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Common belief in rationality in games with unawareness☆

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ABSTRACT

This paper investigates static games with unawareness, where players may be unaware of some of the choices that can be made. That is, different players may have different *views* on the game. We propose an epistemic model that encodes players' belief hierarchies on choices and views, and use it to formulate the basic reasoning concept of *common belief in rationality* . We do so for two scenarios: one in which we only limit the possible views that may enter the players' belief hierarchies, and one in which we fix the players' belief hierarchies on views. For both scenarios we design a recursive elimination procedure that yields for every possible view the choices that can rationally be made under common belief in rationality.

1. Introduction

A standard assumption in game theory is that all ingredients of the game – the players, their choices and their utility functions – are perfectly transparent to everybody involved. However, there are many situations of interest in which players may not be fully informed about some of these ingredients. For instance, a player may be uncertain about the precise utility functions of his opponents. Such situations may be modeled as *games with incomplete information,* and Harsanyi (1967–1968) opened the door towards a formal analysis of this class of games. In some cases the lack of information may even be more basic, as a player may be unaware of certain choices that can be made, or may even be unaware of the presence of certain players in the game. Harsanyi (1967–1968, pp.167–168) argued that unawareness of choices can also be modeled within the framework of incomplete information, by assigning a very low utility to the choices that players are unaware of. But conceptually this still seems very different from being truly unaware of these choices.[1] This type of situations, where players are unaware of certain choices or the presence of certain players in the game, has recently given rise to the study of *games with unawareness.* For an overview of the relatively young literature in this field, see Schipper (2014).

In terms of reasoning there is a crucial difference between these two classes of games. In a game with incomplete information, a player may not be informed about the true utility function of an opponent, yet at the same time may reason about all the possible utility functions that this opponent may have. And he may reason about an opponent reasoning about all the possible utility functions that some third player may have, and so on. That is, if we list all the possible utility functions that the players may have, then there is no limit to the players' reasoning about these utility functions.

The same is not true for games with unawareness, however. If a player is unaware of an opponent's choice c, then he cannot reason about other players who are aware of c. In a sense, the choice c is not part of his language, or state space, and hence this choice c cannot enter at any level of his reasoning. These endogenous constraints on the players' reasoning constitute the key factor that distinguishes games with unawareness from other classes of games.

At the same time, this reasoning about the level of unawareness of other players is at the central stage of games with unawareness. Indeed, if a player in a game with unawareness must decide what to do, then he must base his choice not only on his own (possibly partial) view of the game, but also on what he believes about the opponents' views of the game, what he believes that his opponents believe about the views of other players, and so on. In other words, a player holds a belief hierarchy on the players' views of the game, and bases his choice upon this belief hierarchy.

In that light, the reasoning of players in games with unawareness is considerably more complex than in standard games, as a player must form beliefs about the opponents' choices *and* the opponents' views, where his beliefs about the opponents' choices will depend on his belief about their views. In the literature,

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[1] See Hu and Stuart (2001) and Meier and Schipper (2014, p.227) for a discussion of this issue.

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the reasoning about views has typically been disentangled from the reasoning about choices, as most models for games with unawareness *exogenously* specify a belief hierarchy on views for every player. The strategic reasoning is then modeled by using an equilibrium or rationalizability concept that assumes these exogenously given belief hierarchies on views.

In this paper we take a different approach by *combining* the players' reasoning about views and choices into *one* belief hierarchy that models both. More precisely, we propose a model of static games with unawareness that no longer fixes the players' belief hierarchies on views, but where we only limit the possible views that may enter the players' belief hierarchies. We impose no restrictions, however, on how these views enter the belief hierarchies, or what probabilities these views receive at the various levels of a belief hierarchy. Subsequently, we encode the players' belief hierarchies on *choices and views* by an appropriately designed epistemic model with types. Types in this epistemic model thus simultaneously describe the players' reasoning about views and their strategic reasoning — something that proves to be very convenient for an epistemic analysis. Another difference with most of the existing literature is that we allow for *probabilistic* beliefs about the opponents' views, and not only deterministic beliefs. We find this important, as a player who is truly uncertain about the level of unawareness of his opponent may well ascribe positive probability to *various* possible views for this opponent. Such probabilistic beliefs on views can naturally be captured by our choice of an epistemic model.

We use our epistemic model to investigate the strategic reasoning of players in games with unawareness, which is the main purpose of this paper. To do so, we focus on the central yet basic reasoning concept of *common belief in rationality* (Spohn, 1982; Brandenburger and Dekel, 1987; Tan and Werlang, 1988) which in standard two-player static games characterizes *rationalizability* (Bernheim, 1984; Pearce, 1984), while characterizing *correlated rationalizability* (Brandenburger and Dekel, 1987) and the iterated strict dominance procedure in standard static games with two players or more.[2] In the context of games with unawareness, this concept states that a player believes that his opponents choose optimally *given their views of the game,* that a player believes that his opponents believe that the other players choose optimally *given their views of the game,* and so on. It turns out that this concept can naturally be formulated within the language of our epistemic model which, as we saw, encodes belief hierarchies on choices and views.

A natural question is whether we can find a recursive elimination procedure à la *iterated strict dominance* that characterizes precisely those choices that can rationally be made under common belief in rationality. We indeed propose such a procedure, and call it *iterated strict dominance for unawareness.* The main difference with the standard strict dominance procedure is that, in every round and for every player, we eliminate choices for *every possible view* that this player can hold in the game. More precisely, at a given view v_i for player *i* we first eliminate those choices for opponent *j* that have not survived the previous round for *any possible view of player j* that player *i* can reason about when holding the view v_i. Subsequently, at view v_i we eliminate for player *i* those choices that are strictly dominated, given the current set of opponents' choices.

We show in Theorem 5.2 that this procedure selects, for every player and every view, precisely those choices that this player can make with this particular view under common belief in rationality. Since the procedure always yields a non-empty output, it immediately follows that for every static game with unawareness there is for every player and every view at least one belief hierarchy on choices and views that expresses common belief in rationality.

Moreover, the output of the procedure is shown to be independent of the order and speed by which we eliminate the choices at the various views. This enables us to define a variant of the procedure, called *bottom-up procedure,* in which we start by doing all the eliminations for the smallest views, after which we do all the eliminations at the views that only contain smallest views as subviews, and so on. Since this alternative procedure merely corresponds to a change of the order and speed of elimination, it yields precisely the same output. The main advantage of the bottom-up procedure is that it allows us to do the eliminations on a view-by-view basis, making it more attractive from a practical point of view.

The procedure *iterated strict dominance for unawareness* is very similar to the *generalized iterated strict dominance procedure* (Bach and Perea, 2020) which has been designed for static games with *incomplete information.* The main difference is that in the latter procedure, choices are being eliminated at every possible *utility function* that a player can have in the game, instead of at every possible *view* that a player can hold.

As a second step, we reconcile the concept of common belief in rationality with the common assumption that the players' belief hierarchies on views are fixed. The new concept then selects, for every view and every fixed belief hierarchy on views, those choices that a player can rationally make under common belief in rationality if he holds this particular view and belief hierarchy on views. Also for this concept we design a recursive elimination procedure, called *iterated strict dominance with fixed beliefs on views,* that yields precisely these choices. See our Theorem 6.2.

