

# Reasoning about Your Own Future Mistakes\*

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## Abstract

We propose a model of reasoning in dynamic games in which a player, at each information set, holds a conditional belief about his own future choices and the opponents' future choices. These conditional beliefs are assumed to be *cautious*, that is, the player never completely rules out any feasible future choice by himself or the opponents. We impose the following key conditions: (a) a player always believes that he will choose rationally in the future, (b) a player always believes that his opponents will choose rationally in the future, and (c) a player deems his own mistakes infinitely less likely than the opponents' mistakes. Common belief in these conditions leads to the new concept of *strong sequential rationalizability*. We show that strongly sequentially rationalizable strategies exist in every finite dynamic game. We prove, moreover, that strong sequential rationalizability constitutes a refinement of both *perfect rationalizability* (a rationalizability analogue to Selten's (1975) perfect equilibrium) and *procedural quasi-perfect rationalizability* (a rationalizability analogue to van Damme's (1984) quasi-perfect equilibrium). As a consequence, it avoids both weakly dominated strategies in the normal form and strategies containing weakly dominated actions in the agent normal form.

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## 1 Introduction

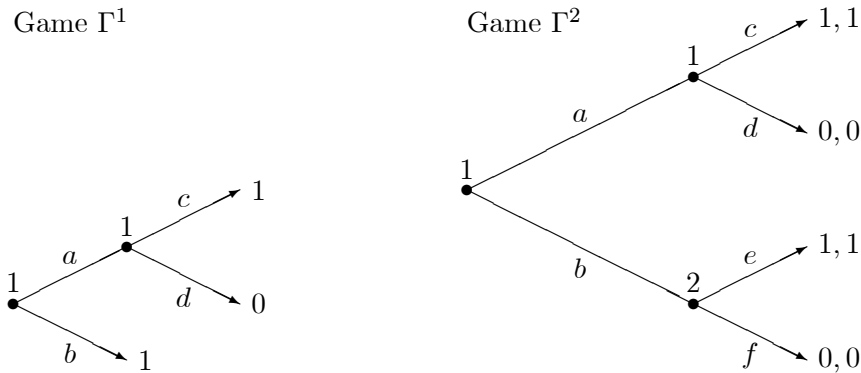
When reasoning in a game, it is natural to assume that your opponents – and even you yourself – may make mistakes with some small probability. This assumption, to which we often refer

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**Figure 1:** Belief about your own future mistakes

as *cautious reasoning*, has first been implemented by Selten (1975) in his concept of *perfect equilibrium*. The main idea is that a player, at each of his information sets in a dynamic game, assigns a – possibly infinitesimal – strictly positive probability to each of the opponents’ choices and to each of his own future choices. Since then, Selten’s idea of cautious reasoning has inspired many other concepts in game theory, such as *proper equilibrium* (Myerson (1978)), *permissibility* (Brandenburger (1992), Börgers (1994)) and *proper rationalizability* (Schuhmacher (1999), Asheim (2001)) for static games, and *quasi-perfect equilibrium* (van Damme (1984)) and *quasi-perfect rationalizability* (Asheim and Perea (2005)) for dynamic games.

Among these, perfect equilibrium is the only concept in which a player believes that he may make mistakes *himself* in the future. Indeed, in each of the other concepts the cautious reasoning of a player only concerns the strategy choices by his *opponents*, but not the choices by himself. As an illustration, consider the game  $\Gamma^1$  in the left-hand panel of Figure 1, where player 1 is the only player. If player 1 fears that, by mistake, he may choose  $d$  in the future, then his unique best choice is to go for strategy  $b$ . This is the only perfect equilibrium strategy for player 1, whereas the other concepts above also allow for strategy  $(a, c)$ . Intuitively,  $b$  seems the only plausible choice for player 1 here. Indeed, why would player 1 risk making a future mistake by choosing  $a$ , if by choosing  $b$  he achieves the maximum possible utility with no risk of making future mistakes? This argument speaks for perfect equilibrium in this game.

At the same time, perfect equilibrium allows a player to believe that his *own* future mistakes are more likely than the opponents’ mistakes. Consider, for instance, the game  $\Gamma^2$  in the right-hand panel of Figure 1. According to perfect equilibrium, player 1 is free to believe that his own future mistake  $d$  is more likely than player 2’s mistake  $f$ , which allows player 1 to go for strategy  $b$ . The other concepts above uniquely select player 1’s strategy  $(a, c)$ , since under these concepts player 1 believes that he will not make mistakes himself. In this particular game,  $(a, c)$  seems the only plausible strategy for player 1. Indeed, believing that your own future mistakes

are more likely than those of your opponents strikes us as rather counterintuitive.

Summarizing, none of the established concepts above uniquely selects player 1’s intuitive choice in both of the games  $\Gamma^1$  and  $\Gamma^2$ . In fact, we are not aware of any concept in the published game theory literature – be it an equilibrium concept or rationalizability concept – that filters player 1’s plausible choice in both of the games  $\Gamma^1$  and  $\Gamma^2$ . This raises the question: Can we develop a new concept that does? This is precisely the question we wish to answer in this paper.

Blume and Meier (2021) have taken a first important step by developing the new concept of *strong sequential equilibrium*. Like perfect equilibrium and quasi-perfect equilibrium, they simulate mistakes by trembles – that is, sequences of full-support probability distributions that converge to the strategy profile under consideration. When evaluating the strategy of player  $i$ , Blume and Meier (2021) also consider trembles in player  $i$ ’s own strategy, similarly to how perfect equilibrium proceeds. However, the key condition in strong sequential equilibrium is that player  $i$  deems the trembles in his own strategy much smaller, in fact *infinitely smaller*, than the trembles in the opponents’ strategies. By this feature, strong sequential equilibrium selects the “right” choice for player 1 in both games  $\Gamma^1$  and  $\Gamma^2$  above.

In this paper, we develop a *rationalizability* concept, termed *strong sequential rationalizability*, that shares some of the key properties of strong sequential equilibrium. More precisely, in strong sequential rationalizability a player holds, at each of his information sets, a cautious belief about his opponents’ strategies and a cautious belief about his own continuation strategy. Here, by *cautious* we mean that the player never rules out any feasible opponent’s strategy, and never rules out any of his own continuation strategies, although he may deem some of these strategies infinitely more likely than other strategies.

We impose the following key conditions on these beliefs: (a) a player always believes that he will himself choose rationally in the future, given his future beliefs about the opponents’ strategies and his future beliefs about himself; (b) a player always believes that his opponents will choose rationally in the future; and (c) a player always deems his own future mistakes infinitely less likely than the mistakes by others.<sup>1</sup> In (a) and (b), when we say that a player believes that he, or another player, chooses rationally, we mean that he only assigns infinitesimal probability to all suboptimal strategies. Common belief in the properties (a), (b) and (c) throughout the game yields the concept of strong sequential rationalizability. Formally, we define the concept by the recursive elimination of strategies and beliefs in the game (see Section 3).<sup>2</sup> In Theorem 3.1 we show that this elimination procedure always yields a non-empty output in every finite dynamic

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<sup>1</sup>Alternatively, one could consider a weaker concept in which a player believes that his own mistakes are less likely, but not necessarily *infinitely* less likely, than those of the opponents. In this paper we adopt the stronger version. This is analogous to proper equilibrium and proper rationalizability, where a player deems more costly mistakes by the opponents infinitely less likely than less costly mistakes.

<sup>2</sup>A problem we would like to explore in the near future is characterizing strong sequential rationalizability *epistemically*. That is, what are precisely the epistemic conditions on the players’ beliefs that lead us to strong sequential rationalizability? Our current formulation of this concept is by means of a non-epistemic recursive elimination procedure, although the conditions of “belief in your own future rationality” and “deeming your own mistakes least likely” already have some epistemic flavour.

game, and hence strongly sequentially rationalizable strategies always exist. The existence proof, which can be found in Appendix B, is constructive.

Property (b) shows that, in terms of reasoning, strong sequential rationalizability is very similar to *common belief in future rationality* (Perea (2014)), in which players also always believe that their opponents will choose rationally in the future. In that sense, strong sequential rationalizability is a true backward induction concept in which the players only reason about the game that lies ahead.

By virtue of the properties (a), (b) and (c) above, especially property (c), the concept of strong sequential rationalizability uniquely selects the “right” choice for player 1 in both of the games  $\Gamma^1$  and  $\Gamma^2$  above. Indeed, in game  $\Gamma^1$  strong sequential rationalizability uniquely selects strategy  $b$  for player 1, as player 1 does not rule out his own future mistake  $d$ . The selection here thus coincides with that of perfect equilibrium. In game  $\Gamma^2$ , player 1’s reasoning in line with strong sequential rationalizability is as follows: Player 1 deems it possible that he himself will make the mistake  $d$  and that player 2 will make the mistake  $f$ , but by (c) he deems player 2’s mistake infinitely more likely than his own mistake. Therefore, player 1 will go for strategy  $(a, c)$ , which is also the prediction of quasi-perfect equilibrium and quasi-perfect rationalizability.

In this paper we show that this is not a coincidence: In every dynamic game, strong sequential rationalizability is a refinement of both *perfect rationalizability* (see Remark 1 in Section 3) and *procedural quasi-perfect rationalizability* (see Theorem 4.1). Here, by perfect rationalizability we mean the rationalizability analogue to perfect equilibrium, whereas procedural quasi-perfect rationalizability is a slight weakening of quasi-perfect rationalizability which we introduce in this paper. The main difference between the two versions of quasi-perfect rationalizability is that the former is defined by means of a recursive procedure, whereas the latter is characterized by epistemic conditions. But both concepts are based on the same principles: a player always believes in the opponents’ future rationality, believes that the opponents make mistakes with infinitesimal probability, but believes that he himself will make no mistakes in the future. Typically, both concepts coincide behaviorally, but there are dynamic games where the former is behaviorally weaker than the latter.

The two results mentioned above thus show that, in every dynamic game, strong sequential rationalizability inherits the desirable properties of both perfect rationalizability (that you take into account your own future mistakes) and procedural quasi-perfect rationalizability (that you focus primarily on the *opponents’* mistakes). As a consequence, strong sequential rationalizability rules out strategies containing weakly dominated *actions* in the agent normal form, as well as weakly dominated *strategies* in the normal form. This follows from the fact that perfect rationalizability avoids strategies containing weakly dominated actions in the agent normal form, and procedural quasi-perfect rationalizability avoids weakly dominated strategies in the normal form.

Remarkably, the result that strong sequential rationalizability is a refinement of both perfect rationalizability and procedural quasi-perfect rationalizability does not have an equilibrium counterpart. As is shown in Blume and Meier (2021), strong sequential equilibrium is a re-

finement of quasi-perfect equilibrium but not of perfect equilibrium. In fact, it cannot be a refinement of both as the sets of quasi-perfect equilibria and perfect equilibria may be disjoint.

Differences between the various cautious reasoning concepts listed above only appear when there are relevant ties in the dynamic game. If we consider, for instance, a dynamic game with perfect information without relevant ties, then all the above mentioned concepts (except for permissibility) would single out the unique backward induction strategy for each of the players. More generally, if we take a dynamic game (with or without perfect information) where the backward dominance procedure (Perea (2014)) singles out a unique strategy for every player, then all these concepts would also coincide, as they all constitute refinements of the backward dominance procedure. That is, to appreciate the difference between the above mentioned concepts we must necessarily turn to games with relevant ties.

Compared to Blume and Meier’s (2021) strong sequential equilibrium concept, the concept of strong sequential rationalizability differs in various dimensions. First, strong sequential rationalizability is not an equilibrium concept, and hence does not impose “correct beliefs assumptions” stating that a player must believe that his opponents are correct about his beliefs, or that player  $i$  must believe that player  $j$  has the same belief about player  $k$  as player  $i$  has. Strong sequential equilibrium, on the other hand, does impose such correct beliefs conditions. Second, beliefs about the opponents’ choices and mistakes and beliefs about your own future choices and mistakes are explicitly modelled in strong sequential rationalizability, whereas these are only implicitly present – in terms of trembles of the equilibrium strategies – in strong sequential equilibrium. Finally, we use *non-standard probability distributions* (Robinson (1973), Hammond (1994) and Halpern (2010)) with infinitesimals to model beliefs about mistakes, which greatly simplifies the presentation and analysis in our case. In contrast, Blume and Meier start by using the traditional framework of converging sequences of full-support *standard* probability distributions, but subsequently provide a characterization in terms of non-standard probabilities.

This leaves the question why we did not opt for *lexicographic probability systems* (Blume, Brandenburger and Dekel (1991)) to model such cautious beliefs, as is common nowadays in epistemic game theory. The reason is that lexicographic probability systems are not expressive enough for our purposes: In the concept of strong sequential rationalizability, player  $i$  holds, at each of his information sets, both (a) a cautious belief about his *own* future choices, and (b) a cautious belief about the *opponents’* future choices. To determine which choice is optimal for player  $i$  at that information set, we must take the “product” of the belief about his own choices and the belief about the opponents’ choices. Taking such products is not well-defined for lexicographic probability distributions, whereas it comes for free when using non-standard probability distributions.

The outline of this paper is as follows. In Section 2 we introduce the necessary notation for dynamic games and provide a brief overview of non-standard analysis, which will be sufficient for understanding the main body of this paper. In Section 3 we introduce the new concept of strong sequential rationalizability, show its existence, and observe that it is a refinement of perfect rationalizability. In Section 4 we introduce a slight weakening of the quasi-perfect

rationalizability concept, called *procedural quasi-perfect rationalizability*, and show that strong sequential rationalizability is a refinement of procedural quasi-perfect rationalizability. Section 5 concludes with some final remarks. Appendix A gives a more extensive treatment of non-standard analysis, which is needed for some of the proofs. Appendices B and C contain the proofs for Sections 3 and 4, respectively. Appendix D, finally, explores the relation between our notion of procedural quasi-perfect rationalizability and quasi-perfect rationalizability as defined in Asheim and Perea (2005).

## 2 Definitions

In this section we introduce the notation for dynamic games, and provide a short overview of non-standard analysis which is needed for our definitions of *strong sequential rationalizability* and *quasi-perfect rationalizability*.

### 2.1 Dynamic Games

In this paper we will focus on finite dynamic games with complete information that allow for simultaneous moves and imperfect information. The restriction to complete information is merely for the sake of simplicity. The definition below, and the concept presented in this paper, can easily be extended to games with incomplete information. Moreover, to keep our notation and definitions simple we exclude moves of nature in the definition that follows. However, our definition can easily be generalized to situations that involve moves of nature. Formally, a *finite dynamic game* is a tuple

$$G = (I, X, Z, (X_i)_{i \in I}, (C_i(x))_{i \in I, x \in X_i}, (H_i)_{i \in I}, (u_i)_{i \in I})$$

where

- (a)  $I = \{1, 2, \dots, n\}$  is the finite set of *players*;
- (b)  $X$  is the finite set of *histories*, consisting of *non-terminal* and *terminal* histories. At every non-terminal history, one or more players must make a choice, whereas at every terminal history the game ends. By  $\emptyset$  we denote the history that marks the beginning of the game;
- (c)  $Z \subseteq X$  is the set of *terminal* histories;
- (d)  $X_i \subseteq X$  is the set of *non-terminal* histories where player  $i$  must make a choice. For a given non-terminal history  $x$ , we denote by  $I(x) := \{i \in I \mid x \in X_i\}$  the set of *active* players at  $x$ . We allow  $I(x)$  to contain more than one player, that is, we allow for *simultaneous moves*. At the same time, we require  $I(x)$  to be non-empty for every non-terminal history  $x$ ;
- (e)  $C_i(x)$  is the finite set of *choices* available to player  $i$  at a history  $x \in X_i$ ;
- (f)  $H_i$  is the collection of *information sets* for player  $i$ . Every information set  $h \in H_i$  is a subset of histories in  $X_i$  such that  $\cup_{h \in H_i} h = X_i$  and  $h \cap h' = \emptyset$  for every two different  $h, h' \in H_i$ . That is,  $H_i$  is a partition of  $X_i$ . Moreover, we assume that  $C_i(x) = C_i(y)$  whenever  $x, y$  belong

to the same information set in  $H_i$ , and  $C_i(x) \cap C_i(y) = \emptyset$  whenever  $x$  and  $y$  belong to different information sets in  $H_i$ . The interpretation of an information set  $h \in H_i$  is that player  $i$  at  $h$  knows that a history in  $h$  has been realized. However, if  $h$  contains more than one history, player  $i$  does not know which of these histories has been realized. Hence, we allow for *imperfect information*;

(g)  $u_i : Z \rightarrow \mathbb{R}$  is player  $i$ 's *utility function*, assigning to every terminal history  $z \in Z$  some utility  $u_i(z)$ .

For practical purposes, we assume that all players are active at the beginning of the game  $\emptyset$ , that is,  $I(\emptyset) = I$ . If, in reality, player  $i$  does not choose at  $\emptyset$ , then we define  $C_i(\emptyset)$  to be a singleton.

For every non-terminal history  $x$  and choice combination  $(c_i)_{i \in I(x)}$  in  $\times_{i \in I(x)} C_i(x)$ , we denote by  $x' = (x, (c_i)_{i \in I(x)})$  the (terminal or non-terminal) history that immediately follows this choice combination at  $x$ . In this case, we say that  $x'$  immediately follows  $x$ . We say that a history  $x$  *follows* a non-terminal history  $x'$  if there is a sequence of histories  $x^1, \dots, x^K$  such that  $x^1 = x'$ ,  $x^K = x$ , and  $x^{k+1}$  immediately follows  $x^k$  for all  $k \in \{1, \dots, K-1\}$ . A history  $x$  is said to *weakly follow*  $x'$  if either  $x$  follows  $x'$  or  $x = x'$ . In the obvious way, we can then also define what it means for  $x$  to (*weakly*) *precede* another history  $x'$ . Analogously, for two information sets  $h$  and  $h'$ , we say that  $h$  (*weakly*) follows  $h'$  if there is some  $x \in h$  and  $x' \in h'$  such that  $x$  (*weakly*) follows  $x'$ . We assume that the dynamic game has *non-overlapping information sets*, that is, for every two information sets  $h, h'$  it is never the case that  $h$  weakly follows  $h'$  and  $h'$  follows  $h$ .

In view of (f), we can write  $C_i(h)$  to denote the (unique) set of choices that player  $i$  has available at information set  $h \in H_i$ . We assume *perfect recall*, that is, for every information set  $h \in H_i$  and every two histories  $x, y \in H_i$ , the sequence of player  $i$  choices leading to  $x$  is the same as the sequence of player  $i$  choices leading to  $y$ . In particular, since different information sets in  $H_i$  prescribe disjoint sets of available choices, the sequence of player  $i$  information sets on the path to  $x$  and on the path to  $y$  must be the same. That is, player  $i$  always remembers, at each of his information sets  $h \in H_i$ , the choices he made in the past and the information he had in the past.

