Expected Utility as an Expression of Linear Preference Intensity*

Andrés Perea†
Maastricht University

This version: August 2023

Abstract

In a decision problem or game we typically fix the person's utilities but not his beliefs. What, then, do these utilities represent? To explore this question we assume that the decision maker holds a *conditional preference relation* – a mapping that assigns to every possible probabilistic belief a preference relation over his choices. We impose a list of axioms on such conditional preference relations that is both necessary and sufficient for admitting an expected utility representation. Most of these axioms express the idea that the decision maker's preference intensity between two choices changes linearly with the belief. Finally, we show that under certain conditions the relative utility differences are unique across the different expected utility representations.

JEL Classification: C72, D81

Keywords: Expected utility, decision problems, games, conditional preference relation, preference intensity, weak dominance.

1 Introduction

In a decision problem or game, we typically write down the decision maker's (DM) utilities, but not his beliefs. The interpretation is that these fixed utilities induce the DM's preferences for *all possible beliefs* he

*A previous version of this paper carried the title "A foundation for expected utility in decision problems and games". A special word of gratitude goes to Emiliano Catonini, Stephan Jagau and Peter Wakker for their very extensive and detailed feedback on this paper. I would also like to thank Geir Asheim, Christian Bach, Rubén Becerril, Giacomo Bonanno, Richard Bradley, Gabriel Frahm, Ángel Hernándo-Veciana, Belén Jerez, Shuige Liu, Andrew Mackenzie, Martin Meier, Rineke Verbrugge, Marco Zaffalon and Gabriel Ziegler for their useful suggestions and comments. Thanks also to some associate editors and referees for their valuable comments on earlier versions. Finally, I am grateful to the seminar audiences at the University of Glasgow, Maastricht University, Universidad Carlos III de Madrid, and the One World Mathematical Game Theory Seminar, for their constructive feedback. I declare there is no conflict of interest.

†*Address:* Dept. of Quantitative Economics and EpiCenter, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. *E-mail:* a.perea@maastrichtuniversity.nl *Web:* http://www.epicenter.name/Perea/

could possibly have. It thus imposes some consistency between the preference relations that the DM would hold for the various beliefs he could entertain. But how can we express this consistency in terms of basic principles?

To answer this question, Gilboa and Schmeidler (2003) developed a framework where the DM holds a preference relation over the available acts for *every* possible probabilistic belief about the states. In this paper, we refer to this object as a *conditional preference relation.* By imposing certain axioms on conditional preference relations, Gilboa and Schmeidler single out those that have a *diversified expected utility* representation – a utility matrix where no row is weakly dominated by, or equivalent to, an affine combination of at most three other rows. The key axiom in their characterization is *diversity,* which states that for every strict ordering of at most four acts there must be a belief for which this ordering obtains.

But what can we say about those conditional preference relations that have an expected utility representation, but not a diversified one? This is an important question, because many utility matrices are non-diversified, and many natural conditional preference relations violate the diversity axiom. Indeed, the diversity axiom rules out all cases where some act is weakly dominated by another act, all scenarios with two states and more than two acts, all scenarios with three states and more than three acts, and many other plausible situations as well.

The purpose of this paper is to fill that gap, by providing a list of axioms that is both necessary and sufficient for the conditional preference relation having an expected utility representation – diversified or not. If there are no weakly dominated acts, then Theorem 2.1 shows that expected utility can be characterized if we replace Gilboa and Schmeidler's diversity axiom by two new axioms: *three choice linear preference intensity* and *four choice linear preference intensity.* Both axioms reveal the idea that the *intensity* with which the decision maker prefers one choice to another changes *linearly* with his belief. More precisely, the first axiom concerns three choices and argues that on two parallel lines of beliefs, the preference intensity between two choices will change at the same rate. This results in a formula that relates the beliefs on these two parallel lines where the decision maker is indifferent between the various pairs of choices. The second axiom concerns four choices, and argues that on a line of beliefs the relative change rates of the preference intensities between the different pairs of choices must respect the chain rule. Also this axiom is expressed in terms of a formula, which relates the beliefs on the line where the decision maker is indifferent between the various pairs of choices. Both axioms may be viewed as testable consequences of the idea that the preference intensity between choices changes linearly with the belief.

Somewhat surprisingly perhaps, the case where some acts are weakly dominated by other acts proves to be much more difficult. In this case, we extend the axioms above to *signed* beliefs involving negative "probabilities". That is, we move our analysis to areas outside the belief simplex. Although negative "probabilities" do not exist in the mind of the decision maker, the axioms still have intuitive content as they impose conditions on how the preference intensities between various pairs of choices change as the beliefs inside the belief simplex change. Moreover, we introduce two new axioms which deal with scenarios where the preference intensity between two choices is constant across all beliefs. This set of axioms is shown to be necessary and sufficient for admitting an expected utility representation in the general case.

If there are no weakly dominated acts, and there is a belief where the decision maker is indifferent between some, but not all, choices (provided there are at least three choices), then it is shown in Proposition 2.2 that the utility differences are unique up to a positive multiplicative constant. In that case, the utility differences between two choices a and b may be viewed as expressing the decision maker's "preference intensity" between a and b. This is similar to the approaches by Anscombe and Aumann (1963) and Wakker (1989), where the axioms of state independence and state independent preference intensity, respectively, guarantee that the utility difference between two consequences is the same at every state, and may be viewed as expressing the "preference intensity" between these consequences.

This paper is organized as follows. In Section 2 we introduce the notion of a conditional preference relation, and derive the representation theorem for the case when there are no weakly dominated acts. Section 3 treats the general case, where weakly dominated acts are allowed. We conclude with a discussion in Section 4. All the proofs, and the necessary mathematical definitions, can be found in the appendix.

2 Case of No Weak Dominance

In this section we formally introduce conditional preference relations as the primitive notion of our model, and subsequently impose some axioms on these. For the case where no act is weakly dominated by another act, we show that these axioms characterize those conditional preference relations that admit an expected utility representation.

2.1 Conditional Preference Relations

In line with Gilboa and Schmeidler (2003), the primitive object in this paper is that of a *conditional preference relation* – a mapping that assigns to every probabilistic belief over the states a preference relation over the available choices. In this paper, we also refer to such choices as *acts.* In fact, we will use the terms acts and choices interchangeably. Consider a decision maker (DM) who must choose from a finite set of acts A. The final outcome depends not only on the act $a \in A$, but also on the realization of a state $s \in S$ from a finite set of states S. We assume that the decision maker first forms a probabilistic belief p on S, which then induces a preference relation $\succsim_p$ on A. Formally, a *preference relation* $\succsim_p$ on A is a binary relation $\succsim_p \subseteq A \times A$. If $(a, b) \in \succsim_p$ we write $a \succsim_p b$, and the interpretation is that the DM weakly prefers act a to act b if his belief is p. In the following definition, we denote by $\Delta(S)$ the set of probability distributions on S.

Definition 2.1 (Conditional preference relation) *Consider a finite set of acts A and a finite set of states S. A conditional preference relation on (A, S) is a mapping $\succsim$ that assigns to every probabilistic belief $p \in \Delta(S)$ a preference relation $\succsim_p$ on A.*

For two acts a and b, we write that $a \sim_p b$ if $a \succsim_p b$ and $b \succsim_p a$. Similarly, we write $a \succ_p b$ if $a \succsim_p b$ but not $b \succsim_p a$. For two acts $a, b \in A$ we define the sets of beliefs $P_{a\sim b} := \{p \in \Delta(S) \mid a \sim_p b\}$, $P_{a\succ b} := \{p \in \Delta(S) \mid a \succ_p b\}$ and $P_{a\succsim b} := \{p \in \Delta(S) \mid a \succsim_p b\}$. We say that (a) *$a$ strictly dominates b* under $\succsim$ if $a \succ_p b$ for all $p \in \Delta(S)$; (b) *a weakly dominates b* under $\succsim$ if $a \succsim_p b$ for all $p \in \Delta(S)$, and $a \succ_p b$ for at least one $p \in \Delta(S)$; (c) a is *equivalent* to b under $\succsim$ if $a \sim_p b$ for all $p \in \Delta(S)$.

In the remainder of this paper we will assume that the conditional preference relation does not have equivalent acts. In the discussion section we will briefly explain how our analysis can easily be extended to cover equivalent acts.

An *expected utility representation* can be defined as follows.

Definition 2.2 (Expected-utility representation) *A conditional preference relation $\succsim$ has an expected utility representation if there is a utility function $u : A \times S \to \mathbf{R}$ such that for every belief $p \in \Delta(S)$ and every two acts $a, b \in A$,*

$$a \succsim_p b \text{ if and only if } \sum_{s \in S} p(s) \cdot u(a, s) \geq \sum_{s \in S} p(s) \cdot u(b, s).$$

In this case, we say that the conditional preference relation $\succsim$ is *represented* by the utility function u. For a given vector $v \in \mathbf{R}^S$ we use the notation $u(a, v) := \sum_{s \in S} v(s) \cdot u(a, s)$. Hence, the condition above can be written as $a \succsim_p b$ if and only if $u(a, p) \geq u(b, p)$.

2.2 Regularity Axioms

We will start by reviewing some very basic axioms that have already been introduced in Gilboa and Schmeidler (2003), and to which we refer as *regularity axioms.*

Axiom 2.1 (Completeness) *For every belief p and any two acts $a, b \in A$, either $a \succsim_p b$ or $b \succsim_p a$.*

Axiom 2.2 (Transitivity) *For every belief p and every three acts $a, b, c \in A$ with $a \succsim_p b$ and $b \succsim_p c$, it holds that $a \succsim_p c$.*

Axiom 2.3 (Continuity) *For every two different acts $a, b \in A$ and every two beliefs $p \in P_{a \succ b}$ and $q \in P_{b \succ a}$, there is some $\lambda \in (0, 1)$ such that $(1 - \lambda)p + \lambda q \in P_{a \sim b}$.*

Axiom 2.4 (Preservation of indifference) *For every two different acts $a, b \in A$ and every two beliefs $p \in P_{a \sim b}$ and $q \in P_{a \sim b}$, we have that $(1 - \lambda)p + \lambda q \in P_{a \sim b}$ for all $\lambda \in (0, 1)$.*

Axiom 2.5 (Preservation of strict preference) *For every two different acts $a, b \in A$ and every two beliefs $p \in P_{a \succsim b}$ and $q \in P_{a \succ b}$, we have that $(1 - \lambda)p + \lambda q \in P_{a \succ b}$ for all $\lambda \in (0, 1)$.*

Completeness and transitivity together resemble the *ranking* axiom in Gilboa and Schmeidler (2003). Our definition of continuity is formally different from Gilboa and Schmeidler's (2003) version, but reveals the same idea. When taken together, our axioms of preservation of indifference and preservation of strict preference correspond precisely to Gilboa and Schmeidler's (2003) axiom of *combination.*

It can be shown that for the case of two acts, the regularity axioms are both necessary and sufficient for a conditional preference relation having an expected utility representation. A proof can be found in the appendix (see Lemma 5.4).

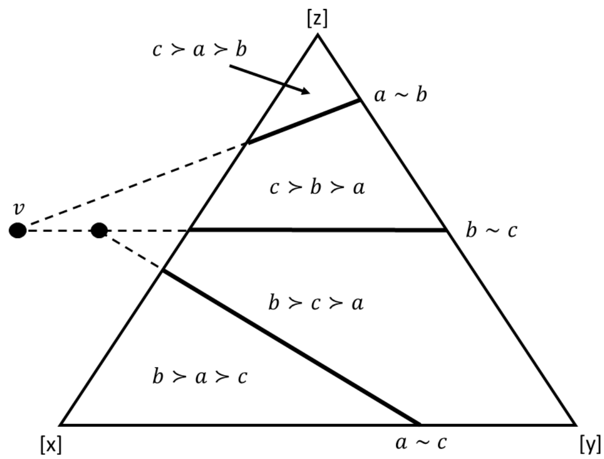

Figure 1: Why regularity axioms are not sufficient

2.3 Three Choice and Four Choice Linear Preference Intensity

If there are more than two acts, the regularity axioms no longer suffice to guarantee an expected utility representation. To see this, consider the conditional preference relation $\succsim$ represented by Figure 1. The area within the triangle represents the set $\Delta(S)$ of all probabilistic beliefs on $S = \{x, y, z\}$, with the probability 1 beliefs $[x], [y]$ and $[z]$ as the extreme points. The two-dimensional plane represents all the vectors in $\mathbf{R}^S$ where the sum of the coordinates is 1, containing the belief simplex $\Delta(S)$ as a subset. It may be verified that $\succsim$ satisfies all the regularity axioms. Yet, there is no expected utility representation for $\succsim$. To see why, suppose there would be a utility function u that represents $\succsim$. Then, the induced expected utilities of a and b must be equal on the set $span(P_{a\sim b})$, which denotes the linear span of $P_{a\sim b}$, the expected utilities of b and c must be equal on the set $span(P_{b\sim c})$ and the expected utilities of a and c must be equal on the set $span(P_{a\sim c})$, also at vectors that lie outside the belief simplex. But then, the expected utilities of a and c must be the same at the vector v where $span(P_{a\sim b})$ and $span(P_{b\sim c})$ intersect, which is impossible since v does not belong to $span(P_{a\sim c})$. This insight leads us to introduce further axioms which *do* guarantee an expected utility representation, at least when no act weakly dominates another act.

Consider three acts a, b, c, and a line L of beliefs as depicted in Figure 2. Here, $int_{b\succ a}$ denotes, somewhat informally, the intensity by which the DM prefers b to a, and similarly for $int_{c\succ a}$.[1] Moreover, p_{ab}, p_{bc} and

[1] On purpose, we are not being very precise about the notion of preference intensity here, since we like to treat it as an intuitive idea rather than a formal concept. There are some special cases where the idea of preference intensity can be made more precise, however. Consider, for instance, two acts a and b, two states x and y, and suppose that the DM prefers a to b at $[x]$, prefers b to a at $[y]$, and becomes indifferent between a and b at the belief p_{ab} between $[x]$ and $[y]$. If the conditional preference relation satisfies the regularity axioms, the DM will prefer a to b for all beliefs that assign probability at most $p_{ab}(y)$ to y, and will prefer b to a for all beliefs that assign probability at most $p_{ab}(x)$ to x. Since the ratio between the sizes of these two

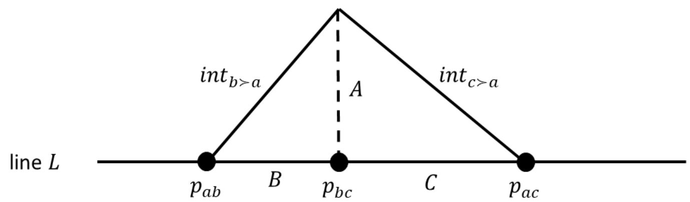

Figure 2: Linear preference intensity on a line of beliefs

p_{ac} are beliefs on the line where the DM is indifferent between a and b, between b and c, and between a and c, respectively. As such, the intensity $int_{b\succ a}$ is 0 at p_{ab}, the intensity $int_{c\succ a}$ is 0 at p_{ac}, and the two intensities are the same at p_{bc}.

If we assume that the preference intensities change linearly with the belief, then the change rates $\frac{\Delta int_{b\succ a}}{\Delta p}$ and $\frac{\Delta int_{c\succ a}}{\Delta p}$ are constant on the line L. Moreover, it can be seen from the figure that $\frac{\Delta int_{b\succ a}}{\Delta p} = \frac{A}{B}$ and $\frac{\Delta int_{c\succ a}}{\Delta p} = -\frac{A}{C}$, which implies that

$$\frac{\Delta int_{b\succ a}}{\Delta int_{c\succ a}} = -\frac{C}{B}. \tag{2.1}$$

Take a state s such that the probability of s is not constant on the line l. Then, we know that

$$-\frac{C}{B} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)}.$$

Together with (2.1) we thus conclude that

$$\frac{\Delta int_{b\succ a}}{\Delta int_{c\succ a}} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)}. \tag{2.2}$$

Now, consider a line L' of beliefs that is parallel to L, with beliefs p'_{ab}, p'_{bc} and p'_{ac} where the DM is indifferent between the respective acts. If the DM's preference intensities change linearly with the belief, then the relative change rate $\Delta int_{b\succ a}/\Delta int_{c\succ a}$ should be the same on the parallel lines L and L'. In view of (2.2) we thus conclude that

$$\frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)} = \frac{p'_{ac}(s) - p'_{bc}(s)}{p'_{ab}(s) - p'_{bc}(s)},$$

which implies that

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).$$

areas of beliefs is $p_{ab}(y)/p_{ab}(x)$, it seems natural to assume that the preference intensity between a and b at x is $p_{ab}(y)/p_{ab}(x)$ as large as the preference intensity between b and a at y.

This equality will be the content of the axiom *three choice linear preference intensity.*

To state this axiom formally, we need the following definitions. A *line of beliefs* is a subset $L \subseteq \Delta(S)$ such that $L = \{(1-\lambda)p + \lambda q \mid \lambda \in [0,1]\}$ for some beliefs $p, q \in \Delta(S)$. Two lines of beliefs L and L' are *parallel* if for every $p, q \in L$ and every $p', q' \in L'$ there is some $\mu \in \mathbf{R}$ with $p - q = \mu(p' - q')$.

Axiom 2.6 (Three choice linear preference intensity) *For every three acts a, b, c, for every line L of beliefs with beliefs p_{ab}, p_{bc}, p_{ac} where the DM is indifferent between the respective acts and which contains a belief where the DM is not indifferent between any of these acts, every line L' parallel to L with beliefs $p'_{ab}, p'_{bc}, p'_{ac}$ where the DM is indifferent between the respective acts and which contains a belief where the DM is not indifferent between any of these acts, it holds that*

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s))$$

for every state s.

It turns out that this axiom can be verified in an easy way if the conditional preference relation has no weakly dominated choices: In this case, it is equivalent to checking that every vector v (possible outside the belief simplex) which is in both $span(P_{a\sim b})$ and $span(P_{b\sim c})$, must also be in $span(P_{a\sim c})$.

Proposition 2.1 (Characterization of three choice linear preference intensity) *Consider a conditional preference relation $\succsim$ that has no weakly dominated choices and that satisfies the regularity axioms. Then, $\succsim$ satisfies three choice linear preference intensity, if and only if, for every three choices a, b, c it holds that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$.*

The conditional preference relation in Figure 1 clearly violates this property, since the vector v in that figure belongs to both $span(P_{a\sim b})$ and $span(P_{b\sim c})$, but not to $span(P_{a\sim c})$. Hence, in view of the result above, it cannot satisfy three choice linear preference intensity.

The property above may be seen as a strong version of transitivity of the indifference relation: If a belief p belongs to both $P_{a\sim b}$ and $P_{b\sim c}$ then, by transitivity of the indifference relation, p will also belong to $P_{a\sim c}$. The property above states that this relation must also hold outside the belief simplex, where $P_{a\sim b}, P_{b\sim c}$ and $P_{a\sim c}$ are replaced by $span(P_{a\sim b}), span(P_{b\sim c})$ and $span(P_{a\sim c})$, respectively.

We will now show that in the case of four choices or more, the linearity of preference intensity implies yet another testable condition. Consider four choices a, b, c, d, a line of beliefs L, and beliefs $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}, p_{cd}$ on that line where the DM is indifferent between the respective choices. Then, we know by (2.2) that

$$\frac{\Delta int_{b\succ a}}{\Delta int_{c\succ a}} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)}, \quad \frac{\Delta int_{c\succ a}}{\Delta int_{d\succ a}} = \frac{p_{ad}(s) - p_{cd}(s)}{p_{ac}(s) - p_{cd}(s)} \text{ and } \frac{\Delta int_{b\succ a}}{\Delta int_{d\succ a}} = \frac{p_{ad}(s) - p_{bd}(s)}{p_{ab}(s) - p_{bd}(s)}. \tag{2.3}$$

Since, by the chain rule, it holds that

$$\frac{\Delta int_{b\succ a}}{\Delta int_{d\succ a}} = \frac{\Delta int_{b\succ a}}{\Delta int_{c\succ a}} \cdot \frac{\Delta int_{c\succ a}}{\Delta int_{d\succ a}},$$

it follows by (2.3) that

$$\frac{p_{ad}(s) - p_{bd}(s)}{p_{ab}(s) - p_{bd}(s)} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)} \cdot \frac{p_{ad}(s) - p_{cd}(s)}{p_{ac}(s) - p_{cd}(s)}.$$

By cross-multiplication, we thus obtain the following testable condition.