To define this procedure we first encode the given belief hierarchy on views by means of an epistemic model with types, similar to the one mentioned above. The difference is that there is no reference to choices in this epistemic model, only to views. Types in this epistemic model are called *view-types,* as they encode belief hierarchies on views only. More precisely, every view-type in the model can be identified with a view and a probability distribution on the opponents' view-types. The new procedure is more refined as above, as it now eliminates, in every round and for every player, choices at every possible *view-type* for that player. Moreover, at a given view-type r_i for player *i*, the first-order beliefs that can be eliminated at r_i are based on the probability distribution that r_i induces on the opponents' view-types. In that sense, the procedure is closely related to the *interim correlated rationalizability procedure* (Dekel et al., 2007) for static games with *incomplete information.* The key difference is that in the latter procedure, choices are being eliminated at pairs of *utility functions* and belief hierarchies on *utility functions*, whereas in this paper choices are eliminated at pairs of *views* and belief hierarchies on *views.*

With these two procedures we thus characterize the behavioral consequences of common belief in rationality in games with unawareness in two scenarios: the basic scenario where we only limit the possible *views* that may enter the players' belief hierarchies, but where no other restrictions are imposed, and a scenario where the belief hierarchies on views are fixed. Moreover, if the belief hierarchies on views are fixed and *deterministic*, then our procedure becomes equivalent to the *extensive-form rationalizability* procedure in Heifetz et al. (2013b) when applied to *static* games with unawareness. Our analysis is also closely related to Feinberg (2021) who investigates the concept of *rationalizability* for static games with unawareness. Most other papers on

[2] The difference between rationalizability (Bernheim, 1984; Pearce, 1984) and correlated rationalizability (Brandenburger and Dekel, 1987) is that the former, in games with three players or more, requires player *i*'s belief about opponent *j*'s choice to be independent from his belief about another opponent *k*'s choice. The latter concept does not impose this independence condition.

games with unawareness investigate *equilibrium* concepts instead of rationalizability concepts.

The rest of the paper is organized as follows. In Section 2 we provide our definition of static games with unawareness. In Section 3 we encode belief hierarchies on choices and views by means of an epistemic model with types, and use it to formally define common belief in rationality for static games with unawareness in Section 4. In Section 5 we present the *iterated strict dominance procedure for unawareness* and show that it characterizes the behavioral consequences of common belief in rationality. In Section 6 we impose a fixed belief hierarchy on views for every player, present the *iterated strict dominance procedure with fixed beliefs on views,* and show that it characterizes the behavioral consequences of common belief in rationality with fixed belief hierarchies on views. In Section 7 we relate our work to other papers on unawareness in the literature. We conclude in Section 8. The Appendix contains all proofs, and shows how to formally derive belief hierarchies on views from types in an epistemic model.

2. Static games with unawareness

In this paper we restrict to *static games,* and focus on unawareness about the possible *choices* that the players can make. That is, a player may be unaware of certain choices that he, or his opponents, can make in the game. Feinberg (2021) allows players, in addition, to be unaware of some of the other *players* in the game. Such unawareness, however, will not be part of our framework.

Before we can analyze games with unawareness, we must first establish how we *describe* the possible unawareness of players about some of the choices in the game. We will do so by defining, for every player, a collection of *partial descriptions* of the full game, which contain some – but not necessarily all – possible choices that can be made. These partial descriptions will be called the possible *views* that the player can hold. Every view can thus be interpreted as a personal, and possibly incomplete, perception of the full game.

Formally, a *static game* is a tuple $G = (C_i, u_i)_{i \in I}$ where I is a finite set of players, C_i is a finite set of choices, and $u_i : \times_{j \in I} C_j \to \mathbf{R}$ is a utility function for every player i. A *view* of the game G is a tuple $v = (D_i)_{i \in I}$ where $D_i \subseteq C_i$ is a non-empty, possibly reduced set of choices for every player i. The interpretation is that a player with view $v = (D_i)_{i \in I}$ is only aware of the choices in D_i for every player i. We implicitly assume that a player with view $v = (D_i)_{i \in I}$ believes that the utilities induced by the choice combinations in v coincide with those of the game G. For that reason, it is not necessary to specify new utility functions for a view. For any two views $v = (D_i)_{i \in I}$ and $v' = (D'_i)_{i \in I}$ we write $v \subseteq v'$ if $D_i \subseteq D'_i$ for all players i. In this case, we say that view v is *contained* in view v'. That is, all choices considered possible in v are also considered possible in v'. An important principle in this – and any other – paper on unawareness is that a player with view v can only reason about views that are contained in v.

We can now define a static game with unawareness as a tuple consisting of a full static game, containing all choices that the players can possibly make, and for every player a finite collection of possible views of the full game.

Definition 2.1 (*Static Game with Unawareness*)**.** A static game with unawareness is a tuple $G^u = (G^{base}, (V_i)_{i \in I})$ where G^{base} is a static game, and V_i is a non-empty finite collection of views for player i of the game G^{base}. Moreover, for every player i, every view v_i in V_i, and every opponent $j \neq i$ there must be a view in V_j that is contained in v_i.

Here, we refer to G^{base} as the *base game.* The condition above thus guarantees that for every possible view $v_i \in V_i$ that player i can have, there is for every opponent j at least one view $v_j \in V_j$ that player i can reason about. This property plays a key role in this paper. Moreover, it implies that every "smallest" view will always be shared by all the players in the game. To see this, consider a smallest view $v_i \in V_i$ in the game, meaning that no other view $v_j \in V_j$ for any player j is strictly contained in v_i. By the property above, it must then necessarily hold that $v_i \in V_j$ for every player j, and hence every smallest view v_i is shared by all the players.

Unlike most other definitions in game theory, not all ingredients in a static game with unawareness are commonly known among the players. In particular, if player i holds a certain view v_i, he will not be aware of – and hence, not know of – the existence of views in the model that are not contained in v_i. This will be illustrated below in Example 1.

It is still possible that a player with view v feels he is missing something, but cannot state exactly what. That is, he believes that the actual set of choices is larger than his view, but cannot describe exactly what choices are missing. In the literature, this is known as "awareness of unawareness". However, this type of unawareness will not be modeled in this paper.

Note that, for every player i, the collection of views V_i need not contain *all* possible views of the game G^{base}. By considering *limited* collections of views, we put restrictions on the possible belief hierarchies on views that we allow for. Indeed, for every player i we restrict to belief hierarchies in which i only deems possible views in V_j for every opponent j, believes that every opponent j only deems possible views in V_k for every player $k \neq j$ (possibly equal to i himself), and so on.

