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Advances

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A Simple Bargaining Procedure for the Myerson Value

Abstract: We consider situations where the cooperation and negotiation possibilities between pairs of agents are given by an undirected graph. Every connected component of agents has a value, which is the total surplus the agents can generate by working together. We present a simple, sequential, bilateral bargaining procedure, in which at every stage the two agents in a link, (i,j) bargain about their share from cooperation in the connected component they are part of. We show that this procedure yields the Myerson value (Myerson, 1997) if the marginal value of any link in a connected component is increasing *in the number of links* in that connected component.

Keywords: Myerson value, networks, bargaining, cooperation

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1 Introduction

This paper introduces a non-cooperative bargaining procedure that leads to the Myerson value (Myerson 1977) for network games. Network games range from the seminal work by Myerson (1977), in which cooperative TU games are restricted by communication structures, to the more general specification introduced by Jackson and Wolinsky (1996), in which the value function takes the network as a primitive.[1] The Myerson value has been widely used as an exogenously given payoff scheme in many non-cooperative settings, although the choice of allocation rules, or equivalently, payoff schemes is quite large.[2]

1 See the book by Jackson (2008), Chapter 12, for a more formal definition.

2 Apart from the Myerson value, many other allocation rules have been proposed and studied for network games. See, for example, Amer and Carreras (1995), Bergantiños, Carreras, and García- Jurado (1993), Borm, Owen, and Tijs (1992), Calvo, Lasaga, and van den Nouweland (1999), Hamiache (1999), Herings, van der Laan, and Talman (2008), Jackson (2005), Slikker (2005), Slikker, Gilles, Norde, and Tijs (2005), Talman and Yamamoto (2008) and Vázquez-Brage, García-Jurado, and Carreras (1996).

The Myerson value is relatively easy to compute. Nevertheless, this is not the only reason for its popularity. The Myerson value holds appealing properties when applied to network formation settings.[3] Namely, some non-cooperative network formation procedures have a potential when the payoffs for players are equal to the Myerson value in each of the possible resulting networks.[4] A similar result is known about the Shapley value and non-cooperative coalition formation procedures.[5] In addition, Jackson (2003) shows that there always exists a pairwise stable network relative to the Myerson value and, more importantly, following improving paths with respect to the Myerson value always leads to a pairwise stable network. These are very important features of the Myerson value, since the existence of pairwise stable networks or them being a "sink" of a dynamic network formation process is not always granted.

The bargaining procedure we propose, consisting of a sequence of bilateral negotiations between players, leads to the Myerson value of the final network if the value function is *link-convex*, or, in other words, if the marginal value of a link to a connected component is increasing in the presence of other links inside that component. The procedure works as follows. We first fix a graph representing the links, or pairs of agents, that are allowed to bargain directly, and a rule of order on them. When a link is called to play, the corresponding two players simultaneously bid to become a proposer of a take-it-or-leave-it offer to the other player. In case of acceptance by the other player, both players cooperate and commit to the proposed pair of payoffs, and the mechanism turns to the next link given by the rule of order. In case of rejection, the proposer has to pay his announced bid to the other player and either player loses the possibility to bargain (and cooperate) directly with each other any time later in the procedure. This means that the procedure starts all over again by fixing a new graph, namely the current graph after deleting the link representing the pair of players not reaching an agreement, and a new rule of order over such a new graph. The procedure continues until all the links in the original graph have been considered. The final result of this procedure is a series of bilateral agreements on payoffs and a graph, possibly empty, which includes all links that have reached an agreement and therefore will cooperate to extract the value.

3 See for example Aumann and Myerson (1988), Dutta, van den Nouweland, and Tijs (1998), Garrat and Qin (2003a) and Garrat and Qin (2003b), Pin (2011), Slikker et al. (2000), Slikker and van den Nouweland (2000) and Slikker and van den Nouweland (2002).

4 For a definition of games with potential, see Monderer and Shapley (1996).

5 See Qin (1996).

Given the sequential nature of our bargaining procedure, there is a first-mover advantage. Two players agreeing on payoffs that are too high can induce a future disagreement when they do not leave enough surplus on the table for the players that bargain later in the procedure. In order to avoid this type of agreement, we need to impose link-convexity on the value function. Indeed, we show that link-convexity of the value function implies that the Myerson value for all agents in a component increases by adding any link to this component. Therefore, if link-convexity is satisfied, there is no incentive for inducing a later disagreement, as the resulting payoffs of the bargaining procedure after a rejection would be smaller for everybody. This allows us to prove that, under the assumption that the surplus from cooperation is link-convex, there is a unique subgame perfect equilibrium outcome in pure strategies to the above explained procedure such that (i) all proposed payoffs are accepted and (ii) the final payoffs for the players coincide with the Myerson value applied to the starting graph.

