every extensive form game the collection of KM-stable sets of the normal form. In order to prove the following theorem, we need the definition of invariant solutions which can be found in Mertens (1987), Theorem 2 (b).[2] As to distinguish it from invariance under player splittings, we call this concept *Mertens invariance*.

A normal form solution φ is called *Mertens invariant* if for every two games Γ, Γ^* with the same player set and every linear, payoff preserving

[2] Mertens did not use the word invariance in this paper.

function f mapping the strategy space of Γ onto the strategy space of Γ^* we have

$$\varphi(\Gamma^*) = \{f(S) \mid S \in \varphi(\Gamma)\} \tag{4.1}$$

and for every $T \in \varphi(\Gamma^*)$ it holds that

$$\bigcup\{S \in \varphi(\Gamma) \mid f(S) = T\} = f^{-1}(T). \tag{4.2}$$

Theorem 4.7. *The KM-stability concept satisfies the player splitting property.*

Proof: Let Γ be an extensive form game, π an independent player splitting on the structure of Γ and Γ' a game which is equivalent to Γ^π. Let Γ^* be the normal form game with the same player set as Γ in which the strategy space of player i is the product of the mixed strategy spaces of the corresponding agents ij. Let f be the payoff preserving, linear function from the strategy space of Γ onto the strategy space of Γ^* which is obtained by taking the marginals. Since it is shown in Mertens (1987), section 4.2.5 that $\mathscr{KM}$ is Mertens invariant, it follows that the Kohlberg-Mertens stable sets of Γ and Γ^* satisfy equations (4.1) and (4.2).

It remains to show that the KM-stable sets of Γ^* and Γ' are the same. However, this follows from the fact that the perturbed games of Γ^* and Γ' are the same as is shown in Mertens (1989), proof of Theorem 4. Consequently, the KM-stable sets of Γ and Γ' satisfy equations (2.1) and (2.2). □

In 1989, Mertens introduced a new stability concept called *Mertens stable sets*. Since it would require too much space to give an exact description of Mertens stable sets we refer to Mertens (1989) for a precise definition. Mertens (1989) showed that the Mertens stability concept has the player splitting property.

5. Invariance under general player splittings

The player splitting property can be viewed as a tool which is used to investigate the behavior of solutions under independent player splittings. Among the solution concepts considered, all but one are robust against independent player splittings. This result supports the idea, stated by Mertens (1989), that independent player splittings do not really change the strategic abilities of the players. On the other hand, Mertens argues that dependent player splittings *do* change the strategic situation of the game. A natural question which arises is whether this statement can be supported by the behavior of solution concepts under dependent player splittings.

In order to answer this question, we introduce a definition of invariance of solutions under player splittings. This definition mainly follows the idea of the player splitting property: the solutions should not change by applying the player splitting. In the next section, we show that each of the solution concepts considered above is not invariant under *any* dependent player splitting. More precisely, for every extensive form structure and every dependent player splitting, we can find payoffs for the terminal nodes such that the solution of the new game, obtained by player splitting, is different from the solution of the

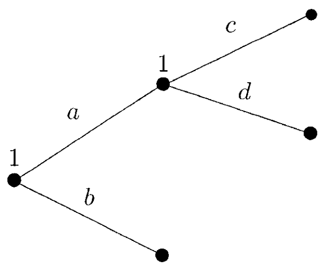

Fig. 2.

original game. Consequently, the class of independent player splittings can be regarded as the *largest* class of player splittings leaving the strategic situation of the game unchanged.

Definition of invariance under player splittings

Let $\mathcal{S}$ be an extensive form structure and π a player splitting on $\mathcal{S}$. In order to compare the solutions in Γ and Γ^π, we need an onto function f which transforms every mixed strategy profile in Γ into an equivalent mixed strategy profile of the new game Γ^π. By the latter, we mean that f transforms every mixed strategy p_i in Γ into an equivalent mixed strategy vector $(q_{ij})_{j \in J(i)}$ in Γ^π. Such a function is called a *transformation function*.

In the extreme case where π assigns a different agent to every information set, the mixed strategy vectors in Γ^π are exactly the behavior strategies in Γ. Kuhn (1953) has shown that every mixed strategy has an equivalent behavior strategy if and only if the extensive form structure has perfect recall. Since, in particular, we want the transformation function f to exist for such 'maximal' player splittings, we assume from now on that $\mathcal{S}$ satisfies perfect recall.