The reasons for imposing such restrictions are two-fold. First, for a specific game-theoretic context, some views just make more sense than other views, and it thus seems reasonable to restrict to the more plausible views. Second, the concept of common belief in rationality, which is the main object of study in this paper, would hardly have any bite if we were to allow for all possible views. In that case, every choice that would be optimal for at least *some* belief about the opponents' choices could be rationalized under the concept of common belief in rationality. To see this, consider some choice c_i for player i that is optimal for some belief b_i about the opponents' choices. If all views are allowed, then player i is free to believe that every opponent only has one available choice, that every opponent believes that every other player only has one available choice, and so on. In that way, we can trivially embed the belief b_i in a belief hierarchy on choices and views that expresses common belief in rationality, thus rationalizing the choice c_i under common belief in rationality. However, by imposing some restrictions on the possible belief hierarchies on views, we may be able to derive some non-trivial behavioral consequences from common belief in rationality. In Section 6 we will impose further restrictions on the players' belief hierarchies on views by assuming a *unique* belief hierarchy on views for every player.

As another special case of our model, one may select the "full view" $(C_i)_{i \in I}$ as the only possible view for every player. In that case, the game with unawareness would reduce to a traditional static game in which all players agree that the game being played is G^{base} and no other.

The model that perhaps comes closest to ours is Meier and Schipper (2014), which focuses on static games with unawareness and incomplete information. Some other models of unawareness, such as Rêgo and Halpern (2012), Heifetz et al. (2013b) and Feinberg (2021),[3] are explicitly about dynamic games. An additional

3 Additional papers that model games with unawareness can be found in Section 7.

Table 1
"A day at the beach", modeled as a game with unawareness.

	G^{base}	Faraway	Distant	Nextdoor	Closeby			
Base game	Faraway	0,0	4,1	4,4	4,3			
	Distant	3,2	0,0	3,4	3,3			
	Nextdoor	2,2	2,1	0,0	2,3			
	Closeby	1,2	1,1	1,4	0,0			
	v_1	Faraway	Distant	Nextdoor	Closeby	v_1'	Nextdoor	Closeby
Your views	Faraway	0	4	4	4	Nextdoor	0	2
	Distant	3	0	3	3	Closeby	1	0
	Nextdoor	2	2	0	2			
	Closeby	1	1	1	0			
	v_2	Faraway	Distant	Nextdoor	Closeby	v_2'	Nextdoor	Closeby
Barbara's views	Faraway	0	2	2	2	Nextdoor	0	4
	Distant	1	0	1	1	Closeby	3	0
	Nextdoor	4	4	0	4			
	Closeby	3	3	3	0			

difference between our model and the three models above is that the latter fix for every player a view and a *belief hierarchy on views,* whereas we do not. That is, these papers exogenously describe, for every player, the view he holds on the game, what the player believes about the opponents' views, what he believes about the opponents' beliefs about the views by the other players, and so on. In contrast, we allow players to hold any view and any belief hierarchy on views they wish, as long as these only use views from the collections $(V_i)_{i \in I}$. As we already said, the case of fixed belief hierarchies on views will be explored in Section 6.

Moreover, in our model we allow such belief hierarchies on views to be *probabilistic,* whereas Feinberg (2021) restricts to deterministic belief hierarchies on views. Rêgo and Halpern (2012) and Heifetz et al. (2013b), in turn, do allow for probabilistic belief hierarchies on views through the introduction of chance moves.

A last difference we wish to outline is that the models by Rêgo and Halpern (2012) and Heifetz et al. (2013b) were specifically designed for *dynamic* games with unawareness. But their definitions capture static games as a special case.

We now illustrate the definition of a static game with unawareness by means of an example.

Example 1 (*A Day at the Beach*)**.** You and Barbara can go to four possible beaches: the *Nextdoor Beach,* the *Closeby Beach,* the *Faraway Beach* and the *Distant Beach.* The first two beaches are close to the hotel, whereas the latter two are more remote and hard to find. You happen to know about the two remote beaches, but are unsure whether Barbara is aware of these. The question is: To which beach do you go?

As to the utilities, suppose you had an argument with Barbara yesterday, and therefore you would both prefer to avoid each other by going to different beaches today. Assume, moreover, that you prefer the *Faraway Beach* to the *Distant Beach*, the *Distant Beach* to the *Nextdoor Beach*, and the *Nextdoor Beach* to the *Closeby Beach*. You know that Barbara prefers the *Nextdoor Beach* to the *Closeby Beach*, and suspect that she prefers the *Closeby Beach* to the *Faraway Beach*, and the *Faraway Beach* to the *Distant Beach* in case she is aware of the two remote beaches.

This situation can be represented by the game with unawareness in Table 1, where G^{base} is the base game, $V_1 = \{v_1, v_1'\}$ contains the views for you and $V_2 = \{v_2, v_2'\}$ contains the views for Barbara that are relevant for the situation at hand. In the base game, your choices are in the rows and Barbara's choices are in the columns. In both of your possible views v_1 and v_1', your choices are in the rows and Barbara's choices in the columns. In the corresponding cells we have put your utilities. In Barbara's views v_2 and v_2' we have put her choices in the rows and your choices in the columns, and have written the induced utilities for her in the cells. This is a general convention we adopt for depicting views of a player *i*: we always put *i*'s choices in the rows, the opponents' choice combinations in the columns, and the induced utilities for player *i* in the corresponding cells.

The view v_1 represents your actual view, in which you are aware of all four beaches. Since you are unsure whether Barbara is aware of the two remote beaches or not, you believe that Barbara's view is either v_2 or v_2'. If you believe that Barbara's view is v_2', you must necessarily believe that Barbara believes that your view is v_1'. Indeed, if Barbara is not aware of the two remote beaches, she cannot even reason about the possibility that you are aware of these remote beaches. It may be verified that Table 1 yields a well-defined static game with unawareness, meeting the condition on views as specified in Definition 2.1.

Note that this scenario allows for multiple belief hierarchies on views for you, provided your view is v_1, and similarly for Barbara if her view is v_2. Indeed, if your view is v_1, then one possible belief hierarchy on views is that you believe that Barbara holds the view v_2, that you believe that Barbara believes that you hold the view v_1, and so on. Another belief hierarchy could be that you believe that Barbara's view is v_2', that you believe that Barbara believes that you hold the view v_1', and so on.

However, if your view is v_1', then you can only reason about the view v_2' for Barbara and the view v_1' for yourself. Hence, the only possible belief hierarchy on views would be the one where you believe that Barbara's view is v_2', believe that Barbara believes that your view is v_1', and so on.

Note that not all the ingredients of this game with unawareness are common knowledge among the two players. Consider, for instance, the case where Barbara's view is v_2'. Then, she only has mental access to the views v_1' and v_2' in the model, and is not even aware of the existence of the other views v_1 and v_2.

Finally, we wish to mention that a specific view v_i for player *i* only specifies the choices – for himself, but also for the opponents – that he is aware of himself, but does not specify what player *i* believes about the awareness of other players. For instance, the view v_1 above only tells us that you are aware of the four beaches yourself. In particular, you are aware of four possible choices for Barbara. This does not mean, however, that you believe that Barbara is aware of these four choices also.

3. Epistemic models

In this section we will introduce epistemic models with types for games with unawareness, as a convenient way to encode belief hierarchies about the players' choices and views in the game. We start by laying out the definition, and discussing some of its key properties. Afterwards, we illustrate it by means of the example "A day at the beach". Finally, we relate it to other definitions in the literature.