The need for the value function to be convex can also be found in the implementation literature dealing with the Shapley value.[6] The Demand Commitment Game (DCG) is a well-known game form that Winter (1994) and Dasgupta and Chiu (1998), in two different versions, have proven to lead to the Shapley value in subgame perfect equilibrium. The DCG consists basically of players sequentially sending a demand and committing to join any coalition that would provide them with such a demand.[7] Both versions of the procedure, which differ on how to order players, require the TU game to be convex in order to implement the Shapley value in any subgame perfect equilibrium. Convexity of the TU game implies that the Shapley value to each agent increases if a new agent, or a new set of agents, joins the coalition. Therefore, if the TU game is convex, no agent has an incentive of demanding too much, which would induce some agents out of the grand coalition. In a recent working paper, De Fontenay and Gans (2007) implement the Myerson value in a bilateral sequential setting. Instead of assuming perfect information and convexity on the value function, they assume that (i) only breakdowns, and not agreed-upon

6 In the case of the literature implementing the Shapley value, the value function is defined as a function of coalitions of players. In the case of the Myerson value, the value function is mainly a function of the links in a network.

7 Currarini and Morelli (2000) and Slikker and van den Nouweland (2001) adapt the DCG to network settings. In Currarini and Morelli (2000) players' demands are not contingent on network structures, while Slikker and van den Nouweland (2001) consider a simultaneous version of the DCG where demands are link-contingent, i.e., players send a claim or demand per link in which they would be willing to participate.

payoffs, are common knowledge, and (ii) players hold "passive beliefs" about unobservable actions (i.e., the agreed-upon payoffs inside the links in which they are not participating). Passive beliefs mean that players believe that all unobservable actions are in concordance with equilibrium behavior and, in case of receiving an offer out of equilibrium, they do not update beliefs about the unobservable actions. Therefore, there is no room for foreseeing future rejections as a result of a current (possible) out-of-equilibrium agreements, which, as mentioned above, was causing the need of a convexity-type requirement on the value function.

Finally, Slikker (2007) has shown that a modification of the bidding-for-the-surplus (BFS) obtains the Myerson value for monotonic value functions.[8] As it is a procedure based on the BFS, it mainly differs from our procedure in three points. First, we propose a procedure that is based on sequential and bilateral bargaining, while the BFS is not. Second, in the BFS, prices are always paid in equilibrium, whereas in our mechanism the announced prices are paid only if a proposal is rejected. In particular, no prices are paid in the unique subgame perfect equilibrium outcome. Finally, in case of a rejection, the BFS mechanism for network games deletes all the links of the proposer, while our procedure only eliminates the link for which the players have failed to reach an agreement. Mutuswami Pérez-Castrillo, and Wettstein (2004) extend the BFS mechanism to the provision of public goods and network formation. In the latter context, the main difference with respect to the original BFS mechanism is that the proposer announces, in addition to a monetary offer, a coalition and a connected graph on such coalition.[9]

Our procedure here, compared to the BFS explained above, has the advantage of simplicity from the players' point of view. Players are not

8 The BFS procedure was first introduced by Pérez-Castrillo and Wettstein (2001) and consists of a two-stage negotiation process in which all currently active players first simultaneously bid against all the other players for becoming a proposer. After net bids are paid and received, the proposer, who is the player with the highest net bid, makes a feasible proposal to all of the active players, who sequentially accept or reject the proposal. If they all accept, then the set of active players cooperate together and distribute the value as proposed. If one player rejects then the proposer leaves the game and the mechanism starts all over again. In this case, the set of active players coincides with the previous set of active players excluding the player whose proposal has just been rejected. An alternative mechanism that also implements the Shapley value is provided by Hart and Mas-Colell (1996)

9 For the sake of completeness, Vidal-Puga and Bergantiños (2003) apply the BFS mechanism to implement the Owen value, which is an extension of the Shapley value to cooperative situations where players are organized in a-priori unions.

required to compute and send an offer to every other player, and, in the event of being a proposer, they only send offers to players with whom they can potentially cooperate, and they do so sequentially. The bid is only paid in case of a rejection, still allowing to balance bargaining power without the need of making a transaction. Final payoffs are also easy to compute since in equilibrium they are equal to offers, as no bid prices have to be paid. Compared to the DCG explained further above, the advantage of our mechanism is that it is a bilateral procedure, arguably fitting better in network contexts. Since the value from cooperation is generated by bilateral relationships, it seems natural to allow the participants in each bilateral relationship to bargain directly. As such, the presence or absence of a link in the network goes hand in hand with the agreement or disagreement by the two participating players on how to distribute payoffs. Furthermore, mechanisms that are sequential and bilateral create a direct, positive relationship between the amount of bilateral relationships and bargaining participation, since the bigger the amount of a player's bilateral relationships is, the more frequently will he be called to play.

This paper is organized as follows. In Section 2, we lay out the model of a cooperative game with network structure, introduce the Myerson value, and offer some examples of link-convexity. In Section 3, we describe the bargaining procedure. In Section 4, we present and prove the main result, stating that the bargaining procedure uniquely leads to the Myerson value, provided link-convexity is satisfied. There is a web appendix available online that contains an illustration of the bargaining procedure by means of a three-player example, the proof that, in a link-convex game, the Myerson value of a player in a connected component decreases if we delete a link from this component, and a proof of the claim that the last example in Section 2.4 is link-convex.