Axiom 2.7 (Four choice linear preference intensity) *For every four choices a, b, c, d, and for every line L of beliefs with beliefs $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}, p_{cd}$ where the DM is indifferent between the respective choices, and such that L contains a belief where the DM is not indifferent between any of these choices, it holds that*

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{cd}(s)) \cdot (p_{ad}(s) - p_{bd}(s)) = (p_{ab}(s) - p_{bd}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \cdot (p_{ad}(s) - p_{cd}(s)).$$

for every state s.

2.4 Representation Theorem

If there are no weakly dominated acts, then the axioms we have gathered so far are not only necessary, but also sufficient, for an expected utility representation. We thus obtain the following representation result.

Theorem 2.1 (No weakly dominated choices) *Consider a finite set of acts A, a finite set of states S, and a conditional preference relation $\succsim$ on (A, S) such that no act weakly dominates another act under $\succsim$. Then, $\succsim$ has an expected utility representation, if and only if, it satisfies completeness, transitivity, continuity, preservation of indifference, preservation of strict preference, three choice linear preference intensity and four choice linear preference intensity.*

In Section 3 we will see that these axioms may not be sufficient for an expected utility representation if there are weakly dominated acts.

2.5 Unique Relative Utility Differences

So far, we have explored the case where no acts are weakly dominated, and we have identified a system of axioms that is both necessary and sufficient for an expected utility representation. But how unique is this representation? As we will see below, the expected utility differences are "typically" unique up to a positive multiplicative constant.

Proposition 2.2 *Consider a finite set of acts A, a finite set of states S, and a conditional preference relation $\succsim$ on (A, S), such that it admits an expected utility representation, no act weakly dominates another act under $\succsim$, and in the case of at least three acts there is a belief where the DM is indifferent between some, but not all, acts. Then, for every two utility functions u, v that represent $\succsim$ there is some $\alpha > 0$ such that $v(a, s) - v(b, s) = \alpha \cdot (u(a, s) - u(b, s))$ for all $a, b \in A$ and all $s \in S$.*

Under the conditions of the proposition, there would be exactly $|S|+1$ degrees of freedom for choosing a representing utility function: $|S|$ degrees because we can choose the utilities for one of the choices freely at each of the $|S|$ states, and another degree of freedom because the utility differences at each of the states may be multiplied by the same positive number without changing the induced conditional preference relation.

Moreover, under these conditions the utility difference $u(a,p)-u(b,p)$ at a belief p, which is unique up to a positive multiplicative constant, may be viewed as expressing the "preference intensity" between a and b at p. The conditions above thus guarantee that the relative preference intensities are unique. As an example, suppose that $0 < u(a,x)-u(b,x) = 2 \cdot (u(b,y)-u(a,y))$. Then, the DM will be indifferent between a and b at the belief $1/3[x]+2/3[y]$,[2] which seems to reflect that the intensity by which the DM prefers a to b at x is twice the intensity by which he prefers b to a at y. This indeed corresponds to the fact that the utility difference between a and b at x is twice as large as at y, in absolute terms. However, we will not enter the debates on whether such utility differences, or preference intensities, can be interpreted as reflecting neo-classical cardinal utility (see, for instance, Baccelli and Mongin (2016), Baumol (1958) and Moscati (2018)).

The above interpretation of the utility differences may no longer hold, however, if the conditions in the proposition above are not satisfied. Suppose there are three acts a, b and c, two states x and y, and let $\succsim$ be such that $a \succ_p b \succ_p c$ if $p(x) > 1/2$, $a \sim_p b \sim_p c$ if $p(x) = 1/2$, and $c \succ_p b \succ_p a$ if $p(x) < 1/2$. Hence, the three indifference sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are all equal to $\{1/2[x]+1/2[y]\}$, and thus there is no belief where the DM is indifferent between some, but not all, acts. Note that the utility functions u, v given by $u(a,x)=3, u(b,x)=2, u(c,x)=0, u(a,y)=-3, u(b,y)=-2, u(c,y)=0$ and $v(a,x)=3, v(b,x)=1, v(c,x)=0, v(a,y)=-3, v(b,y)=-1, v(c,y)=0$ both represent $\succsim$. Yet, the utility differences in u and v differ by more than just a multiplicative constant. The reason is that in this case, $\succsim$ does not provide us with sufficiently many data to derive the DM's preference intensity over the three acts at the various beliefs. A similar phenomenon may arise if there are weakly dominated acts.

3 Case of Weak Dominance

In this section we start with an example showing that the previous axioms may no longer guarantee an expected utility representation if weakly dominated acts are allowed. This leads us to introduce a new system of axioms. The first axioms are translations of the previous axioms to so-called *signed* beliefs, which allow for negative "probabilities". The last few axioms deal with scenarios where the preference intensity between two choices is *constant* across all beliefs. It is shown that the new axiom system so obtained is both necessary and sufficient for an expected utility representation in the general case.

3.1 Why Previous Axioms are Not Sufficient

Consider the conditional preference relation $\succsim$ in Figure 3, where b strictly dominates c and d strictly dominates a. It may be verified that $\succsim$ satisfies the regularity axioms. Moreover, it trivially satisfies three

[2] Here, $[x]$ denotes the denegerate belief that assigns probability 1 to the state $[x]$. Similarly for $[y]$.

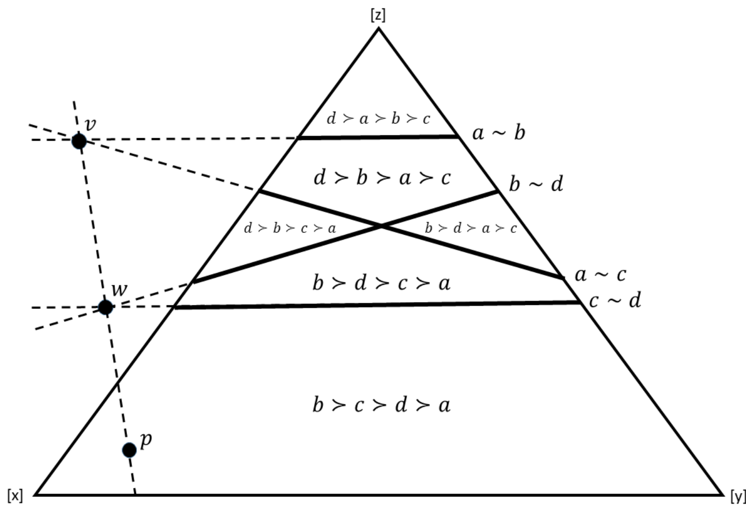

Figure 3: Why previous axioms are not sufficient for general case

choice linear preference intensity since for every three choices there is at least one pair for which the DM is never indifferent between the choices in that pair. Also, it trivially satisfies four choice linear preference intensity since the DM is never indifferent between a and d, and never indifferent between b and c.

Despite this, $\succsim$ does not have an expected utility representation. Indeed, if the utility function u were to represent $\succsim$, then the utility of b would be equal to the utility of c at the points v and w outside the belief simplex. This would imply that the utility of b would be equal to the utility of c at the belief p. However, the DM strictly prefers b to c at p, which is a contradiction.

3.2 Signed Indifference Beliefs

This naturally begs the question: What is "wrong" with the conditional preference relation in Figure 3? We will see that it violates the idea of linear preference intensity once we step *outside the belief simplex.*

To see what we mean by this, consider a line of beliefs L, three choices a, b, c, and their relative preference intensities on L as depicted in Figure 4. In the picture we have extended the line L outside the belief simplex. The part inside the belief simplex is the line segment between the two dotted lines. Note that on the line L inside the belief simplex, the intensity by which the DM prefers c to a is always higher than the intensity by which he prefers b to a, and hence the DM always prefers c to b on the part of the line L inside the belief simplex. In particular, the DM is never indifferent between b and c on the part of the line L inside the belief simplex.

However, if we extend the line L and the preference intensities $int_{c \succ a}$, $int_{b \succ a}$ outside the belief simplex, then there is a point on this line, q_{bc}, where the "preference intensity" between c and a is equal to the "preference intensity" between b and a. We put "preference intensity" between quotes here because we refer

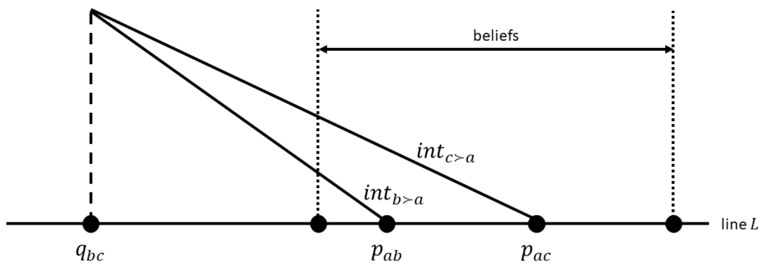

Figure 4: Relative change of linear preference intensities

to a point q_{bc} which is not a belief. Such a point q_{bc} is called a *signed* belief as it involves negative values, but the sum of all values is still equal to 1. As the "preference intensities" between b and a and between c and a are the same at q_{bc}, we could say that the DM is "indifferent" at the signed belief q_{bc}.

The real meaning of this signed indifference belief q_{bc} is that it determines the relative change rate of the preference intensities between b and a and between c and a inside the belief simplex. Indeed, similarly to what we have seen in Section 2.3, it can be derived from Figure 4 that

$$\frac{\Delta int_{b \succ a}}{\Delta int_{c \succ a}} = \frac{p_{ac}(s) - q_{bc}(s)}{p_{ab}(s) - q_{bc}(s)}$$

whenever the probability of state s is not constant on the line L.

Based on this insight, it becomes meaningful to extend conditional preference relations to *signed* beliefs outside the belief simplex. This yields the definition of a *signed* conditional preference relation below. Formally, a *signed belief* is a vector $q \in \mathbf{R}^S$ where $\sum_{s \in S} q(s) = 1$. It thus allows for negative values $q(s)$ at some states.

Definition 3.1 (Signed conditional preference relation) *A signed conditional preference relation* $\succsim^*$ *assigns to every signed belief* q *some preference relation* $\succsim^*_q$ *over the acts.*

In the following subsection we will extend each of the previous axioms to signed conditional preference relations.

3.3 Extending Axioms to Signed Beliefs

The axioms we have seen so far can be extended to *signed* conditional preference relations, as will be shown below. As an additional piece of notation, let $Q_{a \sim^* b}$ be the set of signed beliefs q with $a \sim^*_q b$. The sets $Q_{a \succsim^* b}$ and $Q_{a \succ^* b}$ are defined analogously. Moreover, by a line of *signed* beliefs we mean a set $L = \{(1-\lambda)q + \lambda r \mid \lambda \in \mathbf{R}\}$ where q, r are some signed beliefs. Two lines of signed beliefs L, L' are called *parallel* if for every q, r in L, and every q', r' in L', there is some $\mu \in \mathbf{R}$ with $q - r = \mu(q' - r')$.

Definition 3.2 (Extension of axioms to signed beliefs) *The following axioms are direct extensions of our previous axioms to signed conditional preference relations:*

Completeness: *For every signed belief q and any two acts $a, b \in A$, either $a \succsim_q^* b$ or $b \succsim_q^* a$.*

Transitivity: *For every signed belief q and every three acts $a, b, c \in A$ with $a \succsim_q^* b$ and $b \succsim_q^* c$, it holds that $a \succsim_q^* c$.*

Continuity: *For every two different acts $a, b \in A$ and every two signed beliefs $q \in Q_{a \succ^* b}$ and $r \in Q_{b \succ^* a}$, there is some $\lambda \in (0,1)$ such that $(1-\lambda)q + \lambda r \in Q_{a \sim^* b}$.*

Preservation of indifference: *For every two different acts $a, b \in A$ and every two signed beliefs $q, r \in Q_{a \sim^* b}$, we have that $(1-\lambda)q + \lambda r \in Q_{a \sim^* b}$ for all $\lambda \in (0,1)$.*

Preservation of strict preference: *For every two different acts $a, b \in A$ and every two signed beliefs $q \in Q_{a \succsim^* b}$ and $r \in Q_{a \succ^* b}$, we have that $(1-\lambda)q + \lambda r \in Q_{a \succ^* b}$ for all $\lambda \in (0,1)$.*

Three choice linear preference intensity: *For every three acts a, b, c, for every line L of signed beliefs with signed beliefs q_{ab}, q_{bc}, q_{ac} where the DM is "indifferent" between the respective acts and which contains a signed belief where the DM is not indifferent between any of these acts, every line L' parallel to L with signed beliefs $q'_{ab}, q'_{bc}, q'_{ac}$ where the DM is "indifferent" between the respective acts and which contains a signed belief where the DM is not indifferent between any of these acts, it holds that*

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q'_{ac}(s) - q'_{bc}(s)) = (q'_{ab}(s) - q'_{bc}(s)) \cdot (q_{ac}(s) - q_{bc}(s))$$

for every state s.

Four choice linear preference intensity: *For every four acts a, b, c, d, and for every line L of signed beliefs with signed beliefs $q_{ab}, q_{ac}, q_{ad}, q_{bc}, q_{bd}, q_{cd}$ where the DM is "indifferent" between the respective acts and which contains a signed belief where the DM is not indifferent between any of these acts, it holds that*

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q_{ac}(s) - q_{cd}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) = (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{cd}(s)).$$

for every state s.

Let us now go back to the conditional preference relation $\succsim$ in Figure 3, for which we have argued that no expected utility representation exists. It turns out that $\succsim$ cannot be extended to a signed conditional preference relation $\succsim^*$ that satisfies all of the axioms above. To see this, note that, in view of preservation of indifference and preservation of strict preference, $Q_{a \sim^* b}$ and $Q_{a \sim^* c}$ must be the linear extensions of $P_{a \sim b}$ and $P_{a \sim c}$ outside the belief simplex, and thus they meet at the signed belief v in Figure 3. By transitivity of $\succsim^*$, it would then follow that $v \in Q_{b \sim^* c}$. By a similar argument, it also would follow by the axioms that $w \in Q_{b \sim^* c}$. But then, by preservation of indifference and preservation of strict preference, $Q_{b \sim^* c}$ must be the line of signed beliefs that goes through v and w. In particular, $Q_{b \sim^* c}$ intersects the belief simplex, which means that there are beliefs at which the DM is indifferent between b and c. This is a contradiction, since b strictly dominates c. Thus, $\succsim$ cannot be extended to a signed conditional preference relation $\succsim^*$ that satisfies all of the axioms.

3.4 Constant Preference Intensity

On top of the axioms above, we need some further axioms which deal with situations where the preference intensity between two choices is "constant".

Definition 3.3 (Constant preference intensity) *A signed conditional preference relation $\succsim^*$ exhibits* ***constant preference intensity*** *between two choices a and b if either $a \succ^*_q b$ for all signed beliefs q, or $b \succ^*_q a$ for all signed beliefs q.*

In terms of expected utility, this means that the expected utility difference between a and b is constant. A necessary consequence of expected utility is "transitivity of constant preference intensity".

Axiom 3.1 (Transitive constant preference intensity) *For every three acts a, b, c where $\succsim^*$ exhibits constant preference intensity between a and b and between b and c, there must also be constant preference intensity between a and c.*

If there is constant preference intensity between two or more acts, then this will also have consequences for the formula of *four choice linear preference intensity.* To see this, consider four choices a, b, c, d, and a line L of signed beliefs with associated indifference beliefs. Suppose now that the preference intensity between c and d is constant. Then, the preference intensity between a and c and the preference intensity between a and d will only differ by a constant. In particular, the change rate of the preference intensity between a and c will be the same as between a and d. Thus,

$$\frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ d}(q))} = \frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ c}(q))}.$$

Since

$$\frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ d}(q))} = \frac{q_{ad}(s) - q_{bd}(s)}{q_{ab}(s) - q_{bd}(s)} \text{ and } \frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ c}(q))} = \frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)}$$

it follows that

$$\frac{q_{ad}(s) - q_{bd}(s)}{q_{ab}(s) - q_{bd}(s)} = \frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)}.$$

We thus obtain the formula

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) = (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s)). \tag{3.1}$$

Suppose, in addition, that the preference intensity between a and b would also be constant. Thus, the preference intensities between a and b, and between c and d, would both be constant. On a line of signed beliefs the preference intensities between the various pairs of choices would then yield a picture similar to that in Figure 5. From the picture it can clearly be seen that

$$q_{ac}(s) - q_{bc}(s) = q_{ad}(s) - q_{bd}(s) \tag{3.2}$$

for every state s. The formulas (3.1) and (3.2) lead to the following axiom.

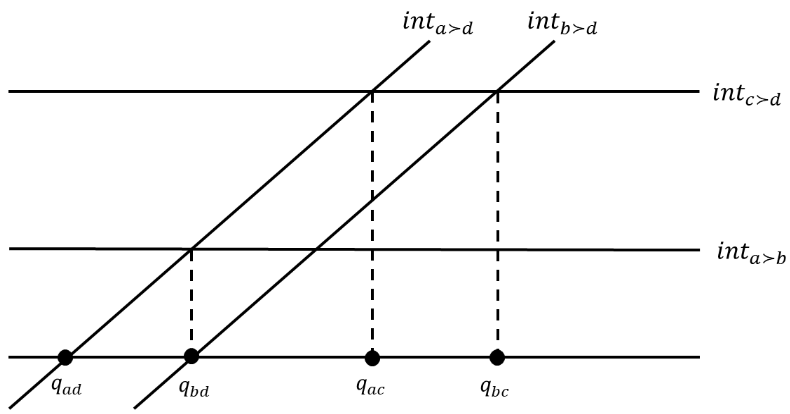

Figure 5: Four choice linear preference intensity with constant preference intensity

Axiom 3.2 (Four choice linear preference intensity with contant preference intensity) *For every line of signed beliefs L, and for every four choices a, b, c, d such that there is a signed belief on this line where the DM is not "indifferent" between any pair of choices in $\{a, b, c, d\}$, the following holds:*

(a) if there is a constant preference intensity between c and d, but not between the other five pairs of choices, then for every five signed beliefs $q_{ab}, q_{ac}, q_{ad}, q_{bc}$ and q_{bd} on the line L where the DM is "indifferent" between the respective choices, it holds for every state s that

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) = (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s));$$

(b) if there is a constant preference intensity between a and b, and between c and d, but not between the other four pairs of choices, then for every four signed beliefs q_{ac}, q_{ad}, q_{bc} and q_{bd} on the line L where the DM is "indifferent" between the respective choices, it holds for every state s that

$$q_{ac}(s) - q_{bc}(s) = q_{ad}(s) - q_{bd}(s).$$

3.5 Representation Theorem

It turns out that the axioms we have established in section are both necessary and sufficient for an expected utility representation in the general case. We thus obtain the following general representation result.

Theorem 3.1 (Expected utility representation) *Consider a finite set of acts A, a finite set of states S, and a conditional preference relation $\succsim$ on (A, S). Then, $\succsim$ has an expected utility representation, if and only if, $\succsim$ can be extended to a signed conditional preference relation $\succsim^*$that satisfies completeness,* *transitivity, continuity, preservation of indifference, preservation of strict preference, three choice linear preference intensity, four choice linear preference intensity, transitive constant preference intensity and four choice linear preference intensity with constant preference intensity.*

Since most of these axioms may be viewed as instances of linear preference intensity, the result above shows that we may interpret expected utility as an expression of linear preference intensity.

4 Discussion

(a) Comparison with Savage. One important difference with the framework of Savage (1954) is that we view the DM's belief as a primitive notion, from which we can derive his preference relation over acts. This is precisely how a conditional preference relation is defined: It takes the belief as an input, and delivers the preferences over acts as an output. One of the beautiful features of the Savage framework is that the DM's belief can be derived from his preferences over acts. That is, Savage views the DM's preferences over acts as the primitive notion, which then induces his belief. There has been a long-standing debate about which of the two, belief or preferences, should be taken as the primitive object, and we do not want to enter this debate here. But the logic that underlies our framework is that the DM first reasons himself towards a belief, then forms his preferences over acts based on this belief, which finally allows him to make a choice based on this preference relation.

Another difference with Savage lies in the role of the utility function. In our model, the utility function generates the DM's preferences over acts for *all* possible beliefs over the states. As the Savage axiom system leads to a unique probabilistic belief over states, the utility function in the Savage framework can only be viewed in combination with this specific belief.

A final difference we would like to stress concerns the uniqueness of the utility representation. Recall from Proposition 2.2 that in the absence of weakly dominated acts there are $|S| + 1$ degrees of freedom for the utility function in our framework, provided there is a belief where the DM is indifferent between some, but not all, acts in the case of at least three acts. Unless all acts are equivalent, this is also the smallest number of degrees of freedom possible. There may be more degrees of freedom, up to $|A| \cdot |S|$, which would be the case if every act strictly dominates, or is strictly dominated by, another act.