3.1. Definition

The idea of *common belief in rationality* (Spohn, 1982; Brandenburger and Dekel, 1987; Tan and Werlang, 1988) is that a player believes that every opponent chooses optimally given his view, that he believes that every opponent believes that every other player chooses optimally given his view, and so on. In order to formally define this idea for static games with unawareness, we must specify (i) what a player believes about the possible choices and views of his opponents, (ii) what he believes about the opponents' beliefs about their opponents' choices and views, and so on. Such belief hierarchies can be encoded by means of epistemic models with types, where every type holds a view and a probabilistic belief about the opponents' choices and types.

To formally define the notion of an epistemic model, we need the following pieces of notation. For every finite set X, we denote by $\Delta(X)$ the set of probability distributions on X. Now, consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$. Then, by $V := \cup_{i\in I} V_i$ we denote the collection of all views in G^u. For a given view $v = (D_i)_{i\in I}$, we denote by $C_i(v) := D_i$ the set of choices for player i that a player with view v can reason about. By $C_{-i}(v) := \times_{j\neq i} C_j(v)$ we denote the set of opponents' choice combinations that player i can reason about while having the view v.

Definition 3.1 (*Epistemic Model*). Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$. An epistemic model for G^u is a tuple $M = (T_i, \hat{v}_i, b_i)_{i\in I}$ where, for every player i,

(a) T_i is a finite set of types,
(b) the view mapping $\hat{v}_i$ assigns to every type $t_i \in T_i$ some view $\hat{v}_i(t_i) \in V_i$. For a given type t_i and a player $j \neq i$, we denote by $T_j(\hat{v}_i(t_i))$ the set of types $t_j \in T_j$ with view $\hat{v}_j(t_j) \subseteq \hat{v}_i(t_i)$, and define $T_{-i}(\hat{v}_i(t_i)) := \times_{j\neq i} T_j(\hat{v}_i(t_i))$,
(c) the belief mapping b_i assigns to every type $t_i \in T_i$ some probabilistic belief $b_i(t_i) \in \Delta(C_{-i}(\hat{v}_i(t_i)) \times T_{-i}(\hat{v}_i(t_i)))$,
(d) for every type $t_i \in T_i$, the belief $b_i(t_i)$ only assigns positive probability to opponents' choice-type pairs (c_j, t_j) where $c_j \in C_j(\hat{v}_j(t_j))$.

Condition (c) thus guarantees that a type t_i only assigns probabilities to opponents' choices and views that he is aware of. In fact, we assume that type t_i is *unaware* of all opponents' choices c_j that are not in $C_j(\hat{v}_i(t_i))$ and all opponents' types that are not in $T_j(\hat{v}_i(t_i))$. In other words, the choices, views and types in the epistemic model that are not contained in his own view $\hat{v}_i(t_i)$ are not in t_i's vocabulary.

By the same condition (c), a type t_i also believes this to be true for all opponents' types t_j that it can reason about: For all such opponents' types t_j, type t_i believes that t_j is unaware of all choices, views and types that are outside its view $\hat{v}_j(t_j)$. This follows by applying condition (c) to type t_j. By applying this argument recursively, we conclude that type t_i, within its entire belief hierarchy, will only consider types t_j which are unaware of all choices, views and types that are outside its view $\hat{v}_j(t_j)$.

As such, the complete epistemic model may be viewed as a description from the modeler's point of view, whereas the various types in the model may only have mental access to a *part* of the epistemic model.[4]

Condition (d), finally, reflects the fact that a player j with type t_j can only make choices that are contained in his own view $\hat{v}_j(t_j)$. However, if t_j's view is contained in t_i's view then, possibly, player i with type t_i could still reason about choices for j that player j with type t_j is not aware of.

Now, consider for player i a type t_i^* in the epistemic model. Then, we can derive for t_i^* a full infinite belief hierarchy about the choices and views of the players. Indeed, t_i^*'s first-order belief about the opponents' choices would simply be the marginal of the probability distribution $b_i(t_i^*)$ on $C_{-i}(\hat{v}_i(t_i))$. As $b_i(t_i^*)$ induces a probability distribution on $T_{-i}(\hat{v}_i(t_i))$, and every opponent's type holds a view, we can also derive t_i^*'s first-order belief about the opponents' views. Hence, for t_i^* we can derive its first-order belief about the opponents' choices and views in this way.

The second-order beliefs for t_i^* can be derived as follows. Similarly as above, we can derive for every opponent's type t_j its first-order belief about the other players' choices and views. As t_i^* has a belief $b_i(t_i^*)$ about the opponents' types, we can derive the belief that t_i^* has about the first-order belief that every opponent j has about the other players' choices and views. This yields the second-order belief that type t_i^* has. By continuing in this fashion we can also derive the third-order belief, and all higher-order beliefs, for the type t_i^*, representing t_i^*'s belief hierarchy about the players' choices and views. How this works precisely will be illustrated later by means of an example.

The belief hierarchy of type t_i^* will have some natural properties, which follow from the definition of the epistemic model. Condition (c) guarantees, for instance, that at all layers of the belief hierarchy, t_i^* will only consider views and choices that are in his vocabulary, that t_i^* believes that all other players will, throughout their entire belief hierarchy, only consider choices and views that are in their vocabulary, and so on. In fact, as stated above, t_i^* is not aware of any choices, views and types that go beyond his own view $\hat{v}_i(t_i)$, and in his belief hierarchy he will only consider opponents' types that share this property.

In particular, within his belief hierarchy type t_i^* will only reason about views that are smaller (or equal) than his own, believes that other players share this property, believes that other players believe that other players share this property, and so on. We refer to this property as *common belief in smaller views.* Similar properties can be found in other papers on games with unawareness. Indeed, this condition corresponds to condition C2 in Rêgo and Halpern (2012), condition I4 in Heifetz et al. (2013b), condition (iii)(a) from Definition 1 in Heinsalu (2014), the "confinement" condition in Heifetz et al. (2013a) and Meier and Schipper (2014), and Condition 2 in Feinberg (2021).

The condition of *common belief in smaller views* is the main ingredient that distinguishes epistemic models for unawareness from epistemic models for incomplete information. In the latter scenario no condition of this kind is needed, as a player with a certain utility function has in principle mental access to *all* other utility functions in the model, and hence no restrictions need to be imposed on belief hierarchies on utility functions. One could say that, *a priori,* all belief hierarchies on utility functions are equally plausible. This is not the case for games with unawareness.

3.2. Example

As an illustration of an epistemic model, consider the one in Table 2 for the game "A day at the beach". It may be verified that conditions (c) and (d) in Definition 3.1 are satisfied. The beliefs for the types should be read as follows: Type t_1^* for you assigns probability 1 to the event that Barbara chooses *Nextdoor Beach* and has type t_2. Type t_2' for Barbara is not aware of the views v_1 and v_2, and hence of the existence of *Faraway Beach* and *Distant Beach,* and assigns probability 0.6 to the event that you choose *Nextdoor Beach* and have type t_1', and assigns probability 0.4 to the

[4] More precisely, take the perspective of a player i with type t_i and view $v = \hat{v}_i(t_i)$. Then, this player can reason about all types t_j for opponent j where $\hat{v}_j(t_j) \subseteq v$, and can reason about each of his own types t_i' where $\hat{v}_i(t_i') \subseteq v$. The latter is important, since player i may reason about an opponent j who reasons about i's (that is, his own) type.