We view a strategy for player  $i$  as a *plan of action* (Rubinstein (1991)), assigning choices only to those histories  $h \in H_i$  that are not precluded by previous choices. Formally, consider a collection of information sets  $\hat{H}_i \subseteq H_i$ , and a mapping  $s_i : \hat{H}_i \rightarrow \cup_{h \in \hat{H}_i} C_i(h)$  assigning to every information set  $h \in \hat{H}_i$  some available choice  $s_i(h) \in C_i(h)$ . We say that an information set  $h \in H$  is *reachable* under  $s_i$  if at every information set  $h' \in \hat{H}_i$  preceding  $h$ , the choice  $s_i(h')$  is the unique choice that leads to  $h$ . The mapping  $s_i : \hat{H}_i \rightarrow \cup_{h \in \hat{H}_i} C_i(h)$  is called a *strategy* if  $\hat{H}_i$  contains exactly those information sets in  $H_i$  that are reachable under  $s_i$ .

By  $S_i$  we denote the set of strategies for player  $i$ . For every information set  $h \in H$  and player  $i$ , we denote by  $S_i(h)$  the set of strategies for player  $i$  under which  $h$  is reachable. Similarly, for a given strategy  $s_i$  we denote by  $H_i(s_i)$  the collection of information sets in  $H_i$  that are reachable under  $s_i$ .

## 2.2 Non-Standard Numbers

The analysis of non-standard numbers was initiated by Robinson (1973). It has later been incorporated into the analysis of games by Hammond (1994) and Halpern (2010), who review non-standard analysis and connect it to conditional and lexicographic probability systems.

Consider a number  $\varepsilon > 0$  with the property that  $\varepsilon < a$  for every strictly positive real number  $a \in \mathbf{R}$ ,  $a > 0$ . The number  $\varepsilon$  is called an *infinitesimal*. Following Robinson (1973), Hammond (1994) and Halpern (2010), let  $\mathbf{R}(\varepsilon)$  be the smallest field that includes all real numbers and the infinitesimal  $\varepsilon$ . That is,  $\mathbf{R}(\varepsilon)$  contains all numbers  $a$  that can be written as

$$a = \frac{a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots + a_K\varepsilon^K}{b_0 + b_1\varepsilon + b_2\varepsilon^2 + \dots + b_K\varepsilon^K},$$

where  $a_k, b_k \in \mathbf{R}$  for all  $k \in \{0, \dots, K\}$ ,  $b_k \neq 0$  for some  $k \in \{0, \dots, K\}$ , and where either  $a_0 \neq 0$  or  $b_0 \neq 0$ . In other words,  $\mathbf{R}(\varepsilon)$  contains all fractions of finite polynomials in  $\varepsilon$ . Numbers in  $\mathbf{R}(\varepsilon)$  are called *non-standard*.

A non-standard number  $a \in \mathbf{R}(\varepsilon)$  is *finite* if there is a strictly positive real number  $b \in \mathbf{R}$ ,  $b > 0$  with  $-b < a < b$ . Every finite non-standard number  $a \in \mathbf{R}(\varepsilon)$  can uniquely be written as

$$a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots,$$

where  $a_k \in \mathbf{R}$  for every  $k \geq 0$ . (See Appendix A for a proof). By  $st(a) := a_0$  we denote the *standard part* of  $a$ , which is the real number that is “closest” to  $a$ . By  $v(a)$  we denote the smallest index  $k$  for which  $a_k \neq 0$ , and call it the *valuation* of  $a$ . For two finite numbers  $a, b \in \mathbf{R}(\varepsilon)$ , we say that  $a$  is of *infinitely smaller size* than  $b$  if  $v(a) > v(b)$ . We use the term “infinitely smaller size” rather than the more familiar “infinitely smaller” because we also apply it to negative numbers. For instance,  $\varepsilon$  is of infinitely smaller size than  $-1$ , although  $\varepsilon$  is not smaller than  $-1$ .

## 2.3 Non-Standard Probability Distributions

Consider a finite set  $X$ . A *non-standard probability distribution* on  $X$  is a function  $p : X \rightarrow \mathbf{R}(\varepsilon)$  such that  $p(x) \geq 0$  for all  $x \in X$  and  $\sum_{x \in X} p(x) = 1$ . We say that  $p$  is *cautious* on  $X$  if  $p(x) > 0$  for all  $x \in X$ . We say that  $p$  *believes* an event  $E \subseteq X$  if  $\sum_{x \in E} p(x)$  has standard part 1. Consider, for instance, the set  $X = \{x, y, z\}$  and the non-standard probability distribution  $p$  on  $X$  with  $p(x) = 1 - \varepsilon - \varepsilon^2$ ,  $p(y) = \varepsilon$  and  $p(z) = \varepsilon^2$ . Note that  $p$  is cautious, since every probability is strictly positive. Moreover,  $\varepsilon$  is of infinitely smaller size than  $1 - \varepsilon - \varepsilon^2$ , and  $\varepsilon^2$  is of infinitely smaller size than  $\varepsilon$ . Hence,  $p$  can be interpreted as a cautious belief in which you deem event  $\{x\}$  infinitely more likely than  $\{y\}$ , and event  $\{y\}$  infinitely more likely than  $\{z\}$ , while deeming each of these three events possible. Note that  $p$  believes the event  $\{x\}$ .

For a subset  $Y \subseteq X$  with  $\sum_{x \in Y} p(x) > 0$ , the *conditional* probability distribution on  $Y$  induced by  $p$  is the non-standard probability distribution  $p_Y$  on  $Y$  given by

$$p_Y(x) := \frac{p(x)}{\sum_{y \in Y} p(y)}$$



for every  $x \in Y$ .

A more extensive treatment of non-standard analysis, which is needed for some of the proofs, can be found in Appendix A.

### 3 Strong Sequential Rationalizability

The main ideas behind *strong sequential equilibrium* (Blume and Meier (2021)) are that a player (a) is cautious about the opponents' behavior and his own behavior, that is, he assigns a – possibly infinitesimal – strictly positive probability to every opponent's strategy and every continuation strategy by himself, (b) always believes that his opponents and he himself will choose rationally in the future, and (c) believes that he may make mistakes himself in the future, but deems his own future mistakes much less likely – in fact, infinitely less likely – than the mistakes by his opponents. In this section we attempt to incorporate this idea in a rationalizability concept that we call *strong sequential rationalizability*. We first give a formal definition of strong sequential rationalizability, subsequently prove that strongly sequentially rationalizable strategies always exist, and finally illustrate the concept by means of an example.

For every player  $i$ , let  $B_i^{self}$  be the set of cautious non-standard probability distributions on the set  $S_i$  of  $i$ 's own strategies. A member  $b_i^{self} \in B_i^{self}$  will be interpreted as a *belief* that player  $i$  has about his own future choices in the game. Hence, every belief in  $B_i^{self}$  always deems each of his own future choices possible. The assumption that a player holds a belief about his own future choices seems natural, but is certainly not standard in game theory. Battigalli, di Tillio and Samet (2013) and Battigalli and de Vito (2019) are some of the few papers that incorporate beliefs about own choices in a game-theoretic setting.

Moreover, let  $B_i^{opp}$  be the set of cautious non-standard probability distributions on the set  $S_{-i}$  of *opponents'* strategy combinations, where  $S_{-i} := \times_{j \neq i} S_j$ . A member  $b_i^{opp} \in B_i^{opp}$  represents a *belief* of player  $i$  about the opponents' strategies. By definition, every belief in  $B_i^{opp}$  deems each of the opponents' strategy combinations possible. By  $B_i$  we denote the set of belief pairs  $b_i = (b_i^{self}, b_i^{opp})$  where  $b_i^{self} \in B_i^{self}$  and  $b_i^{opp} \in B_i^{opp}$ .

Consider an information set  $h \in H_i$  and a choice  $c_i \in C_i(h)$  available at  $h$ . By  $S_i(h, c_i)$  we denote the set of strategies  $s_i \in S_i(h)$  with  $s_i(h) = c_i$ . By  $S_{-i}(h) := \{s_{-i} \in S_{-i} \mid \text{there is some } s_i \in S_i \text{ such that } (s_i, s_{-i}) \text{ reaches a history in } h\}$  we denote the set of opponents' strategy combinations that are possible when  $h$  is reached. For a given belief pair  $b_i = (b_i^{self}, b_i^{opp})$ , let  $b_i^{self}(h, c_i)$  be the induced conditional belief on  $S_i(h, c_i)$ , and let  $b_i^{opp}(h)$  be the induced conditional belief on  $S_{-i}(h)$ . By

$$u_i(c_i, b_i, h) := \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

we denote the expected (non-standard) utility at information set  $h$  of making choice  $c_i$  under the belief pair  $b_i$ . Here,  $z(s_i, s_{-i})$  is the outcome reached by the strategy combination  $(s_i, s_{-i})$ .

Note that defining  $u_i(c_i, b_i, h)$  would not be possible by using lexicographic probability systems, as we need to take the product of player  $i$ 's belief  $b_i^{self}(h, c_i)$  over his own future choices and player  $i$ 's belief  $b_i^{opp}(h)$  over the opponents' strategies. If we would use lexicographic probability systems, it is not clear how to define such product.

A choice  $c_i \in C_i(h)$  is *locally rational* for the belief pair  $b_i = (b_i^{self}, b_i^{opp})$  at information set  $h \in H_i$  if

$$u_i(c_i, b_i, h) \geq u_i(c'_i, b_i, h) \text{ for all } c'_i \in C_i(h).$$

For every information set  $h$  we define

$$S_i^{rat}(b_i, h) := \{s_i \in S_i(h) \mid s_i(h') \text{ locally rational for } b_i \text{ at every } h' \in H_i(s_i) \text{ following } h\}.$$

We say that the belief pair  $b_i = (b_i^{self}, b_i^{opp})$  *believes in his own future rationality* if for every information set  $h \in H_i$  and every choice  $c_i \in C_i(h)$ , the induced conditional belief  $b_i^{self}(h, c_i)$  believes  $S_i^{rat}(b_i, h)$ . Note that at information set  $h \in H_i$ , player  $i$  need not believe that his choice at  $h$  is optimal. Indeed, it may well be that  $c_i$  is suboptimal. The definition above only requires player  $i$  to believe at  $h \in H_i$  that his own choices *strictly following*  $h$  are optimal. This condition is similar to the notion of *optimal planning* in Battigalli and de Vito (2019).

We say that  $b_i$  *deems his own mistakes least likely*<sup>3</sup> if for every information set  $h \in H_i$ , every choice  $c_i \in C_i(h)$ , every strategy  $s_i \in S_i(h, c_i)$  and every strategy combination  $s_{-i} \in S_{-i}$  we have that

$$v[b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i))] > v(b_i^{opp}(s_{-i})).$$

Recall from Section 2.2 that for a non-standard number  $a = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots$ , the valuation  $v(a)$  is the lowest index  $k$  with  $a_k \neq 0$ . That is, the infinitesimal mistake part  $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i))$  in the belief about  $i$ 's own strategy is of infinitely smaller size than each of the belief probabilities  $b_i^{opp}(s_{-i})$  about the opponent's strategies. Also here it is crucial that we use non-standard probability distributions instead of lexicographic probability systems, as we need to compare the size of two possibly infinitesimal probabilities. Lexicographic probability systems are not expressive enough to do this.

We now define *strong sequential rationalizability* by recursively defining sets of strategies  $S_i^k(h)$ , for every information set  $h$ , and sets of belief pairs  $B_i^k$ , as follows. Remember that  $\emptyset$  is the information set that marks the beginning of the game.

**Definition 3.1 (Strong sequential rationalizability)** (*Initial step*) Set  $S_i^0(h) := S_i(h)$  and

$$B_i^0 : = \{b_i \in B_i \mid b_i \text{ believes in his own future rationality and} \\ \text{deems own mistakes least likely}\}$$

---

<sup>3</sup>A more accurate description would be “deems his own mistakes infinitely less likely than the opponents’ mistakes”. However, we use “least likely” instead, as to keep the description short.

for all players  $i$  and all information sets  $h$ .<sup>4</sup>

(Inductive step) Let  $k \geq 1$ , and suppose that  $S_i^{k-1}(h)$  and  $B_i^{k-1}$  have been defined for all players  $i$  and all information sets  $h$ . Then define, for every player  $i$  and every information set  $h$ ,

$$S_i^k(h) := \{s_i \in S_i^{k-1}(h) \mid \text{there is some } b_i \in B_i^{k-1} \text{ such that } s_i(h') \text{ is locally rational for } b_i \text{ at every } h' \in H_i(s_i) \text{ weakly following } h\}$$

and

$$B_i^k := \{b_i = (b_i^{self}, b_i^{opp}) \in B_i^{k-1} \mid b_i^{opp}(h) \text{ believes } S_{-i}^k(h) \text{ for all } h \in H_i\}.$$

A strategy  $s_i \in S_i$  is called *strongly sequentially rationalizable* if  $s_i \in S_i^k(\emptyset)$  for all  $k \geq 0$ .

Here,  $S_{-i}^k(h)$  need not be equal to  $\prod_{j \neq i} S_j^k(h)$ . By construction, if  $k \geq 1$  and a belief in  $B_i^k$  assigns, at information set  $h \in H_i$ , a non-infinitesimal probability to an opponent's strategy  $s_j$ , then  $s_j$  must be in  $S_j^k(h)$ , and hence there must be a belief in  $B_j^{k-1}$  for which  $s_j$  is optimal from  $h$  onwards. In other words, a belief in  $B_i^k$  believes, at every information set, that each opponent will choose rationally now and in the future. This resembles the idea of *belief in the opponents' future rationality* as formalized in Perea (2014). Similar ideas can be found in Penta (2015) and Baltag, Smets and Zvesper (2009).

Note that in the definition of strong sequential rationalizability, the beliefs for player  $i$  can always be summarized by a pair  $(b_i^{self}, b_i^{opp})$ . Thus, we implicitly assume that player  $i$ 's belief about his own future choices are always independent from his belief about the opponents' future choices.

One important question, of course, is whether in every finite dynamic game we can find for every player at least one strategy that is strongly sequentially rationalizable. The answer to this question is “yes”, as will be shown by the following theorem.

**Theorem 3.1 (Existence)** *For every finite dynamic game, and every player  $i$ , there is at least one strategy for player  $i$  that is strongly sequentially rationalizable.*

The proof, which is constructive, can be found in Appendix B.

If in the definition of strong sequential rationalizability we drop the condition of “own mistakes being deemed least likely” in  $B_i^0$ , then we obtain a rationalizability analogue to Selten's (1975) perfect equilibrium which we call *perfect rationalizability*.

**Definition 3.2 (Perfect rationalizability)** (Initial step) Set  $S_i^0(h) := S_i(h)$  and

$$B_i^0 := \{b_i \in B_i \mid b_i \text{ believes in his own future rationality}\}$$

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<sup>4</sup>Here, and at other places in the definition, information set  $h$  need not belong to player  $i$ .

for all players  $i$  and all information sets  $h$ .

(Inductive step) Let  $k \geq 1$ , and suppose that  $S_i^{k-1}(h)$  and  $B_i^{k-1}$  have been defined for all players  $i$  and all information sets  $h$ . Then define, for every player  $i$  and every information set  $h$ ,

$$S_i^k(h) := \{s_i \in S_i^{k-1}(h) \mid \text{there is some } b_i \in B_i^{k-1} \text{ such that } s_i(h') \text{ is locally rational for } b_i \text{ at every } h' \in H_i(s_i) \text{ weakly following } h\}$$

and

$$B_i^k := \{b_i = (b_i^{self}, b_i^{opp}) \in B_i^{k-1} \mid b_i^{opp}(h) \text{ believes } S_{-i}^k(h) \text{ for all } h \in H_i\}.$$

A strategy  $s_i \in S_i$  is called *perfectly rationalizable* if  $s_i \in S_i^k(\emptyset)$  for all  $k \geq 0$ .

The following observation is an immediate consequence of the definitions.

**Remark 1** *Every strategy that is strongly sequentially rationalizable, is also perfectly rationalizable.*

This remark thus states that strong sequential rationalizability inherits all the desirable properties from perfect equilibrium, except for the ‘‘correct beliefs’’ conditions that separate it from perfect rationalizability.

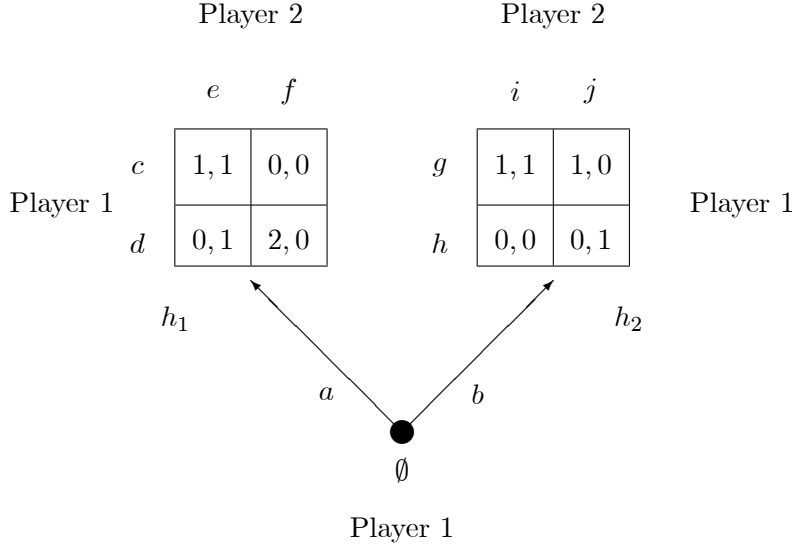
The other direction in the remark above is not true, as can be seen from the game  $\Gamma^2$  in Figure 1. In that game, the strategy  $b$  is perfectly rationalizable but not strongly sequentially rationalizable. Indeed, if player 1 believes that his own mistake  $d$  is infinitely less likely than the opponent’s mistake  $f$ , then he must go for  $(a, c)$ .

We will now illustrate the strong sequential rationalizability procedure by means of an example.

**Example 1: Illustration of strong sequential rationalizability procedure.**

Consider the game in Figure 2. This game starts at  $\emptyset$ , where player 1 can choose between  $a$  and  $b$ , and player 2 has a unique choice which we do not explicitly model. If player 1 chooses  $a$  the game reaches information set  $h_1$ , where players 1 and 2 can simultaneously choose between  $c$  and  $d$ , and between  $e$  and  $f$ , respectively. If he chooses  $b$  instead, the game reaches  $h_2$  where players 1 and 2 can simultaneously choose between  $g$  and  $h$ , and between  $i$  and  $j$ , respectively.