In the Savage framework, on the other hand, the utility representation is always unique up to a positive affine transformation, leaving only two degrees of freedom. The reason is that a DM in the Savage framework holds preferences over *all possible* mappings from states to consequences, providing us with "more data" that restrict the possible utilities compared to a DM in our framework. However, the two degrees of freedom in Savage's framework are only possible because Savage's axiom of small event continuity implies that there are infinitely many states. We assume only finitely many states, but our "richness of data" comes from the fact that a conditional preference relation specifies a preference relation for infinitely many beliefs (if there are at least two states). Most comments here also apply to the framework in Anscombe and Aumann (1963).

(b) Related foundations for expected utility in decision problems and games. The foundation for expected utility that is closest to ours is by Gilboa and Schmeidler (2003). As already stressed in the introduction, their axiom system singles out those conditional preference relations that can be represented by a *diversified* utility function, and the crucial axiom in their analysis is *diversity.* The diversity axiom by Gilboa and Schmeidler may be viewed as a "richness" condition on the set of states, and seems plausible if the number of states is very large, or even (countably or uncountably) infinite, as is allowed by the Gilboa-Schmeidler framework. In contrast, we mainly concentrate on settings like finite games where, tyically, the number of states is relatively small. In such scenarios, the diversity axiom seems overly restrictive. Our axiom system, in turn, imposes no such richness condition on the set of states, and puts no restrictions on the utility matrix that can be used to represent the conditional preference relation.

Jagau (2022) shows that the regularity axioms, together with the axioms of *constant preference intensity* and *transitive preference sensitivity,* are necessary and sufficient for an expected utility representation if there are no weakly dominated acts. Constant preference intensity and transitive preference sensitivity are strongly based on our axioms of three choice linear preference intensity and four choice linear preference intensity, respectively.

Perea (2020) proves that the regularity axioms, together with the axiom *existence of a uniform preference increase,* are both necessary and sufficient for an expected utility representation. The existence of a uniform preference increase states that from the conditional preference relation at hand, one should be able to increase the preference intensity between a fixed choice and each of the other choices by a uniform amount.

Luce and Raiffa (1957)'s formulation of a decision problem under uncertainty is rather similar to ours, in that they view the DM's sets of actions and states as primitive notions. On top of this, they assume a consequence mapping, assigning to every act and state the consequence that results. Battigalli, Cerreia-Vioglio, Maccheroni and Marinacci (2017) show how the Anscombe-Aumann model can be reconciled with the Luce-Raiffa framework, by letting the DM hold preferences over mixed actions in the Luce-Raiffa model, and proposing an axiomatic characterization of expected utility within this setup.

Fishburn (1976) and Fishburn and Roberts (1978) concentrate on games, and assume that every player holds a preference relation over the combinations of randomized choices – or mixed strategies – of all the players. Combinations of mixed strategies may be viewed as lotteries with objective probabilities on the set of possible (pure) choice combinations in the game. By imposing certain axioms on these preference relations over mixed strategy combinations, they are able to identify those that admit an expected utility representation. It may thus be viewed as a generalization of von Neumann and Morgenstern's (1947) axiomatic characterization of expected utility for lotteries. The crucial difference with our approach is that we do not consider randomizations over choices, and that we use conditional preference relations as the primitive, rather than preferences over lotteries with objective probabilities.

In Aumann and Drèze (2002), a game is modelled as a mapping that assigns to every choice combination by the players a lottery over consequences for each of the players. The DM (a player in the game) is then assumed to hold a preference relation on the probability distributions over such mappings. Aumann and Drèze (2005) take a different approach, by supposing that the DM in a game holds a preference relation on lotteries which are defined over his own choices and over the possible consequences in the game. In both papers, it is shown that certain axioms on the preference relation lead to an expected utility representation that involves a unique, or essentially unique, probabilistic belief for the DM about the opponents' choice combinations. In that sense, these results are similar to Savage (1954).

Mariotti (1995) points out that a DM in Savage (1954) is required to hold preferences over acts that do not belong to his actual decision problem, and finds this problematic. Mariotti (1995) goes even further, and shows that certain game-theoretic principles are inconsistent with the axioms of completeness and monotonicity in Savage's framework, thus establishing a degree of "incompatibility" between games on the one hand and the framework of Savage on the other hand.

(c) Comparison with case-based decision theory. Case-based decision theory, as originally formulated in Gilboa and Schmeidler (1995), assumes that the DM evaluates an act based on how this act performed in previous decision problems. More precisely, assume that C represents the collection of decision problems, or cases, the DM faced in the past, and that $s(c)$ measures the similarity of decision problem c to the present decision problem. Then, the desirability of an act a in the present decision problem is measured by $\sum_{c \in C} s(c) \cdot u(a, c)$, where $u(a, c)$ is the utility that selecting act a generated in decision problem c.

Our framework can be embedded into case-based decision theory as follows: If a conditional preference relation is represented by a utility function u, then the desirability of an act a in the present decision problem, for a given $p \in \Delta(S)$, is given by $\sum_{s \in S} p(s) \cdot u(a, s)$. Now suppose that the states s represent decision problems that the DM faced in the past, and that $p(s)$ measures the similarity of problem s to the decision problem he is facing now. Then, the measure for the desirability of act a resembles exactly that in Gilboa and Schmeidler (1995).

Alternatively, one could still interpret p as a probabilistic belief over states, and identify every state s with the degenerate belief $[s]$ that assigns probability 1 to s. Suppose that, for some reason, the DM has had each of these degenerate beliefs $[s]$ in the past, and remembers the utility $u(a, s)$ that each act a generated under that belief. Then, every belief $[s]$ can be viewed as a case in the Gilboa-Schmeidler framework. If the DM's actual belief is p, then the belief probability $p(s)$ can be viewed as the similarity of the actual belief p to the past belief $[s]$. Also in this scenario, the measure for the desirability of act a in the actual problem, with the actual belief p, coincides with that of the Gilboa-Schmeidler framework.

(d) Utility differences as preference intensities. In Proposition 2.2 we have shown that under certain conditions, the utility differences are unique up to a positive multiplicative constant. In that case, the expected utility difference between two acts a and b at a state s may be interpreted as the "preference intensity" between a and b at the state s. This is similar to how utility differences are interpreted in Anscombe and Aumann (1963) and Wakker (1989). The *state independence* axiom in Anscombe and Aumann (1963) states that the preference relation over objective lotteries on consequences must be independent of the state. This implies, in turn, that the utility differences between two consequences must be the same at every state, and these may be viewed as expressing the "preference intensity" between the two consequences.

The key condition in Wakker's (1989) axiom system is *state independent preference intensity.* The main idea is that the "preference intensity" between two consequences c_1 and c_2 at a state s can be measured by taking two acts, where one is strictly preferred to the other, and replacing the two acts at state s by c_1 and c_2, respectively, such that the DM becomes indifferent between the two new acts. State independent preference intensity requires that if the preference intensities between c_1 and c_2 and between c_3 and c_4 coincide at one state, then they must coincide at all states. In that case, the utility difference between two consequences will always be the same at all states, and may thus be viewed as expressing the "preference intensity" between the two consequences.[3]

(e) Linear preference intensity. The axiomatic characterizations in this paper show that expected utility may be viewed as an expression of linear preference intensity. Indeed, some of the regularity axioms for two choices, the axioms of three and four choice linear preference intensity for more than two choices, and their extensions to signed beliefs, represent consequences of scenarios where the preference intensity between two choices changes linearly with the belief. But how natural is this idea of linear preference intensity? From a behavioral and empirical point of view, one could conduct behavioral experiments to test these axioms. On a more theoretical basis, the idea states that (i) the change in preference intensity should only depend on the *change* in belief, not on the particular initial and final belief, thereby revealing a specific type of invariance, and (ii) for a given direction of belief change, the change in preference intensity must be proportional to the size of the belief change. Conceptually, it thus represents the simplest possible way in which the preference intensity can vary with the belief. A problem, of course, is that preference intensity cannot be measured directly, but many of the axioms represent verifiable properties that logically follow from the assumption of linear preference intensity.

(f) Testability of the axioms. The axioms we provide for the scenario when there are *no weakly dominated acts* are all empirically testable. Three choice and four choice linear preference intensity, for instance, describe how the sets of beliefs where the DM is indifferent between the various pairs of choices must relate to each other. This changes when we move to the scenario where there are *weakly dominated acts.* The axioms then impose conditions on extensions of the conditional preference relations outside the belief simplex. As a consequence, it will no longer be possible to test these axioms directly, as beliefs outside the belief simplex cannot be observed. However, as we argued before, the axioms still have intuitive content, as they describe how the preference intensities between the various pairs of choices must change if we change the belief *inside* the belief simplex. This raises the question: Can these axioms be replaced by alternative conditions that *are* directly empirically verifiable? At this moment I would not know how this can be done, and I therefore leave this as an open question here.

(g) Equivalent acts. In this paper we have restricted attention to scenarios where no two acts are equivalent. In fact, our entire analysis can easily be extended to the case where equivalent acts are allowed. Suppose we start with a set of acts A where some acts are equivalent. Then, we can partition A into equivalence classes $\{A_1, A_2, ..., A_K\}$ with representative acts $a_1, a_2, ..., a_K$, and subsequently restrict the conditional preference relation $\succsim$ to the set $A' = \{a_1, a_2, ..., a_K\}$, resulting in a new conditional preference relation $\succsim'$. Then, Theorem 3.1 can be generalized as follows: The conditional preference relation $\succsim$ has an expected utility representation, if and only if, $\succsim'$ satisfies the conditions in Theorem 3.1. The proof is easy: If $\succsim'$ satisfies the conditions in the theorem, then by the same theorem it is represented by a utility function u. Extend u to a utility function v on $A \times S$ by setting $v(a, s) := u(a_k, s)$ for all acts $a \in A$ and all $s \in S$, where $a \in A_k$. Clearly, v will then represent $\succsim$. In the same way, the other results in this paper can also be extended to cases that allow for equivalent acts.

[3] Also in vNM-settings, expected utility differences are often interpreted as representing preference intensities. See, for instance, Börgers and Postl (2009), which focusus on voting scenarios between two parties.

5 Appendix

5.1 Mathematical Definitions

In this section we introduce the mathematical definitions and notation needed for this paper, mainly from linear algebra. For a finite set X, we denote by $\mathbf{R}^X$ the set of all functions $v : X \to \mathbf{R}$. Scalar multiplication and addition on $\mathbf{R}^X$ are defined in the usual way: For a function $v \in \mathbf{R}^X$ and a number $\lambda \in \mathbf{R}$, the function $\lambda \cdot v$ is given by $(\lambda \cdot v)(x) = \lambda \cdot v(x)$ for all $x \in X$. Similarly, for functions $v, w \in \mathbf{R}^X$, the sum $v + w$ is given by $(v + w)(x) = v(x) + w(x)$ for all $x \in X$. The set $\mathbf{R}^X$ together with these two operations constitutes a *linear space,* and elements in $\mathbf{R}^X$ are called *vectors.* By $\underline{0}$ we denote the vector in $\mathbf{R}^X$ where $\underline{0}(x) = 0$ for all $x \in X$.

A subset $V \subseteq \mathbf{R}^X$ is called a *linear subspace* of $\mathbf{R}^X$ if for every $v, w \in V$ and every $\alpha, \beta \in \mathbf{R}$, we have that $\alpha v + \beta w \in V$. For a subset $V \subseteq \mathbf{R}^X$, we denote by

$$span(V) := \{\sum_{k=1}^{K} \alpha_k v_k \mid K \geq 1,\ \alpha_k \in \mathbf{R} \text{ and } v_k \in V \text{ for all } k \in \{1, ..., K\}\}$$

the set of all (finite) *linear combinations* of elements in V, and call it the (*linear) span* of V. Here, $\sum_{k=1}^{K} \alpha_k v_k$ is called a *linear combination* of the vectors $v_1, ..., v_K$. A linear combination $v = \lambda_1 v_1 + ... + \lambda_K v_K$, where $v_1, ..., v_K \in \mathbf{R}^X$ and $\lambda_1, ..., \lambda_K \in \mathbf{R}$, is called a *convex combination* if $\lambda_1, ..., \lambda_K \geq 0$ and $\lambda_1 + ... + \lambda_K = 1$.

The set $span(V)$ is always a linear subspace, and if V itself is a linear subspace then $span(V) = V$. Vectors $v_1, ..., v_K \in \mathbf{R}^X$ are called *linearly independent* if none of the vectors is a linear combination of the other vectors. The set of vectors $\{v_1, ..., v_K\}$ is a *basis* for V if $v_1, ..., v_K$ are linearly independent, and $span(\{v_1, ..., v_K\}) = V$. Every basis for V has the same number of vectors, and this number is called the *dimension* of V, denoted by $\dim(V)$. If $V = \{\underline{0}\}$, then $\dim(V) = 0$.

A *probability distribution* on X is a vector $p \in \mathbf{R}^X$ such that $\sum_{x \in X} p(x) = 1$ and $p(x) \geq 0$ for all $x \in X$. The set of probability distributions on X is denoted by $\Delta(X)$. For a given element $x \in X$, we denote by $[x]$ the probability distribution in $\Delta(X)$ where $[x](x) = 1$ and $[x](y) = 0$ for all $y \in X \backslash \{x\}$. A probability distribution p has *full support* if $p(x) > 0$ for all $x \in X$.

For every two vectors $v, w \in \mathbf{R}^X$, the *vector product* is given by $v \cdot w := \sum_{x \in X} v(x) w(x)$. A *hyperplane* is a set of the form $H = \{v \in \mathbf{R}^X \mid v \cdot w = c\}$, where $w \in \mathbf{R}^X \backslash \{\underline{0}\}$ and $c \in \mathbf{R}$. If $c = 0$ then H is a linear subspace of dimension $|X| - 1$, where $|X|$ denotes the number of elements in X.

5.2 Proofs of Section 2

In this subsection we will prove Proposition 2.1, Theorem 2.1 and Proposition 2.2. Before doing so, we first derive some preparatory results. The first characterizes the span of the set of beliefs where the DM is indifferent between a and b.

Lemma 5.1 (Span of an indifference set) *Consider a conditional preference relation $\succsim$ that satisfies preservation of indifference, and two choices a and b. Then,*

$$span(P_{a \sim b}) = \{\lambda_1 p_1 + \lambda_2 p_2 \mid p_1, p_2 \in P_{a \sim b} \text{ and } \lambda_1, \lambda_2 \in \mathbf{R}\}.$$

Proof. Let

$$A := \{\lambda_1 p_1 + \lambda_2 p_2 \mid p_1, p_2 \in P_{a\sim b} \text{ and } \lambda_1, \lambda_2 \in \mathbf{R}\}.$$

We will show that $span(P_{a\sim b}) = A$. Clearly, $A \subseteq span(P_{a\sim b})$. Hence, it remains to show that $span(P_{a\sim b}) \subseteq A$. Take some $p \in span(P_{a\sim b})$. Then, there are some beliefs $p_1, ..., p_k, p_{k+1}, ..., p_{k+m} \in P_{a\sim b}$ and numbers $\lambda_1, ..., \lambda_k, \lambda_{k+1}, ..., \lambda_{k+m} > 0$ such that

$$p = \lambda_1 p_1 + ... + \lambda_k p_k - \lambda_{k+1} p_{k+1} - ... - \lambda_{k+m} p_{k+m}. \tag{5.1}$$

Let $\alpha_1 := \lambda_1 + ... + \lambda_k$ and $\alpha_2 := \lambda_{k+1} + ... + \lambda_{k+m}$. If $\alpha_1 > 0$ and $\alpha_2 > 0$, then define the vectors

$$q_1 := \frac{\lambda_1}{\alpha_1} p_1 + ... + \frac{\lambda_k}{\alpha_1} p_k \text{ and } q_2 := \frac{\lambda_{k+1}}{\alpha_2} p_{k+1} + ... + \frac{\lambda_{k+m}}{\alpha_2} p_{k+m}.$$

It may be verified that q_1 and q_2 are convex combinations of beliefs in $P_{a\sim b}$. Hence, by repeatedly using preservation of indifference, it follows that $q_1, q_2 \in P_{a\sim b}$. By (5.1) we have that $p = \alpha_1 q_1 - \alpha_2 q_2$, and thus $p \in A$.

If $\alpha_1 > 0$ and $\alpha_2 = 0$, then we have that $p = \alpha_1 q_1 + 0 \cdot q_1$, which is in A. The case when $\alpha_1 = 0$ and $\alpha_2 > 0$ is similar. Finally, when $\alpha_1 = 0$ and $\alpha_2 = 0$, then $p = 0 \cdot p_1 + 0 \cdot p_2$ for two arbitrary beliefs $p_1, p_2 \in P_{a\sim b}$, and hence $p \in A$.

In general, we thus see that every $p \in span(P_{a\sim b})$ is also in A, and thus $span(P_{a\sim b}) \subseteq A$. Together with the observation above that $A \subseteq span(P_{a\sim b})$, we conclude that $span(P_{a\sim b}) = A$. This completes the proof.■

The second preparatory result contains some further properties of the set of beliefs where the DM is indifferent between a and b, gathered in Lemma 5.2. In this lemma, we denote by $S_{a\sim b}$ the set of states s where $a \sim_{[s]} b$. Moreover, we say that there are preference reversals between a and b if there are beliefs $p, q \in \Delta(S)$ such that $a \succ_p b$ and $b \succ_q a$.

Lemma 5.2 (Linear structure of indifference sets) *Suppose there are two choices, a and b, and n states. Consider a conditional preference relation $\succsim$ that satisfies the regularity axioms. Then, the following properties hold:*

(a) $P_{a\sim b} = span(P_{a\sim b}) \cap \Delta(S)$;

(b) if $\succsim$ has preference reversals between a and b, then $span(P_{a\sim b})$ is a hyperplane with dimension $n-1$, and there is a full support belief $p \in P_{a\sim b}$ with $p(s) > 0$ for all $s \in S$;

(c) if a weakly dominates b under $\succsim$ then $P_{a\sim b} = \{p \in \Delta(S) \mid \sum_{s \in S_{a\sim b}} p(s) = 1\}$.

Proof. (a) Clearly, $P_{a\sim b} \subseteq span(P_{a\sim b}) \cap \Delta(S)$. It remains to show that $span(P_{a\sim b}) \cap \Delta(S) \subseteq P_{a\sim b}$. Take some $p \in span(P_{a\sim b}) \cap \Delta(S)$. Then, by Lemma 5.1, there are beliefs $p_1, p_2 \in P_{a\sim b}$ and numbers λ_1, λ_2 such that $p = \lambda_1 p_1 + \lambda_2 p_2$. Since $p \in \Delta(S)$, we must have that $\sum_{s \in S} p(s) = 1$. Moreover, as p_1, p_2 are beliefs, it holds that $\sum_{s \in S} p_1(s) = \sum_{s \in S} p_2(s) = 1$. But then, it must be that $\lambda_1 + \lambda_2 = 1$.

Suppose first that $\lambda_1 = 0$. Then, $\lambda_2 = 1$, and hence $p = p_2$, which is in $P_{a\sim b}$. The case where $\lambda_2 = 0$ is similar. Assume next that $\lambda_1, \lambda_2 > 0$. As $\lambda_1 + \lambda_2 = 1$, it follows that p is a convex combination of p_1 and p_2, which are both in $P_{a\sim b}$. By preservation of indifference, it follows that $p \in P_{a\sim b}$.

Suppose now that $\lambda_1 > 0$ and $\lambda_2 < 0$. Since $\lambda_1 + \lambda_2 = 1$, it must be that $\lambda_1 > 1$. Hence, we have that

$$p_1 = \frac{1}{\lambda_1}p - \frac{\lambda_2}{\lambda_1}p_2 = \frac{1}{\lambda_1}p + (1 - \frac{1}{\lambda_1})p_2 \tag{5.2}$$

since $\lambda_2 = 1 - \lambda_1$. As $\lambda_1 > 1$, it follows that p_1 is a convex combination of p and p_2, where p_1 and p_2 are both in $P_{a\sim b}$.