2 The model

2.1 Cooperation in graphs and value functions

Let $N = \{1, ..., n\}$ be the set of players. The bilateral negotiation possibilities among players are given by an undirected graph g with N being the set of nodes. Such a graph g consists of a set of undirected links (i, j), and the interpretation is that players i and j can negotiate directly if and only if $(i, j) \in g$. The coalitions in N which will eventually cooperate are the connected components of N in g. Every connected component S in g has a value $w(S, g)$, which is the total surplus from cooperation for S, if the cooperation structure is given by g. This value is assumed to be perfectly transferable among players in S.

Let G be the set of all possible undirected graphs on N. For every graph g and coalition S, we define the restriction of g to S as $g|_S = \{(i,j) \in g : i \in S \text{ and } j \in S\}$. Note that $g|_S \subseteq g$ and $g|_N = g$. A coalition $R \subseteq S$ is called a *connected component* of S in g if: (1) for every two players in R, there is a path, that is, a set of consecutive links, in $g|_S$ connecting them, and (2) for any player i in R and any player j not in R, there is no path in $g|_S$ that connects them. Let $S|g$ be the set of connected components of S induced by g. Similarly, we may define $N|g$ as the set of connected components of N in g. Note that $N|g$ is a partition of N. A graph g is *connected* if $N|g = \{N\}$.

A *value function* is a map w that assigns to every graph g and every connected component S in g a value $w(S,g)$. Following Jackson and Wolinsky (1996), we assume throughout the paper that the value $w(S,g)$ does not depend on the cooperation structure outside S. That is, $w(S,g) = w(S,g')$ whenever $g|_S = g'|_S$. However, we allow $w(S,g)$ to depend on the cooperation structure inside S, hence, $w(S,g)$ may differ from $w(S,g')$ if $g|_S$ and $g'|_S$ are different. From now on, the value function w is assumed to be fixed.

2.2 The Myerson value

An *allocation rule* is a function y that assigns to every graph $g \in G$ some payoff vector $y(g) \in \Re^n$. An allocation rule is called *component efficient* if for every graph $g \in G$ and every connected component $S \in N|g$, we have

$$\sum_{i \in S} y_i(g) = w(S,g).$$

Let $g \backslash (i,j)$ be the graph that results after deleting the link (i,j) from g. An allocation rule y is called *fair* (Myerson (1977)) if for every graph $g \in G$ and every link $(i,j) \in g$, it holds that

$$y_i(g) - y_i(g \backslash (i,j)) = y_j(g) - y_j(g \backslash (i,j)).$$

By fairness, we thus impose that two players who cooperate directly gain or lose the same amount by dissolving this cooperation.

Jackson and Wolinsky (1996) show that there is a unique component efficient and fair allocation rule, which can be written as the Shapley value of some auxiliary TU-game to be described below. This is an extension of an earlier result by Myerson (1977), who restricted attention to value functions w with the property that $w(S,g)$ is independent of the cooperation structure inside and outside S.

For every graph $g \in G$, let $[N, U_g]$ be the auxiliary TU-game in which the characteristic function U_g assigns to every coalition S the value

$$U_g(S) = \sum_{R \in S|g} w(R, g|_S).$$

Intuitively, $U_g(S)$ is the total surplus from cooperation that players in S may obtain if the cooperation structure is restricted to g and no cooperation with players outside S is possible. Note that if S is a connected component in g, then $U_g(S) = w(S, g)$.

Theorem 2.1 (*Myerson 1977 and Jackson and Wolinsky 1996*). *There is a unique component efficient and fair allocation rule, namely the allocation rule assigning to every graph g the Shapley value of* $[N, U_g]$.

Throughout the paper, the fair and component efficient allocation rule will be referred to as the Myerson value, and will be denoted by m. More precisely, by $m_i(g)$, we denote the payoff for player i if the Myerson value is applied to graph g.

2.3 Link-convexity

For the remainder of this paper, we shall restrict our attention to a particular class of value functions, namely those satisfying *link-convexity*. In words, link-convexity states that the marginal contribution of a link to a connected component is increasing in the number of links inside this component. Formally, we have the following definition.

Definition 2.2 *A value function w is called link-convex if*

$$U_g(S) - U_{g\backslash l^1}(S) > U_{g\backslash l^2}(S) - U_{g\backslash\{l^1, l^2\}}(S),$$

for every graph g, *every connected component* $S \in N|g$ *and every two links* l^1, l^2 *in* $g|_S$.

Note that link convexity implies that

$$U_g(S) > U_{g\backslash l}(S), \tag{1}$$

for every graph g, connected component S in g and link $l \in g|_S$. This is obtained by considering the case $l^1 = l^2$ in the definition of link-convexity.

We now derive a result that proves to be important for our main theorem. One can show, namely, that for a link-convex value function, deleting a link from a connected component leads to a strictly lower Myerson value payoff for all players in that component.

Lemma 2.3 *Let w be a link-convex value function. Then,* $m_i(w, g) > m_i(w, g \backslash l)$ *for every graph g, connected component* $S \in N|g$*, player* $i \in S$ *and link* $l \in g|_S$.

The proof for this lemma can be found in the web appendix.