In the analysis that follows we do not explicitly consider player 2’s beliefs at  $\emptyset$  as these are not relevant for our purposes. Remember that  $b_1^{self}$  denotes player 1’s cautious belief about his own strategy choice, whereas his cautious belief about player 2’s strategy choice is denoted by  $b_1^{opp}$ . Then,  $b_1 = (b_1^{self}, b_1^{opp})$  constitutes player 1’s belief about his own behavior and player 2’s behavior. Let  $b_1^{self}(\emptyset, a)$  be player 1’s conditional belief about his own future choice at  $h_1$ , and  $b_1^{self}(\emptyset, b)$  his conditional belief about his own future choice at  $h_2$ . As an abbreviation, we denote



**Figure 2:** Illustration of strong sequential rationalizability

by  $b_1^{opp}(e) := b_1^{opp}(e, i) + b_1^{opp}(e, j)$  the probability that player 1 assigns to player 2 choosing  $e$  at  $h_1$ , and similarly for  $b_1^{opp}(f)$ . Then, by definition, player 1's expected utility of choosing  $a$  at  $\emptyset$  is

$$\begin{aligned}
 u_1(a, b_1, \emptyset) &= b_1^{self}(\emptyset, a)(a, c) \cdot b_1^{opp}(e) \cdot 1 + b_1^{self}(\emptyset, a)(a, d) \cdot b_1^{opp}(f) \cdot 2 \\
 &= 1 - b_1^{self}(\emptyset, a)(a, d) \cdot (1 - 3b_1^{opp}(f)) - b_1^{opp}(f).
 \end{aligned} \tag{3.1}$$

Here, we have used the fact that  $b_1^{self}(\emptyset, a)(a, c) = 1 - b_1^{self}(\emptyset, a)(a, d)$ , and that  $b_1^{opp}(e) = 1 - b_1^{opp}(f)$ . Similarly, player 1's expected utility of choosing  $b$  at  $\emptyset$  is

$$u_1(b, b_1, \emptyset) = b_1^{self}(\emptyset, b)(b, g) = 1 - b_1^{self}(\emptyset, b)(b, h). \tag{3.2}$$

**Initial step.** Note that  $(b, g)$  is player 1's only optimal strategy at  $h_2$ , and hence  $(b, h)$  is a mistake. As in  $B_1^0$  player 1 must believe in his own future rationality, we have that  $v(b_1^{self}(\emptyset, b)(b, h)) > 0$  for all  $b_1 \in B_1^0$ .

**Step 1.** Clearly,  $S_1^1(h_2) = \{(b, g)\}$ . By construction, every  $b_2 \in B_2^1$  must believe  $S_1^1(h_2) = \{(b, g)\}$ , and therefore  $v(b_2^{opp}(h_2)(b, h)) > 0$  for all  $b_2 \in B_2^1$ .

Similarly, note that  $e$  is player 2's only optimal choice at  $h_1$ , and hence  $f$  is a mistake. We thus conclude that  $S_2^1(h_1) = \{(e, i), (e, j)\}$ . By definition, every  $b_1 \in B_1^1$  must believe  $S_2^1(h_1)$  at  $h_1$ , and hence  $v(b_1^{opp}(f)) > 0$  for all  $b_1 \in B_1^1$ .

Remember that  $v(b_1^{self}(\emptyset, b)(b, h)) > 0$  for all  $b_1 \in B_1^0$ . Therefore,  $st(b_1^{self}(\emptyset, b)(b, h)) = 0$ . As every  $b_1 \in B_1^0$  deems his own mistakes least likely, we see that

$$v(b_1^{self}(\emptyset, b)(b, h)) = v(b_1^{self}(\emptyset, b)(b, h) - st(b_1^{self}(\emptyset, b)(b, h))) > v(b_1^{opp}(f))$$

for all  $b_1 \in B_1^0$ , and hence in particular for all  $b_1 \in B_1^1$ . Together with the insight above that  $v(b_1^{opp}(f)) > 0$  for all  $b_1 \in B_1^1$ , we conclude that

$$v(b_1^{self}(\emptyset, b)(b, h)) > v(b_1^{opp}(f)) > 0 \text{ for all } b_1 \in B_1^1. \quad (3.3)$$

**Step 2.** We have seen that  $v(b_2^{opp}(h_2)(b, h)) > 0$  for all  $b_2 \in B_2^1$ . Hence,  $i$  is player 2's unique optimal choice at  $h_2$  for every belief  $b_2 \in B_2^1$ . As  $e$  is player 2's unique optimal choice at  $h_1$  for any belief, we conclude that  $S_2^2(\emptyset) = \{(e, i)\}$ .

We now turn to player 1's beliefs. By combining (3.1), (3.2) and (3.3), it holds for every  $b_1 \in B_1^1$  that

$$\begin{aligned} u_1(a, b_1, \emptyset) &= 1 - b_1^{self}(\emptyset, a)(a, d) \cdot (1 - 3b_1^{opp}(f)) - b_1^{opp}(f) \\ &< 1 - b_1^{opp}(f) < 1 - b_1^{self}(\emptyset, b)(b, h) = u_1(b, b_1, \emptyset). \end{aligned}$$

Hence,  $b$  is the only optimal choice for player 1 at  $\emptyset$  for every belief  $b_1 \in B_1^1$ . As  $g$  is the only optimal choice for player 1 after  $b$  for every  $b_1 \in B_1^1$ , we conclude that  $S_1^2(\emptyset) = \{(b, g)\}$ .

We thus see that  $S_1^2(\emptyset) = \{(b, g)\}$  and  $S_2^2(\emptyset) = \{(e, i)\}$ . By Theorem 3.1, there is at least one strategy for player 1 and 2 that survives the procedure, and hence  $(b, g)$  and  $(e, i)$  must be the *only* strategies for player 1 and 2 that survive the procedure. Therefore,  $(b, g)$  and  $(e, i)$  are the unique strongly sequentially rationalizable strategies for players 1 and 2 in this game.

It turns out that that these are also the only quasi-perfectly rationalizable strategies in this game. However, strategy  $(a, c)$  for player 1 is *perfectly rationalizable*, whereas it is not strongly sequentially rationalizable. The reason is that according to perfect rationalizability, player 1 is free to believe that his own mistakes are more likely than player 2's mistakes. In particular, player 1 is free to believe that his mistake  $h$  after choosing  $b$  is much more likely than his own mistake  $d$ , and player 2's mistake  $f$ , after choosing  $a$ . In that case, it would be optimal for player 1 to choose  $a$  at  $\emptyset$  and  $c$  at  $h_1$ .  $\square$

## 4 Relation to Procedural Quasi-Perfect Rationalizability

In this section we propose *procedural quasi-perfect rationalizability* as a non-equilibrium counterpart to van Damme's (1984) *quasi-perfect equilibrium*. We will show in Appendix D that quasi-perfect rationalizability as defined in Asheim and Perea (2005) is always a refinement of procedural quasi-perfect rationalizability, and that there are dynamic games where both concepts differ in terms of the strategies induced. Yet, in "most" dynamic games the two concepts yield exactly the same sets of strategies.

Like strong sequential rationalizability, procedural quasi-perfect rationalizability is defined by iteratively eliminating strategies and beliefs from the game. The main idea that distinguishes (procedural) quasi-perfect rationalizability and quasi-perfect equilibrium from strong sequential rationalizability, is that a player believes, at each of his information sets, that his opponents will always make mistakes with some positive infinitesimal probability, but that he will not make mistakes himself. Similarly to strong sequential rationalizability, both (procedural) quasi-perfect rationalizability and quasi-perfect equilibrium are still based on the assumption that players deem all opponents' strategies possible, and that a player, at each of his information sets, believes in the opponents' future rationality.

We will show that strong sequential rationalizability is always a *refinement* of procedural quasi-perfect rationalizability. That is, every strategy that is strongly sequentially rationalizable is also procedurally quasi-perfectly rationalizable. The other direction is not true, as can be seen from the game  $\Gamma^1$  in Figure 1. Indeed, in that game the strategy  $(a, c)$  is procedurally quasi-perfectly rationalizable but not strongly sequentially rationalizable, since  $(a, c)$  induces the risk of making the mistake  $d$  in the future (that is, at the second node). In behavioral terms, this is precisely the key difference between the two concepts: Strong sequential rationalizability may additionally rule out strategies that are inferior because of the risk of *own* future mistakes.

Asheim and Perea (2005)'s definition of quasi-perfect rationalizability differs both methodologically and behaviorally from procedural quasi-perfect rationalizability. Instead of using a procedure that recursively eliminates strategies and beliefs from the game, Asheim and Perea use *belief hierarchies* as a primitive notion to define quasi-perfect rationalizability. That is, Asheim and Perea do not only consider first-order beliefs about the opponents' strategies, as we do in procedural quasi-perfect rationalizability, but also explore the players' *second-order* beliefs about the opponents' beliefs about the strategies of others, and *higher-order beliefs* as well. Asheim and Perea encode such belief hierarchies by means of epistemic models with types and lexicographic beliefs, and impose epistemic conditions on such belief hierarchies which give rise to their definition of quasi-perfect rationalizability.

In Appendix D we explore, in detail, the formal relation between procedural quasi-perfect rationalizability and quasi-perfect rationalizability. We show that every strategy that is quasi-perfectly rationalizable is also procedurally quasi-perfectly rationalizable, but not *vice versa*. Hence, quasi-perfect rationalizability is stronger than procedural quasi-perfect rationalizability. Intuitively, the key difference is the following: According to procedural quasi-perfect rationalizability, if a player  $i$  at information set  $h$  deems an opponent's strategy  $s_j$  most plausible, then there is a belief  $b_j$  that survives all rounds and for which the strategy  $s_j$  is optimal from  $h$  onwards. Quasi-perfect rationalizability requires more: If player  $i$ , at information set  $h$ , deems an opponent's belief  $b_j$  possible, and at (a *possibly different*) information set  $h'$  deems the opponent's strategy  $s_j$  most plausible given the opponent's belief  $b_j$ , then the strategy  $s_j$  must be optimal for  $b_j$  from  $h'$  onwards. Here, when we say that player  $i$  deems strategy  $s_j$  most plausible at  $h$ , we mean that there is no other strategy  $s'_j \in S_j(h)$  that  $b_i$  deems infinitely more likely than  $s_j$ . Hence, according to quasi-perfect rationalizability, the opponent's belief  $b_j$  that

player  $i$  assigns to his opponent at information set  $h$  is not only used to justify his behavior from  $h$  onwards, but also to form his belief about  $j$ 's behavior at information sets that do not follow  $h$ . In that sense, quasi-perfect rationalizability imposes restrictions that go beyond belief in the opponents' future rationality. In contrast, procedural quasi-perfect rationalizability only imposes rationality restrictions that are in line with belief in the opponents' future rationality.

In order to formally introduce procedural quasi-perfect rationalizability, we need the following additional notation and definitions. As before, let  $B_i^{opp}$  be the set of cautious non-standard probability distributions on the set of opponents' strategy combinations  $S_{-i}$ . For a belief  $b_i^{opp} \in B_i^{opp}$  and information set  $h \in H_i$ , let  $b_i^{opp}(h)$  be the induced conditional probability distribution on  $S_{-i}(h)$ . Consider an information set  $h \in H_i$  and a strategy  $s_i \in S_i(h)$ . By

$$u_i(s_i, b_i^{opp}(h)) := \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

we denote the *expected utility* at information set  $h \in H_i$  of choosing strategy  $s_i$  under the conditional belief  $b_i^{opp}(h)$ . We say that the strategy  $s_i \in S_i(h)$  is *globally rational* for  $b_i^{opp}$  at  $h$  if

$$u_i(s_i, b_i^{opp}(h)) \geq u_i(s'_i, b_i^{opp}(h)) \text{ for all } s'_i \in S_i(h).$$

Strategy  $s_i$  is *globally rational* for  $b_i^{opp}$  if  $s_i$  is globally rational for  $b_i^{opp}$  at every  $h \in H_i(s_i)$ .

In *procedural quasi-perfect rationalizability*, we recursively define sets of strategies  $S_i^k(h)$ , for all information sets  $h$ , and sets of non-standard beliefs  $B_i^{opp,k}$ , as follows.

**Definition 4.1 (Procedural quasi-perfect rationalizability)** (*Initial step*) Set  $S_i^0(h) := S_i(h)$  and  $B_i^{opp,0} := B_i^{opp}$  for all players  $i$  and all information sets  $h$ .

(*Inductive step*) Let  $k \geq 1$ , and suppose that  $S_i^{k-1}(h)$  and  $B_i^{opp,k-1}$  have been defined for all players  $i$  and all information sets  $h$ . Then define, for every player  $i$  and every information set  $h$ ,

$$S_i^k(h) := \{s_i \in S_i^{k-1}(h) \mid \text{there is some } b_i^{opp} \in B_i^{opp,k-1} \text{ such that } s_i \text{ is globally rational for } b_i^{opp} \text{ at every } h' \in H_i(s_i) \text{ weakly following } h\}$$

and

$$B_i^{opp,k} := \{b_i \in B_i^{opp,k-1} \mid b_i^{opp}(h) \text{ believes } S_{-i}^k(h) \text{ for all } h \in H_i\}.$$

A strategy  $s_i \in S_i$  is *procedurally quasi-perfectly rationalizable* if  $s_i \in S_i^k(\emptyset)$  for all  $k \geq 0$ .

Similar to strong sequential rationalizability, also procedural quasi-perfect rationalizability embodies the idea of *belief in the opponents' future rationality* as proposed by Perea (2014). Indeed, if a belief in  $B_i^{opp,k}$  assigns, at information set  $h \in H_i$ , a non-infinitesimal probability to an opponent's strategy  $s_j$ , then  $s_j$  must be in  $S_j^k(h)$ , and hence there must be some belief



in  $B_j^{opp,k-1}$  for which  $s_j$  is optimal from  $h$  onwards. The crucial difference between procedural quasi-perfect rationalizability and strong sequential rationalizability is that in the latter concept, a player also deems possible future mistakes by himself, whereas in the first concept a player only takes into account mistakes by his opponents. However – and that is crucial – under the latter concept the player deems his own mistakes infinitely less likely than the opponents’ mistakes. Due to this last property, it can be shown that every strongly sequentially rationalizable strategy is also procedurally quasi-perfectly rationalizable.

**Theorem 4.1 (Relation with procedural quasi-perfect rationalizability)** *Every strongly sequentially rationalizable strategy is procedurally quasi-perfectly rationalizable.*

The other direction is not true. Indeed, in game  $\Gamma^1$  from Figure 1 the strategy  $(a, c)$  is procedurally quasi-perfectly rationalizable but not strongly sequentially rationalizable. The proof of Theorem 4.1 can be found in Appendix C. What makes the proof challenging is that strong sequential rationalizability and procedural quasi-perfect rationalizability are defined in fundamentally different ways: In strong sequential rationalizability a player holds beliefs about his own future choices, and optimality of a strategy is defined *locally*, on a choice-by-choice basis. That is, optimality requires that at every information set the prescribed choice is locally optimal, given the player’s belief about his own future choices and given his belief about the opponents’ strategies. In contrast, procedural quasi-perfect rationalizability does not involve beliefs about the player’s own future choices, and optimality is defined *globally*. That is, optimality requires that at every information set the player’s strategy is optimal, given his belief about the opponents’ strategies. A key step in the proof is to show that, given a fixed belief, a sequence of locally optimal *choices* in the strong sequential rationalizability concept always yields a globally optimal *strategy* in the procedural quasi-perfect rationalizability concept. See Corollary 8.1 in Appendix C.

In Appendix D we show that Theorem 4.1 is no longer true if we replace procedural quasi-perfect rationalizability by quasi-perfect rationalizability as defined in Asheim and Perea (2005). Indeed, we provide a counterexample where some strategy is strongly sequentially rationalizable, but not quasi-perfectly rationalizable. Apparently, the extra conditions that quasi-perfect rationalizability imposes relative to procedural quasi-perfect rationalizability, as discussed at the beginning of this section, are not shared by strong sequential rationalizability. At the same time, quasi-perfect rationalizability is not a refinement of strong sequential rationalizability, as can be seen from the game  $\Gamma^1$  in Figure 1. In that game, strategy  $(a, c)$  is quasi-perfectly rationalizable, but not strongly sequentially rationalizable.

It is easily seen that procedural quasi-perfect rationalizability is a refinement of the backward dominance procedure in Perea (2014). Indeed, every strategy that is procedurally quasi-perfectly rationalizable also survives the backward dominance procedure, but not *vice versa*. The key difference between the two procedures is that the backward dominance procedure does not impose cautious reasoning, as a player, at each of his information sets, is free to assign probability

0 to certain opponents' strategies. Perea (2014) has shown that in every game with perfect information without relevant ties, the only strategies that survive the backward dominance procedure are the backward induction strategies. In the light of Theorem 4.1 it thus follows that in every such game, the only procedurally quasi-perfectly rationalizable (and hence the only strongly sequentially rationalizable) strategies are the backward induction strategies. This also holds for the perfectly rationalizable strategies.

Since we have seen in the previous section that strong sequential rationalizability is a refinement of perfect rationalizability, it follows from Theorem 4.1 that strong sequential rationalizability refines both perfect rationalizability and procedural quasi-perfect rationalizability. Hence, it inherits all the desirable properties that perfect rationalizability and procedural quasi-perfect rationalizability display. Yet, it adds the requirement that a player deems his own future mistakes infinitely less likely than his opponents' mistakes, without completely discarding his own future mistakes as procedural quasi-perfect rationalizability does.

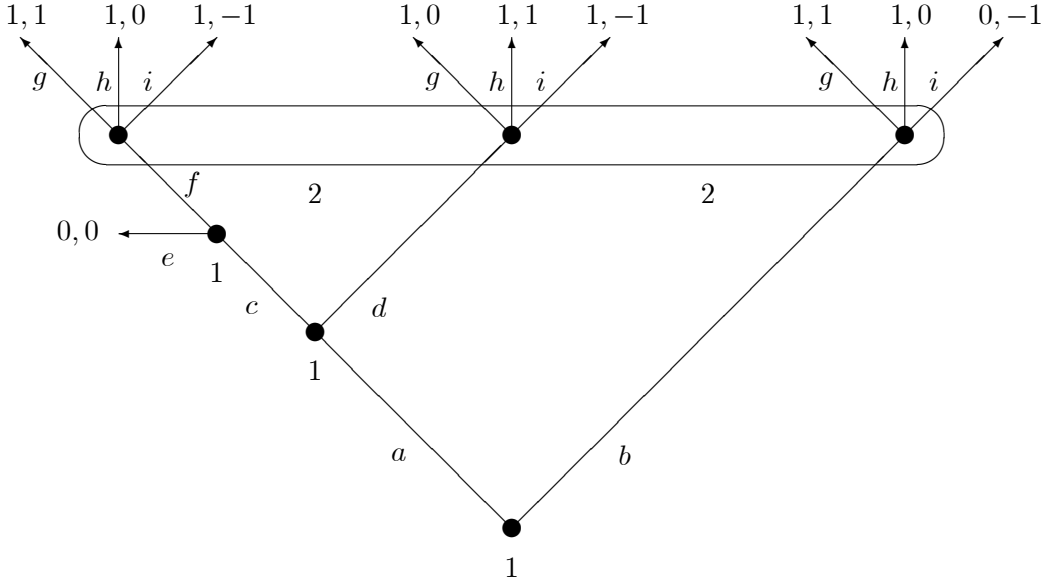
**Corollary 4.1 (Relation with perfect and procedural quasi-perfect rationalizability)**  
*Every strongly sequentially rationalizable strategy is both perfectly rationalizable and procedurally quasi-perfectly rationalizable.*

In particular, strong sequential rationalizability rules out both strategies containing weakly dominated *actions* in the agent normal form, as well as weakly dominated *strategies* in the normal form. This follows from the fact that perfect rationalizability avoids strategies containing weakly dominated actions in the agent normal form, and procedural quasi-perfect rationalizability avoids weakly dominated strategies in the normal form.