We will show that p must be in $P_{a\sim b}$. Suppose, on the contrary, that $p \notin P_{a\sim b}$. Assume, without loss of generality, that $p \in P_{a\succ b}$. Then, it follows from (5.2) and preservation of strict preference that $p_1 \in P_{a\succ b}$, which is a contradiction. Hence, $p \in P_{a\sim b}$. The case where $\lambda_1 < 0$ and $\lambda_2 > 0$ is similar. In general, we conclude that every $p \in span(P_{a\sim b}) \cap \Delta(S)$ is also in $P_{a\sim b}$. Hence, $span(P_{a\sim b}) \cap \Delta(S) \subseteq P_{a\sim b}$. As we have already seen that $P_{a\sim b} \subseteq span(P_{a\sim b}) \cap \Delta(S)$, we have that $P_{a\sim b} = span(P_{a\sim b}) \cap \Delta(S)$.

(b) Suppose that $\succsim$ has preference reversals on $\{a, b\}$. Then, there must be a state x where $a \succ_x b$, and another state y where $b \succ_y a$. Here, we write $a \succ_x b$ as a short-cut for $a \succ_{[x]} b$. By continuity, there must be a belief $p_2 = (1 - \lambda_2)[x] + \lambda_2[y]$ on the line segment between $[x]$ and $[y]$ where $a \sim_{p_2} b$. Now, let the remaining states be numbered $s_3, ..., s_n$ such that

$$\begin{gathered} a \succ_{s_k} b \text{ for all } k \in \{3, ..., m\}, \\ b \succ_{s_k} a \text{ for all } k \in \{m+1, ..., m+l\}, \text{ and} \\ a \sim_{s_k} b \text{ for all } k \in \{m+l+1, ..., n\}. \end{gathered}$$

We choose (i) for every $k \in \{3, ..., m\}$ a belief $p_k = (1 - \lambda_k)[s_k] + \lambda_k[y]$ with $a \sim_{p_k} b$, (ii) for every $k \in \{m+1, ..., m+l\}$ a belief $p_k = (1 - \lambda_k)[s_k] + \lambda_k[x]$ with $a \sim_{p_k} b$, and (iii) for every $k \in \{m+l+1, ..., n\}$ the belief $p_k = [s_k]$ with $a \sim_{p_k} b$.

We will now show that $p_2, ..., p_n$ are linearly independent. Take some numbers $\alpha_2, ..., \alpha_n$ such that $\sum_{k=2}^n \alpha_k \cdot p_k = \underline{0}$. By construction, this sum is equal to

$$\begin{aligned} &\alpha_2((1 - \lambda_2)[x] + \lambda_2[y]) + \sum_{k=3}^{m} \alpha_k((1 - \lambda_k)[s_k] + \lambda_k[y]) + \sum_{k=m+1}^{m+l} \alpha_k((1 - \lambda_k)[s_k] + \lambda_k[x]) + \sum_{k=m+l+1}^{n} \alpha_k[s_k] \\ &= \left(\alpha_2(1 - \lambda_2) + \sum_{k=m+1}^{m+l} \alpha_k\lambda_k\right)[x] + \left(\alpha_2\lambda_2 + \sum_{k=3}^{m} \alpha_k\lambda_k\right)[y] + \sum_{k=3}^{m+l} \alpha_k(1 - \lambda_k)[s_k] + \sum_{k=m+l+1}^{n} \alpha_k[s_k] = \underline{0}. \end{aligned}$$

As the vectors $[x], [y], [s_3], ..., [s_n]$ are linearly independent, and $0 < \lambda_k < 1$ for all $k \in \{2, ..., m+l\}$, it follows that $\alpha_k = 0$ for all $k \in \{3, ..., n\}$. This, in turn, implies that also $\alpha_2 = 0$. Hence, the indifference beliefs $p_2, ..., p_n \in P_{a\sim b}$ are linearly independent.

As a consequence, the dimension of $span(P_{a\sim b})$ is at least $n - 1$. The dimension of $span(P_{a\sim b})$ cannot be n, since otherwise we would have that $span(P_{a\sim b}) = \mathbf{R}^S$, and hence, by (a), $P_{a\sim b} = \mathbf{R}^S \cap \Delta(S) = \Delta(S)$.

This would contradict the assumption that there are preference reversals between a and b. We thus conclude that the dimension of $span(P_{a\sim b})$ must be $n-1$, and therefore $span(P_{a\sim b})$ is a hyperplane.

To show that $P_{a\sim b}$ contains a belief p with $p(s) > 0$ for every state s, consider the vector $p := \frac{1}{n-1}p_2 + ... + \frac{1}{n-1}p_n$. It may be verified that p is a belief. Moreover, by construction of the beliefs $p_2, ..., p_n$, we have that $p(s) > 0$ for all states s.

(c) Let $A = \{p \in \Delta(S) \mid \sum_{s\in S_{a\sim b}} p(s) = 1\}$. To show that $P_{a\sim b} \subseteq A$, take some $p \in P_{a\sim b}$. Assume, contrary to what we want to show, that $p \notin A$. Then, $p(s) > 0$ for some $s \in S_{a\succ b}$, where $S_{a\succ b}$ is the set of states t with $a \succ_t b$. As $p = \sum_{s\in S_{a\sim b}} p(s) \cdot [s] + \sum_{s\in S_{a\succ b}} p(s) \cdot [s]$ it follows by preservation of indifference and preservation of strict preference that $p \in P_{a\succ b}$. This is a contradiction to the assumption that $p \in P_{a\sim b}$. We thus conclude that $p \in A$. Hence, $P_{a\sim b} \subseteq A$. The inclusion $A \subseteq P_{a\sim b}$ follows directly by preservation of indifference. We thus see that $P_{a\sim b} = A$. This completes the proof. ■

The third preparatory result provides sufficient conditions for an expected utility representation between two choices.

Lemma 5.3 (Sufficient conditions for expected utility representation) *Consider a conditional preference relation $\succsim$ that satisfies the regularity axioms, two choices a and b, and a utility function u. Suppose that $\succsim$ has preference reversals between a and b, and that there are n states. If there is a belief p^* with $a \succ_{p^*} b$ and $u(a, p^*) > u(b, p^*)$, and $n-1$ linearly independent vectors $v_1, ..., v_{n-1}$ in $span(P_{a\sim b})$ with $u(a, v_k) = u(b, v_k)$ for all $k \in \{1, ..., n-1\}$, then u represents $\succsim$ on $\{a, b\}$.*

Proof. Let $P_{u(a)=u(b)}$ be the set of beliefs p with $u(a, p) = u(b, p)$, and similarly for $P_{u(a)>u(b)}$. To show that u represents $\succsim$ on $\{a, b\}$, it is thus sufficient to show that $P_{a\sim b} = P_{u(a)=u(b)}$ and $P_{a\succ b} = P_{u(a)>u(b)}$.

We start by showing that $P_{a\sim b} = P_{u(a)=u(b)}$. Consider the set $V_{u(a)=u(b)} := \{v \in \mathbf{R}^S \mid u(a, v) = u(b, v)\}$. It may be verified that $V_{u(a)=u(b)}$ is a linear space. Moreover, $P_{u(a)=u(b)} = V_{u(a)=u(b)} \cap \Delta(S)$. We now show that $span(P_{a\sim b}) = V_{u(a)=u(b)}$. We first prove that $span(P_{a\sim b}) \subseteq V_{u(a)=u(b)}$. In Lemma 5.2 (b) we have seen that $span(P_{a\sim b})$ has dimension $n-1$. Since the vectors $v_1, ..., v_{n-1}$ in $span(P_{a\sim b})$ are linearly independent, we conclude that $\{v_1, ..., v_{n-1}\}$ is a basis of $span(P_{a\sim b})$. Take some $v \in span(P_{a\sim b})$. Then, we can write $v = \lambda_1 v_1 + ... + \lambda_{n-1} v_{n-1}$ for some numbers $\lambda_1, ..., \lambda_{n-1}$. Since $v_k \in V_{u(a)=u(b)}$ for all $k \in \{1, ...n-1\}$ and $V_{u(a)=u(b)}$ is a linear subspace, it follows that $v \in V_{u(a)=u(b)}$. Thus, $span(P_{a\sim b}) \subseteq V_{u(a)=u(b)}$.

We now show that $V_{u(a)=u(b)} = span(P_{a\sim b})$. Since $V_{u(a)=u(b)}$ is a linear subspace of $\mathbf{R}^S$, its dimension can be at most n. Moreover, as $span(P_{a\sim b}) \subseteq V_{u(a)=u(b)}$ and $span(P_{a\sim b})$ has dimension $n-1$, the dimension of $V_{u(a)=u(b)}$ is at least $n-1$. Suppose, contrary to what we want to prove, that $V_{u(a)=u(b)} \neq span(P_{a\sim b})$. Then, the dimension of $V_{u(a)=u(b)}$ must be n, and hence $V_{u(a)=u(b)} = \mathbf{R}^S$. However, this is a contradiction since $u(a, p^*) > u(b, p^*)$, and hence $p^* \notin V_{u(a)=u(b)}$. We thus conclude that $V_{u(a)=u(b)} = span(P_{a\sim b})$. Since $P_{u(a)=u(b)} = V_{u(a)=u(b)} \cap \Delta(S)$ and, by Lemma 5.2 (a), $P_{a\sim b} = span(P_{a\sim b}) \cap \Delta(S)$, we conclude that $P_{a\sim b} = P_{u(a)=u(b)}$.

We next prove that $P_{a\succ b} = P_{u(a)>u(b)}$. Let p^* be the belief where $a \succ_{p^*} b$ and $u(a, p^*) > u(b, p^*)$. Consider the set

$$A := \{p \in \Delta(S) \mid \text{there is no } \lambda \in [0, 1] \text{ with } (1-\lambda)p + \lambda p^* \in P_{a\sim b}\}.$$

We show that $P_{a \succ b} = A$. To prove that $P_{a \succ b} \subseteq A$, take some $p \in P_{a \succ b}$. Since $p^* \in P_{a \succ b}$ it follows by preservation of strict preference that $(1-\lambda)p + \lambda p^* \in P_{a \succ b}$ for every $\lambda \in [0,1]$, and hence $p \in A$. Thus, $P_{a \succ b} \subseteq A$.

To show that $A \subseteq P_{a \succ b}$, take some $p \in A$. Suppose that $p \notin P_{a \succ b}$. Since $p \in A$, we must have that $p \notin P_{a \sim b}$, and hence $p \in P_{b \succ a}$. By continuity, there must then be some $\lambda \in (0,1)$ with $(1-\lambda)p + \lambda p^* \in P_{a \sim b}$. This, however, contradicts the assumption that $p \in A$. Hence, $p \in P_{a \succ b}$, which yields $A \subseteq P_{a \succ b}$. Altogether, we conclude that $P_{a \succ b} = A$.

We next show that $P_{u(a)>u(b)} = A$. Since $P_{a \sim b} = P_{u(a)=u(b)}$, it follows that

$$A = \{p \in \Delta(S) \mid \text{there is no } \lambda \in [0,1] \text{ with } (1-\lambda)p + \lambda p^* \in P_{u(a)=u(b)}\}.$$

As $p^* \in P_{u(a)>u(b)}$ by construction, it can be shown in a similar same way as above that $P_{u(a)>u(b)} = A$. As such, $P_{a \succ b} = A = P_{u(a)>u(b)}$.

Since $P_{a \sim b} = P_{u(a)=u(b)}$ and $P_{a \succ b} = P_{u(a)>u(b)}$, the utility function u represents $\succsim$ on $\{a, b\}$. This completes the proof. ■

The following result contains an axiomatic characterization of expected utility for the case of two choices.

Lemma 5.4 (Characterization of expected utility for two choices) *Consider a set A consisting of two acts, a finite set of states S, and a conditional preference relation $\succsim$ on (A, S). Then, $\succsim$ has an expected utility representation, if and only if, it satisfies completeness, transitivity, continuity, preservation of indifference and preservation of strict preference.*

Proof of Lemma 5.4. Suppose first that $\succsim$ has an expected utility representation u. Then, it can easily be verified that $\succsim$ satisfies the regularity axioms. We leave this to the reader.

Assume next that $\succsim$ satisfies the regularity axioms. We will show that $\succsim$ has an expected utility representation. We distinguish three cases: (a) there are preference reversals between a and b, (b) a weakly dominates b, and (c) b weakly dominates a. For the remainder of this proof, we assume that the number of states is n.

(a) Suppose that there are preference reversals between a and b. Since we know from Lemma 5.2 (b) that $span(P_{a \sim b})$ has dimension $n-1$, there are $n-1$ linearly independent beliefs $p_1, ..., p_{n-1} \in P_{a \sim b}$. Moreover, there must be some state x with $a \succ_{[x]} b$. As $[x] \notin P_{a \sim b}$, we know from Lemma 5.2 (a) that $[x] \notin span(P_{a \sim b})$, and hence the beliefs $p_1, ..., p_{n-1}, [x]$ are linearly independent. Fix some number $\alpha < u(a, x)$, and find the unique utilities $\{u(b, s) \mid s \in S\}$ such that $u(b, x) = \alpha < u(a, x)$ and $u(b, p_k) = u(a, p_k)$ for all $k \in \{1, ... n-1\}$. By Lemma 5.3 it then follows that u represents $\succsim$.

(b) Suppose that a weakly dominates b. Choose a utility function u such that, for every state s, we have $u(a, s) > u(b, s)$ when $[s] \in P_{a \succ b}$, and $u(a, s) = u(b, s)$ when $[s] \in P_{a \sim b}$. It then follows by Lemma 5.2 (c) that $P_{a \sim b} = P_{u(a)=u(b)}$. Since every belief p is either in $P_{a \sim b}$ or $P_{a \succ b}$, it follows that $P_{a \succ b} = P_{u(a)>u(b)}$. We thus conclude that the utility function u represents $\succsim$.

(c) This proof is similar to that for (b). The proof is hereby complete. ■

The following result guarantees the existence of a line of beliefs with certain properties.

Lemma 5.5 (Line containing three indifference beliefs) *Consider a conditional preference relation $\succsim$ that has preference reversals for all pairs of choices, and satisfies the regularity axioms. Then, for every three choices a, b, c, there is a line of beliefs that contains full support beliefs p_{ab}, p_{ac}, p_{bc} where the DM is indifferent between the respective choices, and that contains a belief where the DM is not indifferent between any of these three choices.*

Proof. Suppose first that there is a full support belief $p \in P_{a\sim b} \cap P_{b\sim c}$. Then, by transitivity, $p \in P_{a\sim c}$. We can then choose a line of beliefs through p that contains a belief where the DM is not indifferent between any of the three choices. Such a line will satisfy the statement in the lemma.

Assume next that there is no full support belief in $P_{a\sim b} \cap P_{b\sim c}$. By transitivity, there will be no full support belief in $P_{a\sim b} \cap P_{a\sim c}$ or $P_{b\sim c} \cap P_{a\sim c}$ either. Let $\Delta^+(S)$ be the set of full support beliefs. Then, the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ will be pairwise disjoint on $\Delta^+(S)$. As, by Lemma 5.2 (a), these indifference sets are the intersections of hyperplanes with $\Delta(S)$, it must be that one of these indifference sets is "in between" the other two. Suppose, without loss of generality, that $P_{b\sim c}$ is in between $P_{a\sim b}$ and $P_{a\sim c}$. By Lemma 5.2 (b), there is a full support belief $p_{ab} \in P_{a\sim b}$ and a full support belief $p_{ac} \in P_{a\sim c}$. Let l be the line of beliefs that goes through p_{ab} and p_{ac}. As the set $P_{b\sim c}$ is in between $P_{a\sim b}$ and $P_{a\sim c}$, there must be a belief $p_{bc} \in P_{b\sim c}$ on the line l between p_{ab} and p_{ac}. Moreover, p_{bc} is a full support belief, since p_{ab} and p_{ac} are full support beliefs. Finally, the full support beliefs p_{ab} and p_{ac} can be chosen such that l contains a belief where the DM is not indifferent between any of the three choices. The line l thus satisfies the requirements of the lemma. This completes the proof. ■

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Consider a conditional preference relation $\succsim$ that has no weakly dominated choices and satisfies the regularity axioms. Since we exclude equivalent choices, it must be that $\succsim$ has preference reversals between every pair of choices.

(a) Assume first that $\succsim$ satisfies three choice linear preference intensity. Consider three choices a, b and c. We must show that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$. Take some $q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$. We distinguish two cases: (1) $\sum_{s\in S} q(s) \neq 0$, and (2) $\sum_{s\in S} q(s) = 0$.

Case 1. Assume that $\sum_{s\in S} q(s) \neq 0$. Then, there is some number $\lambda \neq 0$ such that $\hat{q} := \lambda q$ satisfies $\sum_{s\in S} \hat{q}(s) = 1$. Moreover, $\hat{q} \in span(P_{a\sim b}) \cap span(P_{b\sim c})$ also. By Lemma 5.5 there is a line l that contains full support beliefs $p_{ab} \in P_{a\sim b}, p_{bc} \in P_{b\sim c}$ and $p_{ac} \in P_{a\sim c}$. Then, there is some $\varepsilon \in (0, 1)$ small enough such that (i) the vectors $p'_{ab} := (1-\varepsilon)p_{ab} + \varepsilon\hat{q}$, $p'_{bc} := (1-\varepsilon)p_{bc} + \varepsilon\hat{q}$ and $p' := (1-\varepsilon)p_{ac} + \varepsilon\hat{q}$ are all in $\Delta(S)$, and (ii) the line l' through p'_{ab} and p'_{bc} contains a belief $p'_{ac} \in P_{a\sim c}$. Since $p'_{ab} - p'_{bc} = (1-\varepsilon) \cdot (p_{ab} - p_{bc})$, we conclude that the lines l and l' are parallel. Moreover, the lines l and l' can be chosen such that they contain beliefs where the DM is not indifferent between any of the three choices. Hence, by preservation of strict preference, p'_{ac} is the unique belief in $P_{a\sim c}$ on the line l'. Also, the lines l and l' can be chosen such that the probability of no state is constant on l or l'.

Recall that $\hat{q} \in span(P_{a\sim b})$. Thus, we conclude that $p'_{ab} \in span(P_{a\sim b}) \cap \Delta(S)$. By Lemma 5.2 (a) it follows that $p'_{ab} \in P_{a\sim b}$. As $\hat{q} \in span(P_{b\sim c})$ it can be shown in a similar way that $p'_{bc} \in P_{b\sim c}$.

Recall that $p' := (1-\varepsilon)p_{ac} + \varepsilon\hat{q}$. We will now show that $p'_{ac} = p'$. Suppose first that $p_{ab} = p_{bc}$. Then, by transitivity, $p_{ac} = p_{ab} = p_{bc}$. Moreover, by definition of p'_{ab} and p'_{bc} it follows that $p'_{ab} = p'_{bc}$, and hence by transitivity we must have that $p'_{ac} = p'_{ab} = p'_{bc}$. Thus, $p' = (1-\varepsilon)p_{ac} + \varepsilon\hat{q} = (1-\varepsilon)p_{ab} + \varepsilon\hat{q} = p'_{ab} = p'_{ac}$.

Suppose now that $p_{ab} \neq p_{bc}$. Then, by transitivity, the beliefs p_{ab}, p_{bc} and p_{ac} are pairwise different. By definition of p'_{ab} and p'_{bc}, we then have that $p'_{ab} \neq p'_{bc}$. Hence, by transitivity, the beliefs p'_{ab}, p'_{bc} and p'_{ac} are pairwise different. By three choice linear preference intensity, we have for every state s that

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)). \tag{5.3}$$

Note that, by definition, $(p'_{ab}(s) - p'_{bc}(s)) = (1-\varepsilon)(p_{ab}(s) - p_{bc}(s))$. Since the beliefs p_{ab}, p_{bc} and p_{ac} are pairwise different, the beliefs p'_{ab}, p'_{bc} and p'_{ac} are pairwise different, and no state has constant probability on the lines l and l', it follows together with (5.3) that $(p'_{ac}(s) - p'_{bc}(s)) = (1-\varepsilon)(p_{ac}(s) - p_{bc}(s))$, and thus

$$p'_{ac}(s) = (1-\varepsilon)(p_{ac}(s) - p_{bc}(s)) + p'_{bc}(s) = (1-\varepsilon)p_{ac}(s) + \varepsilon\hat{q}(s) = p'(s).$$

As this holds for every state s, we conclude that $p'_{ac} = p'$. Thus, the belief $p' = (1-\varepsilon)p_{ac} + \varepsilon\hat{q}$ is in $P_{a\sim c}$. As such, $\hat{q} = \frac{1}{\varepsilon}p' + (1 - \frac{1}{\varepsilon})p_{ac} \in \mathit{span}(P_{a\sim c})$, which implies that $q \in \mathit{span}(P_{a\sim c})$ also.