2.4 Examples for link-convexity

We conclude this section with some examples of value functions that satisfy link-convexity.[10] It is obvious that a value function w of the form $w(g, S) = f(|g|_S|)$, where $f(.)$ is a strictly convex function and $|g|_S|$ is the number of links in $g|_S$, will be link-convex. Other relatively simple value functions w that are link-convex have a product structure, for example as in $w(S, g) = \prod_{l \in g|_S} c(l)$ or in $w(S, g) = |S| \prod_{l \in g|_S} c(l)$, where $c(l)$ is a positive number attributed to link l, as long as $c(l) > 2$.

The form $w(S, g) = \prod_{l \in g|_S} c(l)$, for every g and every $S \in N|g$, represents for example a production function. Consider a group N of scientists that may work together on a project. Each scientist is involved in bilateral cooperations with other scientists, represented by a graph g. As every scientist has his own field of expertise, every bilateral cooperation, or link, yields a type of knowledge that can only be generated by this specific cooperation. For every link l, the function $c(l)$ reflects the contribution of the bilateral cooperation l to the project. The total scientific output of the group S corresponds to a Cobb–Douglas production function in the input variables $c(l)$ generated by the various bilateral cooperations inside S.

The form $w(S, g) = |S| \prod_{l \in g|_S} c(l)$, for every g and every $S \in N|g$, represents for example a profit function. Consider a group N of firms producing a homogeneous good in a perfectly competitive market. Each firm may be engaged in bilateral R&D agreements with other firms. Each of these agreements are represented by links in a graph g and they translate into the reduction of production costs by developing new, more efficient technologies. Suppose that $C_i(x_i, g)$ denotes the total cost for firm i of producing quantity x_i when the collection of R&D agreements is equal to g. Fix $C_i(x_i, g) = x_i^2$ for every disconnected firm i in g and assume that the ratio $\frac{C_i(x_i, g \backslash l)}{C_i(x_i, g)}$, representing the cost reduction induced by the R&D cooperation l, is independent of

10 In addition to these examples, van den Nouweland and Borm (1991) find, in a context of TU games with communication structures, necessary and sufficient conditions for the resulting game U_g to be link-convex when the underlying TU game is convex.

the graph g, the firm i, and the quantity x_i. Let us therefore fix $\alpha(l) = \frac{C_i(x_i, g \setminus l)}{C_i(x_i, g)}$ for each l in g. It is easy to check that all these ingredients result in

$$C_i(x_i, g) = \frac{x_i^2}{\prod_{l \in g|_S} \alpha(l)},$$

for every firm i in a connected component S. Assume that each firm independently chooses its profit maximizing output and the price of the good is normalized to 1. Then, if w represents total profits in a connected component we would obtain the product structure we were describing above, with[11]

$$w(S, g) = \frac{|S|}{4} \prod_{l \in g|_S} \alpha(l),$$

when S contains at least two firms, and $w(\{i\}, g) = \frac{1}{4}$ when i is disconnected.

Finally, we can obtain a link-convex value function with a quadratic expression by considering a simplified version of the setting introduced by Goyal and Joshi (2003) on networks of research collaboration between firms. A node in N represents a firm and a link in the network represents a research collaboration that results in a reduction of marginal costs, as in the previous example above. Following Goyal and Joshi (2003), the marginal cost of a firm i participating in a network g of research collaborations is given by

$$c_i(g) = \gamma_0 - \gamma d_i(g), \qquad [2]$$

where $d_i(g)$ is the *degree* or number of links in the network g that agent i has, and both γ_0 and γ are strictly greater than zero. Assume that each firm i enjoys a monopoly in a market where demand is given by $p_i^d = \alpha_i - Q_i$, with $\alpha_i > \gamma_0$. Given optimal monopoly quantities, each firm obtains a profit equal to

$$\pi_i(g) = \left[\frac{\alpha_i - \gamma_0 + \gamma d_i(g)}{2}\right]^2, \qquad [3]$$

when the network of research collaborations is equal to g.

11 Optimal quantity for firm i in a connected component S is given by

$$x_i(g) = \frac{1}{2} \prod_{l \in g|_S} \alpha(l),$$

and therefore its profit is given by

$$\pi_i(g) = \frac{1}{4} \prod_{l \in g|_S} \alpha(l).$$

Firms in a component have to bear costs from building those collaboration agreements in $g|_S$. Assuming that such costs constitute a strictly concave function C increasing on the number of links of a component S, with $\alpha_i - \gamma_0 \geq C(1) - C(0)$, we obtain that

$$w(S, g) = \sum_{i \in S} \pi_i(g) - C(|g|_S|), \qquad [4]$$

for each component $S \in N|g$, which is link-convex. The details of why this value function is link-convex can be found at the end of the web appendix.[12]

3 The bargaining procedure

3.1 Description

We now present a simple bargaining procedure that leads to the Myerson value. Let N be the set of players and g a graph representing the bilateral negotiation possibilities. For every connected component S, we start the following procedure: Fix an ordering of the links in $g|_S$ such that, whenever *(i, j)* is not the last link in S, then either i or j (or both) are involved in some future link. For every link *(i, j)*, the negotiation between i and j consists of two stages:

Bidding Stage. Both players simultaneously make non-negative bids b_i and b_j.