Interestingly, Corollary 4.1 does not have an equilibrium counterpart. In Blume and Meier (2021) it is shown that strong sequential equilibrium is a refinement of quasi-perfect equilibrium but not of perfect equilibrium. Even more, there are dynamic games where the sets of quasi-perfect equilibria and perfect equilibria are disjoint, rendering Corollary 4.1 impossible in an equilibrium framework. Apparently, the correct beliefs assumption which is implicit in equilibrium concepts renders the logic behind perfect equilibrium and quasi-perfect equilibrium incompatible. However, in view of Corollary 4.1, both logics become compatible once we drop the correct beliefs assumption.

In each of the examples we have seen so far, the other direction of Corollary 4.1 was also true. Indeed, in each of those examples every strategy that was both perfectly rationalizable and procedurally quasi-perfectly rationalizable was also strongly sequentially rationalizable. It is easily seen that this direction is always true in every game with perfect information and without relevant ties, as in such games the three concepts above all uniquely select the backward induction strategies for the players. Whether this direction is still true for games with perfect information and relevant ties is still an open question to us.

However, as the following example will show, the opposite direction of the corollary is not generally true for games with imperfect information. There are games where a strategy is



**Figure 3:** Combining perfect and procedural quasi-perfect rationalizability does not lead to strong sequential rationalizability

both perfectly rationalizable and procedurally quasi-perfectly rationalizable, but not strongly sequentially rationalizable.

**Example 2: Combining perfect and procedural quasi-perfect rationalizability does not lead to strong sequential rationalizability.**

Consider the game in Figure 3. Then, strategy  $g$  for player 2 is both perfectly rationalizable and procedurally quasi-perfectly rationalizable, but not strongly sequentially rationalizable. To see that  $g$  is perfectly rationalizable, note first that strategy  $b$  is perfectly rationalizable for player 1 if he believes that his own mistake ( $c, e$ ) is much more likely than player 2's mistake  $i$ . Hence, under perfect rationalizability, player 2 may assign at his information set a high probability to player 1 choosing  $b$ , which makes  $g$  optimal for player 2.

To see that  $g$  is procedurally quasi-perfectly rationalizable, note that under procedural quasi-perfect rationalizability player 1 believes that he will not make mistakes himself, and hence strategy  $(a, c, f)$  will be among his optimal strategies. Therefore, player 2 may assign at his information set a high probability to player 1 choosing  $(a, c, f)$ , which makes  $g$  optimal for player 2.

Under strong sequential rationalizability, however, player 1 believes that he will make the mistake  $(c, e)$  with positive probability, but he deems the probability of player 2 making the mistake  $i$  much higher. As such, the only strongly sequentially rationalizable strategy for player

1 is  $(a, d)$ . Hence, player 2 must at his information set assign a high probability to player 1 choosing  $(a, d)$ , which implies that player 2 must choose  $h$ . That is,  $h$  is the only strongly sequentially rationalizable strategy for player 2. In particular,  $g$  is not strongly sequentially rationalizable.  $\square$

## 5 Concluding Remarks

This paper presents a new rationalizability concept for dynamic games, based on three principles: (a) a player always believes in his own future rationality and in the opponents' future rationality, (b) a player always believes that he himself, and his opponents, will make mistakes with positive infinitesimal probability, and (c) a player deems his own mistakes infinitely less likely than those of his opponents. The concept may open the door to variations, or extensions, across different dimensions.

First, it could be extended – probably without major difficulties – to dynamic games with incomplete information. Second, condition (a), which guarantees that the concept has a backward induction flavour, could be replaced, for instance, by *strong belief in the opponents' rationality* (Battigalli and Siniscalchi (2002)), turning it into a forward induction concept. Third, as discussed in Footnote 1, the concept could be weakened if condition (c) is replaced by the weaker requirement that a player deems his own mistakes less likely, but not necessarily infinitely less likely, than those of his opponents.

On a more technical level, this paper shows that non-standard probabilities can be a very convenient tool for modelling cautious reasoning in games. One advantage compared to lexicographic probabilities is that the product of two probabilities can be taken without any problems. This is crucial in this paper, where a player at a given information set needs to “take the product” of his belief about his own future mistakes and his belief about the opponents' future mistakes.

## 6 Appendix A: Non-Standard Analysis

### 6.1 Non-Standard Numbers

Recall that the field of non-standard numbers  $\mathbf{R}(\varepsilon)$  contains all numbers  $a$  that can be written as

$$a = \frac{a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots + a_K\varepsilon^K}{b_0 + b_1\varepsilon + b_2\varepsilon^2 + \dots + b_K\varepsilon^K}, \quad (6.1)$$

where  $a_k, b_k \in \mathbf{R}$  for all  $k \in \{0, \dots, K\}$ ,  $b_k \neq 0$  for some  $k \in \{0, \dots, K\}$ , and where either  $a_0 \neq 0$  or  $b_0 \neq 0$ . We call the non-standard number  $a \in \mathbf{R}(\varepsilon)$  *finite* if there is some number  $b \in \mathbf{R}$  such that  $|a| < b$ . It is easily seen that  $a$  is finite, if and only if,  $b_0 \neq 0$ . We now show that every finite non-standard number can be written as a (possibly infinite) polynomial in  $\varepsilon$ . Since  $b_0 \neq 0$ , we

can write the denominator in (6.1) as

$$b_0 \left( 1 + \frac{b_1}{b_0} \varepsilon + \frac{b_2}{b_0} \varepsilon^2 + \dots + \frac{b_K}{b_0} \varepsilon^K \right).$$

Moreover, by the property of  $\varepsilon$  we know that  $|\frac{b_k}{b_0} \varepsilon^k| < (\frac{1}{2})^k$  for every  $k \in \{1, \dots, K\}$ , and hence

$$\left| \frac{b_1}{b_0} \varepsilon + \frac{b_2}{b_0} \varepsilon^2 + \dots + \frac{b_K}{b_0} \varepsilon^K \right| \leq \sum_{k=1}^K \left| \frac{b_k}{b_0} \varepsilon^k \right| < \sum_{k=1}^K \left( \frac{1}{2} \right)^k < 1.$$

But then, by the formula for geometric series it immediately follows that

$$\left( 1 + \frac{b_1}{b_0} \varepsilon + \frac{b_2}{b_0} \varepsilon^2 + \dots + \frac{b_K}{b_0} \varepsilon^K \right)^{-1} = 1 + \sum_{m=1}^{\infty} (-1)^m \left( \frac{b_1}{b_0} \varepsilon + \frac{b_2}{b_0} \varepsilon^2 + \dots + \frac{b_K}{b_0} \varepsilon^K \right)^m.$$

Combining this with (6.1) then yields

$$a = \frac{1}{b_0} \left( a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots + a_K \varepsilon^K \right) \left( 1 + \sum_{m=1}^{\infty} (-1)^m \left( \frac{b_1}{b_0} \varepsilon + \frac{b_2}{b_0} \varepsilon^2 + \dots + \frac{b_K}{b_0} \varepsilon^K \right)^m \right)$$

which is a power series in  $\varepsilon$ . We thus conclude that every *finite* number  $a \in \mathbf{R}(\varepsilon)$  can be written as

$$a = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots, \tag{6.2}$$

where  $a_k \in \mathbf{R}$  for every  $k \geq 0$ . We call this a *power series representation* of the number  $a$ . Below, we will show that this power series representation is unique.

## 6.2 Properties of Non-Standard Numbers

In this subsection we will investigate some important properties of finite non-standard numbers. First, we show that the sign of a non-standard number is fully determined by the sign of the leading coefficient in the power series representation (6.2). This property thus illustrates the lexicographic nature of the power series representation of non-standard numbers, as the leading coefficient  $a_k$  turns out to be “infinitely more important” than the collection of all the coefficients that follow.

**Lemma 6.1 (Leading coefficient determines sign)** *Consider a finite number  $a \in \mathbf{R}(\varepsilon)$  where  $a = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots$ . Then,  $a > 0$  if and only if there is some  $k \geq 0$  with  $a_k > 0$  and  $a_m = 0$  for all  $m < k$ .*

**Proof.** For the “if” part, suppose that  $a_k > 0$  and  $a_m = 0$  for all  $m < k$ . Hence,

$$a = a_k \varepsilon^k + a_{k+1} \varepsilon^{k+1} + \dots$$

where  $a_k > 0$ . By the property of  $\varepsilon$  we know that

$$|a_m| \varepsilon^m < a_k \left(\frac{1}{2}\right)^m \varepsilon^k$$

for every  $m \geq k + 1$ . Hence,

$$\left| \sum_{m=k+1}^{\infty} a_m \varepsilon^m \right| \leq \sum_{m=k+1}^{\infty} |a_m| \varepsilon^m < \sum_{m=k+1}^{\infty} a_k \left(\frac{1}{2}\right)^m \varepsilon^k \leq a_k \varepsilon^k,$$

which immediately implies that  $a > 0$ .

For the “only if” part, assume that  $a > 0$ . If  $a_k = 0$  for all  $k \geq 0$ , then  $a = 0$ , which would be a contradiction. Hence, there must be some  $k \geq 0$  with  $a_k \neq 0$  and  $a_m = 0$  for all  $m < k$ . If  $a_k < 0$ , then it follows by the “if” part above that  $a < 0$ , which would be a contradiction. Hence, we conclude that  $a_k > 0$ . ■

The lemma above really is the key result in this section, as all other properties follow rather directly from this lemma. A first consequence of Lemma 6.1 is that a non-standard number is 0 precisely when all coefficients in the power series representation are equal to 0.

**Lemma 6.2 (Zero has unique representation)** *Consider a finite number  $a \in \mathbf{R}(\varepsilon)$  where  $a = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots$ . Then,  $a = 0$  if and only if  $a_k = 0$  for all  $k \geq 0$ .*

**Proof.** The “if” direction is trivial. For the “only if” direction, assume that  $a = 0$ . Contrary to what we want to show, assume that there is some  $k \geq 0$  with  $a_k \neq 0$  and  $a_m = 0$  for all  $m < k$ . If  $a_k > 0$  then it follows from Lemma 6.1 that  $a > 0$ , which would be a contradiction. If  $a_k < 0$  then it follows by Lemma 6.1 that  $a < 0$ , which would also be a contradiction. Hence, we conclude that  $a_k = 0$  for all  $k \geq 0$ . ■

This lemma implies that for every finite non-standard number  $a$ , the power series representation is unique. Indeed, suppose that

$$a = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots = b_0 + b_1 \varepsilon + b_2 \varepsilon^2 + \dots .$$

Then,

$$(a_0 - b_0) + (a_1 - b_1) \varepsilon + (a_2 - b_2) \varepsilon^2 + \dots = 0,$$

which implies by Lemma 6.2 that  $a_k = b_k$  for all  $k \geq 0$ . We can therefore refer to (6.2) as *the* power series representation of the non-standard number  $a$ .

If  $a \neq 0$ , we denote by  $v(a)$  the smallest index  $k$  for which  $a_k \neq 0$ , and call it the *valuation* of the number  $a$ . We set  $v(0) := \infty$ . For two finite non-standard numbers  $a$  and  $b$  we say that  $a$  is of *infinitely smaller size* than  $b$  if  $v(a) > v(b)$ . We use the term “infinitely smaller size” rather than the more familiar “infinitely smaller” because we also apply it to negative numbers. For instance,  $\varepsilon$  is of infinitely smaller size than  $-1$ , although  $\varepsilon$  is not smaller than  $-1$ . Note that, by definition,  $0$  is of infinitely smaller size than any finite non-zero non-standard number.

For a finite non-standard number  $a$  with power series representation (6.2), and a given  $k \geq 0$ , we call

$$\text{trunc}^k(a) = a_0 + a_1\varepsilon + \dots + a_k\varepsilon^k$$

the  $k$ -th order truncation of  $a$ . The  $0$ -th order truncation  $a_0 \in \mathbf{R}$  is also called the *standard part* of  $a$ , and is denoted by  $st(a)$ . Hence,  $st(a)$  is the unique real number that is closest to  $a$ .

A consequence of Lemma 6.2 and Lemma 6.1 is that the  $k$ -th order truncation of  $a$  will either be zero, or have the same sign as  $a$ . This property will be important for our proofs.

**Lemma 6.3 (Truncation has the same sign )** *Consider a finite number  $a \in \mathbf{R}(\varepsilon)$  with  $a \geq 0$ . Then,  $\text{trunc}^k(a) \geq 0$  for every  $k \geq 0$ .*

**Proof.** Suppose first that  $a = 0$ . Then, by Lemma 6.2,  $a_k = 0$  for all  $k \geq 0$ , and hence  $\text{trunc}^k(a) = 0$  for all  $k \geq 0$ .

Assume next that  $a > 0$ . Then, by Lemma 6.1, there is some  $r$  with  $a_r > 0$  and  $a_m = 0$  for all  $m < r$ . If  $k < r$ , then  $\text{trunc}^k(a) = 0$ . If  $k \geq r$ , then  $\text{trunc}^k(a) > 0$  by Lemma 6.1. ■

In the following subsection we will use the properties above to investigate non-standard probability distributions.

### 6.3 Non-Standard Probability Distributions

Consider a finite set  $X$ . A *non-standard probability distribution* on  $X$  is a function  $p : X \rightarrow \mathbf{R}(\varepsilon)$  such that  $p(x) \geq 0$  for all  $x \in X$  and  $\sum_{x \in X} p(x) = 1$ . By  $\Delta^{ns}(X)$  we denote the set of non-standard probability distributions on  $X$ . Such non-standard probability distributions will often be interpreted as *beliefs*. We therefore use the terms “non-standard probability distribution” and “belief” interchangeably in this paper. For two elements  $x$  and  $y$  in  $X$ , we say that  $p$  deems  $x$  *infinitely more likely than*  $y$  if  $p(y)$  is of infinitely smaller size than  $p(x)$ .

Consider a non-standard probability distribution  $p$  on  $X$ . For a subset  $Y \subseteq X$  with  $\sum_{x \in Y} p(x) > 0$ , the *conditional* probability distribution on  $Y$  induced by  $p$  is the non-standard probability distribution  $p_Y$  on  $Y$  given by

$$p_Y(x) := \frac{p(x)}{\sum_{y \in Y} p(y)}$$

for every  $x \in Y$ .

We say that  $p$  is *cautious* on  $X$  if  $p(x) > 0$  for all  $x \in X$ , such that conditional probability distributions can be formed for every subset  $Y \subseteq X$ . We call  $p$  a *standard* probability distribution on  $X$  if  $p(x) \in \mathbf{R}$  for all  $x \in X$ , and the set of standard probability distributions on  $X$  is denoted by  $\Delta(X)$ . A *standard zero-sum distribution* on  $X$  is a function  $f : X \rightarrow \mathbf{R}$  with  $\sum_{x \in X} f(x) = 0$ .

Consider a non-standard probability distribution  $p$  on  $X$ . From above we know that every probability  $p(x)$  has a unique power series representation

$$p(x) = p_0(x) + p_1(x)\varepsilon + p_2(x)\varepsilon^2 + \dots ,$$

where  $p_k(x) \in \mathbf{R}$  for every  $k \geq 0$ . As  $\sum_{x \in X} p(x) = 1$ , it follows that

$$\left( \sum_{x \in X} p_0(x) - 1 \right) + \varepsilon \left( \sum_{x \in X} p_1(x) \right) + \varepsilon^2 \left( \sum_{x \in X} p_2(x) \right) + \dots = 0.$$

By Lemma 6.2 we thus conclude that

$$\sum_{x \in X} p_0(x) = 1 \text{ and } \sum_{x \in X} p_k(x) = 0 \text{ for all } k \geq 1.$$

Moreover, since  $p(x) \geq 0$  for every  $x \in X$ , it follows by Lemma 6.3 that  $p_0(x) = \text{trunc}^0(p(x)) \geq 0$  for every  $x \in X$ . Hence, we conclude that  $p_0 := (p_0(x))_{x \in X}$  is a standard probability distribution in  $\Delta(X)$ , and that  $p_k := (p_k(x))_{x \in X}$  is a standard zero-sum distribution on  $X$ .

As such, every non-standard probability distribution  $p$  on  $X$  can uniquely be written as

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots \tag{6.3}$$

where  $p_0 \in \Delta(X)$  is a standard probability distribution on  $X$ , and  $p_k$  is a standard zero-sum distribution on  $X$  for every  $k \geq 1$ . We call this the (unique) *power series representation* of the non-standard probability distribution  $p$ . This representation will be important for our game-theoretic analysis later on.

By  $st(p) := p_0$  we denote the standard part of the non-standard probability distribution  $p$ . For a subset  $Y \subseteq X$  we say that  $p$  *believes*  $Y$  if  $st(\sum_{x \in Y} p(x)) = 1$ . By the representation (6.3), this thus means that  $\sum_{x \in Y} p_0(x) = 1$ .

Consider an element  $x \in X$ . As  $p(x) \geq 0$ , it follows from (6.3) and Lemma 6.3 that  $\text{trunc}^k(p(x)) = p_0(x) + \varepsilon p_1(x) + \dots + \varepsilon^k p_k(x) \geq 0$  for every  $k \geq 0$ . Hence,  $p_0 + \varepsilon p_1 + \dots + \varepsilon^k p_k$  is a non-standard probability distribution on  $X$  as well.

Suppose that  $p$  is cautious on  $X$ , and let  $k$  be the minimal index such that the truncated non-standard probability distribution  $p_0 + \varepsilon p_1 + \dots + \varepsilon^k p_k$  is cautious on  $X$ . Then,  $p_0 + \varepsilon p_1 + \dots + \varepsilon^k p_k$  is called the *minimal cautious truncation* of  $p$  on  $X$ .