Case 2. Assume next that $\sum_{s\in S} q(s) = 0$. Let $V_0 := \{v \in \mathbf{R}^S \mid \sum_{s\in S} v(s) = 0\}$. We distinguish two subcases: (2.1) $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \nsubseteq V_0$, and (2.2) $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \subseteq V_0$.

Case 2.1. Assume that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \nsubseteq V_0$. Then, there is some $r \in \mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c})$ with $\sum_{s\in S} r(s) \neq 0$. Hence, we know by Case 1 that $r \in \mathit{span}(P_{a\sim c})$. Moreover, as $q, r \in \mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c})$, we conclude that $q - r \in \mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c})$ also, with $\sum_{s\in S}(q-r)(s) \neq 0$. Hence, by Case 1, also $q - r \in \mathit{span}(P_{a\sim c})$. As $q = r + (q-r)$, and both r and $q - r$ are in $\mathit{span}(P_{a\sim c})$, it follows that $q \in \mathit{span}(P_{a\sim c})$.

Case 2.2. Suppose that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \subseteq V_0$. It can be shown that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) = \mathit{span}(P_{a\sim b}) \cap V_0$. To see this, note first that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \subseteq \mathit{span}(P_{a\sim b}) \cap V_0$, since $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \subseteq V_0$. Moreover, we also know that $\mathit{span}(P_{a\sim b}) \neq \mathit{span}(P_{b\sim c})$, since otherwise $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c})$ would contain beliefs in $P_{a\sim b}$ which would clearly not be in V_0. Since, by Lemma 5.2 (b), $\mathit{span}(P_{a\sim b})$ and $\mathit{span}(P_{b\sim c})$ are linear subspaces of dimension $n-1$, it follows that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c})$ is a linear subspace of dimension $n-2$. Now, consider the linear subspace $\mathit{span}(P_{a\sim b}) \cap V_0$. Clearly, $\mathit{span}(P_{a\sim b}) \neq V_0$, since $\mathit{span}(P_{a\sim b})$ contains beliefs in $P_{a\sim b}$ which are not in V_0. Since $\mathit{span}(P_{a\sim b})$ and V_0 are linear subspaces of dimension $n-1$, it follows that $\mathit{span}(P_{a\sim b}) \cap V_0$ is a linear subspace of dimension $n-2$. Since $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \subseteq \mathit{span}(P_{a\sim b}) \cap V_0$ and both linear subspaces have the same dimension, $n-2$, both spaces must be equal. Hence, $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) = \mathit{span}(P_{a\sim b}) \cap V_0$.

Moreover, it must be that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{a\sim c}) \subseteq V_0$ also. To see this, assume on the contrary that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{a\sim c}) \nsubseteq V_0$. Then, it would follow from Case 2.1 that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{a\sim c}) \subseteq \mathit{span}(P_{b\sim c})$, and thus $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{a\sim c}) \subseteq \mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) \subseteq V_0$. This would be a contradiction. Hence, we conclude that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{a\sim c}) \subseteq V_0$. It can then be shown, in the same way as above, that $\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{a\sim c}) = \mathit{span}(P_{a\sim b}) \cap V_0$.

By combining the latter two equalities, we get

$$\mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{b\sim c}) = \mathit{span}(P_{a\sim b}) \cap V_0 = \mathit{span}(P_{a\sim b}) \cap \mathit{span}(P_{a\sim c}),$$

which implies that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$. As $q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$, it follows that $q \in span(P_{a\sim c})$. This completes the proof of (a).

(b) Suppose now that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$ for all three choices a, b, c. We must show that $\succsim$ satisfies three choice linear preference intensity. Consider two parallel lines of beliefs l, l' that (i) contain beliefs where the DM is not indifferent between any two choices from $\{a, b, c\}$, (ii) where l contains indifference beliefs $p_{ab} \in P_{a\sim b}, p_{bc} \in P_{b\sim c}$ and $p_{ac} \in P_{a\sim c}$, and (iii) l' contains indifference beliefs $p'_{ab} \in P_{a\sim b}, p'_{bc} \in P_{b\sim c}$ and $p'_{ac} \in P_{a\sim c}$. Let l_{ab} be the line through p_{ab} and p'_{ab}, let l_{bc} be the line through p_{bc} and p'_{bc}, and l_{ac} the line through p_{ac} and p'_{ac}. Note that all these lines belong to the same two-dimensional plane: the plane that goes through l and l'.

Assume first that the lines l_{ab}, l_{bc} and l_{ac} are all parallel. Then, there is a vector q such that $p'_{ab} = p_{ab} + q$, $p'_{bc} = p_{bc} + q$ and $p'_{ac} = p_{ac} + q$. As a consequence, for every state s,

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).$$

Hence, the formula for three choice linear preference intensity is satisfied.

Assume next that the lines l_{ab}, l_{bc} and l_{ac} are not all parallel. Without loss of generality, we suppose that l_{ab} and l_{bc} are not parallel. Since these two lines lie in the same two-dimensional plane, they must intersect at a unique vector q. Since q lies on l_{ab}, which goes through p_{ab} and p'_{ab} in $P_{a\sim b}$, we conclude that $q \in span(P_{a\sim b})$. Similarly, as q lies on l_{bc}, it follows that $q \in span(P_{b\sim c})$. Since we assume that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$, we conclude that $q \in span(P_{a\sim c})$ too.

Let V be the two-dimensional plane that goes through the lines l and l'. Since, by condition (i) above, l and l' contain beliefs where the DM is not indifferent between a and c, it follows that $span(P_{a\sim c}) \cap V = l_{ac}$. As $q \in span(P_{a\sim c}) \cap V$, we conclude that q lies on the line l_{ac}.

As q lies on l_{ab}, l_{bc} and l_{ac}, the beliefs p_{ab}, p_{bc}, p_{ac} lie on l, the beliefs p'_{ab}, p'_{bc} and p'_{ac} lie on l', and the lines l and l' are parallel, there is a unique number λ such that $p'_{ab} = (1-\lambda)q + \lambda p_{ab}$, $p'_{bc} = (1-\lambda)q + \lambda p_{bc}$ and $p'_{ac} = (1-\lambda)q + \lambda p_{ac}$. Hence, for every state s we have that

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = \lambda \cdot (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).$$

Thus, the formula for three choice linear preference intensity is satisfied. We therefore conclude that $\succsim$ satisfies three choice linear preference intensity. This completes the proof. ■

In our last preparatory result, we characterize the span of an indifference set $P_{a\sim b}$ in case of an expected utility representation.

Lemma 5.6 (Span of indifference set under utility representation) *Consider a conditional preference relation $\succsim$ with an expected utility representation u. Suppose there are preferene reversals between choices a and b. Then,*

$$span(P_{a\sim b}) = \{q \in \mathbf{R}^S \mid u(a, q) = u(b, q)\}.$$

Proof. Let $A := \{q \in \mathbf{R}^S \mid u(a,q) = u(b,q)\}$. We first show that $span(P_{a\sim b}) \subseteq A$. Take some $q \in span(P_{a\sim b})$. Then, by Lemma 5.1, there are $p_1, p_2 \in P_{a\sim b}$ and numbers λ_1, λ_2 such that $q = \lambda_1 p_1 + \lambda_2 p_2$. As $u(a,p_1) = u(b,p_1)$ and $u(a,p_2) = u(b,p_2)$, it follows that $u(a,q) = u(b,q)$, and hence $q \in A$. Thus, $span(P_{a\sim b}) \subseteq A$. By Lemma 5.2 (b) we know that $span(P_{a\sim b})$ has dimension $n-1$. Since A is a linear subspace with dimension $n-1$ also, and $span(P_{a\sim b}) \subseteq A$, it must be that $span(P_{a\sim b}) = A$. This completes the proof. ■

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. (a) Suppose first that $\succsim$ has an expected utility representation u. From Lemma 5.4, we know that $\succsim$ satisfies the regularity axioms.

To show three choice linear preference intensity it suffices, in view of Proposition 2.1, to show that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$ for all three choices a, b, c. Take some $q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$. Then, by Lemma 5.1, there are $p_{ab}^1, p_{ab}^2 \in P_{a\sim b}$, $p_{bc}^1, p_{bc}^2 \in P_{b\sim c}$ and numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$ such that $q = \lambda_1 p_{ab}^1 + \lambda_2 p_{ab}^2 = \mu_1 p_{bc}^1 + \mu_2 p_{bc}^2$. As $u(a, p_{ab}^1) = u(b, p_{ab}^1)$ and $u(a, p_{ab}^2) = u(b, p_{ab}^2)$, it follows that $u(a,q) = u(b,q)$. In a similar fashion, it follows that $u(b,q) = u(c,q)$, and hence $u(a,q) = u(c,q)$. By Lemma 5.6 it thus follows that $q \in span(P_{a\sim c})$. Hence, $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$, which implies by Proposition 2.1 that $\succsim$ satisfies three choice linear preference intensity.

We finally show four choice linear preference intensity. Consider a line of beliefs l, and four choices a, b, c, d such that there is a belief on the line where the DM is not indifferent between any pair of choices in $\{a, b, c, d\}$. Moreover, let $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}$ and p_{cd} be corresponding indifference beliefs on this line. Consider some state s. If the probability of s is constant on the line l, then the formula for four choice linear preference intensity holds trivially.

We therefore assume from now on that the probability of s is not constant on l, so that every belief on l is uniquely given by the probability it assigns to s. Suppose that $p_{ab} = p_{ac}$. Then, by transitivity, it must be that $p_{ab} = p_{ac} = p_{bc}$, and the formula for four choice linear preference intensity would hold trivially. Similarly, the formula would trivially hold if $p_{ab} = p_{ad}$ or $p_{ac} = p_{ad}$.

We now assume that p_{ab}, p_{ac}, p_{ad} are pairwise different. Then, by transitivity, p_{bc} is different from p_{ab} and p_{ac}, the belief p_{bd} is different from p_{ab} and p_{ad}, and the belief p_{cd} is different from p_{ac} and p_{ad}.

Consider two arbitrary, but different, beliefs p_1, p_2 on l, and define

$$\Delta(u(a) - u(b)) := (u(a,p_1) - u(b,p_1)) - (u(a,p_2) - u(b,p_2)).$$

As there is a belief on the line where the DM is indifferent between a and b, and another belief on the line where the DM is not, we must have that $\Delta(u(a) - u(b)) \neq 0$. In a similar way, we define $\Delta(u(a) - u(c))$ and $\Delta(u(a) - u(d))$.

By applying the arguments from Section 4.3 to expected utility differences, instead of preference intensity, it follows that

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(c))} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)}. \tag{5.4}$$

Recall that also $\Delta(u(a)-u(c)) \neq 0$. Moreover, since $p_{ab} \neq p_{bc}$ and the belief on the line is uniquely given by its probability on s, we have that $p_{ab}(s) \neq p_{bc}(s)$. Thus, the two ratios above are well-defined. In a similar fashion, it follows that

$$\frac{\Delta(u(a)-u(c))}{\Delta(u(a)-u(d))} = \frac{p_{ad}(s)-p_{cd}(s)}{p_{ac}(s)-p_{cd}(s)} \text{ and } \frac{\Delta(u(a)-u(b))}{\Delta(u(a)-u(d))} = \frac{p_{ad}(s)-p_{bd}(s)}{p_{ab}(s)-p_{bd}(s)}. \tag{5.5}$$

As, by definition,

$$\frac{\Delta(u(a)-u(b))}{\Delta(u(a)-u(d))} = \frac{\Delta(u(a)-u(b))}{\Delta(u(a)-u(c))} \cdot \frac{\Delta(u(a)-u(c))}{\Delta(u(a)-u(d))},$$

it follows by (5.4) and (5.5) that the formula for four choice linear preference intensity obtains. Thus, $\succsim$ satisfies four choice linear preference intensity.

(b) Suppose that $\succsim$ satisfies the regularity axioms, three choice linear preference intensity and four choice linear preference intensity. If there are only two choices, then we know from Lemma 5.4 that there is an expected utility representation. We therefore assume, from now on, that there are at least three choices.

To show that $\succsim$ has an expected utility representation, we distinguish two cases: (1) $P_{a\sim b} = P_{c\sim d}$ for every two pairs of choices $\{a,b\}$ and $\{c,d\}$, and (2) $P_{a\sim b} \neq P_{c\sim d}$ for some pairs of choices $\{a,b\}$ and $\{c,d\}$.

Case 1. Suppose that $P_{a\sim b} = P_{c\sim d}$ for every two pairs of choices $\{a,b\}$ and $\{c,d\}$. Let $A := P_{a\sim b}$ for some pair of choices $\{a,b\}$. Note that $A \neq \Delta(S)$, as we assume that no two choices are equivalent under $\succsim$. Since we also assume that no choice weakly dominates another choice, there will be preference reversals between all pairs of choices. Let x be a state where $[x] \notin A$. Hence, $[x] \notin P_{a\sim b}$ for every two choices a and b. By transitivity, we can order the choices $c_1, c_2, ..., c_K$ such that

$$c_1 \succ_{[x]} c_2 \succ_{[x]} c_3 \succ_{[x]} ... \succ_{[x]} c_K.$$

Choose numbers $v_1, ..., v_K$ with $v_1 > v_2 > ... > v_K$.

For choice c_1, set $u(c_1, x) = v_1$, and set the utilities $u(c_1, s)$ for states $s \neq x$ arbitrarily.

By Lemma 5.2 (b) we know that $span(A)$ has dimension $n-1$, where n is the number of states. Let $\{p_1, ..., p_{n-1}\}$ be a basis for $span(A)$. As $[x] \notin span(A)$, we know that $\{p_1, ..., p_{n-1}, [x]\}$ is a basis for $\mathbf{R}^S$. For every choice c_k with $k \geq 2$ find the unique utilities $u(c_k, s)$ such that

$$u(c_k, p_1) = u(c_1, p_1), ..., u(c_k, p_{n-1}) = u(c_1, p_{n-1}) \text{ and } u(c_k, x) = v_k. \tag{5.6}$$

We will show that the utility function u represents $\succsim$.

Take two choices a, b with $a \succ_{[x]} b$. Then, by construction of the utility function, we have that $u(a, p_k) = u(b, p_k)$ for all $k \in \{1, ..., n-1\}$, and $u(a, x) > u(b, x)$. As $\{p_1, ..., p_{n-1}\}$ is a basis for $span(P_{a\sim b})$, we know that $p_1, ..., p_{n-1}$ are linearly independent. It thus follows by Lemma 5.3 that u represents $\succsim$ on the pair of choices $\{a, b\}$. As this holds for every pair of choices $\{a, b\}$, we conclude that u represents $\succsim$.

Case 2. Suppose that $P_{a\sim b} \neq P_{c\sim d}$ for some pairs of choices $\{a,b\}$ and $\{c,d\}$. Then, there must be some choices a, b, c such that $P_{a\sim c} \neq P_{b\sim c}$. To see this, suppose on the contrary that $P_{a\sim c} = P_{b\sim c}$ for all three choices a, b, c. Then, take two arbitrary pairs of choices $\{a, b\}$ and $\{c, d\}$ where $\{a, b\} \cap \{c, d\} = \emptyset$. By assumption we would then have that $P_{a\sim b} = P_{b\sim c} = P_{c\sim d}$, and hence $P_{a\sim b} = P_{c\sim d}$ for all pairs $\{a, b\}$ and $\{c, d\}$. This would be a contradiction. Hence, $P_{a\sim c} \neq P_{b\sim c}$ for some choices a, b, c.

Now take some choice d different from a, b and c, if it exists. Then, either $P_{a\sim d} \neq P_{b\sim d}$ or $P_{a\sim d} \neq P_{c\sim d}$. To see this, suppose on the contrary that $P_{a\sim d} = P_{b\sim d} = P_{c\sim d}$. Define $A := P_{a\sim d} = P_{b\sim d} = P_{c\sim d}$. Since, by transitivity, $P_{a\sim d} \cap P_{b\sim d} \subseteq P_{a\sim b}$ and $P_{b\sim d} \cap P_{c\sim d} \subseteq P_{b\sim c}$, it follows that $A \subseteq P_{a\sim b} \cap P_{b\sim c}$. Thus, $span(A) \subseteq span(P_{a\sim b}) \cap span(P_{b\sim c})$. However, since $P_{a\sim c} \neq P_{b\sim c}$ we have, by transitivity, that $P_{a\sim b} \neq P_{b\sim c}$. As, by Lemma 5.2 (b), both $span(P_{a\sim b})$ and $span(P_{b\sim c})$ have dimension $n-1$, it must be that $span(P_{a\sim b}) \cap span(P_{b\sim c})$ has dimension $n-2$. However, A has dimension $n-1$, and hence it cannot be that $span(A) \subseteq span(P_{a\sim b}) \cap span(P_{b\sim c})$. We thus obtain a contradiction, and conclude that either $P_{a\sim d} \neq P_{b\sim d}$ or $P_{a\sim d} \neq P_{c\sim d}$.

Based on the two insights above, we can order the choices $c_1, c_2, ..., c_K$ such that $P_{c_3\sim c_1} \neq P_{c_3\sim c_2}$, and for every $k \geq 4$ either $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$ or $P_{c_k\sim c_1} \neq P_{c_k\sim c_3}$. Let the utilities for c_1 and c_2 be given as in the proof of Lemma 5.4. For the other choices, we define their utilities according to the following procedure:

Utilities for c_3: By Lemma 5.2 (b), there are $n-1$ linearly independent beliefs $p_1, ..., p_{n-1} \in P_{c_3\sim c_1}$. Choose a belief $p_n \in P_{c_3\sim c_2} \backslash P_{c_3\sim c_1}$. Note that this is possible since $P_{c_3\sim c_1} \neq P_{c_3\sim c_2}$, and because of Lemma 5.2 (a) and (b). By Lemma 5.2 (a), $p_n \notin span(P_{c_3\sim c_1})$, and hence $p_1, ..., p_{n-1}, p_n$ are linearly independent. Find the unique utilities $\{u(c_3, s) \mid s \in S\}$ such that

$$u(c_3, p_m) = u(c_1, p_m) \text{ for all } m \in \{1, ..., n-1\}, \text{ and } u(c_3, p_n) = u(c_2, p_n). \tag{5.7}$$

Utilities for $c_4, ..., c_K$. For every $k \geq 4$, inductively define the utilities for c_k as follows. From above, we know that either $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$ or $P_{c_k\sim c_1} \neq P_{c_k\sim c_3}$. Suppose that $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$. Like above, we can choose linearly independent beliefs $p_1, ..., p_{n-1}, p_n$ where $p_1, ..., p_{n-1} \in P_{c_k\sim c_1}$ and $p_n \in P_{c_k\sim c_2} \backslash P_{c_k\sim c_1}$. Find the unique utilities $\{u(c_k, s) \mid s \in S\}$ such that

$$u(c_k, p_m) = u(c_1, p_m) \text{ for all } m \in \{1, ..., n-1\}, \text{ and } u(c_k, p_n) = u(c_2, p_n). \tag{5.8}$$

If $P_{c_k\sim c_1} \neq P_{c_k\sim c_3}$, the utilities can be defined analogously,

We will now show that these utilities represents the conditional preference relation $\succsim$.

We prove, by induction on k, that u represents $\succsim$ on $\{c_1, ..., c_k\}$. For $k = 2$ we know this is true, in the light of the proof of Lemma 5.4.

Suppose now that $k \geq 3$, and that u represents $\succsim$ on $\{c_1, ..., c_{k-1}\}$. We must show that u represents $\succsim$ on all pairs $\{c_k, c_m\}$ where $m \in \{1, ..., k-1\}$.

We start by showing that u represents $\succsim$ on $\{c_k, c_1\}$. Assume, without loss of generality, that $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$. Then, by (5.7) and (5.8) we know that $u(c_k, p_n) = u(c_2, p_n)$. As $p_n \in P_{c_k\sim c_2} \backslash P_{c_k\sim c_1}$, we may assume, without loss of generality, that $p_n \in P_{c_k\succ c_1}$. As $p_n \in P_{c_k\sim c_2}$, it follows that $p_n \in P_{c_2\succ c_1}$, and hence $u(c_2, p_n) > u(c_1, p_n)$. As, by (5.7) and (5.8), $u(c_k, p_n) = u(c_2, p_n)$, we conclude that $u(c_k, p_n) > u(c_1, p_n)$. Thus, $p_n \in P_{c_k\succ c_1}$ is such that $u(c_k, p_n) > u(c_1, p_n)$. Together with (5.7) and (5.8), we conclude from Lemma 5.3 that u represents $\succsim$ on $\{c_k, c_1\}$.