Proposal Stage. The player with the highest bid proposes a payoff pair (x_i, x_j) which the other player can accept or reject. If the player accepts, we move to the next link. If the player rejects the link, *(i, j)* is removed, the proposer pays his bid to the other player and the procedure starts all over again for the reduced graph $g \backslash (i,j)$. In case both players have chosen the same bid in the bidding stage, both will propose with equal probability.

At the end, we reach a graph $g' \subseteq g$, possibly the empty graph, where all bilateral negotiations have led to an agreement. For every connected component S' in g' with at least two players, let $X(S')$ be the sum of all agreed upon payoffs

12 Alternatively, if we assume that there is perfect competition in each node i, then the consumer surplus in each market is a quadratic function of the number of links that each firm i has, namely $CS_i^{PC} = \frac{1}{2}(\alpha_i - \gamma_0 + \gamma d_i(g))^2$. If w is the sum of all market surpluses minus an increasing, concave function C on the number of links inside a component, we would also obtain a link-convex value function.

of all the players in S'. If $X(S')$ does not exceed the total value $w(S', g')$, then every player $i \in S'$ obtains, next to the bids received or paid as a result of possible previous rejections, the sum of all the agreed upon payoffs to i. If $X(S')$ exceeds the total value $w(S', g')$, no player in S' receives anything next to the bids received or paid during the negotiation process. If $S' = \{i\}$, then player i obtains, next to the bids received or paid as a result of possible previous rejections, his stand-alone value $w(\{i\}, g')$.

3.2 Order of bilateral negotiations

In our procedure, we have imposed a restriction on the order of the bilateral negotiations. Namely, within a connected component S, if i and j negotiate and are not the last ones to negotiate, then either i or j (or both) will negotiate at least once more in the future. Put equivalently, if i and j are not the last ones to negotiate within S, then it should not be the case that both i and j will leave the negotiations immediately afterwards. We call such orders of bilateral negotiations regular.

If orders of bilateral negotiations are not regular, then there is no reason to expect a fair allocation of the surplus. Let us see why this is the case. Assume that within some connected component S, players i and j negotiate directly and that both of them leave the negotiations immediately afterwards. If some other players in S still negotiate in the future, i and j will grab all the surplus, and leave the remaining negotiators just the minimal amount so that they would accept. This, of course, would never yield a fair allocation, and hence the Myerson value will surely not result as a subgame perfect equilibrium if we do not impose our regularity condition on the order of bilateral negotiations.

The question remains whether we can always find a regular order of bilateral negotiations. It can easily be verified that this is indeed always possible. Namely, at every stage in the procedure we can choose the next link (i,j) in such a way that the set of remaining links (including (i,j)) is connected. If we do so, then, unless (i,j) is the last link, either i or j will be connected to a future link, and hence either i or j will negotiate at least once more. So, we obtain a regular order.

4 The main result

We now prove that the bargaining procedure always leads to the Myerson value if the value function w is link-convex.

Theorem 4.1 *Let w be a link-convex value function and g a graph. Then, there is a unique subgame perfect equilibrium outcome for the bargaining procedure, and in this outcome the total payoffs for all the players coincide with the Myerson value at g.*

Proof. We show the statement by induction on the number of links in g. Assume that the graph is empty. Then, by construction of the procedure, every player receives his stand-alone value, which is also the Myerson value of the empty graph.

Now consider a graph g and assume that for any subgraph g', the procedure implements the Myerson value. By the construction of the bargaining procedure it is clear that we can apply the procedure to each connected component in g separately. We therefore assume, without loss of generality, that the full player set N is a connected component within g.

Suppose now that the procedure reaches the link (i, j), that is, i and j must negotiate. Let N^{out} be the players that will not negotiate anymore, and let $N^{\text{in}} = N \backslash N^{\text{out}}$ be those players that will negotiate at least once more, starting from link (i,j). Let X^{out} denote the sum of total payoffs claimed so far by the players in N^{out}, and let M^{out} be the sum of the Myerson value payoffs for the players in N^{out}. We prove the following claim.

Claim. *Consider the subgame that starts at link (i, j), and where the players in N^{out} have together claimed a total payoff of X^{out}. Suppose that, for every remaining link (k, l), we have that $X^{\text{out}} < M^{\text{out}} + |N^{\text{in}}|(m_k(g) - m_k(g \backslash (k, l)))$. Then, there is a unique subgame perfect equilibrium outcome in this subgame, where every player $k \in N^{\text{in}}$ will receive a total payoff*

$$X_k = m_k(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|}.$$

Proof of the Claim. We prove the claim by induction on the number of links that follow (i,j). To start with, assume that (i,j) is the last link. Suppose that $X^{\text{out}} < M^{\text{out}} + |N^{\text{in}}|(m_i(g) - m_i(g \backslash (i,j)))$, that is,

$$X^{\text{out}} < M^{\text{out}} + 2(m_i(g) - m_i(g \backslash (i, j))).$$

Assume that players i and j have claimed total payoffs of X_i and X_j so far. If i wins the bidding stage, proposes (x_i, x_j), and player j rejects, j's total payoff will be $m_j(g \backslash (i,j)) + b_i$. Here, we use the induction assumption that if the link gets broken, the bargaining procedure leads to the Myerson value for $g \backslash (i,j)$. On the other hand, if player j accepts, his total payoff will be $X_j + x_j$. We will now show that player i will choose the x_j such that $X_j + x_j = m_j(g \backslash (i,j)) + b_i$.