## 7 Appendix B: Proof of Theorem 3.1

Remember that  $B_i^k$  denotes the set of belief pairs  $(b_i^{self}, b_i^{opp})$  for player  $i$  that survives round  $k$  of the procedure. Similarly,  $S_i^k(h)$  is the set of strategies for player  $i$  that survive round  $k$  of the procedure at information set  $h$ . By construction,  $B_i^k \subseteq B_i^{k-1}$  and  $S_i^k(h) \subseteq S_i^{k-1}(h)$  for all  $k \geq 1$ . Since the collection of information sets is finite, and the set of strategies  $S_i(h)$  is finite for every player  $i$  and every information set  $h$ , the procedure must terminate within finitely many steps. To prove the existence of strongly sequentially rationalizable strategies, it is therefore sufficient to show that  $B_i^k$  and  $S_i^k(h)$  are always non-empty for every player  $i$ , every information set  $h$  and every  $k \geq 0$ . We prove so by induction on  $k$ .

For  $k = 0$  we have that  $S_i^0(h) = S_i(h)$ , and hence  $S_i^0(h)$  is non-empty. To prove that  $B_i^0$  is non-empty, we must show that there is a belief pair  $(b_i^{self}, b_i^{opp})$  for player  $i$  that believes in his own future rationality and deems his own mistakes least likely.

Take an arbitrary cautious non-standard probability distribution  $b_i^{opp}$  on the set  $S_{-i}$  of opponents' strategy combinations. For every opponents' strategy combination, let  $v(b_i^{opp}(s_{-i}))$  be the valuation of the probability  $b_i^{opp}(s_{-i})$ , as defined in Section 5.2. That is, if  $b_i^{opp}(s_{-i}) = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots$ , then  $v(b_i^{opp}(s_{-i}))$  is the smallest number  $m$  such that  $a_m \neq 0$ . Let  $v := \max\{v(b_i^{opp}(s_{-i})) \mid s_{-i} \in S_{-i}\}$  be the maximal valuation of  $b_i^{opp}(s_{-i})$  across all opponents' strategy combinations  $s_{-i} \in S_{-i}$ .

We now define the cautious non-standard probability distribution  $b_i^{self}$  on the set  $S_i$  of  $i$ 's own strategies by a backward induction construction, as follows. For every  $m \geq 0$ , let  $H_i^m$  be the collection of information sets in  $H_i$  that are followed by at most  $m$  consecutive information sets in  $H_i$ .

We start by considering all information sets in  $H_i^0$ , that is, player  $i$  information sets that are not followed by any other player  $i$  information set. Consider an information set  $h \in H_i^0$ , and let  $c_i^*(h)$  be an optimal choice for player  $i$  at  $h$  given the conditional belief  $b_i^{opp}(h)$ . Let  $\sigma_{ih}$  be the cautious non-standard probability distribution on the set of available choices  $C_i(h)$  given by

$$\sigma_{ih}(c_i) := \begin{cases} 1 - (|C_i(h)| - 1) \cdot \varepsilon^{v+1}, & \text{if } c_i = c_i^*(h) \\ \varepsilon^{v+1}, & \text{if } c_i \neq c_i^*(h) \end{cases} . \quad (7.1)$$

Now let  $m \geq 1$  and consider some  $h \in H_i^m$ . Suppose that the choice  $c_i^*(h')$  and the cautious non-standard probability distribution  $\sigma_{ih'}$  have been defined at all  $h' \in H_i^l$  where  $l \leq m - 1$ . In particular,  $c_i^*(h')$  and  $\sigma_{ih'}$  have been defined for all information sets  $h' \in H_i$  following  $h$ . For every choice  $c_i \in C_i(h)$ , let  $u_i(c_i, ((\sigma_{ih'})_{h' \in H_i: h' \succ h}, b_i^{opp}), h)$  be the expected utility of making choice  $c_i$  at  $h$ , given the conditional belief  $b_i^{opp}(h)$  about the opponents' strategy combinations, and given the non-standard probability distributions  $\sigma_{ih'}$  on  $i$ 's own choices at  $h'$  for every  $h' \in H_i$  that follows  $h$ . Above, we have used the expression " $h' \succ h$ " as a shortcut for " $h'$  follows  $h$ ". Let  $c_i^*(h)$  be an optimal choice for player  $i$  at  $h$ , that is,

$$u_i(c_i^*(h), ((\sigma_{ih'})_{h' \in H_i: h' \succ h}, b_i^{opp}), h) \geq u_i(c_i, ((\sigma_{ih'})_{h' \in H_i: h' \succ h}, b_i^{opp}), h) \text{ for all } c_i \in C_i(h). \quad (7.2)$$

Moreover, let  $\sigma_{ih}$  be the cautious non-standard probability distribution on the set of available choices  $C_i(h)$  given by

$$\sigma_{ih}(c_i) := \begin{cases} 1 - (|C_i(h)| - 1) \cdot \varepsilon^{v+1}, & \text{if } c_i = c_i^*(h) \\ \varepsilon^{v+1}, & \text{if } c_i \neq c_i^*(h) \end{cases} . \quad (7.3)$$

By induction on  $m$  we have thus defined, for every information set  $h \in H_i$ , the cautious non-standard probability distribution  $\sigma_{ih}$  on  $C_i(h)$ .

Let  $b_i^{self}$  be the cautious non-standard probability distribution on  $i$ 's own strategies given by

$$b_i^{self}(s_i) := \prod_{h \in H_i(s_i)} \sigma_{ih}(s_i(h)) \text{ for every } s_i \in S_i. \quad (7.4)$$

We will now show that  $b_i = (b_i^{self}, b_i^{opp})$  believes in his own future rationality and deems his own mistakes least likely.

To prove that  $b_i$  believes in his own future rationality, we must show that  $b_i^{self}(h, c_i)$  believes  $S_i^{rat}(b_i, h)$  for every  $h \in H_i$  and  $c_i \in C_i(h)$ . Take some  $h \in H_i$  and  $c_i \in C_i(h)$ . By (7.4) we conclude that

$$u_i(c_i, b_i, h) = u_i(c_i, ((\sigma_{ih'})_{h' \in H_i: h' \succ h}, b_i^{opp}), h). \quad (7.5)$$

By (7.5) and (7.2) it then follows that

$$u_i(c_i^*(h), b_i, h) \geq u_i(c_i, b_i, h) \text{ for all } c_i \in C_i(h). \quad (7.6)$$

Let  $s_i^*(h, c_i)$  be the unique strategy in  $S_i(h, c_i)$  such that  $s_i^*(h, c_i)$  prescribes the optimal choice  $c_i^*(h')$  at every  $h' \in H_i(s_i^*(h, c_i))$  not weakly preceding  $h$ . Hence, in particular,  $s_i^*(h, c_i)$  prescribes the optimal choice  $c_i^*(h')$  at every  $h' \in H_i(s_i^*(h, c_i))$  following  $h$ . Then, by (7.6) applied to every  $h' \in H_i(s_i^*(h, c_i))$  following  $h$ , we conclude that  $s_i^*(h, c_i) \in S_i^{rat}(b_i, h)$ .

Moreover, by (7.4) and (7.3) we conclude that the standard part of the conditional non-standard probability distribution  $b_i^{self}(h, c_i)$  assigns probability 1 to  $s_i^*(h, c_i) \in S_i^{rat}(b_i, h)$ . This implies that  $b_i^{self}(h, c_i)$  believes  $S_i^{rat}(b_i, h)$ . As this is true for every  $h \in H_i$  and  $c_i \in C_i(h)$ , we conclude that  $b_i$  believes in his own future rationality.

We next prove that  $b_i = (b_i^{self}, b_i^{opp})$  deems his own mistakes least likely. Consider an information set  $h \in H_i$  and a choice  $c_i \in C_i(h)$ . From (7.4) and (7.3) we see that for every strategy  $s_i \in S_i(h, c_i)$ , the infinitesimal mistake part  $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i))$  has a valuation which is at least  $v+1$ . On the other hand, the non-standard probability  $b_i^{opp}(s_{-i})$  has a valuation of at most  $v$  for every opponents' strategy combination  $s_{-i}$ , by definition of  $v$ . We thus conclude that  $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i)(s_i))$  is of infinitely smaller size than  $b_i^{opp}(s_{-i})$ . As this holds for every information set  $h \in H_i$ , choice  $c_i \in C_i(h)$ , strategy  $s_i \in S_i(h, c_i)$  and opponents' strategy combination  $s_{-i} \in S_{-i}$ , we know that  $b_i = (b_i^{self}, b_i^{opp})$  deems his own mistakes least likely.

Overall, we have thus constructed a belief  $b_i = (b_i^{self}, b_i^{opp})$  that believes in his own future rationality and deems his own mistakes least likely. Hence, by definition,  $b_i \in B_i^0$ , which implies that  $B_i^0$  is non-empty.

Now, let  $k \geq 1$  and assume that  $S_i^{k-1}(h)$  and  $B_i^{k-1}$  are non-empty for every player  $i$  and every information set  $h$ . Consider some player  $i$  and information set  $h$ . We will show that  $S_i^k(h)$  and  $B_i^k$  are non-empty.

To show that  $S_i^k(h)$  is non-empty, take some  $b_i = (b_i^{self}, b_i^{opp})$  in  $B_i^{k-1}$ . This is possible since we assume that  $B_i^{k-1}$  is non-empty. At every information set  $h' \in H_i$ , let  $c_i[h']$  be a locally rationally choice for  $b_i = (b_i^{self}, b_i^{opp})$  at  $h'$ . Let  $s_i$  be a strategy in  $S_i(h)$  such that  $s_i(h') = c_i[h']$  for every  $h' \in H_i(s_i)$  weakly following  $h$ . Then, by construction,  $s_i \in S_i^k(h)$ , and hence  $S_i^k(h)$  is non-empty.

We will next construct a belief  $b_i = (b_i^{self}, b_i^{opp})$  in  $B_i^k$ . We start by defining  $b_i^{opp}$ . That is, we must find a belief  $b_i^{opp}$  such that  $b_i^{opp}(h)$  believes  $S_{-i}^k(h)$  for all  $h \in H_i$ . For every opponent  $j \neq i$ , let  $b_j$  be an arbitrary belief pair in  $B_j^{k-1}$ . This is possible since we assume that  $B_j^{k-1}$  is non-empty. For every information set  $h \in H_i$  let  $s_{-i}[h] = (s_j[h])_{j \neq i}$  be an opponents' strategy combination in  $S_{-i}(h)$  with the following property: For every opponent  $j \neq i$  and every information set  $h' \in H_j(s_j)$  that does not precede  $h$ , the choice  $(s_j[h])(h')$  is locally rational for  $b_j$  at  $h'$ . Clearly, such a strategy  $s_j[h]$  can always be found. Then, by construction,  $s_j[h] \in S_j^k(h')$  for every  $h'$  that does not precede  $h$  and such that  $s_j[h] \in S_j(h')$ . Hence,

$$s_{-i}[h] \in S_{-i}^k(h') \text{ for every } h' \text{ that does not precede } h \quad (7.7)$$

and such that  $s_{-i}[h] \in S_{-i}(h')$ .

Since  $s_{-i}[h] \in S_{-i}(h)$ , it follows in particular that  $s_{-i}[h] \in S_{-i}^k(h)$ .

Let  $h_i^0, h_i^1, \dots, h_i^M$  be a numbering of the information sets of player  $i$  which respects their precedence ordering. That is, if  $h_i^l$  precedes  $h_i^m$  then  $l < m$ . Hence, it must be that  $h_i^0 = \emptyset$ . Let  $b_i^{opp}$  be the cautious non-standard belief about the opponents' strategy combinations given by

$$b_i^{opp}(s_{-i}) = \begin{cases} 1 - a, & \text{if } s_{-i} = s_{-i}[h_i^0] \\ \varepsilon^m, & \text{if } s_{-i} \neq s_{-i}[h_i^0], \text{ and } m \in \{1, \dots, M\} \text{ is minimal with } s_{-i} = s_{-i}[h_i^m], \\ \varepsilon^{M+1}, & \text{otherwise,} \end{cases}$$

where  $a$  is chosen such that  $\sum_{s_{-i} \in S_{-i}} b_i^{opp}(s_{-i}) = 1$ . Hence,  $st(a) = 0$ .

We will now show that  $b_i^{opp}(h)$  believes  $S_{-i}^k(h)$  for all  $h \in H_i$ . Take some arbitrary  $h \in H_i$ , and let  $h = h_i^m$ . Let  $l \in \{0, 1, \dots, M\}$  be the smallest number such that  $s_{-i}[h_i^l] \in S_{-i}(h_i^m)$ . Then, by construction, the standard part of the conditional belief  $b_i^{opp}(h_i^m)$  assigns probability 1 to  $s_{-i}[h_i^l]$ . That is,  $b_i^{opp}(h_i^m)$  believes  $\{s_{-i}[h_i^l]\}$ . Since  $s_{-i}[h_i^m] \in S_{-i}(h_i^m)$ , we know that  $l \leq m$ , and hence  $h_i^m$  does not precede  $h_i^l$ . We thus conclude that  $s_{-i}[h_i^l] \in S_{-i}(h_i^m)$  and that  $h_i^m$  does not precede  $h_i^l$ . But then, by (7.7),  $s_{-i}[h_i^l] \in S_{-i}^k(h_i^m)$ . As the conditional belief  $b_i^{opp}(h_i^m)$  believes

$\{s_{-i}[h_i^l]\}$ , we conclude that  $b_i^{opp}(h_i^m)$  believes  $S_{-i}^k(h_i^m)$ . This holds for every  $m$ , and hence  $b_i^{opp}(h)$  believes  $S_{-i}^k(h)$  for all  $h \in H_i$ .

In this way, we can construct a cautious non-standard belief  $b_i^{opp}$  on the opponents' strategy combinations such that  $b_i^{opp}(h)$  believes  $S_{-i}^k(h)$  for all  $h \in H_i$ . With  $b_i^{opp}$  at hand, we can then define the belief  $b_i^{self}$  in the same way as above, guaranteeing that  $b_i = (b_i^{self}, b_i^{opp})$  believes in his own future rationality and deems his own mistakes least likely. Hence,  $b_i \in B_i^0$ . Since, moreover,  $b_i^{opp}(h)$  believes  $S_{-i}^k(h)$  for all  $h \in H_i$ , we conclude that  $b_i \in B_i^k$ . We have thus shown that  $B_i^k$  is non-empty.

By induction on  $k$ , it follows that  $S_i^k(h)$  and  $B_i^k$  are always non-empty for every player  $i$ , every information set  $h$  and every  $k \geq 0$ . In particular,  $S_i^k(\emptyset)$  is always non-empty for all  $k \geq 0$ . Since the procedure terminates within finitely many steps, it follows that for every player  $i$  there is at least one strongly sequentially rationalizable strategy.  $\blacksquare$

## 8 Appendix C: Proof of Theorem 4.1

To prove Theorem 4.1, we proceed by three preparatory steps.

For the first step, consider a belief pair  $(b_i^{self}, b_i^{opp})$  in  $B_i$ . For every information set  $h \in H_i$  and choice  $c_i \in C_i(h)$ , let  $st(b_i^{self}(h, c_i))$  be the standard part of the conditional belief  $b_i^{self}(h, c_i)$  on  $S_i(h, c_i)$ . Moreover, let  $tr(b_i^{opp})$  be the minimal cautious truncation of the cautious belief  $b_i^{opp}$  on  $S_{-i}$ , as defined in Section 5.3. For every  $h \in H_i$ , this truncated belief  $tr(b_i^{opp})$  induces a conditional cautious belief  $tr(b_i^{opp})(h)$  on  $S_{-i}(h)$ . We say that a choice  $c_i^* \in C_i(h)$  is *locally rational* for  $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$  at  $h$  if

$$\begin{aligned} & \sum_{s_i \in S_i(h, c_i^*)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i^*))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\ & \geq \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \end{aligned}$$

for every  $c_i \in C_i(h)$ .

**Lemma 8.1 (Truncation preserves local rationality)** *Let  $(b_i^{self}, b_i^{opp})$  be a belief pair in  $B_i$  that deems own mistakes least likely,  $h \in H_i$  an information set for player  $i$ , and  $c_i^* \in C_i(h)$  a choice for player  $i$  at  $h$ . If  $c_i^*$  is locally rational for  $(b_i^{self}, b_i^{opp})$  at  $h$ , then  $c_i^*$  is also locally rational at  $h$  for the truncated beliefs  $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$ .*

**Proof.** Let the power series representation of the belief  $b_i^{opp}$  on  $S_{-i}$ , as defined in Section 5.3, be given by

$$b_i^{opp} = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots,$$

and let the minimal cautious truncation on  $S_{-i}$  be

$$tr(b_i^{opp}) = p_0 + \varepsilon p_1 + \dots + \varepsilon^K p_K.$$

By definition of the minimal cautious truncation, there must be some  $s_{-i}^* \in S_{-i}$  such that

$$b_i^{opp}(s_{-i}^*) = \varepsilon^K p_K(s_{-i}^*) + \varepsilon^{K+1} p_{K+1}(s_{-i}^*) + \dots \quad (8.1)$$

with  $p_K(s_{-i}^*) > 0$ .

Then, the conditional beliefs at  $h$  induced by  $b_i^{opp}$  and  $tr(b_i^{opp})$  are given by

$$b_i^{opp}(h)(s_{-i}) = \frac{1}{a} (p_0(s_{-i}) + \varepsilon p_1(s_{-i}) + \varepsilon^2 p_2(s_{-i}) + \dots) \quad (8.2)$$

for every  $s_{-i} \in S_{-i}(h)$ , where  $a := \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(s_{-i})$ , and

$$tr(b_i^{opp})(h)(s_{-i}) = \frac{1}{b} (p_0(s_{-i}) + \varepsilon p_1(s_{-i}) + \dots + \varepsilon^K p_K(s_{-i})) \quad (8.3)$$

for every  $s_{-i} \in S_{-i}(h)$ , where  $b := \sum_{s_{-i} \in S_{-i}(h)} tr(b_i^{opp})(s_{-i})$ .

For every choice  $c_i \in C_i(h)$ , let the power series representation of the conditional belief  $b_i^{self}(h, c_i)$  on  $S_i(h, c_i)$  be given by

$$b_i^{self}(h, c_i) = q_0^{c_i} + \varepsilon q_1^{c_i} + \varepsilon^2 q_2^{c_i} + \dots ,$$

which implies that

$$b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i))(s_i) = \varepsilon q_1^{c_i}(s_i) + \varepsilon^2 q_2^{c_i}(s_i) + \dots \quad (8.4)$$

for every  $s_i \in S_i(h, c_i)$ .