We next show that u represents $\succsim$ on $\{c_k, c_2\}$. As $P_{c_k \sim c_1} \neq P_{c_k \sim c_2}$, it follows by transitivity that $P_{c_k \sim c_1} \neq P_{c_1 \sim c_2}$. Hence, it follows by Lemma 5.2 (a) and (b) that $span(P_{c_k \sim c_1}) \cap span(P_{c_1 \sim c_2})$ has dimension $n-2$. Take a basis $\{q_1, ..., q_{n-2}\}$ for $span(P_{c_k \sim c_1}) \cap span(P_{c_1 \sim c_2})$. By Proposition 2.1 we know that $span(P_{c_k \sim c_1}) \cap span(P_{c_1 \sim c_2}) \subseteq span(P_{c_k \sim c_2})$, and hence the vectors $q_1, ..., q_{n-2}$ are all in $span(P_{c_k \sim c_2})$. As each of these vectors q_m is in $span(P_{c_k \sim c_1})$, it follows by Lemma 5.1 that q_m can be written as $q_m = \lambda_1 r_1 + \lambda_2 r_2$, where $\lambda_1, \lambda_2 \in \mathbf{R}$ and $r_1, r_2 \in P_{c_k \sim c_1}$. Since u represents $\succsim$ on $\{c_k, c_1\}$, we know that $u(c_k, r_1) = u(c_1, r_1)$ and $u(c_k, r_2) = u(c_1, r_2)$, which implies that $u(c_k, q_m) = u(c_1, q_m)$. As q_m is also in $span(P_{c_1 \sim c_2})$, and u represents $\succsim$ on $\{c_1, c_2\}$, it follows in a similar way that $u(c_1, q_m) = u(c_2, q_m)$. Hence, we conclude that

$$u(c_k, q_m) = u(c_2, q_m) \text{ for all } m \in \{1, ..., n-2\} \text{ and } u(c_k, p_n) = u(c_2, p_n), \tag{5.9}$$

where the last equality follows from (5.7) and (5.8). Moreover, as $p_n \notin P_{c_k \sim c_1}$, we know that $p_n \notin span(P_{c_k \sim c_1}) \cap span(P_{c_1 \sim c_2})$, and hence the $n-1$ vectors above are linearly independent.

Since $P_{c_k \sim c_1} \neq P_{c_k \sim c_2}$, there is a belief $p \in P_{c_k \sim c_1} \backslash P_{c_k \sim c_2}$. Assume, without loss of generality, that $p \in P_{c_k \succ c_2}$. As $p \in P_{c_k \sim c_1}$, it follows by transitivity that $p \in P_{c_1 \succ c_2}$. As u represents $\succsim$ on $\{c_k, c_1\}$ and $\{c_1, c_2\}$, it follows that $u(c_k, p) = u(c_1, p)$ and $u(c_1, p) > u(c_2, p)$, which implies that $u(c_k, p) > u(c_2, p)$. Hence, there is some belief p with $P_{c_k \succ c_2}$ and $u(c_k, p) > u(c_2, p)$. Together with (5.9) and Lemma 5.3, we conclude that u represents $\succsim$ on $\{c_k, c_2\}$.

We finally show that u represents $\succsim$ on $\{c_k, c_m\}$ for every $m \in \{3, ..., k-1\}$. Take some $m \in \{3, ..., k-1\}$. Then, necessarily, $k \geq 4$. To abbreviate the notation, we define $span_{ml} := span(P_{c_m \sim c_l})$ for every $m, l \in \{1, ..., k\}$. We distinguish two cases: (1) $span_{k1} \cap span_{1m} \neq span_{k2} \cap span_{2m}$ or $span_{k1} \cap span_{1m} \neq span_{k3} \cap span_{3m}$, and (2) $span_{k1} \cap span_{1m} = span_{k2} \cap span_{2m}$ and $span_{k1} \cap span_{1m} = span_{k3} \cap span_{3m}$.

Case 1. Assume, without loss of generality, that $span_{k1} \cap span_{1m} \neq span_{k2} \cap span_{2m}$. Since, by Lemma 5.2 (b), the four linear spans have dimension $n-1$, it follows that the two intersections have dimension $n-2$ or $n-1$. Moreover, as the two intersections are different, we conclude that

$$A := span[(span_{k1} \cap span_{1m})) \cup (span_{k2} \cap span_{2m})]$$

has dimension $n-1$ or n. Moreover, we know from Proposition 2.1 that $span_{k1} \cap span_{1m}$ and $span_{k2} \cap span_{2m}$ are both subsets of $span_{km}$, and hence $A \subseteq span_{km}$ also. As c_k and c_m are not equivalent, A cannot have dimension n, and thus the dimension of A must be $n-1$.

Take a basis $\{q_1, ..., q_{n-1}\}$ for A, where every q_l is either in $span_{k1} \cap span_{1m}$ or in $span_{k2} \cap span_{2m}$. Suppose that q_l is in $span_{k1} \cap span_{1m}$. As u represents $\succsim$ on $\{c_k, c_1\}$ and $\{c_1, c_m\}$, it can be shown in the same way as above that $u(c_k, q_l) = u(c_1, q_l)$ and $u(c_1, q_l) = u(c_m, q_l)$, which implies that $u(c_k, q_l) = u(c_m, c_l)$. If q_l is in $span_{k2} \cap span_{2m}$, it can be shown in a similar way that $u(c_k, q_l) = u(c_m, q_l)$ also. We thus see that

$$q_l \in span_{km} \text{ and } u(c_k, q_l) = u(c_m, q_l) \text{ for every } l \in \{1, ..., n-1\}. \tag{5.10}$$

Since $P_{c_k \sim c_1} \neq P_{c_k \sim c_2}$, either $P_{c_k \sim c_1} \backslash P_{c_k \sim c_m}$ or $P_{c_k \sim c_2} \backslash P_{c_k \sim c_m}$ must be non-empty. Assume, without loss of generality, that $P_{c_k \sim c_1} \backslash P_{c_k \sim c_m}$ is non-empty. Take some $p \in P_{c_k \sim c_1} \backslash P_{c_k \sim c_m}$. Assume, without loss of generality, that $p \in P_{c_k \succ c_m}$. As $p \in P_{c_k \sim c_1}$, it follows by transitivity that $p \in P_{c_1 \succ c_m}$. Since u represents $\succsim$ on

$\{c_k, c_1\}$ and $\{c_1, c_m\}$, we know that $u(c_k, p) = u(c_1, p)$ and $u(c_1, p) > u(c_m, p)$, and thus $u(c_k, p) > u(c_m, p)$. We have thus found a belief $p \in P_{c_k \succ c_m}$ with $u(c_k, p) > u(c_m, p)$. Together with (5.10) and Lemma 5.3 we conclude that u represents $\succsim$ on $\{c_k, c_m\}$.

Case 2. Suppose that $span_{k1} \cap span_{1m} = span_{k2} \cap span_{2m}$ and $span_{k1} \cap span_{1m} = span_{k3} \cap span_{3m}$.

Claim. There are $i, j \in \{1, 2, 3\}$ such that for every triple $a, b, c \in \{c_i, c_j, c_m, c_k\}$ the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different.

Proof of claim. We first show that $P_{c_k\sim c_1} \neq P_{c_1\sim c_m}$. Suppose not. Then, $P_{c_k\sim c_1} = P_{c_1\sim c_m}$ and hence, by transitivity, $P_{c_k\sim c_1} = P_{c_k\sim c_m}$. Thus, $span_{k1} \cap span_{1m} = span_{k1} = span_{km}$. Since $span_{k1} \cap span_{1m} = span_{k2} \cap span_{2m}$, it follows that $span_{k2} \cap span_{2m} = span_{km}$, which can only be if $P_{c_k\sim c_2} = P_{c_2\sim c_m} = P_{c_k\sim c_m}$. As such, $P_{c_k\sim c_1} = P_{c_k\sim c_m} = P_{c_k\sim c_2}$, which contradicts the assumption that $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$. Hence, $P_{c_k\sim c_1} \neq P_{c_1\sim c_m}$. By transitivity, $P_{c_k\sim c_1}, P_{c_1\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different.

As a consequence, $span_{k1} \cap span_{1m}$ has dimension $n - 2$. Since $span_{k1} \cap span_{1m} = span_{k2} \cap span_{2m}$ it follows that $span_{k2} \cap span_{2m}$ has dimension $n - 2$ also, which can only be if $P_{c_k\sim c_2} \neq P_{c_2\sim c_m}$. Thus, by transitivity, $P_{c_k\sim c_2}, P_{c_2\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different. As $span_{k1} \cap span_{1m} = span_{k3} \cap span_{3m}$, it follows in a similar way that $P_{c_k\sim c_3}, P_{c_3\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different also.

Consider the sets $A = \{c_1, c_2, c_m, c_k\}$, $B = \{c_1, c_3, c_m, c_k\}$ and $C = \{c_2, c_3, c_m, c_k\}$. Suppose, contrary to what we want to show, that in each of these sets there is a triple a, b, c such that $P_{a\sim b} = P_{a\sim c} = P_{b\sim c}$. Since, by assumption, $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$, and we have seen above that $P_{c_k\sim c_1}, P_{c_1\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different and $P_{c_k\sim c_2}, P_{c_2\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different, we must have in set A that $P_{c_1\sim c_2} = P_{c_2\sim c_m} = P_{c_1\sim c_m}$.

By a similar argument, we must have in set B that either $P_{c_1\sim c_3} = P_{c_3\sim c_m} = P_{c_1\sim c_m}$ or $P_{c_1\sim c_3} = P_{c_3\sim c_k} = P_{c_1\sim c_k}$. However, if $P_{c_1\sim c_3} = P_{c_3\sim c_m} = P_{c_1\sim c_m}$ then, by the insight above that $P_{c_1\sim c_2} = P_{c_1\sim c_m}$, it would follow that $P_{c_1\sim c_2} = P_{c_1\sim c_3}$, which is a contradiction to the fact that $P_{c_1\sim c_2} \neq P_{c_1\sim c_3}$. We must thus have that $P_{c_1\sim c_3} = P_{c_3\sim c_k} = P_{c_1\sim c_k}$.

By a similar argument, we must have in set C that either $P_{c_2\sim c_3} = P_{c_3\sim c_m} = P_{c_2\sim c_m}$ or $P_{c_2\sim c_3} = P_{c_3\sim c_k} = P_{c_2\sim c_k}$. If $P_{c_2\sim c_3} = P_{c_3\sim c_m} = P_{c_2\sim c_m}$ then, together with the insight above that $P_{c_1\sim c_2} = P_{c_2\sim c_m}$, it would follow that $P_{c_1\sim c_2} = P_{c_2\sim c_3}$. This would contradict the assumption that $P_{c_1\sim c_2} \neq P_{c_2\sim c_3}$. If $P_{c_2\sim c_3} = P_{c_3\sim c_k} = P_{c_2\sim c_k}$ then, together with the fact above that $P_{c_1\sim c_3} = P_{c_3\sim c_k}$, it would follow that $P_{c_1\sim c_3} = P_{c_2\sim c_3}$. This would contradict the assumption that $P_{c_1\sim c_3} \neq P_{c_2\sim c_3}$. We thus arrive at a general contradiction, and hence there are $i, j \in \{1, 2, 3\}$ such that for every triple $a, b, c \in \{c_i, c_j, c_m, c_k\}$ the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different. This completes the proof of the claim.

According to the claim, we can choose $i, j \in \{1, 2, 3\}$ such that for every triple $a, b, c \in \{c_i, c_j, c_m, c_k\}$ the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different. Define the set of choices $D := \{c_i, c_j, c_m, c_k\}$, and let

$$A := span_{ki} \cap span_{im}.$$

We show that A has dimension $n - 2$, that $A \subseteq span(P_{a\sim b})$ for all $a, b \in D$, and that $A = span(P_{a\sim b}) \cap span(P_{c\sim d})$ whenever $P_{a\sim b} \neq P_{c\sim d}$.

Since by the choice of i, j we have that $P_{c_k\sim c_i} \neq P_{c_i\sim c_m}$, it follows that A has dimension $n - 2$. Note by Proposition 2.1 that $A \subseteq span_{km}$. Moreover, as we assume in Case 2 that $span_{ki} \cap span_{im} = span_{kj} \cap span_{jm}$, it follows that $A = span_{ki} \cap span_{im} \cap span_{kj}$, and thus we have by Proposition 2.1 that $A \subseteq span_{ij}$ also. Hence, $A \subseteq span(P_{a\sim b})$ for all $a, b \in D$.

Now, let $P_{a\sim b} \neq P_{c\sim d}$ for some $a, b, c, d \in D$. Then $span(P_{a\sim b}) \cap span(P_{c\sim d})$ has dimension $n-2$. As $A \subseteq span(P_{a\sim b}) \cap span(P_{c\sim d})$ and A has dimension $n-2$ as well, it must be that $A = span(P_{a\sim b}) \cap span(P_{c\sim d})$.

Let $\Delta^+(S) := \{p \in \Delta(S) \mid p(s) > 0 \text{ for all } s \in S\}$ be the set of full support beliefs. We distinguish two cases: (2.1) $A \cap \Delta^+(S)$ is empty, and (2.2) $A \cap \Delta^+(S)$ is non-empty.

Case 2.1. Suppose that $A \cap \Delta^+(S)$ is empty. Recall from Lemma 5.2 (b) that each of the indifference sets $P_{a\sim b}$, where $a, b \in D$, has a full support belief in $\Delta^+(S)$, and thus $P_{a\sim b} \cap \Delta^+(S)$ is non-empty. Moreover, recall from above that $P_{a\sim b} \cap P_{c\sim d} = A$ whenever $P_{a\sim b} \neq P_{c\sim d}$. As $A \cap \Delta^+(S)$ is empty, it follows that $P_{a\sim b} \cap P_{c\sim d} \cap \Delta^+(S)$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$ and $a, b, c, d \in D$.

Let $\{P_1, ..., P_R\}$ be the collection of pairwise different indifference sets that remains if from $\{P_{a\sim b} \mid a, b \in D\}$ we remove all duplicate sets. Since i, j have been chosen such that for every triple a, b, c in D the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different, we know that $R \geq 3$.

As $P_{a\sim b} \cap P_{c\sim d} \cap \Delta^+(S)$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$, it follows that the sets $P_1 \cap \Delta^+(S), ..., P_R \cap \Delta^+(S)$ are pairwise disjoint. Moreover, we have seen that each of the latter sets are non-empty. Since $span(P_1), ..., span(P_R)$ are hyperplanes of dimension $n-1$, we can order the sets $P_1, ..., P_R$ such that $P_2 \cap \Delta^+(S), ..., P_{R-1} \cap \Delta^+(S)$ are in between $P_1 \cap \Delta^+(S)$ and $P_R \cap \Delta^+(S)$. Take some $p_1 \in P_1 \cap \Delta^+(S)$ and $p_R \in P_R \cap \Delta^+(S)$, and let l be the line through p_1 and p_R. Then, the corresponding line segment from p_1 to p_R is included in $\Delta^+(S)$. As $P_2 \cap \Delta^+(S), ..., P_{R-1} \cap \Delta^+(S)$ are in between $P_1 \cap \Delta^+(S)$ and $P_R \cap \Delta^+(S)$, the line l contains for every $r \in \{2, ..., R-1\}$ a unique belief p_r in P_r. In particular, for every pair of choices a, b in D, there is a unique belief $p_{ab} \in P_{a\sim b}$ on the line l, and the line l contains a belief where the DM is not indifferent between any of the choices in D.

Recall that for every triple a, b, c in D the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different. As $P_{a\sim b} \cap P_{c\sim d} \cap \Delta^+(S)$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$, we must have for every triple a, b, c in D that p_{ab}, p_{ac} and p_{bc} are pairwise different.

Let s be a state such that the probability of s is not constant on the line l. By four choice linear preference intensity, we have that

$$\frac{p_{ac}(s) - p_{cd}(s)}{p_{ad}(s) - p_{cd}(s)} = \frac{(p_{ab}(s) - p_{bd}(s))(p_{ac}(s) - p_{bc}(s))}{(p_{ab}(s) - p_{bc}(s))(p_{ad}(s) - p_{bd}(s))}, \tag{5.11}$$

where $a := c_i$, $b := c_j$, $c := c_m$ and $d := c_k$. Note that both fractions are well-defined since $p_{ad} \neq p_{cd}$, $p_{ab} \neq p_{bc}$ and $p_{ad} \neq p_{bd}$. Moreover, as p_{ac}, p_{ad}, p_{cd} are pairwise different, we have that $p_{ac}(s) - p_{cd}(s) \neq p_{ad}(s) - p_{cd}(s)$, and hence the fraction on the lefthand side is not equal to 1. As such, the fraction on the righthand side is not equal to 1 either. Let this fraction on the righthand side be called F. Then, by (5.11), p_{cd} is the unique belief on l where

$$p_{cd}(s) = \frac{F \cdot p_{ad}(s) - p_{ac}(s)}{F - 1}. \tag{5.12}$$

Remember that $A \subseteq span(P_{c\sim d})$, that A has dimension $n-2$, and that $span(P_{c\sim d})$ has dimension $n-1$. Let $\{q_2, ..., q_{n-1}\}$ be a basis for A. As $p_{cd} \in P_{c\sim d}$ is not in A, we conclude that $\{p_{cd}, q_2, ..., q_{n-1}\}$ is a basis for $span(P_{c\sim d})$.

Now, let $\succsim^u$ be the conditional preference relation generated by the utility function u. We have already seen that u represents $\succsim$ on all pairs of choices in $\{a, b, c, d\}$, except $\{c, d\}$. In particular, we thus know that

$$u(a, p_{ab}) = u(b, p_{ab}),\ u(a, p_{ac}) = u(c, p_{ac}),\ u(a, p_{ad}) = u(d, p_{ad}),\ u(b, p_{bc}) = u(c, p_{bc}) \text{ and } u(b, p_{bd}) = u(d, p_{bd}).$$

As we have seen in part (a) of the proof that $\succsim^u$ satisfies four choice linear preference intensity, the unique belief on the line l where the DM is indifferent between c and d under $\succsim^u$ is given by (5.12). Therefore,

$$u(c, p_{cd}) = u(d, p_{cd}). \tag{5.13}$$

Recall that $A = span(P_{d\sim a}) \cap span(P_{a\sim c})$. As u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, it follows that $u(d, v) = u(a, v)$ and $u(a, v) = u(c, v)$ for every $v \in span(P_{d\sim a}) \cap span(P_{a\sim c})$. Therefore, $u(c, v) = u(d, v)$ for every $v \in A$. In particular,

$$u(c, q_k) = u(d, q_k) \text{ for every } k \in \{2, ..., n-1\}, \tag{5.14}$$

where $\{q_2, ..., q_{n-1}\}$ is a basis for A. Moreover, we have seen that $\{p_{cd}, q_2, ..., q_{n-1}\}$ is a basis for $span(P_{c\sim d})$.

Recall that $A = span(P_{c_k\sim c_1}) \cap span(P_{c_1\sim c_m})$ has dimension $n-2$, and thus $P_{c_k\sim c_1} \neq P_{c_1\sim c_m}$. Thus, $P_{d\sim a} \neq P_{a\sim c}$. We can thus choose some $p \in P_{d\sim a} \backslash P_{a\sim c}$. Assume, without loss of generality, that $p \in P_{a\succ c}$. By transitivity, we then have that $p \in P_{d\succ c}$. Since u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, we know that $u(d, p) = u(a, p)$ and $u(a, p) > u(c, p)$, and hence

$$u(d, p) > u(c, p) \text{ for some } p \in P_{d\succ c}. \tag{5.15}$$

In view of (5.13), (5.14) and (5.15), it follows by Lemma 5.3 that u represents $\succsim$ on $\{c, d\} = \{c_k, c_m\}$.

Case 2.2. Suppose that $A \cap \Delta^+(S)$ is non-empty. Then, there is some full support belief p^* in A, with $p^*(s) > 0$ for all states s. As we have seen that $A \subseteq span(P_{a\sim b})$ for all $a, b \in D$, it follows that $p^* \in P_{a\sim b}$ for all pairs $a, b \in D$.

Since we have seen that A has dimension $n-2$, the linear subspace A is contained in some hyperplane containing the zero vector. Hence, there is some vector $n^A \in \mathbf{R}^S$ such that

$$n^A \cdot v = 0 \text{ for all } v \in A. \tag{5.16}$$

Moreover, we can choose the vector n^A such that for every pair $a, b \in D$ there is some $p \in P_{a\sim b}$ with $n^A \cdot p \neq 0$.