If player i offers x_j such that $X_j + x_j > m_j(g\backslash(i,j)) + b_i$ then player j obtains strictly more by accepting than by rejecting, therefore, he accepts. Player i obtains in this case $X_i + x_i = U_g(N) - X^{\text{out}} - (X_j + x_j)$.

If x_j is such that $X_j + x_j = m_j(g\backslash(i,j)) + b_i$, then player j is indifferent between accepting or rejecting the offer. Let us admit then the possibility for mixed strategies and fix q as the probability that j accepts this offer $x_j = m_j(g\backslash(i,j)) + b_i - X_j$. Player i obtains in this case $X_i + x_i = q[U_g\ (N) - X^{\text{out}} - m_j(g\backslash(i,j))] + (1-q)m_i(g\backslash(i,j)) - b_i$.

Finally, if x_j is such that $X_j + x_j < m_j(g\backslash(i,j)) + b_i$, then player j rejects the offer and player i obtains $m_i(g\backslash(i,j)) - b_i$.

We show that q, the probability that player j accepts if $X_j + x_j = m_j(g\backslash(i,j)) + b_i$, must be 1. Suppose, namely, that q is strictly less than 1. Let $\varepsilon > 0$ be such that

$$\epsilon < (1-q)\left[U_g(N) - X^{\text{out}} - m_i(g\backslash(i,j)) - m_j(g\backslash(i,j))\right].$$

Then, player i would be strictly better off by offering any x_j' such that $X_j + x_j' = m_j(g\backslash(i,j)) + b_i + \varepsilon$ than by offering x_j with $X_j + x_j = m_j(g\backslash(i,j)) + b_i$. But, in that case, there is no best response for player i because for any offer x_j' with $X_j + x_j' = m_j(g\backslash(i,j)) + b_i + \epsilon$, there is a smaller offer $x_j'' < x_j'$ that player j would still accept with probability 1. Therefore, there is no subgame perfect equilibrium with $q < 1$.

Hence, in subgame perfect equilibrium, player i offers x_j with $X_j + x_j = m_j(g\backslash(i,j)) + b_i$, and player j accepts that offer with probability 1, even if we allow for mixed strategies at this stage. Hence, player j's total payoff is

$$X_j + x_j = m_j(g\backslash(i,j)) + b_i, \tag{5}$$

and player i's total payoff would be the remaining amount, which is

$$\begin{aligned} X_i + x_i &= U_g(N) - X^{\text{out}} - (X_j + x_j) \\ &= U_g(N) - X^{\text{out}} - m_j(g\backslash(i,j)) - b_i. \end{aligned} \tag{6}$$

This indicates that player i's total payoff is decreasing in his own bid when he is the proposer.

Similarly, if player j is the proposer, player i's total payoff is

$$X_i + x_i = m_i(g\backslash(i,j)) + b_j, \tag{7}$$

and player j's total payoff is

$$X_j + x_j = U_g(N) - X^{\text{out}} - m_i(g\backslash(i,j)) - b_j. \tag{8}$$

Hence, also player j's total payoff is decreasing in his own bid when he is the proposer.

This implies that, in equilibrium, players i and j must choose the same bid. Namely, if $b_i > b_j$, player i could improve his total payoff by lowering his bid a little and still become the proposer. Therefore, players i and j will both become the proposer with probability one half.

Furthermore, the common bid b must be such that both players are indifferent between being the proposer and the respondent. If a player strictly preferred being the proposer, then he could improve his total payoff by raising his bid a little and become the proposer with probability one. If he strictly preferred being the respondent, he could improve his total payoff by lowering his bid a little and become the respondent with probability one. It can easily be shown that such a common bid exists and is unique: If we take $b_i = b_j = b$ and set (5) equal to (8), we get the same solution for b as when we set (6) equal to (7), namely

$$b = \frac{U_g(N) - X^{\text{out}} - m_i(g\backslash(i,j)) - m_j(g\backslash(i,j))}{2}.$$

As, $U_g(N) = M^{\text{out}} + m_i(g) + m_j(g)$, we have that

$$\begin{aligned} b &= \frac{M^{\text{out}} - X^{\text{out}} + m_i(g) + m_j(g) - m_i(g\backslash(i,j)) - m_j(g\backslash(i,j))}{2} \\ &= m_i(g) - m_i(g\backslash(i,j)) + \frac{M^{\text{out}} - X^{\text{out}}}{2}. \end{aligned} \qquad [9]$$

Here, the second equality follows from the fact that $m_j(g) - m_j(g\backslash(i,j)) = m_i(g) - m_i(g\backslash(i,j))$. Since, by assumption, $X^{\text{out}} < M^{\text{out}} + 2(m_i(g) - m_i(g\backslash(i,j)))$, we obtain that $b > 0$. So, the common bid is positive and therefore well-defined.