As  $(b_i^{self}, b_i^{opp})$  deems own mistakes least likely, we must have that  $b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i))(s_i)$  is of infinitely smaller size than  $b_i^{opp}(s_{-i}^*)$  for every  $c_i \in C_i(h)$  and every  $s_i \in S_i(h, c_i)$ . By (8.1) and (8.4) it thus follows that

$$b_i^{self}(h, c_i)(s_i) - st(b_i^{self}(h, c_i))(s_i) = \varepsilon^{K+1} q_{K+1}^{c_i}(s_i) + \varepsilon^{K+2} q_{K+2}^{c_i}(s_i) + \dots ,$$

and hence

$$b_i^{self}(h, c_i) = q_0^{c_i} + \varepsilon^{K+1} q_{K+1}^{c_i} + \varepsilon^{K+2} q_{K+2}^{c_i} + \dots . \quad (8.5)$$

Let  $b_i = (b_i^{self}, b_i^{opp})$ . By (8.2) and (8.5) it follows, for every choice  $c_i \in C_i(h)$ , that

$$\begin{aligned}
u_i(c_i, b_i, h) &= \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} (q_0^{c_i}(s_i) + \varepsilon^{K+1} q_{K+1}^{c_i}(s_i) + \varepsilon^{K+2} q_{K+2}^{c_i}(s_i) + \dots) \cdot \\
&\quad \cdot \frac{1}{a} (p_0(s_{-i}) + \varepsilon p_1(s_{-i}) + \varepsilon^2 p_2(s_{-i}) + \dots) \cdot u_i(z(s_i, s_{-i})) \\
&= \frac{1}{a} \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} (q_0^{c_i}(s_i) p_0(s_{-i}) + \varepsilon q_0^{c_i}(s_i) p_1(s_{-i}) + \dots \\
&\quad + \varepsilon^K q_0^{c_i}(s_i) p_K(s_{-i}) + d^{c_i}(s_i, s_{-i})) \cdot u_i(z(s_i, s_{-i})),
\end{aligned}$$

where  $v(d^{c_i}(s_i, s_{-i})) \geq K+1$ . Remember that  $v(d^{c_i}(s_i, s_{-i}))$  denotes the valuation of  $d^{c_i}(s_i, s_{-i})$ , which is the index of the leading coefficient in the power series representation of  $d^{c_i}(s_i, s_{-i})$ . Moreover, recall that the choice  $c_i^*$  is locally rational for  $b_i = (b_i^{self}, b_i^{opp})$  at  $h$ . Then, for every choice  $c_i \in C_i(h)$ ,

$$\begin{aligned}
u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h) &= \frac{1}{a} \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} ((q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_0(s_{-i}) + \\
&\quad + \varepsilon (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_1(s_{-i}) + \dots \\
&\quad + \varepsilon^K (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_K(s_{-i}) + \hat{d}^{c_i}(s_i, s_{-i})) \cdot u_i(z(s_i, s_{-i})) \tag{8.6}
\end{aligned}$$

where  $v(\hat{d}^{c_i}(s_i, s_{-i})) \geq K+1$ .

Since  $c_i^*$  is locally rational for  $b_i = (b_i^{self}, b_i^{opp})$  at  $h$ , we have that  $u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h) \geq 0$  for all  $c_i \in C_i(h)$ . This implies that  $\frac{a}{b} \cdot (u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h)) \geq 0$ . By Lemma 6.3 we thus know that  $\text{trunc}^K(\frac{a}{b} \cdot (u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h))) \geq 0$ .

By (8.6) we thus conclude that

$$\begin{aligned}
\text{trunc}^K(\frac{a}{b} \cdot (u_i(c_i^*, b_i, h) - u_i(c_i, b_i, h))) &= \frac{1}{b} (\sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} ((q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_0(s_{-i}) + \\
&\quad + \varepsilon (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_1(s_{-i}) + \dots + \varepsilon^K (q_0^{c_i^*}(s_i) - q_0^{c_i}(s_i)) p_K(s_{-i})) \cdot u_i(z(s_i, s_{-i}))) \geq 0
\end{aligned}$$

for every  $c_i \in C_i(h)$ . Hence,

$$\begin{aligned} & \frac{1}{b} \cdot \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} (q_0^{c_i^*}(s_i)p_0(s_{-i}) + \varepsilon q_0^{c_i^*}(s_i)p_1(s_{-i}) + \dots + \varepsilon^K q_0^{c_i^*}(s_i)p_K(s_{-i})) \cdot u_i(z(s_i, s_{-i})) \\ & \geq \frac{1}{b} \cdot \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} (q_0^{c_i}(s_i)p_0(s_{-i}) + \varepsilon q_0^{c_i}(s_i)p_1(s_{-i}) + \dots + \varepsilon^K q_0^{c_i}(s_i)p_K(s_{-i})) \cdot u_i(z(s_i, s_{-i})) \end{aligned}$$

for all  $c_i \in C_i(h)$ . As  $st(b_i^{self}(h, c_i)) = q_0^{c_i}$  for all  $c_i \in C_i(h)$  and  $tr(b_i^{opp})(h)$  is given by (8.3), the above inequality is equivalent to

$$\begin{aligned} & \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i^*))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\ & \geq \sum_{s_i \in S_i(h)} \sum_{s_{-i} \in S_{-i}(h)} st(b_i^{self}(h, c_i))(s_i) \cdot tr(b_i^{opp})(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \end{aligned}$$

for all  $c_i \in C_i(h)$ . This, in turn, means that  $c_i^*$  is locally rational at  $h$  for  $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$ , which was to show.  $\blacksquare$

As a second step, we prove that if a player believes in his own future rationality, then *local* rationality of a strategy at all information sets weakly following information set  $h^*$  implies *global* rationality of this strategy at  $h^*$ . To define this lemma formally, we need some additional notation and definitions. Let  $\hat{b}_i^{self} = (\hat{b}_i^{self}(h, c_i))_{h \in H_i, c_i \in C_i(h)}$ , where  $\hat{b}_i^{self}(h, c_i)$  is a standard probability distribution on  $S_i(h, c_i)$  for every  $h \in H_i$  and every  $c_i \in C_i(h)$ . Moreover, let  $b_i^{opp} \in B_i^{opp}$ . For a given information set  $h \in H_i$  and choice  $c_i \in C_i(h)$ , we define the expected utility

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) := \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} \hat{b}_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})).$$

We call  $c_i$  locally rational for  $(\hat{b}_i^{self}, b_i^{opp})$  at  $h$  if

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) \geq u_i(c'_i, (\hat{b}_i^{self}, b_i^{opp}), h) \text{ for all } c'_i \in C_i(h).$$

We say that  $(\hat{b}_i^{self}, b_i^{opp})$  believes in his own future rationality if for every  $h \in H_i$  and  $c_i \in C_i(h)$ , the standard probability distribution  $\hat{b}_i^{self}(h, c_i)$  only assigns positive probability to strategies  $s_i \in S_i(h, c_i)$  where  $s_i(h')$  is locally rational for  $(\hat{b}_i^{self}, b_i^{opp})$  at every  $h' \in H_i(s_i)$  following  $h$ .

**Lemma 8.2 (When local rationality implies global rationality)** *Let*

$\hat{b}_i^{self} = (\hat{b}_i^{self}(h, c_i))_{h \in H_i, c_i \in C_i(h)}$  *where*  $\hat{b}_i^{self}(h, c_i)$  *is a standard probability distribution on*  $S_i(h, c_i)$  *for every*  $h \in H_i$  *and every*  $c_i \in C_i(h)$ . *Let*  $b_i^{opp} \in B_i^{opp}$  *and assume that*  $(\hat{b}_i^{self}, b_i^{opp})$  *believes in his own future rationality. Let*  $s_i^* \in S_i$  *and*  $h^* \in H_i(s_i^*)$  *such that*  $s_i^*(h)$  *is locally rational for*  $(\hat{b}_i^{self}, b_i^{opp})$  *at every*  $h \in H_i(s_i^*)$  *weakly following*  $h^*$ . *Then,*  $s_i^*$  *is globally rational for*  $b_i^{opp}$  *at*  $h^*$ .

**Proof.** We first introduce some additional notation. For every information set  $h \in H_i$  and every choice  $c_i \in C_i(h)$ , let

$$u_i^{\max}(b_i^{opp}, h) := \max_{s_i \in S_i(h)} u_i(s_i, b_i^{opp}(h))$$

and

$$u_i^{\max}(b_i^{opp}, h, c_i) := \max_{s_i \in S_i(h, c_i)} u_i(s_i, b_i^{opp}(h)).$$

Then, we have that

$$u_i^{\max}(b_i^{opp}, h) = \max_{c_i \in C_i(h)} u_i^{\max}(b_i^{opp}, h, c_i) \quad (8.7)$$

for every  $h \in H_i$ . Moreover, strategy  $s_i$  is globally rational for  $b_i^{opp}$  at  $h \in H_i(s_i)$  if

$$u_i(s_i, b_i^{opp}(h)) = u_i^{\max}(b_i^{opp}, h).$$

For every information set  $h \in H_i$  and every choice  $c_i \in C_i(h)$  we also define

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) := \sum_{s_i \in S_i(h, c_i)} \sum_{s_{-i} \in S_{-i}(h)} \hat{b}_i^{self}(h, c_i)(s_i) \cdot b_i^{opp}(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i})).$$

Hence, strategy  $s_i$  is locally rational for  $(\hat{b}_i^{self}, b_i^{opp})$  at  $h \in H_i(s_i)$  if

$$u_i(s_i(h), (\hat{b}_i^{self}, b_i^{opp}), h) \geq u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h) \text{ for all } c_i \in C_i(h).$$

We prove the statement in the lemma by induction on the number of player  $i$  information sets that follow  $h^*$ . If  $h^*$  is not followed by any player  $i$  information set, then the statement holds because local rationality for  $(\hat{b}_i^{self}, b_i^{opp})$  at  $h^*$  coincides with global rational for  $b_i^{opp}$  at  $h$ .

Suppose now that  $h^*$  is followed by  $k \geq 1$  consecutive player  $i$  information sets, and that the statement holds for every player  $i$  information set that follows  $h^*$ . Consider some choice  $c_i \in C_i(h^*)$ . By  $H_i^+(h^*, c_i)$  we denote the collection of information sets  $h \in H_i$  such that  $h$  weakly follows  $h^*$  and  $c_i$ , and there is no  $h' \in H_i$  preceding  $h$  that also weakly follows  $h^*$  and  $c_i$ . Let  $S_{-i}^{not}(h^*, c_i)$  be the collection of those opponents' strategy combinations  $s_{-i} \in S_{-i}(h^*)$  that after  $h^*$  and  $c_i$  do not lead to any player  $i$  information set.



Then, for every  $c_i \in C_i(h^*)$  we have that

$$\begin{aligned}
u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{s_i \in S_i(h^*, c_i)} \sum_{s_{-i} \in S_{-i}(h^*)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \\
&\quad \cdot \left[ \sum_{h \in H_i^+(h^*, c_i)} \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) + \right. \\
&\quad \left. + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \right] \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \\
&\quad \cdot \left[ \sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \sum_{s_{-i} \in S_{-i}(h)} \frac{b_i^{opp}(h^*)(s_{-i})}{b_i^{opp}(h^*)(S_{-i}(h))} \cdot u_i(z(s_i, s_{-i})) \right. \\
&\quad \left. + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \right] \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \\
&\quad \left[ \sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \sum_{s_{-i} \in S_{-i}(h)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \right. \\
&\quad \left. + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \right] \\
&= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \left[ \sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i(s_i, b_i^{opp}(h)) \right. \\
&\quad \left. + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \right]. \tag{8.8}
\end{aligned}$$

Here, the fourth equality follows from the rules of conditional probabilities.

As  $(\hat{b}_i^{self}, b_i^{opp})$  believes in his own future rationality,  $\hat{b}_i^{self}(h^*, c_i)$  only assigns positive probability to  $s_i \in S_i(h^*, c_i)$  where  $s_i$  is locally rational for  $(\hat{b}_i^{self}, b_i^{opp})$  at every  $h \in H_i^+(h^*, c_i)$ , and every  $h' \in H_i(s_i)$  that follows  $h$ . By the induction assumption, we know that every such  $s_i$  is globally rational at every  $h \in H_i^+(h^*, c_i)$ . Hence,  $\hat{b}_i^{self}(h^*, c_i)$  only assigns positive probability to  $s_i \in S_i(h^*, c_i)$  where

$$u_i(s_i, b_i^{opp}(h)) = u_i^{\max}(b_i^{opp}, h)$$

for every  $h \in H_i^+(h^*, c_i)$ . Together with (8.8) we conclude that

$$\begin{aligned}
u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{s_i \in S_i(h^*, c_i)} \hat{b}_i^{self}(h^*, c_i)(s_i) \cdot \left[ \sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i^{\max}(b_i^{opp}, h) \right. \\
&\quad \left. + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \right] \\
&= \sum_{h \in H_i^+(h^*, c_i)} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i^{\max}(b_i^{opp}, h) + \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, c_i)} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= u_i^{\max}(b_i^{opp}, h^*, c_i).
\end{aligned} \tag{8.9}$$

Here, the last equality follows from the fact that the terminal node  $z(s_i, s_{-i})$  does not depend on the specific  $s_i \in S_i(h^*, c_i)$  if  $s_{-i} \in S_{-i}^{not}(h^*, c_i)$ . Hence, we see that

$$u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) = u_i^{\max}(b_i^{opp}, h^*, c_i) \text{ for all } c_i \in C_i(h^*). \tag{8.10}$$

As  $s_i^*(h^*)$  is locally rational for  $(\hat{b}_i^{self}, b_i^{opp})$  at  $h^*$ , we know that

$$\begin{aligned}
u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \max_{c_i \in C_i(h^*)} u_i(c_i, (\hat{b}_i^{self}, b_i^{opp}), h^*) \\
&= \max_{c_i \in C_i(h^*)} u_i^{\max}(b_i^{opp}, h^*, c_i) = u_i^{\max}(b_i^{opp}, h^*),
\end{aligned} \tag{8.11}$$

where the second equality follows from (8.10) and the last equality from (8.7).

On the other hand, we know by (8.9) that

$$\begin{aligned}
u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{h \in H_i^+(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i^{\max}(b_i^{opp}, h) + \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})).
\end{aligned} \tag{8.12}$$

As  $s_i^*(h)$  is assumed to be locally rational for  $(\hat{b}_i^{self}, b_i^{opp})$  at every  $h \in H_i(s_i^*)$  weakly following  $h^*$ , we know that, for every  $h \in H_i^+(h^*, s_i^*(h^*))$ , the choice  $s_i^*(h')$  is locally rational for  $(\hat{b}_i^{self}, b_i^{opp})$  at every  $h' \in H_i(s_i^*)$  weakly following  $h$ . Hence, by the induction assumption,  $s_i^*$  is globally rational for  $b_i^{opp}$  at every  $h \in H_i^+(h^*, s_i^*(h^*))$ , which means that

$$u_i(s_i^*, b_i^{opp}(h)) = u_i^{\max}(b_i^{opp}, h) \text{ for every } h \in H_i^+(h^*, s_i^*(h^*)). \tag{8.13}$$

By combining (8.12) and (8.13) we obtain that

$$\begin{aligned}
u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) &= \sum_{h \in H_i^+(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(S_{-i}(h)) \cdot u_i(s_i^*, b_i^{opp}(h)) + \\
&\quad + \sum_{s_{-i} \in S_{-i}^{not}(h^*, s_i^*(h^*))} b_i^{opp}(h^*)(s_{-i}) \cdot u_i(z(s_i, s_{-i})) \\
&= u_i(s_i^*, b_i^{opp}(h^*)). \tag{8.14}
\end{aligned}$$

From (8.14) and (8.11) we can thus conclude that

$$u_i(s_i^*, b_i^{opp}(h^*)) = u_i(s_i^*(h^*), (\hat{b}_i^{self}, b_i^{opp}), h^*) = u_i^{\max}(b_i^{opp}, h^*).$$

This means that  $s_i^*$  is globally rational for  $b_i^{opp}$  at  $h^*$ , which was to show. By induction, the proof is thus complete.  $\blacksquare$

As a third step, we are able to derive the following important result by combining Lemma 8.1 and Lemma 8.2. This step will be crucial for proving our main theorem below.

**Corollary 8.1 (From local to global rationality)** *Let  $(b_i^{self}, b_i^{opp})$  be a belief pair in  $B_i$  that deems own mistakes least likely and believes in his own future rationality. Let  $s_i^* \in S_i$  and  $h^* \in H_i(s_i^*)$  such that  $s_i^*$  is locally rational for  $(b_i^{self}, b_i^{opp})$  at every  $h \in H_i(s_i^*)$  that weakly follows  $h^*$ . Then,  $s_i^*$  is globally rational at  $h^*$  for the minimal cautious truncation  $tr(b_i^{opp})$  of  $b_i^{opp}$ .*

**Proof.** Suppose that  $s_i^*$  is locally rational for  $(b_i^{self}, b_i^{opp})$  at every  $h \in H_i(s_i^*)$  that weakly follows  $h^*$ . That is,  $s_i^*(h)$  is locally rational for  $(b_i^{self}, b_i^{opp})$  at every  $h \in H_i(s_i^*)$  that weakly follows  $h^*$ . Since  $(b_i^{self}, b_i^{opp})$  deems own mistakes least likely, it follows from Lemma 8.1 that  $s_i^*(h)$  is also locally rational for the truncated belief pair  $((st(b_i^{self}(h, c_i)))_{c_i \in C_i(h)}, tr(b_i^{opp}))$  at every  $h \in H_i(s_i^*)$  that weakly follows  $h^*$ .

Define  $\hat{b}_i^{self} := (st(b_i^{self}(h, c_i)))_{h \in H_i, c_i \in C_i(h)}$ . We will show that  $(\hat{b}_i^{self}, tr(b_i^{opp}))$  believes in his own future rationality. As the original belief pair  $(b_i^{self}, b_i^{opp})$  believes in his own future rationality, we know that for every  $h \in H_i$  and  $c_i \in C_i(h)$ , the standard part of  $b_i^{self}(h, c_i)$  only assigns positive probability to strategies  $s_i \in S_i(h, c_i)$  where  $s_i(h')$  is locally rational for  $(b_i^{self}, b_i^{opp})$  at every  $h' \in H_i(s_i)$  following  $h$ . By Lemma 8.1 we know that every such  $s_i(h')$  is also locally rational for  $(\hat{b}_i^{self}, tr(b_i^{opp}))$  at  $h'$ . Hence, we conclude that for every  $h \in H_i$  and  $c_i \in C_i(h)$ , the belief  $\hat{b}_i^{self}(h, c_i) = st((b_i^{self}(h, c_i)))$  only assigns positive probability to strategies  $s_i \in S_i(h, c_i)$  where  $s_i(h')$  is locally rational for  $(\hat{b}_i^{self}, tr(b_i^{opp}))$  at every  $h' \in H_i(s_i)$  following  $h$ . Therefore,  $(\hat{b}_i^{self}, tr(b_i^{opp}))$  believes in his own future rationality.