In case the player splitting is maximal, we may use Kuhn's function transforming mixed strategies into equivalent behavior strategies as a candidate for f. If the player splitting is independent, the function f^π defined in Section 2 may serve as a transformation function. However, there may be many transformation functions, as is shown by the following example.

Let π be the trivial player splitting which leaves the extensive form structure unchanged. Then, mapping *bc* onto *bd* and vice versa, while keeping *ac* and *ad* fixed, generates a transformation function.

Now, a solution is said to be invariant under the player splitting π if we can find a transformation function f such that the solution of the orginal game coincides with the solution of the new game, when applying the function f. Formally, we say that the solution φ is *invariant under* π if there is a transformation function f such that for every extensive form game Γ with structure $\mathcal{S}$ and every game Γ' which is equivalent to Γ^π we have

$$\varphi(\Gamma') = \{f(S) \mid S \in \varphi(\Gamma)\} \tag{5.1}$$

and for every $T \in \varphi(\Gamma')$ it holds that

$$\bigcup\{S \in \varphi(\Gamma) \mid f(S) = T\} = f^{-1}(T). \tag{5.2}$$

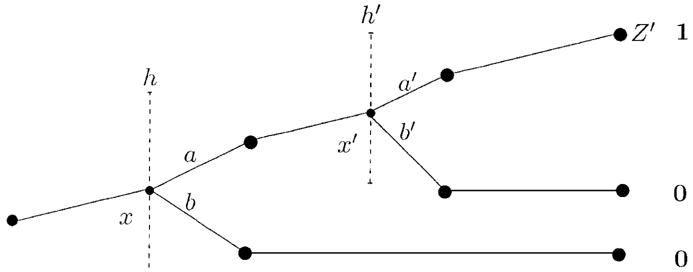

Fig. 3.

If the solution φ is a point valued solution, both conditions are equivalent to

$$\varphi(\Gamma) = f^{-1}(\varphi(\Gamma')).$$

If it holds that

$$\{f(S) \mid S \in \varphi(\Gamma)\} \subset \varphi(\Gamma')$$

we say that φ is *weakly invariant* under π.

From the definition, it is clear that a solution concept satisfying the player splitting property is invariant under every independent player splitting, since the function f^π in Section 2 can be used as transformation function. The converse is not necessarily true, since for a given independent player splitting π, the transformation function f 'justifying' the invariance under π does not have to coincide with the function f^π in Section 2. However, this gap disappears if the solution φ is Mertens invariant (see Section 4 for an exact definition).

6. Behavior of solutions under dependent player splittings

In this section we show that the solution concepts considered in Section 4 are not invariant under any dependent player splitting.

Nash equilibria

Lemma 6.1. *Let $\mathcal{S}$ be an extensive form structure and π a dependent player splitting on $\mathcal{S}$. Then, the Nash equilibrium concept is not invariant under π.*

Proof: Let $\mathcal{S}$ be an extensive form structure and π a dependent player splitting on $\mathcal{S}$. Then there are two different agents ij and ik in $\mathcal{S}^\pi$ which appear both in a certain play of the game. Formally, this means that we can find a path from the root to Z containing two nodes, say $x \in h$ and $x' \in h'$, controlled by agents ij and ik respectively. Let a be the unique action on this path leaving x. Since $|A(h)| \geq 2$, we can find another action, say b at x. Next, choose two different actions a' and b' at x'.

Note that the line from the root to x does not represent one single action. This line covers all the actions and chance moves that are present on the path

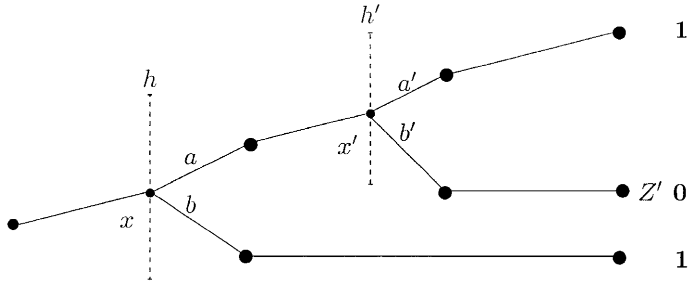

Fig. 4.

to x. The same holds for the line ending at x' and the lines ending at terminal nodes.

By Z' we denote the set of terminal nodes which follow the action a'. Let the payoff function u be given by

$$u_j(z) := \begin{cases} 1 & \text{if } j = i \text{ and } z \in Z' \\ 0 & \text{otherwise} \end{cases}$$

and let Γ be the game $\langle \mathscr{S}, u \rangle$. Choose a pure strategy profile m in Γ^{π} such that

(1) m, chooses all the actions on the path from the root to x', except a and
(2) m chooses the actions b and b'.