We should also check that $b_i = b$ is still a best response to $b_j = b$ even when we allow for mixed strategies at the bidding stage. Note that, for any $b_i < b$ that is chosen with positive probability in any mixed strategy, player i will for sure win the bidding stage when $b_j = b$. In this case, player i will earn a profit smaller than the one he would obtain with $b_i = b$ because payoffs are strictly decreasing in b_i when player i is the proposer. For any $b_i < b$ that is chosen with positive probability in any mixed strategy, player j becomes the proposer with $b_j = b$. In this case, player i obtains exactly the same payoff as when choosing $b_i = b$. Hence, there is no profitable deviation for player i when player j chooses $b_j = b$, even if we consider mixed strategies for player i, because the expected payoff from any of those (possibly mixed) strategies cannot not be bigger than what player i obtains when he is choosing $b_i = b$ (assuming that player j chooses $b_j = b$). The same would be true about player j when player i chooses $b_i = b$, and hence $b_i = b_j = b$ is the bid in a subgame perfect equilibrium in pure strategies.

With all of the above, the total expected payoffs for players i and j equal the total payoffs they obtain by being the respondent, namely

$$X_i + x_i = m_i(g\backslash(i,j)) + b, \text{and } X_j + x_j = m_j(g\backslash(i,j)) + b. \qquad [10]$$

By substituting eqs. [9] into [10], and using the fact that $m_i(g) - m_i(g\backslash(i,j)) = m_j(g) - m_j(g\backslash(i,j))$, we get

$$X_i + x_i = m_i(g) + \frac{M^{\text{out}} - X^{\text{out}}}{2} = m_i(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|},$$

and

$$X_j + x_j = m_j(g) + \frac{M^{\text{out}} - X^{\text{out}}}{2} = m_j(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|}.$$

So, the claim holds if *(i, j)* is the last link.

Consider now a link (i,j) which is not the last link, and suppose that $X^{\text{out}} < M^{\text{out}} + |N^{\text{in}}|(m_k(g) - m_k(g\backslash(k,l)))$ for every remaining link (k,l). Assume also that the claim holds for all the links that follow. There are two possible cases to distinguish, namely that neither i nor j will not leave the negotiations after *(i, j)* or that one of them will.

Case 1. Suppose that neither i nor j will leave the negotiations after (i,j).

In this case, the set of inactive players N^{out} will remain the same after the negotiation at (i, j), and hence so will the claimed amount X^{out}. Therefore, by our induction assumption, the eventual payoffs for the players in N^{in} are not affected at all by the negotiation at (i, j), as long as the offer there is accepted. So, the only objective for players i and j is to make sure that the offer is accepted, and hence the claim follows rather trivially in this case.

Case 2. Suppose that player i will leave the negotiations after (i, j).

Since the order of bilateral negotiations is regular, we know that player j is involved in at least one other remaining link.

Assume that players i and j have claimed total payoffs X_i and X_j so far. If i wins the bidding stage, proposes (x_i, x_j) and player j rejects, then j's total payoff will be

$$m_j(g\backslash(i,j)) + b_i. \qquad [11]$$

If *j* accepts, then his total payoff will be

$$m_j(g) + \frac{(M^{\text{out}} + m_i(g)) - (X^{\text{out}} + X_i + x_i)}{|N^{\text{in}}| - 1}. \qquad [12]$$

Here, we have used the assumption that the claim holds for the link that follows (i, j). Namely, if *j* accepts, then player *i* will have a total payoff equal to $X_i + x_i$. Since player *i* leaves after (i, j), the total claimed amount by the inactive players after *(i, j)* would be $X^{\text{out}} + X_i + x_i$, and the sum of the Myerson value payoffs of the inactive players after *(i, j)* would be $(M^{\text{out}} + m_i(g))$. Moreover, the number of active players after *(i, j)* would be $|N^{\text{in}}| - 1$.

So, if *i* proposes, then he will choose x_i such that (11) and (12) are equal. Hence, player *i*'s total payoff will be

$$X_i + x_i = M^{\text{out}} + m_i(g) - X^{\text{out}} + (|N^{\text{in}}| - 1)(m_j(g) - m_j(g\backslash(i,j)) - b_i), \qquad [13]$$

and player *j*'s total payoff will be

$$m_j(g\backslash(i,j)) + b_i. \qquad [14]$$

We thus see that player *i*'s total payoff is decreasing in his own bid if he is the proposer.

If *j* wins the bidding stage, proposes (x_i, x_j) and *i* rejects, then *i*'s total payoff will be $m_i(g\backslash(i,j)) + b_j$. If *i* accepts, then his total payoff will be $X_i + x_i$, and player *j*'s total payoff will be

$$m_j(g) + \frac{(M^{\text{out}} + m_i(g)) - (X^{\text{out}} + X_i + x_i)}{|N^{\text{in}}| - 1}.$$

So, player *j* chooses x_i such that $X_i + x_i = m_i(g\backslash(i,j)) + b_j$, and hence *j*'s own total payoff is

$$m_j(g) + \frac{(M^{\text{out}} + m_i(g)) - (X^{\text{out}} + m_i(g\backslash(i,j)) + b_j)}{|N^{\text{in}}| - 1}, \qquad [15]$$

whereas *i*'s total payoff is

$$m_i(g\backslash(i,j)) + b_j. \qquad [16]$$

Hence, also player *j*'s total payoff is decreasing in his own bid if he is the proposer.