As such, we conclude that  $s_i^*(h)$  is locally rational for the truncated belief pair  $(\hat{b}_i^{self}, tr(b_i^{opp}))$  at every  $h \in H_i(s_i^*)$  that weakly follows  $h^*$ , and that  $(\hat{b}_i^{self}, tr(b_i^{opp}))$  believes in his own future rationality. By Lemma 8.2 it follows that  $s_i^*$  is globally rational for  $tr(b_i^{opp})$  at  $h^*$ , which was to show.  $\blacksquare$

We are now fully equipped to prove Theorem 4.1.

**Proof of Theorem 4.1.** For every  $k \geq 0$ , every player  $i$  and every  $h \in H_i$ , let  $S_{i,qp}^k(h)$  and  $B_{i,qp}^{opp,k}$  be the sets of strategies and beliefs that survive round  $k$  of the procedural quasi-perfect rationalizability procedure. Similarly, let  $S_{i,ss}^k(h)$  and  $B_{i,ss}^k$  be the sets of strategies and belief pairs that survive round  $k$  of the strong sequential rationalizability procedure. As before, for every  $b_i^{opp} \in B_i^{opp}$  we denote by  $tr(b_i^{opp})$  the minimal cautious truncation of  $b_i^{opp}$  on  $S_{-i}$ . We prove the following claim.

*Claim.* For every  $k \geq 0$ , every player  $i$  and every  $h^* \in H_i$ , (a)  $S_{i,ss}^k(h^*) \subseteq S_{i,qp}^k(h^*)$ , and (b) for every  $(b_i^{self}, b_i^{opp}) \in B_{i,ss}^k$  it holds that  $tr(b_i^{opp}) \in B_{i,qp}^{opp,k}$ .

*Proof of claim.* We prove so by induction on  $k$ . For  $k = 0$  the statement is trivial since  $S_{i,ss}^0(h^*) = S_{i,qp}^0(h^*) = S_i(h^*)$  and  $B_{i,qp}^{opp,0}$  is the set of all cautious non-standard probability distributions on  $S_{-i}$ .

Let  $k \geq 1$ , and suppose that (a) and (b) are true for  $k - 1$ . To show (a) for  $k$ , take some strategy  $s_i^* \in S_{i,ss}^k(h^*)$ . Then, by definition,  $s_i^* \in S_{i,ss}^{k-1}(h^*)$ , and there is some  $(b_i^{self}, b_i^{opp}) \in B_{i,ss}^{k-1}$  such that  $s_i^*(h)$  is locally rational for  $(b_i^{self}, b_i^{opp})$  at every  $h \in H_i(s_i^*)$  weakly following  $h^*$ . Since, by the induction assumption on (a),  $S_{i,ss}^{k-1}(h^*) \subseteq S_{i,qp}^{k-1}(h^*)$ , we know that  $s_i^* \in S_{i,qp}^{k-1}(h^*)$ . Moreover, as  $(b_i^{self}, b_i^{opp}) \in B_{i,ss}^{k-1} \subseteq B_{i,ss}^0$  we know that  $(b_i^{self}, b_i^{opp})$  deems own mistakes least likely and believes in his own future rationality. Since  $s_i^*$  is locally rational for  $(b_i^{self}, b_i^{opp})$  at every  $h \in H_i(s_i^*)$  weakly following  $h^*$ , we thus conclude by Corollary 8.1 that  $s_i^*$  is globally rational for  $tr(b_i^{opp})$  at every  $h \in H_i(s_i^*)$  weakly following  $h^*$ . Moreover, as  $(b_i^{self}, b_i^{opp}) \in B_{i,ss}^{k-1}$ , we know by the induction assumption on (b) that  $tr(b_i^{opp}) \in B_{i,qp}^{opp,k-1}$ .

Summarizing, we see that  $s_i^* \in S_{i,qp}^{k-1}(h^*)$ , and that  $s_i^*$  is globally rational for  $tr(b_i^{opp}) \in B_{i,qp}^{opp,k-1}$  at every  $h \in H_i(s_i^*)$  weakly following  $h^*$ . Hence, by definition,  $s_i^* \in S_{i,qp}^k(h^*)$ . We thus conclude that  $S_{i,ss}^k(h^*) \subseteq S_{i,qp}^k(h^*)$ .

To show (b), take some  $(b_i^{self}, b_i^{opp}) \in B_{i,ss}^k$ . Then, by definition,  $(b_i^{self}, b_i^{opp}) \in B_{i,ss}^{k-1}$  and  $b_i^{opp}(h)$  believes  $S_{-i,ss}^{k-1}(h)$  for every  $h \in H_i$ . By the induction assumption on (b) we already know that  $tr(b_i^{opp}) \in B_{i,qp}^{opp,k-1}$ . Since  $b_i^{opp}(h)$  believes  $S_{-i,ss}^{k-1}(h)$ , the standard part of  $b_i^{opp}(h)$  only assigns positive probability to opponents' strategy combinations  $s_{-i} \in S_{-i,ss}^{k-1}(h)$ . Note that the standard part of  $b_i^{opp}(h)$  is the same as the standard part of  $tr(b_i^{opp})(h)$ . Hence, the standard part of  $tr(b_i^{opp})(h)$  only assigns positive probability to  $s_{-i} \in S_{-i,ss}^{k-1}(h)$ . By the induction assumption

on (a) we know that  $S_{-i,ss}^{k-1}(h) \subseteq S_{-i,qp}^{k-1}(h)$ , and therefore the standard part of  $tr(b_i^{opp})(h)$  only assigns positive probability to  $s_{-i} \in S_{-i,qp}^{k-1}(h)$ . In other words,  $tr(b_i^{opp})(h)$  believes  $S_{-i,qp}^{k-1}(h)$ .

Summarizing, we see that  $tr(b_i^{opp}) \in B_{i,qp}^{opp,k-1}$  and that  $tr(b_i^{opp})(h)$  believes  $S_{-i,qp}^{k-1}(h)$  for every  $h \in H_i$ . Hence, by definition,  $tr(b_i^{opp}) \in B_{i,qp}^{opp,k}$ , as was to show.

By induction on  $k$ , (a) and (b) are true for every  $k \geq 0$ , which completes the proof of the claim.

To prove the theorem, consider some strongly sequentially rationalizable strategy  $s_i^*$  for player  $i$ . Then, by definition,  $s_i^* \in S_{i,ss}^k(\emptyset)$  for all  $k \geq 0$ . Hence, by part (a) of the claim,  $s_i^* \in S_{i,qp}^k(\emptyset)$  for all  $k \geq 0$ , which means that  $s_i^*$  is procedurally quasi-perfectly rationalizable. This completes the proof. ■

## 9 Appendix D: Relation with Quasi-Perfect Rationalizability

In this section we will compare our notion of procedural quasi-perfect rationalizability to quasi-perfect rationalizability as defined by Asheim and Perea (2005). To that purpose, we first review the definition of quasi-perfect rationalizability, and subsequently show that in all games, every quasi-perfectly rationalizable strategy is also procedurally quasi-perfectly rationalizable in our sense. We then show by means of a counterexample that there are procedurally quasi-perfectly rationalizable strategies which are not quasi-perfectly rationalizable. Hence, quasi-perfect rationalizability is a strict refinement of procedural quasi-perfect rationalizability. The same example also demonstrates that even a strongly sequentially rationalizable strategy need not be quasi-perfectly rationalizable.

### 9.1 Quasi-Perfect Rationalizability

We have defined procedural quasi-perfect rationalizability by means of a procedure, that recursively eliminates strategies and beliefs from the game. Asheim and Perea (2005) take a different approach, since they define quasi-perfect rationalizability by looking at *belief hierarchies* encoded by types within an epistemic model. Also, they use *lexicographic beliefs* (Blume, Brandenburger and Dekel (1991)) rather than non-standard beliefs to model cautious reasoning. That is, they take as a primitive not only beliefs about the opponents' strategies, as we do, but also beliefs about the opponents' beliefs about the other players' strategies (second-order beliefs), and higher-order beliefs. Quasi-perfect rationalizability is defined by imposing epistemic conditions on such belief hierarchies. Since we have used non-standard beliefs, rather than lexicographic beliefs, to define strong sequential rationalizability and procedural quasi-perfect rationalizability, we will reproduce the definition of quasi-perfect rationalizability by using non-standard beliefs instead of lexicographic beliefs.

**Definition 9.1 (Epistemic model with non-standard beliefs)** *For a given dynamic game*

$G$ , a finite epistemic model with non-standard beliefs is a tuple  $M = (T_i, \beta_i)_{i \in I}$  such that, for every player  $i$ ,

(a)  $T_i$  is a finite set of types, and

(b)  $\beta_i$  is a function that assigns to every type  $t_i \in T_i$  a non-standard belief  $\beta_i(t_i)$  on  $S_{-i} \times T_{-i}$ .

An epistemic model is used to *encode* non-standard belief hierarchies for the players, including beliefs about the opponents' strategies, beliefs about the opponents' beliefs about their opponents' strategies, and so on. The concept of quasi-perfect rationalizability restricts to types that express common full belief in "caution" and the "event that types induce sequentially rational behavioral strategies". We will now formally define these events.

**Definition 9.2 (Caution)** Consider a finite epistemic model  $M = (T_i, \beta_i)_{i \in I}$  for a dynamic game  $G$ . A type  $t_i \in T_i$  is cautious if for every opponents' type combination  $t_{-i} \in T_{-i}$  with  $\beta_i(t_i)(t_{-i}) > 0$ , it holds that  $\beta_i(t_i)(s_{-i}, t_{-i}) > 0$  for every  $s_{-i} \in S_{-i}$ .

Here,  $\beta_i(t_i)(t_{-i})$  is an abbreviation for the marginal probability  $\beta_i(t_i)(S_{-i} \times \{t_{-i}\})$ . We will use such abbreviations for marginals more often in the remainder of this section. Hence, caution states that if  $t_i$  seems possible a type combination  $t_{-i}$  for his opponents, then he must deem possible every strategy combination for that type combination. In particular,  $t_i$  holds a cautious non-standard belief on the set  $S_{-i}$  of opponents' strategy combinations. Consider a cautious type  $t_i$  and an information set  $h \in H_i$ . By  $\beta_i(t_i, h)$  we denote the induced (cautious) conditional belief on  $S_{-i}(h) \times T_{-i}$ . For every strategy  $s_i \in S_i(h)$  we denote by

$$u_i(s_i, t_i, h) := \sum_{(s_{-i}, t_{-i}) \in S_{-i}(h) \times T_{-i}} \beta_i(t_i, h)(s_{-i}, t_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

the expected (non-standard) utility at  $h$  of choosing strategy  $s_i$  under the conditional belief  $\beta_i(t_i, h)$ . We say that a strategy  $s_i \in S_i(h)$  is *globally rational* for the cautious type  $t_i$  at  $h \in H_i$  if

$$u_i(s_i, t_i, h) \geq u_i(s'_i, t_i, h) \text{ for all } s'_i \in S_i(h).$$

To define what it means for a type to "induce a sequentially rational behavioral strategy" we need some additional terminology. A *behavioral strategy* for player  $i$  is a tuple  $\sigma_i = (\sigma_i(h))_{h \in H_i}$  such that  $\sigma_i(h)$  is a (standard) probability distribution on the set of choices  $C_i(h)$  available for player  $i$  at  $h$ . For a behavioral strategy  $\sigma_i$  and a strategy  $s_i \in S_i$ , let

$$\sigma_i(s_i) := \prod_{h \in H_i(s_i)} \sigma_i(s_i(h)) \tag{9.1}$$

be the induced probability that  $\sigma_i$  assigns to the strategy  $s_i$ . For a given behavioral strategy  $\sigma_i$  and information set  $h \in H_i$ , let  $\sigma_i|_h$  be the behavioral strategy that (i) at every  $h' \in H_i$  preceding

$h$  assigns probability 1 to the unique choice for player  $i$  at  $h'$  leading to  $h$ , and (ii) coincides with  $\sigma_i$  at all other information sets. We say that a behavioral strategy  $\sigma_i$  is *sequentially rational* for a cautious type  $t_i$  if at every information set  $h \in H_i$ , we have that  $\sigma_i|_h(s_i) > 0$  only if  $s_i$  is globally rational for  $t_i$  at  $h$ .

Recall that, for an information set  $h \in H_i$  and a choice  $c_i \in C_i(h)$ , we denote by  $S_i(h, c_i)$  the set of strategies  $s_i \in S_i(h)$  with  $s_i(h) = c_i$ . For a cautious type  $t_i$  and an opponent's type  $t_j$  with  $\beta_i(t_i)(t_j) > 0$ , let  $\sigma_j^{t_i|t_j}$  be the induced behavioral strategy for player  $j$  given by

$$\sigma_j^{t_i|t_j}(h)(c_j) := st \left( \frac{\beta_i(t_i)(S_j(h, c_j) \times \{t_j\})}{\beta_i(t_i)(S_j(h) \times \{t_j\})} \right) \quad (9.2)$$

for every information set  $h \in H_j$  and every choice  $c_j \in C_j(h)$ .

**Definition 9.3 (Inducing sequentially rational behavioral strategies)** *Consider a finite epistemic model  $M = (T_i, \beta_i)_{i \in I}$  for a dynamic game  $G$ . A cautious type  $t_i \in T_i$  induces sequentially rational behavioral strategies if for every opponent  $j \neq i$  and every type  $t_j \in T_j$  with  $\beta_i(t_i)(t_j) > 0$ , the induced behavioral strategy  $\sigma_j^{t_i|t_j}$  is sequentially rational for  $t_j$ .*

We are now ready to define quasi-perfectly rationalizable types as those types that are cautious, induce sequentially rational behavioral strategies, and express common full belief in these two events. Formally, a type  $t_i$  expresses *1-fold* full belief in caution and the event that types induce sequentially rational behavioral strategies if  $\beta_i(t_i)$  only assigns positive (non-standard) probability to opponents' types that are cautious and induce sequentially rational behavioral strategies. For every  $k \geq 2$ , type  $t_i$  expresses *k-fold* full belief in caution and the event that types induce sequentially rational behavioral strategies if  $\beta_i(t_i)$  only assigns positive (non-standard) probability to opponents' types that express  $(k - 1)$ -fold full belief in caution and the event that types induce sequentially rational behavioral strategies. A type  $t_i$  expresses *common* full belief in caution and the event that types induce sequentially rational behavioral strategies if  $t_i$  expresses *k-fold* full belief in caution and the event that types induce sequentially rational behavioral strategies, for every  $k \geq 1$ .

**Definition 9.4 (Quasi-perfect rationalizability)** *Consider a finite epistemic model  $M = (T_i, \beta_i)_{i \in I}$  for a dynamic game  $G$ . A type  $t_i \in T_i$  is quasi-perfectly rationalizable if it is cautious, induces sequentially rational behavioral strategies, and expresses common full belief in caution and the event that types induce sequentially rational behavioral strategies. A strategy  $s_i \in S_i$  is quasi-perfectly rationalizable if there is a finite epistemic model  $M = (T_i, \beta_i)_{i \in I}$  and a quasi-perfectly rationalizable type  $t_i \in T_i$ , such that  $s_i$  is globally rational for  $t_i$  at every  $h \in H_i(s_i)$ .*

In the following subsection we will show that every quasi-perfectly rationalizable strategy is procedurally quasi-perfectly rationalizable in our sense, but not *vice versa*.

## 9.2 Relation Between the Two Quasi-Perfect Rationalizability Concepts

We first show that, in all dynamic games, every strategy that is quasi-perfectly rationalizable is also procedurally quasi-perfectly rationalizable in our sense.

**Theorem 9.1 (Relation with quasi-perfect rationalizability)** *Consider a dynamic game  $G$ . Then, every strategy that is quasi-perfectly rationalizable is also procedurally quasi-perfectly rationalizable.*

**Proof.** Let  $S_i^k(h)$  and  $B_i^{opp,k}$  be the sets of strategies and beliefs that survive round  $k$  of the procedural quasi-perfect rationalizability procedure. Consider a finite epistemic model  $M = (T_i, \beta_i)_{i \in I}$  for  $G$ , as in Asheim and Perea (2005). We prove, by induction on  $k$ , that for every player  $i$ , every quasi-perfectly rationalizable type  $t_i \in T_i$ , and every information set  $h \in H_i$ , we have that (a) every strategy  $s_i \in S_i(h)$  that is globally rational for  $t_i$  at every  $h' \in H_i(s_i)$  weakly following  $h$  is in  $S_i^k(h)$ , and (b) the marginal of  $\beta_i(t_i)$  on  $S_{-i}$  is in  $B_i^{opp,k}$ .

For  $k = 0$  this statement is true because  $S_i^0(h) = S_i(h)$ , the type  $t_i$  is cautious, and  $B_i^{opp,0} = B_i^{opp}$  contains all cautious beliefs on  $S_{-i}$ .

Now let  $k \geq 1$  and suppose that (a) and (b) are true for  $k - 1$  and all players  $i$ . Consider a player  $i$ , a quasi-perfectly rationalizable type  $t_i \in T_i$ , and an information set  $h \in H_i$ . To show (a), take some strategy  $s_i \in S_i(h)$  that is globally rational for  $t_i$  at every  $h' \in H_i(s_i)$  weakly following  $h$ . By the induction assumption on (a) we know that  $s_i \in S_i^{k-1}(h)$ . Let  $b_i^{opp}(t_i)$  be the marginal of  $\beta_i(t_i)$  on  $S_{-i}$ . By the induction assumption on (b) we know that  $b_i^{opp}(t_i) \in B_i^{opp,k-1}$ . Hence,  $s_i \in S_i^{k-1}(h)$  is globally rational for  $b_i^{opp}(t_i) \in B_i^{opp,k-1}$  at every  $h' \in H_i(s_i)$  weakly following  $h$ . This implies that  $s_i \in S_i^k(h)$ , which completes the induction step for (a).

To show (b), let  $b_i^{opp}(t_i)$  be the marginal of  $\beta_i(t_i)$  on  $S_{-i}$ . By the induction assumption on (b) we know that  $b_i^{opp}(t_i) \in B_i^{opp,k-1}$ . To show that  $b_i^{opp}(t_i) \in B_i^{opp,k}$ , we must show that  $b_i^{opp}(t_i)(h)$  believes  $S_{-i}^k(h)$  for all  $h \in H_i$ . That is, we must show that  $st(b_i^{opp}(t_i)(h)(s_{-i})) > 0$  only if  $s_{-i} \in S_{-i}^k(h)$ .