Table 2
An epistemic model for "A day at the beach".

Types	$T_1 = \{t_1^*, t_1^{**}, t_1, t_1'\}$, $T_2 = \{t_2, t_2', t_2''\}$
Beliefs and views for you	$\hat{v}_1(t_1^*) = v_1$ and $b_1(t_1^*) = (\text{Nextdoor}, t_2)$
	$\hat{v}_1(t_1^{**}) = v_1$ and $b_1(t_1^{**}) = (\text{Faraway}, t_2')$
	$\hat{v}_1(t_1) = v_1'$ and $b_1(t_1) = (\text{Nextdoor}, t_2)$
	$\hat{v}_1(t_1') = v_1'$ and $b_1(t_1^*) = (\text{Closeby}, t_2'')$
Beliefs and views for Barbara	$\hat{v}_2(t_2) = v_2'$ and $b_2(t_2) = (\text{Closeby}, t_1)$
	$\hat{v}_2(t_2') = v_2$ and $b_2(t_2') = (0.6) \cdot (\text{Nextdoor}, t_1') + (0.4) \cdot (\text{Closeby}, t_1)$
	$\hat{v}_2(t_2'') = v_2'$ and $b_2(t_2'') = (\text{Nextdoor}, t_1')$

event that you choose *Closeby Beach* and have type t_1. Similarly for the other types.

The belief hierarchy for type t_1^{**} can be derived as follows. Type t_1^{**} believes that Barbara chooses *Faraway Beach* and has type t_2'. As type t_2' has view v_2, the first-order belief for t_1^{**} is that you believe that Barbara chooses *Faraway Beach* and that Barbara has the full view v_2.

Note that Barbara's type t_2' assigns probability 0.6 to the event that you choose *Nextdoor Beach* and have type t_1', and assigns probability 0.4 to the event that you choose *Closeby Beach* and have type t_1. As your types t_1 and t_1' both have view v_1', the second-order belief for type t_1^{**} is that you believe that (i) Barbara assigns probability 0.6 to you choosing *Nextdoor Beach* and probability 0.4 to you choosing *Closeby Beach*, and (ii) Barbara assigns probability 1 to your holding the restricted view v_1'. The higher-order beliefs for t_1^{**} can be derived in a similar fashion. In the same way we can also derive the belief hierarchy for your type t_1^*, which also holds the full view v_1.

3.3. Related definitions

As we have seen, we use an epistemic model to encode a player's belief hierarchy about the possible choices and views in the game. Other papers that use epistemic models to encode belief hierarchies in games with unawareness include Meier and Schipper (2014), Heinsalu (2014) and Guarino (2020).

Meier and Schipper (2014), following Heifetz et al. (2013a), start with a lattice of disjoint state spaces, ordered by "degree of expressiveness", where every state space in this lattice corresponds to a certain awareness level of a player. These state spaces may thus be compared with the different views in our model. For every player there is (i) a choice awareness mapping, assigning to every state the set of choices he is aware of, (ii) a choice mapping, assigning to every state a probability distribution over his choices, and (iii) a belief mapping, assigning to every state a probabilistic belief over the states. When combined, these three mappings induce for every state and every player a belief hierarchy about the players' choices and views in the game.

Heinsalu (2014) shows the existence of a *universal* type space for static games with unawareness. In particular, for every possible belief hierarchy on choices and views there will be a type in this space that generates this belief hierarchy. Guarino (2020) proves the existence of a universal type space for *dynamic* games with unawareness. An important difference with Heinsalu's approach is that Guarino explicitly shows how to construct this universal type space. Also a working paper version of Heifetz et al. (2013b) constructs a universal type space, much along the lines of the Mertens–Zamir construction (Mertens and Zamir, 1985).

Different from the papers above, we do not perform our epistemic analysis with respect to a *fixed* epistemic model, but we rather create for every question we wish to answer a new, tailor made epistemic model which addresses that question. More specifically, we are interested in the following question in this paper: Given a particular view v and choice c_i, can player i rationally choose c_i if his view is v and reasons in accordance with common belief in rationality? To answer this question affirmatively, it is sufficient to design an epistemic model where all views are contained in v, and where there is a type for player i that expresses common belief in rationality (see Section 4) and for which the choice c_i is optimal.

4. Common belief in rationality

In the previous section we have seen how to encode belief hierarchies on choices and views, by means of an epistemic model with types. The next step towards a formal definition of *common belief in rationality* is to define optimal choice for a particular view, and belief in the opponents' rationality. Fix an epistemic model $M = (T_i, \hat{v}_i, b_i)_{i \in I}$. For a given type t_i in M, and a choice $c_i \in C_i(\hat{v}_i(t_i))$ that t_i is aware of, we denote by

$$u_i(c_i, t_i) := \sum_{(c_{-i}, t_{-i}) \in C_{-i}(\hat{v}_i(t_i)) \times T_{-i}(\hat{v}_i(t_i))} b_i(t_i)(c_{-i}, t_{-i}) \cdot u_i(c_i, c_{-i})$$

the *expected utility* induced by choice c_i under t_i's first-order belief about the opponents' choice combinations. We now define what it means for a choice to be optimal for a type t_i.

Definition 4.1 (*Optimal Choice*)**.** Consider an epistemic model $M = (T_i, \hat{v}_i, b_i)_{i \in I}$ and type $t_i \in T_i$. A choice $c_i \in C_i(\hat{v}_i(t_i))$ is optimal for t_i if

$$u_i(c_i, t_i) \geq u_i(c_i', t_i) \text{ for all } c_i' \in C_i(\hat{v}_i(t_i)).$$

We next define what it means to believe in the opponents' rationality. In words, it means that you only deem possible combinations of choices and types for the opponent where the choice is optimal for the type.

Definition 4.2 (*Belief in the Opponents' Rationality*)**.** Consider an epistemic model $M = (T_i, \hat{v}_i, b_i)_{i \in I}$, and a type $t_i \in T_i$. We say that type t_i believes in the opponents' rationality if $b_i(t_i)$ only assigns positive probability to opponents' choice-type pairs (c_j, t_j) where c_j is optimal for t_j.

In the epistemic model of Table 2, it may be verified that all types believe in the opponent's rationality. With this definition at hand, we can now define common belief in rationality in an iterative fashion.

Definition 4.3 (*Common Belief in Rationality*)**.** Consider an epistemic model $M = (T_i, \hat{v}_i, b_i)_{i \in I}$. A type $t_i \in T_i$ expresses 1-fold belief in rationality if it believes in the opponents' rationality. For $k > 1$, we recursively say that a type t_i expresses k-fold belief in rationality if $b_i(t_i)$ only assigns positive probability to opponents' types that express $(k-1)$-fold belief in rationality. A type t_i expresses common belief in rationality if it expresses k-fold belief in rationality for every $k \geq 1$.