Then m is a Nash equilibrium for the game Γ^{π} which gives player i payoff 0.

Let f be an arbitrary transformation function and let $\bar{m} \in f^{-1}(m)$. We show that $\bar{m}$ is not a Nash equilibrium in Γ. Clearly, m is not a Nash equilibrium in Γ since player i can strictly increase his payoff by deviating unilaterally to a pure strategy selecting the actions a and a'. Since $\bar{m}$ and m are payoff equivalent[3] for each player and Nash equilibrium is Mertens invariant (1987), $\bar{m}$ is not a Nash equilibrium. Consequently, the Nash equilibrium correspondence is not invariant under π. □

The other solution concepts

Lemma 6.2. *Let $\mathscr{S}$ be an extensive form structure and π a dependent player splitting on $\mathscr{S}$. Then, perfect equilibria (proper equilibria, KM-stable sets, M-stable sets) are not weakly invariant under π.*

Proof: Let $\mathscr{S}$ be an extensive form structure and π a dependent player splitting on $\mathscr{S}$. Then we can choose $h, h', x, x', a, b, a', b'$ as in Lemma 6.1.

By Z' we denote the set of those terminal nodes following the action b'. Let the payoff function u be given by

[3] Two mixed strategies for a given player are called *payoff equivalent* if they induce the same expected payoff for all players against any mixed strategy profile of the other players.

$$u_j(z) := \begin{cases} 0 & \text{if } j = i \text{ and } z \in Z' \\ 1 & \text{otherwise,} \end{cases}$$

and let $\Gamma = \langle \mathscr{S}, u \rangle$. Choose a pure strategy profile m such that

(1) all the actions on the path to x', including a, are chosen and
(2) the action a' is chosen.

We show that m is a strictly perfect equilibrium in Γ. This follows easily from the fact that player i's strategy in m is a best response against any strategy profile in Γ. Since the other players always receive 1 the strategy profile m is strictly perfect. As a consequence, m is perfect and the set $\{m\}$ is a KM-stable set since every single point set consisting of a strictly perfect equilibrium is a KM-stable set. Moreover, it can be shown that $\{m\}$ is a M-stable set in Γ. Since a M-stable set always contains a proper equilibrium, it follows that m is proper in Γ.

Let π be a player splitting which assigns information sets h and h' to different agents. We prove that for every transformation function f, $f(m)$ is not a perfect equilibrium in Γ^π. Take an arbitrary transformation function f. Let ik be the agent controlling information set h. Since the mixed strategy vector for player i in $f(m)$ should be equivalent to his strategy in m, and all actions leading to x are chosen in m, it follows that $f(m)$ should specify action a at information set h. However, the action a is weakly dominated by b for agent ik in Γ^π, and therefore $f(m)$ can not be perfect in Γ^π. Consequently, $f(m)$ is not proper in Γ^π and the set $f(\{m\})$ is not a KM-stable set and not a M-stable set in Γ^π since KM- and M-stable sets consist solely of perfect equilibria.

Combining all the insights above leads to the conclusion that perfect equilibria, proper equilibria, KM-stable sets and M-stable sets are not weakly invariant under π.[4] □

7. Characterization of independent player splittings and single appearance structures

By combining the results of section 4 and 6, we obtain the following characterizations of the class of independent player splittings.

Theorem 7.1. *The class of independent player splittings is the largest class of player splittings under which*

(a) *Nash equilibria (perfect equilibria, KM-stable sets, M-stable sets) are invariant,*
(b) *proper equilibria are weakly invariant.*

By largest, we mean largest with respect to set inclusion.

Next, we consider a special class of extensive form structures which we call single appearance structures. An extensive form structure is called a *single*

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[4] We thank an anonymous referee for detecting an error in the proof in an earlier version of the paper.

appearance structure if every path in the game tree crosses at most one information set of every player. In other words, each player appears at most one single time in every play of the game. Obviously, the class of single appearance structures can be characterized by stating that it is the largest class of structures for which every player splitting is independent.

Using the theorem above, we can give the following characterizations of the class of single appearance structures in terms of invariance of solution concepts under player splittings.

Theorem 7.2. *The class of single appearance structures is the largest class of extensive form structures $\mathscr{S}$ for which*

(a) *Nash equilibria (perfect equilibria, KM-stable sets, M-stable sets) are invariant under every player splitting on $\mathscr{S}$,*
(b) *proper equilibria are weakly invariant under every player splitting on $\mathscr{S}$.*

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