Consider some information set  $h \in H_i$  and some opponents' strategy combination  $s_{-i}$  such that  $st(b_i^{opp}(t_i)(h)(s_{-i})) > 0$ . We will show that  $s_{-i} \in S_{-i}^k(h)$ . As, by the induction assumption on (b),  $b_i^{opp}(t_i) \in B_i^{opp,k-1}$ , it follows that  $b_i^{opp}(t_i)(h)$  believes  $S_{-i}^{k-1}(h)$ , and hence  $s_{-i} \in S_{-i}^{k-1}(h)$ . Let  $s_{-i} = (s_j)_{j \neq i}$ . To show that  $s_{-i} \in S_{-i}^k(h)$ , we will show that for every opponent  $j \neq i$  there is some  $b_j^{opp} \in B_j^{opp,k-1}$  such that  $s_j$  is globally rational for  $b_j^{opp}$  at every  $h' \in H_j(s_j)$  weakly following  $h$ .

Fix an opponent  $j$ . Since  $st(b_i^{opp}(t_i)(h)(s_j)) > 0$ , the belief  $b_i^{opp}(t_i)$  is the marginal of  $\beta_i(t_i)$  on  $S_{-i}$ , and  $b_i^{opp}(t_i)(h)$  is the induced conditional belief on  $S_{-i}(h)$ , there must be some type  $t_j \in T_j$  with  $\beta_i(t_i)(t_j) > 0$  such that

$$st \left( \frac{\beta_i(t_i)(s_j, t_j)}{\beta_i(t_i)(S_j(h) \times \{t_j\})} \right) > 0. \quad (9.3)$$



Now, let  $b_j^{opp}(t_j)$  be the marginal of  $\beta_j(t_j)$  on  $S_{-j}$ . We show that  $b_j^{opp}(t_j) \in B_j^{opp,k-1}$  and that  $s_j$  is globally rational for  $b_j^{opp}(t_j)$  at every  $h' \in H_j(s_j)$  weakly following  $h$ .

As  $\beta_i(t_i)(t_j) > 0$  and  $t_i$  is quasi-perfectly rationalizable, it must be that  $t_j$  is quasi-perfectly rationalizable as well. Hence, by our induction assumption on (b) we conclude that  $b_j^{opp}(t_j) \in B_j^{opp,k-1}$ .

Consider now some  $h' \in H_j(s_j)$  weakly following  $h$ . We show that  $s_j$  is globally rational for  $b_j^{opp}(t_j)$  at  $h'$ . Take some arbitrary  $h'' \in H_j(s_j)$  weakly following  $h'$ . Then,  $h''$  weakly follows  $h$  and hence  $S_j(h'') \subseteq S_j(h)$ . Moreover,  $s_j \in S_j(h'', s_j(h''))$ . It thus follows by (9.3) that

$$\begin{aligned} st \left( \frac{\beta_i(t_i)(S_j(h''), s_j(h'')) \times \{t_j\}}{\beta_i(t_i)(S_j(h'') \times \{t_j\})} \right) &\geq st \left( \frac{\beta_i(t_i)(S_j(h''), s_j(h'')) \times \{t_j\}}{\beta_i(t_i)(S_j(h) \times \{t_j\})} \right) \\ &\geq st \left( \frac{\beta_i(t_i)(s_j, t_j)}{\beta_i(t_i)(S_j(h) \times \{t_j\})} \right) > 0. \end{aligned}$$

We therefore conclude by (9.2) that

$$\sigma_j^{t_i|t_j}(h'')(s_j(h'')) = st \left( \frac{\beta_i(t_i)(S_j(h''), s_j(h'')) \times \{t_j\}}{\beta_i(t_i)(S_j(h'') \times \{t_j\})} \right) > 0$$

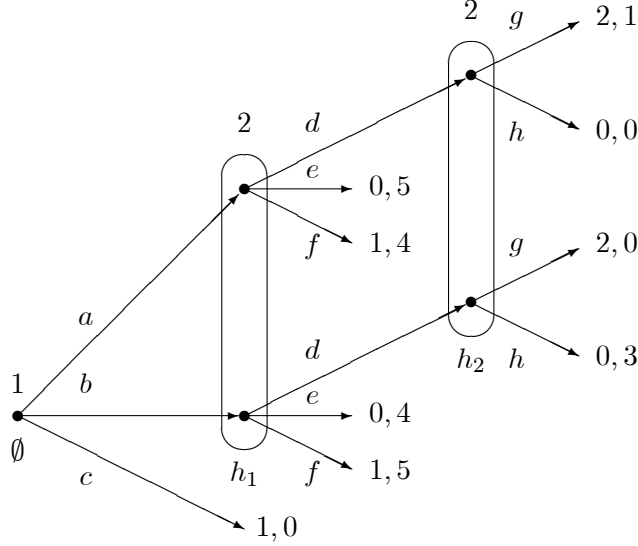
for all  $h'' \in H_j(s_j)$  weakly following  $h'$ . But then, it follows by (9.1) that there is some  $\hat{s}_j \in S_j(h')$  with  $\hat{s}_j(h'') = s_j(h'')$  for all  $h'' \in H_j(s_j)$  weakly following  $h'$  such that

$$\sigma_j^{t_i|t_j}|_{h'}(\hat{s}_j) > 0. \quad (9.4)$$

Since  $t_i$  is quasi-perfectly rationalizable, we know in particular that  $t_i$  induces sequentially rational behavioral strategies. Hence, the induced behavioral strategy  $\sigma_j^{t_i|t_j}$  must be sequentially rational for  $t_j$ . Since by (9.4) we have that  $\sigma_j^{t_i|t_j}|_{h'}(\hat{s}_j) > 0$ , it follows that  $\hat{s}_j$  must be globally rational for  $t_j$  at  $h'$ . Since  $s_j$  and  $\hat{s}_j$  coincide at all  $h'' \in H_j(\hat{s}_j)$  that weakly follow  $h'$ , it follows that also  $s_j$  is globally rational for  $t_j$  at  $h'$ . But then, we conclude that  $s_j$  is globally rational for  $b_j^{opp}(t_j)$  at  $h'$ . As  $h' \in H_j(s_j)$  weakly following  $h$  was chosen arbitrarily, it follows that  $s_j$  is globally rational for  $b_j^{opp}(t_j)$  at every  $h' \in H_j(s_j)$  weakly following  $h$ . Since we have seen that  $b_j^{opp}(t_j) \in B_j^{opp,k-1}$  and  $s_j \in S_j^{k-1}(h)$ , we conclude that  $s_j \in S_j^k(h)$ .

We thus see that  $st(b_i^{opp}(t_i)(h)(s_j)) > 0$  only if  $s_j \in S_j^k$ . Since this holds for every  $h \in H_i$  and every opponent  $j$ , it follows that  $b_i^{opp}(t_i)$  believes  $S_{-i}^k(h)$  for all  $h \in H_i$ . As  $b_i^{opp}(t_i) \in B_i^{opp,k-1}$ , we conclude that  $b_i^{opp}(t_i) \in B_i^{opp,k}$ , which completes the induction step for (b).

By induction on  $k$ , it thus follows that for every player  $i$ , every quasi-perfectly rationalizable type  $t_i \in T_i$ , and every information set  $h \in H_i$ , (a) every strategy  $s_i \in S_i(h)$  that is globally rational for  $t_i$  at every  $h' \in H_i(s_i)$  weakly following  $h$  is in  $S_i^k(h)$  for all  $k \geq 0$ , and (b) the marginal of  $\beta_i(t_i)$  on  $S_{-i}$  is in  $B_i^{opp,k}$  for all  $k \geq 0$ .



**Figure 4:** Procedural quasi-perfect rationalizability does not imply quasi-perfect rationalizability

Now, take a player  $i$ , and a quasi-perfectly rationalizable strategy  $s_i \in S_i$ . Then, there is an epistemic model  $M = (T_i, \beta_i)_{i \in I}$  and a quasi-perfectly rationalizable type  $t_i \in T_i$  such that  $s_i$  is globally rational for  $t_i$  at every  $h \in H_i(s_i)$ . Then, by (a) above,  $s_i \in S_i^k(\emptyset)$  for all  $k \geq 0$ , and hence  $s_i$  is procedurally quasi-perfectly rationalizable. This completes the proof. ■

We next prove, by means of a counter-example, that the opposite direction of this theorem is not true. Consider the dynamic game in Figure 4. Note that player 1 is always indifferent between his strategies  $a$  and  $b$ . We will show that the strategy  $a$  is procedurally quasi-perfectly rationalizable, but not quasi-perfectly rationalizable.

Before we give a formal proof, we first provide an informal intuitive argument. According to procedural quasi-perfect rationalizability, player 1 can rationally choose  $a$  because he may deem player 2's strategy  $f$  infinitely more likely than  $(d, g)$ , strategy  $(d, g)$  infinitely more likely than  $(d, h)$ , and  $(d, h)$  infinitely more likely than  $e$ . Indeed, under such belief player 1 would assign, at the beginning of the game  $\emptyset$ , only non-infinitesimal probability to player 2's strategy  $f$ , which is optimal for player 2 from  $\emptyset$  onwards if player 2 assigns a high probability to player 1 choosing  $b$ .

Such a belief, however, is not possible under the concept of quasi-perfect rationalizability. In order for player 1 to rationally choose  $a$ , he must deem player 2's strategy  $(d, g)$  at least as

likely as  $(d, h)$ . Hence, conditional on information set  $h_2$  being reached, player 1 must assign a non-infinitesimal probability to player 2 choosing  $g$ . According to quasi-perfect rationalizability, this is only possible if player 1 believes that  $g$  is optimal for player 2 at  $h_2$ . Hence, player 1 must believe, conditional on  $h_2$  being reached, that player 2 holds a belief  $b_2$  that assigns probability at least  $3/4$  to player 1 having chosen  $a$ . Under such a belief  $b_2$ , however,  $e$  would be the only optimal strategy for player 2 at  $h_1$ . According to quasi-perfect-rationalizability, player 1 must induce a sequentially rational behavioral strategy for player 2. In particular, conditional on player 2's belief  $b_2$ , and conditional on the information set  $h_1$ , player 1 must only assign non-infinitesimal probability to strategies that are optimal for player 2 under the belief  $b_2$  at  $h_1$ . That is, conditional on player 2's belief  $b_2$ , and conditional on the information set  $h_1$ , player 1 must only assign non-infinitesimal probability to strategy  $e$ . In particular, this means that player 1 must deem player 2's strategy  $e$  infinitely more likely than  $(d, g)$ . However, if that is the case player 1's expected utility from choosing  $a$  will always be lower than 1, and therefore  $a$  cannot be a quasi-perfectly rationalizable strategy.

We will now turn the formal proof. We first show that  $a$  survives the procedural quasi-perfect rationalizability procedure. Let  $S_i^k(h)$  and  $B_i^{opp,k}$  be the sets of strategies and beliefs that survive round  $k$  of the procedural quasi-perfect rationalizability procedure, and define  $S_i^\infty(h) := \bigcap_{k \geq 1} S_i^k(h)$  and  $B_i^{opp,\infty} := \bigcap_{k \geq 1} B_i^{opp,k}$ . Then, it is easily verified that  $S_2^\infty(\emptyset) = \{e, f\}$ . Consider player 1's belief

$$b_1^{opp} := (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot f + \varepsilon \cdot (d, g) + \varepsilon^2 \cdot (d, h) + \varepsilon^3 \cdot e$$

which clearly is in  $B_1^{opp}$ . As

$$st(b_1^{opp}(\emptyset)) = f \text{ where } f \in S_2^\infty(\emptyset)$$

it follows that  $b_1^{opp}(\emptyset)$  believes  $S_2^\infty(\emptyset)$ . Hence,  $b_1^{opp} \in B_1^{opp,\infty}$ .

We now verify that strategy  $a$  is globally rational for  $b_1^{opp}$  at  $\emptyset$ . By construction of the belief  $b_1^{opp}$ ,

$$\begin{aligned} u_1(a, b_1^{opp}(\emptyset)) &= u_1(b, b_1^{opp}(\emptyset)) = (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot 1 + \varepsilon \cdot 2 + \varepsilon^2 \cdot 0 + \varepsilon^3 \cdot 0 \\ &= 1 + \varepsilon - \varepsilon^2 - \varepsilon^3 > 1, \text{ and } u_1(c, b_1^{opp}(\emptyset)) = 1, \end{aligned}$$

and hence  $a$  is indeed globally rational for  $b_1^{opp}$  at  $\emptyset$ . As  $b_1^{opp} \in B_1^{opp,\infty}$ , it follows that  $a \in S_1^\infty(\emptyset)$ , and hence  $a$  is procedurally quasi-perfectly rationalizable.

We next prove that  $a$  is not quasi-perfectly rationalizable. Suppose, contrary to what we want to show, that  $a$  were quasi-perfectly rationalizable. Then, there is an epistemic model  $M = (T_i, \beta_i)_{i \in I}$  for  $G$ , as in Asheim and Perea (2005), and a type  $t_1 \in T_1$ , such that  $t_1$  is quasi-perfectly rationalizable, and  $a$  is globally rational for  $t_1$  at  $\emptyset$ . Let  $\beta_1(t_1)$  be the cautious non-standard belief that  $t_1$  holds on  $S_2 \times T_2$ . For every  $s_2 \in S_2$ , let  $\beta_1(t_1)(s_2)$  be the marginal

probability that  $\beta_1(t_1)$  assigns to  $s_2$ . Then, the expected utility of strategy  $a$  at  $\emptyset$  for type  $t_2$  is given by

$$u_1(a, t_1, \emptyset) = \beta_1(t_1)(f) \cdot 1 + \beta_1(t_1)(d, g) \cdot 2 + \beta_1(t_1)(d, h) \cdot 0 + \beta_1(t_1)(e) \cdot 0. \quad (9.5)$$

Since  $u_1(c, t_1, \emptyset) = 1$  and  $a$  is globally rational for  $t_1$  at  $\emptyset$ , we must have that  $u_1(a, t_1, \emptyset) \geq 1$ , which is only possible if  $\beta_1(t_1)(d, g) \geq \beta_1(t_1)(d, h)$ . Let  $t_2 \in T_2$  be such that

$$\beta_1(t_1)((d, g), t_2) \geq \beta_1(t_1)((d, g), t'_2) \text{ for all } t'_2 \in T_2. \quad (9.6)$$

Since  $\beta_1(t_1)(d, g) \geq \beta_1(t_1)(d, h)$  we must have, by (9.6), that  $\beta_1(t_1)((d, g), t_2)$  is not of infinitely smaller size than  $\beta_1(t_1)((d, h), t_2)$ . This implies, by (9.2), that  $\sigma_2^{t_1|t_2}|_{h_2}(d, g) > 0$ . Since  $t_1$  is quasi-perfectly rationalizable, the behavioral strategy  $\sigma_2^{t_1|t_2}$  must be sequentially rational for  $t_2$ . In particular,  $\sigma_2^{t_1|t_2}|_{h_2}(d, g) > 0$  implies that  $(d, g)$  must be globally rational for  $t_2$  at  $h_2$ . This, in turn, is only possible if  $\beta_2(t_2)(a) \geq \frac{3}{4}$ . Hence, the only strategy that is globally rational for  $t_2$  at  $h_1$  is  $e$ . Since the behavioral strategy  $\sigma_2^{t_1|t_2}$  must be sequentially rational for  $t_2$ , we must have that  $\sigma_2^{t_1|t_2}|_{h_1}(e) = 1$ . Hence, in particular,  $\beta_1(t_1)(e, t_2)$  must be of infinitely larger size than  $\beta_1(t_1)((d, g), t_2)$ . But then, it follows by (9.6) that  $\beta_1(t_1)(e)$  is of infinitely larger size than  $\beta_1(t_1)(d, g)$ . However, this insight, together with (9.5), would imply that  $u_1(a, t_1, \emptyset) < 1$ , and hence  $a$  cannot be globally rational for  $t_1$  at  $\emptyset$ . That is a contradiction. We thus conclude that  $a$  cannot be quasi-perfectly rationalizable. Hence, we have found a strategy  $a$  that is procedurally quasi-perfectly rationalizable, but not quasi-perfectly rationalizable.

In fact, we can show even more in this example. The strategy  $a$  is not only procedurally quasi-perfectly rationalizable, it is even strongly sequentially rationalizable. To see this, let  $S_i^k(h)$  and  $B_i^k$  be the sets of strategies and beliefs that survive round  $k$  of the strong sequential rationalizability procedure, and define  $S_i^\infty(h) := \bigcap_{k \geq 1} S_i^k(h)$  and  $B_i^\infty := \bigcap_{k \geq 1} B_i^k$ . Then, it may be verified that  $S_2^\infty(\emptyset) = \{e, f\}$ . Consider player 1's belief  $b_1 = (b_1^{self}, b_1^{opp})$  where

$$b_1^{self} := \frac{1}{2}(1 - \varepsilon) \cdot a + \frac{1}{2}(1 - \varepsilon) \cdot b + \varepsilon \cdot c$$

and

$$b_1^{opp} := (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot f + \varepsilon \cdot (d, g) + \varepsilon^2 \cdot (d, h) + \varepsilon^3 \cdot e.$$

Then,

$$\begin{aligned} u_1(a, b_1(\emptyset)) &= u_1(b, b_1(\emptyset)) = (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \cdot 1 + \varepsilon \cdot 2 + \varepsilon^2 \cdot 0 + \varepsilon^3 \cdot 0 \\ &= 1 + \varepsilon - \varepsilon^2 - \varepsilon^3 > 1, \text{ and } u_1(c, b_1^{opp}(\emptyset)) = 1, \end{aligned}$$

which implies that choices  $a$  and  $b$  are locally rational for  $b_1$  at  $\emptyset$ . Therefore,  $S_1^{rat}(b_1, \emptyset) = \{a, b\}$ . As  $b_1^{self}(\emptyset)$  believes  $\{a, b\}$ , we conclude that  $b_1$  believes in his own future rationality. Note that

player 1 only makes a choice at  $\emptyset$ , and therefore he trivially deems his own mistakes least likely under the belief  $b_1$ . We thus conclude that  $b_1 \in B_1^0$ .

Since

$$st(b_1^{opp}(\emptyset)) = f \text{ where } f \in S_2^\infty(\emptyset)$$

it follows that  $b_2^{opp}(\emptyset)$  believes  $S_{-1}^\infty(\emptyset)$ . It therefore follows that  $b_1 = (b_1^{self}, b_1^{opp}) \in B_1^\infty$ .

Since we have seen above that  $a$  is locally rational for  $b_1$  at  $\emptyset$ , it follows that  $a \in S_2^\infty(\emptyset)$ , and hence  $a$  is strongly sequentially rationalizable. We have thus found a strategy  $a$  that is strongly sequentially rationalizable but not quasi-perfectly rationalizable. This means that Theorem 4.1 is no longer true if procedural quasi-perfect rationalizability is replaced by quasi-perfect rationalizability.

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