Hence, type t_i believes in the opponents' rationality, believes that the opponents believe in the other players' rationality, and so on. Rational choice under common belief in rationality with a particular view can be defined as follows.

Definition 4.4 (*Rational Choice Under Common Belief in Rationality*). Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$, a view $v_i \in V_i$, and a choice $c_i \in C_i(v_i)$ available at that view. Choice c_i can rationally be made under common belief in rationality with the view v_i if there is an epistemic model $M = (T_j, \hat{v}_j, b_j)_{j \in I}$, and a type $t_i^* \in T_i$ with view $\hat{v}_i(t_i^*) = v_i$, such that t_i^* expresses common belief in rationality, and c_i is optimal for t_i^*.

The definition above thus states that there must be a belief hierarchy for player i with view v_i which expresses common belief in rationality, and such that c_i is optimal given this belief hierarchy and the view v_i. Note that no restrictions are being imposed on the belief hierarchy, beyond the conditions of common belief in rationality, and the fact that the belief hierarchy must be "feasible" given the view v_i. That is, we look at "all possible stories" that are compatible with the view v_i and common belief in rationality, without further restricting the belief hierarchy on views.

To illustrate these notions, consider again the epistemic model from Table 2. As all types believe in the opponent's rationality, it follows that your types t_1^* and t_1^{**} express *common* belief in rationality. Note that *Faraway Beach* is optimal for your type t_1^*, and *Distant Beach* is optimal for your type t_1^{**}. As t_1^* and t_1^{**} both have the full view v_1, it follows that with the view v_1 you can rationally choose *Faraway Beach* and *Distant Beach* under common belief in rationality. In the next section we will see that these are also the *only* choices you can rationally make under common belief in rationality while holding the view v_1.

5. Recursive procedure

In this section we wish to characterize the choices a player can rationally make under common belief in rationality while holding a particular view. To that purpose we introduce a recursive elimination procedure, called *iterated strict dominance for unawareness,* which iteratively eliminates choices from every possible view in the game. We show that the procedure delivers, for every view, exactly those choices that can rationally be made under common belief in rationality with that particular view.

5.1. Definition

To formally define the procedure, we need some additional terminology. A *decision problem* for player i is a pair (D_i, D_{-i}) where $D_i \subseteq C_i$ and $D_{-i} \subseteq C_{-i}$. We say that $c_i \in D_i$ is *strictly dominated* within the decision problem (D_i, D_{-i}) if there is some randomized choice $\rho_i \in \Delta(D_i)$ such that

$$u_i(c_i, c_{-i}) < \sum_{c_i' \in D_i} \rho_i(c_i') \cdot u_i(c_i', c_{-i}) \text{ for all } c_{-i} \in D_{-i}.$$

In the procedure below we start by defining, for every player i and every possible view $v_i \in V_i$, the full decision problem $(C_i(v_i), C_{-i}(v_i))$ that corresponds to the view v_i. Recall that $C_i(v_i)$ is the set of player i's choices and $C_{-i}(v_i)$ the set of opponents' choice combinations that player i is aware of with the view v_i. At every round we then recursively reduce these decision problems at the various views by eliminating choices and opponents' choice combinations.

Definition 5.1 (*Iterated Strict Dominance for Unawareness*). Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$.
(Initial step) For every player i and every view $v_i \in V_i$, define the full decision problem $(C_i^0(v_i), C_{-i}^0(v_i)) := (C_i(v_i), C_{-i}(v_i))$.
(Inductive step) For $k \geq 1$, every player i, and every view $v_i \in V_i$, define

$$C_{-i}^k(v_i) := \{(c_j)_{j \neq i} \in C_{-i}^{k-1}(v_i) \mid \text{ for all } j \neq i \text{ choice } c_j \text{ is in } C_j^{k-1}(v_j) \text{ for some view } v_j \subseteq v_i\},$$

and

$$C_i^k(v_i) := \{c_i \in C_i^{k-1}(v_i) \mid \ c_i \text{ not strictly dominated within the decision problem } (C_i^{k-1}(v_i), C_{-i}^k(v_i))\}.$$

A choice-view pair (c_i, v_i) is said to survive the procedure if $c_i \in C_i^k(v_i)$ for every $k \geq 0$.

Hence, in this procedure we recursively restrict, for every view v_i, the possible beliefs that player i can hold about his opponents' choices, through the sets $C_{-i}^k(v_i)$, and the possible choices that player i can make himself, through the sets $C_i^k(v_i)$. In that sense, it is very similar to the *generalized iterated strict dominance procedure* (Bach and Perea, 2020) for static games with *incomplete information.* The latter procedure recursively restricts such beliefs and choices for every possible *utility function* that player i can have in the game with incomplete information, instead of for every possible *view* in the game, as we do here.

An important difference is that in the case of incomplete information, a player with a certain utility function is able to reason about *all* other utility functions in the model – in a sense, the collection of all utility functions is common knowledge among the players – whereas the same is not true for views in a game with unawareness. Indeed, a player with a certain view can only reason about views in the model that are *contained* in his own view. This fact is reflected in the procedure above, by the way the sets $C_{-i}^k(v_i)$ are defined. Note that in $C_{-i}^k(v_i)$ we only keep those opponents' choices c_j that are in $C_j^{k-1}(v_j)$ for some view v_j that is *contained in* v_i. A similar condition is not present in the generalized iterated strict dominance procedure for games with incomplete information.

Observe that in the special case where V_i only contains the "full view" $(C_j)_{j \in I}$ for every player i, the procedure above reduces to the well-known iterated strict dominance procedure for standard static games without unawareness.

The procedure is also similar to the procedure in Meier and Schipper (2012) that characterizes, for dynamic games with unawareness, the strategies that are extensive-form rationalizable (Heifetz et al., 2013b). Like our procedure, also the one in Meier and Schipper (2012) proceeds by iteratively eliminating strategies that are strictly dominated, although they use *conditional* strict dominance rather than plain strict dominance.

In the following subsection we will show that our procedure always delivers a non-empty set of choices for every possible view, and indeed characterizes precisely those choice-view pairs where the choice is possible for the view under common belief in rationality.

5.2. Non-empty output and characterization result

We first show that the iterated strict dominance procedure for unawareness always yields a non-empty output. More precisely, we show that for every possible view in the game, there is always at least one choice for the respective player that survives the procedure.

Theorem 5.1 (*Non-Empty Output*). *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i and every view $v_i \in V_i$ there is some choice $c_i \in C_i$ such that (c_i, v_i) survives the iterated strict dominance procedure for unawareness.*

We next present the main result in this section, showing that the iterated strict dominance procedure for unawareness selects for every view precisely those choices that can rationally be made under common belief in rationality.