In that case, there is for every pair $a, b \in D$ some $p \in P_{a\sim b}$ with $n^A \cdot p > 0$. To see this, suppose that a, b are such that $n^A \cdot p \leq 0$ for every $p \in P_{a\sim b}$. As there is some $p \in P_{a\sim b}$ with $n^A \cdot p \neq 0$, there must be some $p \in P_{a\sim b}$ with $n^A \cdot p < 0$. Since $p^* \in \Delta^+(S)$, there is some $\lambda > 1$ close enough to 1 such that $q := (1-\lambda)p + \lambda p^* \in \Delta(S)$. Note that $p^* \in A \subseteq span(P_{a\sim b})$ and $p \in P_{a\sim b}$, which implies that $q \in span(P_{a\sim b}) \cap \Delta(S) = P_{a\sim b}$. At the same time we know, by (5.16) and the fact that $p^* \in A$, that $n^A \cdot p^* = 0$. Since $n^A \cdot p < 0$ and $\lambda > 1$, it follows that $n^A \cdot q = (1-\lambda) \cdot (n^A \cdot p) + \lambda \cdot (n^A \cdot p^*) > 0$. Thus,

$$\text{for every } a, b \in D \text{ there is some } p \in P_{a\sim b} \text{ with } n^A \cdot p > 0. \tag{5.17}$$

Let $P^+ := \{p \in \Delta(S) \mid n^A \cdot p > 0\}$. Then, in view of (5.17),

$$P_{a\sim b} \cap P^+ \text{ is non-empty for all } a, b \in D. \tag{5.18}$$

Recall that $P_{a\sim b} \cap P_{c\sim d} = A$ for every $a, b, c, d \in D$ with $P_{a\sim b} \neq P_{c\sim d}$. In view of (5.16) and (5.18) we conclude that $P_{a\sim b} \cap P_{c\sim d} \cap P^+$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$. Hence, $(P_{a\sim b} \cap P^+)$ and $(P_{c\sim d} \cap P^+)$ are disjoint whenever $P_{a\sim b} \neq P_{c\sim d}$. But then, the different sets in $\{P_{a\sim b} \mid a, b \in D\}$ can be numbered $P_1, ..., P_R$, with $R \geq 3$, such that $P_2 \cap P^+, ..., P_{R-1} \cap P^+$ are in between $P_1 \cap P^+$ and $P_R \cap P^+$. In a similar way as in Case 2.1, it can then be shown that u represents $\succsim$ on $\{c_k, c_m\}$.

We thus conclude that u represents $\succsim$ on $\{c_1, ..., c_k\}$. By induction on k, the proof is complete. ■

We next prove Proposition 2.2.

Proof of Proposition 2.2. Let u, v be two different utility representations for $\succsim$. To prove the statement, we distinguish three cases: (1) there are two choices, (2) there are three choices, and (3) there are at least four choices.

Case 1. Suppose there are two choices, a and b. Since there are preference reversals on $\{a, b\}$, there is some $p^* \in P_{a\succ b}$. Define

$$\alpha := \frac{v(a, p^*) - v(b, p^*)}{u(a, p^*) - u(b, p^*)}. \tag{5.19}$$

We show that

$$v(a, p) - v(b, p) = \alpha \cdot (u(a, p) - u(b, p)) \text{ for all beliefs } p \in \Delta(S). \tag{5.20}$$

As there are preference reversals on $\{a, b\}$, it follows by Lemma 5.2 (b) that there are $n - 1$ linearly independent beliefs $p_1, ..., p_{n-1}$ in $P_{a\sim b}$. Moreover, by Lemma 5.2 (a) we know that $p^* \notin span(P_{a\sim b})$. Hence, $\{p_1, ..., p_{n-1}, p^*\}$ are linearly independent, and thus form a basis for $\mathbf{R}^S$. As, by construction, $v(a, p_k) - v(b, p_k) = \alpha \cdot (u(a, p_k) - u(b, p_k)) = 0$ for all $k \in \{1, ..., n-1\}$ and, by (5.19), $v(a, p^*) - v(b, p^*) = \alpha \cdot (u(a, p^*) - u(b, p^*))$, it follows that (5.20) holds for every p in the basis $\{p_1, ..., p_{n-1}, p^*\}$. Now, take some arbitrary belief $p \in \Delta(S)$. Then, $p = \lambda_1 p_1 + ... + \lambda_{n-1} p_{n-1} + \lambda_n p^*$ for some numbers $\lambda_1, ..., \lambda_n$. Thus,

$$\begin{aligned}
v(a, p) - v(b, p) &= \sum_{k=1}^{n-1} \lambda_k \cdot (v(a, p_k) - v(b, p_k)) + \lambda_n \cdot (v(a, p^*) - v(b, p^*)) \\
&= \alpha \cdot \left(\sum_{k=1}^{n-1} \lambda_k \cdot (u(a, p_k) - u(b, p_k)) + \lambda_n \cdot (u(a, p^*) - u(b, p^*)) \right) \\
&= \alpha \cdot (u(a, p) - u(b, p)),
\end{aligned}$$

which establishes (5.20).

Case 2. Suppose there are three choices, a, b and c. Since, by assumption, there is a belief where the DM is indifferent between some, but not all, choices, it must be that $P_{c\sim a} \neq P_{c\sim b}$. Let the number α be given by (5.19). We show, for every two choices $d, e \in \{a, b, c\}$, that

$$v(d,p) - v(e,p) = \alpha \cdot (u(d,p) - u(e,p)) \text{ for all beliefs } p \in \Delta(S). \tag{5.21}$$

By the proof of Case 1, we know that (5.21) holds for the choices a and b. We now show that (5.21) holds for the choices c and a. Let $p_1, ..., p_{n-1} \in \Delta(S)$ be a basis for $span(P_{c\sim a})$. Then,

$$v(c,p_k) - v(a,p_k) = \alpha \cdot (u(c,p_k) - u(a,p_k)) = 0 \text{ for all } k \in \{1, ..., n-1\}. \tag{5.22}$$

Since $P_{c\sim a} \neq P_{c\sim b}$, there is a belief $p_n \in P_{c\sim b} \backslash P_{c\sim a}$. By Lemma 5.2 (a) we must then have that $p_n \notin span(P_{c\sim a})$, and hence $\{p_1, ..., p_{n-1}, p_n\}$ is a basis for $\mathbf{R}^S$. As $p_n \in P_{c\sim b}$, it must be that

$$v(c,p_n) - v(b,p_n) = \alpha \cdot (u(c,p_n) - u(b,p_n)) = 0. \tag{5.23}$$

Moreover, we know from Case 1 that

$$v(b,p_n) - v(a,p_n) = \alpha \cdot (u(b,p_n) - u(a,p_n)). \tag{5.24}$$

If we combine (5.23) and (5.24), we get

$$\begin{aligned} v(c,p_n) - v(a,p_n) &= (v(c,p_n) - v(b,p_n)) + (v(b,p_n) - v(a,p_n)) \\ &= \alpha \cdot (u(c,p_n) - u(b,p_n)) + \alpha \cdot (u(b,p_n) - u(a,p_n)) \\ &= \alpha \cdot (u(c,p_n) - u(a,p_n)). \end{aligned} \tag{5.25}$$

From (5.22) and (5.25) we conclude, in a similar way as in the proof of Case 1, that

$$v(c,p) - v(a,p) = \alpha \cdot (u(c,p) - u(a,p)) \text{ for all beliefs } p.$$

In a similar fashion we can show (5.21) for the choices c and b.

Case 3. Suppose there are at least four choices. By assumption, there is a belief where the DM is indifferent between some, but not all, choices. That is, there are choices a, b, c, d such that $P_{a\sim b} \neq P_{c\sim d}$. Following the proof of Theorem 2.1, it can then be shown that there are three choices a, b and c with $P_{c\sim a} \neq P_{c\sim b}$. Let the number α be given by (5.19). Then, we know by Case 2 that (5.21) holds for every $d, e \in \{a, b, c\}$.

We now show (5.21) for choices d and a, where d is some arbitrary choice not in $\{a, b, c\}$. From the proof of Theorem 2.1 we know that either $P_{d\sim a} \neq P_{d\sim b}$ or $P_{d\sim a} \neq P_{d\sim c}$. Assume, without loss of generality, that $P_{d\sim a} \neq P_{d\sim b}$. Then it can be shown in a similar way as for Case 2 that (5.21) holds for the choices d and a.

Now, take some choice $d \notin \{a, b, c\}$, and some arbitrary choice $e \notin \{a, d\}$. Since we know that (5.21) holds for the choices d and a, and for the choices e and a, it follows that

$$v(d,p) - v(a,p) = \alpha \cdot (u(d,p) - u(a,p)) \text{ for all beliefs } p$$

and

$$v(a,p) - v(e,p) = \alpha \cdot (u(a,p) - u(e,p)) \text{ for all beliefs } p.$$

This implies that

$$\begin{aligned}
v(d,p)-v(e,p) &= (v(d,p)-v(a,p))+(v(a,p)-v(e,p)) \\
&= \alpha\cdot(u(d,p)-u(a,p))+\alpha\cdot(u(a,p)-u(e,p)) \\
&= \alpha\cdot(u(d,p)-u(e,p)) \text{ for all beliefs } p.
\end{aligned}$$

Hence, (5.21) holds for every two choices d, e. This completes the proof. ■

5.3 Proof of Section 3

Before we can prove Theorem 3.1 we need a preparatory result. It describes, for a given signed conditional preference relation meeting the axioms, the structure of the set of signed beliefs for which the DM is "indifferent" between two choices. To formally state the preparatory result, we must introduce some new notions and notation. For a signed conditional preference relation $\succsim^*$ and two choices a and b, we denote by $Q_{a\sim^* b}$ the set of signed beliefs q for which $a \sim_q^* b$. By $\Delta^*(S) := \{q \in \mathbf{R}^S \mid \sum_{s\in S} q(s) = 1\}$ we denote the set of all signed beliefs. Two subsets $Q, Q' \subseteq \Delta^*(S)$ are called *parallel* if there is some vector $v \in \mathbf{R}^S$ such that $Q' = \{q + v \mid q \in Q\}$.

Lemma 5.7 (Signed indifference sets) *Let $\succsim^*$ be a signed conditional preference relation without equivalent choices which satisfies continuity, preservation of indifference and preservation of strict preference.*

(a) Consider two choices a, b such that there is no constant preference intensity between a and b. Then, $span(Q_{a\sim^* b})$ has dimension $|S|-1$, and $Q_{a\sim^* b} = \Delta^*(S) \cap span(Q_{a\sim^* b})$;

(b) Consider three choices a, b, c such that there is constant preference intensity between a and b, but not between a and c, and not between b and c. Then, the sets $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ are parallel.

Proof. **(a)** As there is no constant preference intensity between a and b, there must be signed beliefs q_1 and q_2 such that $a \succ_{q_1}^* b$ and $b \succ_{q_2}^* a$. But then, it can be shown in a similar way as in the proof of Lemma 5.2 (a) that $Q_{a\sim^* b} = \Delta^*(S) \cap span(Q_{a\sim^* b})$ and that $span(Q_{a\sim^* b})$ has dimension $|S|-1$. We therefore omit this proof here.

(b) Suppose that there is constant preference intensity between a and b, but not between a and c, and not between b and c. Then, we know from (a) that $Q_{a\sim^* c} = span(Q_{a\sim^* c}) \cap \Delta^*(S)$ and $Q_{b\sim^* c} = span(Q_{b\sim^* c}) \cap \Delta^*(S)$ where $span(Q_{a\sim^* c})$ and $span(Q_{b\sim^* c})$ both have dimension $|S|-1$. Suppose, contrary to what we want to show, that $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ are not parallel. Then, it must be that $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ intersect, and hence there is some signed belief q which is both in $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$. By transitivity, it would then follow that $q \in Q_{a\sim^* b}$. This, however, is a contradiction, since there is constant preference intensity between a and b, and we exclude equivalent choices. We thus conclude that $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ are parallel. This completes the proof. ■

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. **(a)** Suppose first that $\succsim$ has an expected utility representation u. Let $\succsim^*$ be the signed conditional preference relation where for every signed belief q, and every two choices a and b, we have that $a \succsim_q^* b$ if and only if $u(a,q) \geq u(b,q)$. Then, $\succsim^*$ extends $\succsim$. Similarly to the proofs of Lemma 5.4 and Theorem 2.1, it can then be shown that $\succsim^*$ satisfies the signed beliefs versions of the regularity axioms, three choice preference intensity and four choice preference intensity. Moreover, transitive constant preference intensity and four choice linear preference intensity with constant preference intensity follow rather easily. This proof is therefore left to the reader. Thus, $\succsim$ can be extended to a signed conditional preference relation that satisfies all of the axioms above.

(b) Suppose now that $\succsim$ can be extended to a signed conditional preference relation $\succsim^*$ that satisfies all of the axioms above. We will show that there is a utility function u that represents $\succsim^*$, and thereby represents $\succsim$ as well. We distinguish two cases: (1) for every two choices a, b there is no constant preference intensity between a and b, and (2) there are at least two choices a and b with a constant preference intensity between them.

Case 1. Suppose that, for every two choices a and b, there is no constant preference intensity between a and b. Then, for every two choices a and b there must be signed beliefs q_1 and q_2 such that $a \succ_{q_1}^* b$ and $b \succ_{q_2}^* a$.

Choose some full support belief p^* with $p^*(s) > 0$ for all states s. For every number λ, consider the conditional preference relation $\succsim^\lambda$ where for every two choices a and b, and every belief p,

$$a \succsim_p^\lambda b \text{ if and only if } a \succsim_{(1-\lambda)p^*+\lambda p}^* b.$$

By choosing λ large enough, we can guarantee that $\succsim^\lambda$ has preference reversals between a and b. But then, we can choose λ large enough such that $\succsim^\lambda$ has preference reversals for all pairs of choices.

The reader may verify that $\succsim^\lambda$ satisfies the regularity axioms, three choice linear preference intensity and four choice linear preference intensity. By Theorem 2.1 we then conclude that $\succsim^\lambda$ has an expected utility representation u^λ. Define the utility function u by

$$u(c,s) := (1-1/\lambda)\cdot u^\lambda(c,p^*) + (1/\lambda)\cdot u^\lambda(c,s)$$

for every choice c and state s. We will show that u represents $\succsim$.

Take some arbitrary belief p. Then, $p = (1-\lambda)p^* + \lambda p'$ for the belief $p' := (1-1/\lambda)p^* + (1/\lambda)p$. We conclude, for two arbitrary choices a and b, that

$$\begin{aligned} a \succsim_p b \iff a \succsim_p^* b \iff a \succsim_{(1-\lambda)p^*+\lambda p'}^* b \iff a \succsim_{p'}^\lambda b \iff u^\lambda(a,p') \geq u^\lambda(b,p') \\ \iff u^\lambda(a,(1-1/\lambda)p^* + (1/\lambda)p) \geq u^\lambda(b,(1-1/\lambda)p^* + (1/\lambda)p) \\ \iff (1-1/\lambda)u^\lambda(a,p^*) + (1/\lambda)u^\lambda(a,p) \geq (1-1/\lambda)u^\lambda(b,p^*) + (1/\lambda)u^\lambda(b,p) \iff u(a,p) \geq u(b,p). \end{aligned}$$

Thus, we see that the utility function u represents $\succsim$, which completes the proof of Case 1.

Case 2. Suppose now that there are at least two choices a and b such that $\succsim^*$ exhibits a constant preference intensity between a and b. We start by constructing a set of choices D, as follows. Take an arbitrary choice

$d_1 \in C$. If there is a choice $d_2 \neq d_1$ such that there is no constant preference intensity between d_2 and d_1, then select such a choice d_2. In the next step, if there is a choice $d_3 \neq d_1, d_2$ such that there is no constant preference intensity between d_3 and d_1 and between d_3 and d_2 then select such a choice d_3. Continue in this way until no further choice can be selected in this way. Let $D = \{d_1, ..., d_K\}$ be the resulting set. Then, by construction, there is no constant preference intensity between any two choices in D, and for every choice $c \notin D$ there is a choice $d \in D$ such that there is constant preference intensity between c and d. But we can show even more, as the following claim shows.

Claim. For every choice $c \notin D$ there is exactly one choice $d(c) \in D$ such that there is constant preference intensity between c and $d(c)$.
Proof of claim. Suppose there are two choices $d_1, d_2 \in D$ such that there is a constant preference intensity between c and d_1 and between c and d_2. By transitivity of constant preference intensity, it would then follow that there is a constant preference intensity between d_1 and d_2, which is a contradiction. This completes the proof of the claim.

We distinguish two cases: (2.1) the set D only contains one choice, and (2.2) the set D contains more than one choice.

Case 2.1. Suppose that D only contains one choice, say d. Then, for every choice $c \neq d$, there is constant preference intensity between c and d. By transitivity of constant preference intensity, it would follow that for every two choices $a, b \in C$ we have constant preference intensity between a and b. Consider an arbitrary signed belief q, with the induced ranking $c_1 \succ_q^* c_2 \succ_q^* ... \succ_q^* c_M$. Since there is constant preference intensity between any two choices, this same ranking is induced at *every* signed belief. Take some numbers $\alpha_1 > \alpha_2 > ... > \alpha_M$. Then, the utility function u with $u(c_m, s) := \alpha_m$ for every choice c_m and every state s represents $\succsim^*$, and thereby $\succsim$.

Case 2.2. Suppose that D contains at least two choices. By the claim, there are for every choice $a \notin D$ two choices $d(a), e(a) \in D$ such that there is constant preference intensity between a and $d(a)$, but not between a and $e(a)$. We define the utility function u as follows.

Since there is no constant preference intensity between any two choices in D, we know from Case 1 that there is a utility function v that represents $\succsim^*$ on D. We set $u(d, s) := v(d, s)$ for every choice $d \in D$ and state $s \in S$.

Now take some choice $a \notin D$. As there is no constant preference intensity between a and $e(a)$, there is a signed belief $q_{ae(a)}$ where the DM is "indifferent" between a and $e(a)$. Recall that there is constant preference intensity between a and $d(a) \in D$. We define, for every state s,

$$u(a, s) := u(d(a), s) + u(e(a), q_{ae(a)}) - u(d(a), q_{ae(a)}). \tag{5.26}$$

We show that this utility function u represents $\succsim^*$, by proving that u represents $\succsim^*$ on $\{a, b\}$ for every two choices $a, b \in C$. We distinguish the following cases: (2.2.1) $a, b \in D$, (2.2.2) $a \notin D$ and $b = d(a)$, (2.2.3) $a \notin D$ and $b = e(a)$, (2.2.4) $a \notin D$ and $b \in D \backslash \{d(a), e(a)\}$, and (2.2.5) $a, b \notin D$.

Case 2.2.1. Suppose that $a, b \in D$. Then, u represents $\succsim^*$ on $\{a, b\}$ since v represents $\succsim^*$ on D.

Case 2.2.2. Suppose that $a \notin D$ and $b = d(a)$. Since there is constant preference intensity between a and $d(a)$, it must be that either $a \succ_q^* d(a)$ for all signed beliefs q, or $d(a) \succ_q^* a$ for all signed beliefs q. Assume, without loss of generality, that $a \succ_q^* d(a)$ for all signed beliefs q. Since $e(a) \sim_{q_{ae(a)}}^* a$, it follows that $e(a) \succ_{q_{ae(a)}}^* d(a)$. As u represents $\succsim^*$ on D, we have that $u(e(a), q_{ae(a)}) > u(d(a), q_{ae(a)})$. By (5.26) we conclude that $u(a, q) > u(d(a), q)$ for all signed beliefs q, and hence u represents $\succsim^*$ on $\{a, d(a)\}$.

Case 2.2.3. Assume that $a \notin D$ and $b = e(a)$. Recall from above that $e(a) \sim_{q_{ae(a)}}^* a$. Moreover, by (5.26), we know that $u(a, q_{ae(a)}) = u(e(a), q_{ae(a)})$, and thus $q_{ae(a)} \in Q_{u(a)=u(e(a))}$. Here, we denote by $Q_{u(a)=u(e(a))}$ the set of signed beliefs q where $u(a, q) = u(e(a), q)$. As there is constant preference intensity between a and $d(a)$, but not between a and $e(a)$ and not between $d(a)$ and $e(a)$, we know from Lemma 5.7 (b) that the sets $Q_{a\sim^* e(a)}$ and $Q_{d(a)\sim^* e(a)}$ are parallel. Since, by (5.26), the expected utility difference between a and $d(a)$ is constant across all signed beliefs, we know that also the sets $Q_{u(a)=u(e(a))}$ and $Q_{u(d(a))=u(e(a))}$ are parallel. As u represents $\succsim^*$ on D, we must have that $Q_{d(a)\sim^* e(a)} = Q_{u(d(a))=u(e(a))}$.