Summarizing, we see that for both players, the total payoff when being the proposer is decreasing in the winning bid. But then, both players must choose the same bid b and in such a common bid both players are indifferent between being the proposer and being the respondent. The argument follows as in the proof for the last link, and it can therefore be omitted here. Hence, players i and j will both become the proposer with probability one half.

It is easily verified that such bid b exists: By setting (13) equal to (16) for $b_i = b_j = b$, we obtain the same solution for b as by setting (14) equal to (15), namely

$$b = \frac{M^{\text{out}} - X^{\text{out}} + m_i(g) - m_i(g \backslash (i,j)) + (|N^{\text{in}}| - 1)(m_j(g) - m_j(g \backslash (i,j)))}{|N^{\text{in}}|}.$$

As $m_j(g) - m_j(g \backslash (i,j)) = m_i(g) - m_i(g \backslash (i,j))$, we obtain that

$$b = m_i(g) - m_i(g \backslash (i,j)) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|}, \qquad [17]$$

which is positive since, by assumption, $X^{\text{out}} < M^{\text{out}} + |N^{\text{in}}|(m_i(g) - m_i(g \backslash (i,j)))$. Hence, the common bid b is well-defined.

So, player i's total expected payoff is equal to his expected payoff by being the respondent, which, by eq. [16], is equal to $m_i(g \backslash (i,j)) + b$. By eq. [17], we can then conclude that player i's total expected payoff is

$$m_i(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|}. \qquad [18]$$

Similarly, player j's total expected payoff is equal to $m_j(g \backslash (i,j)) + b$, which, by eq. [17] and the fact that $m_j(g) - m_j(g \backslash (i,j)) = m_i(g) - m_i(g \backslash (i,j))$, is equal to

$$m_j(g) + \frac{M^{out} - X^{out}}{|N^{in}|}. \qquad [19]$$

We now explore what the other active players after *(i, j)* would get. Consider the subgame that starts immediately after (i, j), and let $\tilde{X}^{\text{out}}$, $\tilde{M}^{\text{out}}$, and $\tilde{N}^{\text{in}}$ refer to this subgame. Since player i leaves the negotiations after (i, j), and receives total payoff $m_i(g) + (M^{\text{out}} - X^{\text{out}})/|N^{\text{in}}|$, we have that

$$\tilde{X}^{\text{out}} = X^{\text{out}} + m_i(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|} = m_i(g) + \frac{M^{\text{out}} + (|N^{\text{in}}| - 1)X^{\text{out}}}{|N^{\text{in}}|}.$$

Obviously, $\tilde{M}^{\text{out}} = M^{\text{out}} + m_i(g)$ and $|\tilde{N}^{\text{in}}| = |N^{\text{in}}| - 1$. As, by assumption, $X^{\text{out}} < M^{\text{out}} + |N^{\text{in}}|(m_k(g) - m_k(g\backslash(k,l)))$ for every link (k,l) that comes after (i,j), we may conclude that

$$\begin{aligned}\tilde{X}^{\text{out}} &< m_i(g) + M^{\text{out}} + (|N^{\text{in}}| - 1)(m_k(g) - m_k(g\backslash(k,l))) \\ &= \tilde{M}^{\text{out}} + |\tilde{N}^{\text{in}}|(m_k(g) - m_k(g\backslash(k,l))),\end{aligned}$$

for every link (k,l) that comes after (i,j). But then, by our induction assumption, we may conclude that every active player k after *(i, j)* will receive total payoff

$$m_k(g) + \frac{\tilde{M}^{\text{out}} - \tilde{X}^{\text{out}}}{|\tilde{N}^{\text{in}}|} = m_k(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|}.$$

Together with eqs. [18] and [19], we then obtain that every player $k \in N^{\text{in}}$ receives total payoff

$$m_k(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|},$$

which was to show. By induction, the proof of the claim is complete.

We finally prove the statement of the theorem. Let us move to the beginning of the bargaining procedure, that is, to the first link *(i, j)*. There, obviously, $X^{\text{out}} = M^{\text{out}} = 0$. Since the value function is link-convex, we know by Lemma 2.3 that $m_k(g) > m_k(g\backslash(k,l))$ for every link (k,l). Hence, we have that $X^{\text{out}} < M^{\text{out}} + |N^{\text{in}}|(m_k(g) - m_k(g\backslash(k,l)))$ for every link (k,l). But then, by our Claim, we may conclude that there is a unique subgame perfect equilibrium outcome in the bargaining procedure, where every player k receives total payoff

$$m_k(g) + \frac{M^{\text{out}} - X^{\text{out}}}{|N^{\text{in}}|} = m_k(g).$$

This completes the proof of the theorem. ■

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