Theorem 5.2 (*Characterization of Common Belief in Rationality*)**.** *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i, every view $v_i \in V_i$ and every choice $c_i \in C_i(v_i)$, choice c_i can rationally be made under common belief in rationality with the view v_i, if and only if, (c_i, v_i) survives the procedure of iterated strict dominance for unawareness.*

Consider now the special case where V_i contains all possible views for every player *i*. That is, we do not impose any restrictions on the players' belief hierarchies on views. Then, for every $k \geq 1$ we have that $C^k_{-i}(v_i) = C^0_{-i}(v_i)$ for every player *i* and view v_i, because every opponent's choice c_j in v_i is optimal for a view of player *j* that is contained in v_i and in which c_j appears as the unique choice for player *j*. Consequently, the procedure terminates already at round 1, and every choice $c_i \in C^1_i(v_i)$ survives the procedure at v_i. In view of Theorem 5.2 we thus see that in this case, every choice that is optimal for *some* belief at a certain view can automatically be chosen rationally under *common belief in rationality* with that particular view. Hence, the concept of common belief in rationality is very permissive if we allow for all possible views in the game.

One direction of Theorem 5.2 states that if (c_i, v_i) survives the procedure, then we can always find an epistemic model, and a type $t_i \in T_i$ for player *i* within that epistemic model with view v_i, such that t_i expresses common belief in rationality, and the choice c_i is optimal for t_i. For the construction of this epistemic model we rely on Theorem 5.1, which guarantees that for every player *j*, and every view v_j, there is at least one choice c_j that survives the procedure at v_j.

In particular, this direction implies that for every finite static game with unawareness, we can always construct for every player *i*, and every view v_i, an epistemic model, and a type t_i with view v_i, such that t_i expresses common belief in rationality.

Corollary 5.1 (*Common Belief in Rationality Is Always Possible*)**.** *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i and every view $v_i \in V_i$, there is an epistemic model $M = (T_j, \hat{v}_j, b_j)_{j \in I}$, and a type $t^*_i \in T_i$ with view $\hat{v}_i(t^*_i) = v_i$, such that t^*_i expresses common belief in rationality.*

In other words, for every view it is always possible to reason in accordance with common belief in rationality.

5.3. Example

In this subsection we will illustrate the iterated strict dominance procedure for unawareness by means of the example we discussed above. To save space, we use the abbreviations f, d, n and c for the four beaches.

Example 2 (*Procedure for "A day at the beach"*)**.** Consider the game with unawareness as depicted in Table 1. At the beginning of the procedure we have the full decision problems at the different views, given by

$$\begin{aligned} C^0_1(v_1) &= \{f, d, n, c\}, & C^0_{-1}(v_1) &= \{f, d, n, c\}, \\ C^0_1(v'_1) &= \{n, c\}, & C^0_{-1}(v'_1) &= \{n, c\}, \\ C^0_2(v_2) &= \{f, d, n, c\}, & C^0_{-2}(v_2) &= \{f, d, n, c\} \\ C^0_2(v'_2) &= \{n, c\}, & C^0_{-2}(v'_2) &= \{n, c\}. \end{aligned}$$

Round 1. By definition we have that $C^1_{-1}(v_1) = C^0_{-1}(v_1)$, $C^1_{-1}(v'_1) = C^0_{-1}(v'_1)$, $C^1_{-2}(v_2) = C^0_{-2}(v_2)$ and $C^1_{-2}(v'_2) = C^0_{-2}(v'_2)$. Note that c is strictly dominated for you by the randomized choice $(0.5) \cdot f + (0.5) \cdot d$ within the decision problem $(C^0_1(v_1), C^1_{-1}(v_1))$, and that d is strictly dominated for Barbara by $(0.5) \cdot n + (0.5) \cdot c$ within her decision problem $(C^0_2(v_2), C^1_{-2}(v_2))$. No other choices are strictly dominated in this round. We can therefore eliminate your choice c from $C^0_1(v_1)$ and Barbara's choice d from $C^0_2(v_2)$, yielding the reduced decision problems

$$\begin{aligned} C^1_1(v_1) &= \{f, d, n\}, & C^1_{-1}(v_1) &= \{f, d, n, c\}, \\ C^1_1(v'_1) &= \{n, c\}, & C^1_{-1}(v'_1) &= \{n, c\}, \\ C^1_2(v_2) &= \{f, n, c\}, & C^1_{-2}(v_2) &= \{f, d, n, c\}, \\ C^1_2(v'_2) &= \{n, c\}, & C^1_{-2}(v'_2) &= \{n, c\}. \end{aligned}$$

Round 2. As Barbara's choice d is not in her decision problems at v_2 and v'_2 anymore, we can eliminate Barbara's choice d from your current decision problem at v_1. That is, $C^2_{-1}(v_1) = \{f, n, c\}$ and $C^2_{-1}(v'_1) = \{n, c\}$. Note that we cannot eliminate your choice c from Barbara's decision problems at v_2 and v'_2, since your choice c is still present in $C^1_1(v'_1)$, and your view v'_1 is contained in both v_2 and v'_2. We thus have that $C^2_{-2}(v_2) = \{f, d, n, c\}$ and $C^2_{-2}(v'_2) = \{n, c\}$.

In your decision problem $(C^1_1(v_1), C^2_{-1}(v_1)) = (\{f, d, n\}, \{f, n, c\})$ at v_1, your choice n is strictly dominated by d, and can thus be eliminated from $C^1_1(v_1)$. No other choices can be eliminated in this round. We thus obtain the reduced decision problems

$$\begin{aligned} C^2_1(v_1) &= \{f, d\}, & C^2_{-1}(v_1) &= \{f, n, c\}, \\ C^2_1(v'_1) &= \{n, c\}, & C^2_{-1}(v'_1) &= \{n, c\}, \\ C^2_2(v_2) &= \{f, n, c\}, & C^2_{-2}(v_2) &= \{f, d, n, c\}, \\ C^2_2(v'_2) &= \{n, c\}, & C^2_{-2}(v'_2) &= \{n, c\}. \end{aligned}$$

After this round no further choices can be eliminated at any of the possible views, and hence the procedure terminates at the end of round 2. The choice-view pairs that survive for you are $(f, v_1), (d, v_1), (n, v'_1)$ and (c, v'_1), whereas the choice-view pairs surviving for Barbara are $(f, v_2), (n, v_2), (c, v_2), (n, v'_2)$ and (c, v'_2).

Hence, in view of Theorem 5.2**,** these are exactly the choice-view pairs that are possible under common belief in rationality. That is, under common belief in rationality, you can rationally choose *Faraway Beach* and *Distant Beach* with the view v_1, you can rationally choose *Nextdoor Beach* and *Closeby Beach* with the view v'_1, Barbara can rationally choose *Faraway Beach, Nextdoor Beach* and *Closeby Beach* with the view v_2, and can rationally choose *Nextdoor Beach* and *Closeby Beach* with the view v'_2.