Summarizing, we thus see that (i) $Q_{u(a)=u(e(a))}$ and $Q_{u(d(a))=u(e(a))}$ are parallel, (ii) $Q_{u(d(a))=u(e(a))} = Q_{d(a)\sim^* e(a)}$, and (iii) $Q_{d(a)\sim^* e(a)}$ and $Q_{a\sim^* e(a)}$ are parallel. Thus, $Q_{u(a)=u(e(a))}$ and $Q_{a\sim^* e(a)}$ are parallel. Since $q_{ae(a)}$ is in both $Q_{a\sim^* e(a)}$ and $Q_{u(a)=u(e(a))}$, it follows that $Q_{u(a)=u(e(a))} = Q_{a\sim^* e(a)}$.

Since there is no constant preference intensity between $d(a)$ and $e(a)$, there must be some $q_{d(a)e(a)}$ with $d(a) \sim_{q_{d(a)e(a)}}^* e(a)$. Recall from above that $a \succ_q^* d(a)$ for all signed beliefs q, and thus $a \succ_{q_{d(a)e(a)}}^* d(a)$. Hence, $a \succ_{q_{d(a)e(a)}}^* e(a)$. As u represents $\succsim^*$ on $\{a, d(a)\}$ and $\{d(a), e(a)\}$, we have that $u(a, q_{d(a)e(a)}) > u(d(a), q_{d(a)e(a)})$ and $u(d(a), q_{d(a)e(a)}) = u(e(a), q_{d(a)e(a)})$. This implies $u(a, q_{d(a)e(a)}) > u(e(a), q_{d(a)e(a)})$. We have thus found a belief $q_{d(a)e(a)}$ with $a \succ_{q_{d(a)e(a)}}^* e(a)$ and $u(a, q_{d(a)e(a)}) > u(e(a), q_{d(a)e(a)})$.

As $Q_{u(a)=u(e(a))} = Q_{a\sim^* e(a)}$ it can be shown, in a similar way as in the proof of Lemma 5.4, that u represents $\succsim^*$ on $\{a, e(a)\}$.

Case 2.2.4. Assume that $a \notin D$ and $b \in D\backslash\{d(a), e(a)\}$. We distinguish three cases: (2.2.4.1) $Q_{a\sim^* e(a)}$ is not parallel to $Q_{b\sim^* e(a)}$, (2.2.4.2) $Q_{a\sim^* e(a)}$ is parallel to $Q_{b\sim^* e(a)}$ but $Q_{a\sim^* e(a)} \neq Q_{b\sim^* e(a)}$, and (2.2.4.3) $Q_{a\sim^* e(a)} = Q_{b\sim^* e(a)}$.

Case 2.2.4.1. Suppose that $Q_{a\sim e(a)}$ is not parallel to $Q_{b\sim e(a)}$. Then, there is some signed belief $q_{ab} \in Q_{a\sim e(a)} \cap Q_{b\sim e(a)}$. As $\succsim^*$ is transitive, it follows that $q_{ab} \in Q_{a\sim b}$. Since u represents $\succsim^*$ on $\{a, e(a)\}$ and $\{b, e(a)\}$, we know that $u(a, q_{ab}) = u(e(a), q_{ab}) = u(b, q_{ab})$. We have thus found a signed belief $q_{ab} \in Q_{a\sim b}$ with $q_{ab} \in Q_{u(a)=u(b)}$.

As there is constant preference intensity between a and $d(a)$, but not between a and b and not between b and $d(a)$, we know by Lemma 5.7 (b) that $Q_{a\sim b}$ is parallel to $Q_{b\sim d(a)}$. Moreover, as u represents $\succsim^*$ on $\{b, d(a)\}$, we know that $Q_{b\sim d(a)} = Q_{u(b)=u(d(a))}$. Since, by (5.26), the expected utility between a and $d(a)$ is constant across all signed beliefs, we have that $Q_{u(a)=u(b)}$ is parallel to $Q_{u(b)=u(d(a))}$. Summarizing, we see that (i) $Q_{u(a)=u(b)}$ is parallel to $Q_{u(b)=u(d(a))}$, (ii) $Q_{u(b)=u(d(a))} = Q_{b\sim d(a)}$, and (iii) $Q_{b\sim d(a)}$ is parallel to $Q_{a\sim b}$. Thus, $Q_{u(a)=u(b)}$ is parallel to $Q_{a\sim b}$. Since q_{ab} is both in $Q_{a\sim b}$ and $Q_{u(a)=u(b)}$, we conclude that $Q_{u(a)=u(b)} = Q_{a\sim b}$.

Take some signed belief $q_{d(a)b}$ in $Q_{d(a)\sim^* b}$. Since we assume that $a \succ_q^* d(a)$ for all signed beliefs q, we have that $a \succ_{q_{d(a)b}}^* d(a) \sim_{q_{d(a)b}}^* b$, and thus $a \succ_{q_{d(a)b}}^* b$. As u represents $\succsim^*$ on $\{a, d(a)\}$ and $\{d(a), b\}$, we have that $u(a, q_{d(a)b}) > u(d(a), q_{d(a)b}) = u(b, q_{d(a)b})$. Hence, we have a found a belief $q_{d(a)b}$ with $a \succ^*_{q_{d(a)b}} b$ and $u(a, q_{d(a)b}) > u(b, q_{d(a)b})$. Since $Q_{u(a)=u(b)} = Q_{a\sim b}$, we can use a similar argument as in the proof of Lemma 5.4 to show that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.4.2. Suppose that $Q_{a\sim^* e(a)}$ is parallel to $Q_{b\sim^* e(a)}$ but $Q_{a\sim^* e(a)} \neq Q_{b\sim^* e(a)}$. We show that the sets $Q_{a\sim^* e(a)}, Q_{b\sim^* e(a)}, Q_{a\sim^* b}, Q_{d(a)\sim^* e(a)}$ and $Q_{d(a)\sim^* b}$ must all be parallel. As there is constant preference intensity between a and $d(a)$, but not between a and $e(a)$ and not between $e(a)$ and $d(a)$, it follows by Lemma 5.7 (b) that $Q_{a\sim^* e(a)}$ and $Q_{d(a)\sim^* e(a)}$ are parallel. Similarly, since there is constant preference intensity between a and $d(a)$, but not between a and b and not between b and $d(a)$, it follows by Lemma 5.7 (b) that $Q_{a\sim^* b}$ and $Q_{d(a)\sim^* b}$ are parallel. Moreover, by assumption, $Q_{a\sim^* e(a)}$ is parallel to $Q_{b\sim^* e(a)}$. Now suppose, contrary to what we want to show, that $Q_{a\sim^* b}$ is not parallel to $Q_{a\sim^* e(a)}$. Then, there is some $q \in Q_{a\sim^* b} \cap Q_{a\sim^* e(a)}$ and hence, by transitivity of $\succsim^*$, we have that $q \in Q_{b\sim^* e(a)}$. But then, q is in both $Q_{a\sim^* e(a)}$ and $Q_{b\sim^* e(a)}$, which is impossible since both sets are parallel but not equal. Hence, we must conclude that $Q_{a\sim^* b}$ is parallel to $Q_{a\sim^* e(a)}$. But then, all five sets $Q_{a\sim^* e(a)}, Q_{b\sim^* e(a)}, Q_{a\sim^* b}, Q_{d(a)\sim^* e(a)}$ and $Q_{d(a)\sim^* b}$ are parallel.

Take a line l of signed beliefs that crosses each of these five sets once, and let $q_{ae(a)}, q_{be(a)}, q_{ab}, q_{d(a)e(a)}$ and $q_{d(a)b}$ be the signed beliefs on the line where the DM is "indifferent" between the respective choices. As u represents $\succsim^*$ on $\{a, e(a)\}, \{b, e(a)\}, \{d(a), e(a)\}$ and $\{d(a), b\}$, we conclude that

$$u(a, q_{ae(a)}) = u(e(a), q_{ae(a)}),\ u(b, q_{be(a)}) = u(e(a), q_{be(a)}),$$
$$u(d(a), q_{d(a)e(a)}) = u(e(a), q_{d(a)e(a)}) \text{ and } u(d(a), q_{d(a)b}) = u(b, q_{d(a)b}).$$

Recall that there is constant preference intensity between a and $d(a)$. Since $\succsim^*$ satisfies part (a) of four choice linear preference intensity with constant preference intensity, we know that q_{ab} is uniquely given by the other four signed indifference beliefs. Moreover, as the signed conditional preference relation $\succsim^{*u}$ induced by u also satisfies part (a) of four choice linear preference intensity with constant preference intensity, and coincides with $\succsim^*$ on $\{a, e(a)\}, \{b, e(a)\}, \{d(a), e(a)\}$ and $\{d(a), b\}$, we conclude that $q_{ab} \in Q_{a\sim^{*u} b}$ and hence $u(a, q_{ab}) = u(b, q_{ab})$. Thus, we have found a signed belief $q_{ab} \in Q_{a\sim^* b}$ with $q_{ab} \in Q_{u(a)=u(b)}$.

Since the expected utility difference between a and $d(a)$ is constant across all signed beliefs, we know that (i) $Q_{u(a)=u(b)}$ is parallel to $Q_{u(d(a))=u(b)}$. Moreover, as u represents $\succsim^*$ on $\{d(a), b\}$, we have that (ii) $Q_{u(d(a))=u(b)} = Q_{d(a)\sim^* b}$. Finally, we know that (iii) $Q_{d(a)\sim^* b}$ is parallel to $Q_{a\sim^* b}$. By combining (i), (ii) and (iii), we conclude that $Q_{u(a)=u(b)}$ is parallel to $Q_{a\sim^* b}$. But since we have found a signed belief $q_{ab} \in Q_{a\sim^* b}$ with $q_{ab} \in Q_{u(a)=u(b)}$, it must be that $Q_{u(a)=u(b)} = Q_{a\sim^* b}$.

Now, take some signed belief q with $d(a) \sim^*_q b$. As $a \succ^*_{q'} d(a)$ for all signed beliefs q', we conclude that $a \succ^*_q b$. Since u represents $\succsim^*$ on $\{d(a), b\}$ and $\{a, d(a)\}$, we know that $u(a, q) > u(d(a), q) = u(b, q)$. Hence, we have found some signed belief q with $a \succ^*_q b$ and $u(a, q) > u(b, q)$. Since $Q_{u(a)=u(b)} = Q_{a\sim^* b}$, we can show in a similar way as in the proof of Lemma 5.4 that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.4.3. Assume that $Q_{a\sim^* e(a)} = Q_{b\sim^* e(a)}$. As a and b are not equivalent, it follows by transitivity of $\succsim^*$ that $Q_{a\sim^* b} = Q_{a\sim^* e(a)} = Q_{b\sim^* e(a)}$. Take an arbitrary $q_{ab} \in Q_{a\sim^* b}$. As q_{ab} is in both $Q_{a\sim^* e(a)}$ and $Q_{b\sim^* e(a)}$, and u represents $\succsim^*$ on $\{a, e(a)\}$ and $\{b, e(a)\}$, it follows that $u(a, q) = u(e(a), q) = u(b, q)$. Thus,

$Q_{a\sim^* b} \subseteq Q_{u(a)=u(b)}$. Moreover, since $span(Q_{a\sim^* b})$ and $span(Q_{u(a)=u(b)})$ both have dimension $n-1$, it must be that $Q_{a\sim^* b} = Q_{u(a)=u(b)}$.

Take some signed belief q with $d(a) \sim_q^* b$. Since $a \succ_{q'}^* d(a)$ for all signed beliefs q', we know that $a \succ_q^* b$. As u represents $\succsim^*$ on $\{d(a), b\}$ and $\{a, d(a)\}$, it follows that $u(a,q) > u(d(a),q) = u(b,q)$. Thus, we have found some signed belief q with $a \succ_q^* b$ and $u(a,q) > u(b,q)$.

Summarizing, we see that $Q_{u(a)=u(b)} = Q_{a\sim^* b}$, and there is a signed belief q where $a \succ_q^* b$ and $u(a,q) > u(b,q)$. We can then show in a similar way as in the proof of Lemma 5.4 that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.5. Suppose finally that $a, b \notin D$. We distinguish two cases: (2.2.5.1) $d(a) = d(b)$, and (2.2.5.2) $d(a) \neq d(b)$.

Case 2.2.5.1. Assume that $d(a) = d(b)$. Then, there is constant preference intensity between a and $d(a)$ and between b and $d(a)$. By transitivity of constant preference intensity, there is also constant preference intensity between a and b. That is, either $a \succ_q^* b$ for all signed beliefs q, or $b \succ_q^* a$ for all signed beliefs q. Assume, without loss of generality, that $a \succ_q^* b$ for all signed beliefs q.

Take some choice $c \in D \backslash \{d(a)\}$. Then, we know by the claim that there is no constant preference intensity between a and c, and hence there is a signed belief q with $a \sim_q^* c$. As $a \succ_q^* b$, we know by transitivity of $\succsim^*$ that $c \succ_q^* b$. Since, by the previous cases, u represents $\succsim^*$ on $\{a, c\}$ and $\{b, c\}$, it follows that $u(a,q) = u(c,q) > u(b,q)$. We have thus found a signed belief q with $u(a,q) > u(b,q)$.

Since $d(a) = d(b)$ we know, by construction of the utility function u in (5.26), that the expected utility difference between a and b is constant across all signed beliefs. As we have found a signed belief q with $u(a,q) > u(b,q)$, we conclude that $u(a,q') > u(b,q')$ for all signed beliefs q'. Since $a \succ_{q'}^* b$ for all signed beliefs q', we conclude that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.5.2. Suppose that $d(a) \neq d(b)$. Then, we know by the claim that there is no constant preference intensity between a and $d(b)$, and also not between b and $d(a)$. Since there is constant preference intensity between a and $d(a)$, but not between a and $d(b)$ and not between $d(a)$ and $d(b)$, it follows by Lemma 5.7 (b) that (i) $Q_{d(a)\sim^* d(b)}$ is parallel to $Q_{a\sim^* d(b)}$. In a similar fashion, it follows that (ii) $Q_{d(a)\sim^* d(b)}$ is also parallel to $Q_{b\sim^* d(a)}$.

Moreover, since there is constant preference intensity between a and $d(a)$, but not between b and $d(a)$, it must be, by transitive constant preference intensity, that there is also no constant preference intensity between a and b. But then, since there is constant preference intensity between a and $d(a)$ but not between b and $d(a)$, and not between a and b, it follows by Lemma 5.7 (b) that (iii) $Q_{b\sim^* d(a)}$ is parallel to $Q_{a\sim^* b}$. By combining (i), (ii) and (iii) we conclude that $Q_{a\sim^* b}, Q_{b\sim^* d(a)}, Q_{d(a)\sim^* d(b)}$ and $Q_{a\sim^* d(b)}$ are all parallel.

Take a line l of signed beliefs that cross each of these four parallel sets exactly once, and let $q_{ab}, q_{bd(a)}, q_{d(a)d(b)}$ and $q_{ad(b)}$ be the signed beliefs on this line where the DM is "indifferent" between the respective choices. As there is constant preference intensity between a and $d(a)$, and between b and $d(b)$, and since $\succsim^*$ satisfies part (b) of four choice linear preference intensity with constant preference intensity, we know that q_{ab} is uniquely given by the other three signed "indifference" beliefs.

Now, consider the conditional preference relation $\succsim^{*u}$ induced by the utility function u. Since also $\succsim^{*u}$ satisfies part (b) of four choice linear preference intensity with constant preference intensity, and since, by the previous cases, u represents $\succsim^*$ on $\{b, d(a)\}, \{d(a), d(b)\}$ and $\{a, d(b)\}$, we know that $q_{ab} \in Q_{a\sim^{*u}b}$, and hence $u(a, q_{ab}) = u(b, q_{ab})$. We have thus found a signed belief q_{ab} with $q_{ab} \in Q_{a\sim^* b}$ and $q_{ab} \in Q_{u(a)=u(b)}$.

Since, by (5.26), the expected utility difference between a and $d(a)$ is constant across all signed beliefs, we know that (i) $Q_{u(a)=u(b)}$ is parallel to $Q_{u(b)=u(d(a))}$. Since, by the previous cases, u represents $\succsim^*$ on $\{d(a), b\}$, it follows that (ii) $Q_{u(b)=u(d(a))} = Q_{b\sim^* d(a)}$. Moreover, we have seen above that (iii) $Q_{b\sim^* d(a)}$ is parallel to $Q_{a\sim^* b}$. By combining (i), (ii) and (iii) we conclude that $Q_{u(a)=u(b)}$ is parallel to $Q_{a\sim^* b}$. Since above we have found a signed belief q_{ab} with $q_{ab} \in Q_{a\sim^* b}$ and $q_{ab} \in Q_{u(a)=u(b)}$, it follows that $Q_{a\sim^* b} = Q_{u(a)=u(b)}$.

Now, take some signed belief q with $b \sim^*_q d(a)$. Since we are assuming that $a \succ^*_{q'} d(a)$ for all signed beliefs q', it follows by transitivity of $\succsim^*$ that $a \succ^*_q b$. As u represents $\succsim^*$ on $\{b, d(a)\}$ and $\{a, d(a)\}$, we know that $u(a, q) > u(d(a), q) = u(b, q)$. We have thus found a signed belief q with $a \succ^*_q b$ and $u(a, q) > u(b, q)$. Since $Q_{a\sim^* b} = Q_{u(a)=u(b)}$ we can show, in a similar way as in the proof of Lemma 5.4, that u represents $\succsim^*$ on $\{a, b\}$.

Since we have covered all the possible cases, we conclude that u represents $\succsim^*$ on every pair of choices $\{a, b\}$, and thus u represents $\succsim^*$. Since $\succsim^*$ extends $\succsim$, it follows that u represents $\succsim$. This completes the proof. ■

References

[1] Anscombe, F.J. and R.J. Aumann (1963), A definition of subjective probability, *Annals of Mathematical Statistics* **34,** 199–205.

[2] Aumann, R.J. and J.H. Drèze (2002), Bayes rationality in games, Working paper.

[3] Aumann, R.J. and J.H. Drèze (2005), Assessing strategic risk, Discussion paper 2005-20, Département des Sciences Économiques de l'Université Catholique de Louvain.

[4] Baccelli, J. and P. Mongin (2016), Choice-based cardinal utility: A tribute to Patrick Suppes, *Journal of Economic Methodology* **23,** 268–288.

[5] Battigalli, P., Cerreia-Vioglio, S., Maccheroni, F. and M. Marinacci (2017), Mixed extensions of decision problems under uncertainty, *Economic Theory* **63,** 827–866.

[6] Baumol, W.J. (1958), The cardinal utility which is ordinal, *The Economic Journal* **68,** 665–672.

[7] Bernheim, B.D. (1984), Rationalizable strategic behavior, *Econometrica* **52,** 1007–1028.

[8] Börgers, T. and P. Postl (2009), Efficient compromising, *Journal of Economic Theory* **144,** 2057–2076.

[9] Fishburn, P.C. (1976), Axioms for expected utility in n-person games, *International Journal of Game Theory* **5,** 137–149.

[10] Fishburn, P.C. and F.S. Roberts (1978), Mixture axioms in linear and multilinear utility theories, *Theory and Decision* **9,** 161–171.

[11] Gilboa, I. and D. Schmeidler (1995), Case-based decision theory, *The Quarterly Journal of Economics* **110,** 605–639.

[12] Gilboa, I. and D. Schmeidler (2003), A derivation of expected utility maximization in the context of a game, *Games and Economic Behavior* **44,** 184–194.

[13] Jagau, S. (2022), Additive context-dependent preferences, Manuscript.

[14] Luce, R.D. and H. Raiffa (1957), *Games and Decisions: Introduction and Critical Survey,* Wiley, New York.

[15] Mariotti, M. (1995), Is Bayesian rationality compatible with strategic rationality?, *The Economic Journal* **105,** 1099–1109.

[16] Moscati, I. (2018), *Measuring Utility: From the Marginal Revolution to Behavioral Economics,* Oxford University Press, Oxford, UK.

[17] Perea, A. (2020), A foundation for expected utility in decision problems and games, *Epicenter Working Paper No. 22.*

[18] Savage, L.J. (1954), *The Foundation of Statistics,* Wiley, New York.

[19] von Neumann, J. and O. Morgenstern (1947), *Theory of Games and Economic Behavior*, 2nd edition, Princeton University Press, Princeton, NJ.

[20] Wakker, P.P. (1989), *Additive Representations of Preferences*, Kluwer Academic Publishers, Dordrecht.