5.4. Order independence

Suppose that at every round of the procedure, we would at every view eliminate *some* – but not necessarily *all* – choices that could be eliminated. Would this matter for the final output? The answer, as we will see, is no, provided we do not forget to eliminate a choice forever. In that sense, the iterated strict dominance procedure for unawareness is *order independent.*

To formally define this property, we will first see how the original procedure can be viewed as the iterated application of a *reduction operator.* A *collection of decision problems* is a tuple

$$D = (D_i(v_i), D_{-i}(v_i))_{i \in I, v_i \in V_i}$$

where $D_i(v_i) \subseteq C_i(v_i)$ and $D_{-i}(v_i) \subseteq C_{-i}(v_i)$ for every player i and view $v_i \in V_i$. For two collections of decision problem $D = (D_i(v_i), D_{-i}(v_i))_{i \in I, v_i \in V_i}$ and $E = (E_i(v_i), E_{-i}(v_i))_{i \in I, v_i \in V_i}$, we write $D \subseteq E$ if $D_i(v_i) \subseteq E_i(v_i)$ and $D_{-i}(v_i) \subseteq E_{-i}(v_i)$ for all players i and $v_i \in V_i$.

The *reduction operator* r assigns to every collection of decision problems $D = (D_i(v_i), D_{-i}(v_i))_{i\in I, v_i \in V_i}$ a new collection of decision problems

$$r(D) = (E_i(v_i), E_{-i}(v_i))_{i\in I, v_i\in V_i}$$

where

$$E_{-i}(v_i) := \{(c_j)_{j\neq i} \in D_{-i}(v_i) \mid \text{for all } j \neq i \text{ choice } c_j \text{ is in } D_j(v_j) \text{ for some } v_j \subseteq v_i\}$$

and

$$E_i(v_i) := \{c_i \in D_i(v_i) \mid c_i \text{ not strictly dominated within } (D_i(v_i), E_{-i}(v_i))\}.$$

Then, by definition, *iterated strict dominance for unawareness* corresponds to the iterated application of the reduction operator r to the collection of full decision problems $D^0 = (C_i(v_i), C_{-i}(v_i))_{i\in I, v_i\in V_i}$.

Similarly to Definition 3.1 in Perea (2018), an *elimination order* for iterated strict dominance for unawareness is a sequence of collections of decision problems $(D^0, D^1, \ldots, D^K)$ where (i) D^0 is the collection of full decision problems above, (ii) $r(D^k) \subseteq D^{k+1} \subseteq D^k$ for every $k \in \{0, \ldots, K-1\}$, and (iii) $r(D^K) = D^K$. Here, (ii) states that at every round and at every view, some, but not necessarily all, choices are eliminated that could have been eliminated according to the original procedure. Property (iii) guarantees that no further eliminations are possible at D^K, and hence the elimination order will not forget to eliminate a choice forever. An elimination order can thus be viewed as an alternative, slower way of eliminating choices.

The following result guarantees that choosing a different elimination order will not alter the output of the procedure.

Theorem 5.3 (*Order Independence*)**.** *Let $(D^0, \ldots, D^K)$ and $(E^0, \ldots, E^L)$ be two elimination orders for iterated strict dominance for unawareness. Then, $D^K = E^L$.*

In the following subsection we will present an easy-to-use elimination order which, by the theorem above, will produce exactly the same output as the original procedure.

5.5. Bottom-up procedure

If there are many different views in the game the procedure above may become unattractive from a practical point of view, since at every round we must do the eliminations at all the different views. In such cases we may use an alternative, more efficient procedure that yields exactly the same output. The idea is that we start by visiting the smallest views individually, and perform the well-known iterated elimination of strictly dominated choices there. Next, we turn to the views that only contain the smallest views as subviews, again on a one-by-one basis, and do our eliminations there. At such views v, however, we would always keep a choice c that has survived the eliminations at a smallest view v' contained in v. The reason is that a player with view v may believe that an opponent has the smaller view v' and chooses c. In all subsequent rounds we would visit larger and larger views until we have exhausted all views in the game. This procedure will be called the *bottom-up procedure*.[5] Since it is obtained by choosing an alternative elimination order for iterated strict dominance for unawareness, it will deliver the same output in the light of Theorem 5.3.

To formally define the bottom-up procedure we need some new terminology. A view $v \in V$ is called *smallest* if there is no $v' \in V\backslash\{v\}$ with $v' \subseteq v$. For every view v, we define its *rank* as follows: Every smallest view v has rank 1. Now, suppose that $m \geq 2$, and that the views with ranks $1, \ldots, m-1$ have been identified. Then, a view v has rank m precisely when (i) every subview $v' \in V\backslash\{v\}$ with $v' \subseteq v$ has a rank in $\{1, \ldots, m-1\}$, and (ii) there is at least one subview $v' \in V\backslash\{v\}$ with $v' \subseteq v$ which has rank $m-1$. In the example "A day at the beach", for instance, the views v_1' and v_2' have rank 1, whereas the views v_1 and v_2 have rank 2. For every view v, we denote by $I(v) := \{i \in I \mid v \in V_i\}$ the set of players for whom that view is feasible.

Definition 5.2 (*Bottom-Up Procedure*)**.** Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$ where the highest rank of a view is M.

(Views with rank 1). For every view $v \in V$ with rank 1 we perform the following steps. Define, for every player $i \in I$, the full decision problem $(C_i^0(v), C_{-i}^0(v)) := (C_i(v), C_{-i}(v))$. For every $k \geq 1$ recursively define, for every player i,

$$C_{-i}^k(v) := \times_{j\neq i} C_j^{k-1}(v)$$

and

$$C_i^k(v) := \{c_i \in C_i^{k-1}(v) \mid c_i \text{ not strictly dominated within the decision problem } (C_i^{k-1}(v), C_{-i}^k(v))\}.$$

For every player i, let $C_i^*(v)$ be the set of choices $c_i \in C_i(v)$ with $c_i \in C_i^k(v)$ for all $k \geq 0$. Do this for all views v with rank 1.

(Views with ranks $2, \ldots, M$). Consider some $m \in \{2, \ldots, M\}$, and suppose that $C_i^*(v)$ has been determined for every view $v \in V_i$ with rank $1, 2, \ldots, m-1$ and every player $i \in I(v)$. For every view $v \in V$ with rank m we perform the following steps. Define, for every player $i \in I(v)$, the full decision problem $(C_i^0(v), C_{-i}^0(v)) := (C_i(v), C_{-i}(v))$. For every $k \geq 1$ recursively define, for every player $i \in I(v)$,

$$C_{-i}^k(v) := \{(c_j)_{j\neq i} \in C_{-i}^{k-1}(v) \mid \text{for all } j \neq i, \text{ either } c_j \in C_j^*(v_j) \text{ for some } v_j \subseteq v \text{ with } v_j \neq v, \text{ or } c_j \in C_j^{k-1}(v) \text{ if } j \in I(v)\},$$

and

$$C_i^k(v) := \{c_i \in C_i^{k-1}(v) \mid c_i \text{ not strictly dominated within the decision problem } (C_i^{k-1}(v), C_{-i}^k(v))\}.$$

For every player $i \in I(v)$, let $C_i^*(v)$ be the set of choices $c_i \in C_i(v)$ with $c_i \in C_i^k(v)$ for all $k \geq 0$. Do this for all views v with rank m.

A choice-view pair (c_i, v_i) is said to survive the bottom-up procedure if $c_i \in C_i^*(v_i)$.

By the last condition in Definition 2.1, every smallest view must be shared by all players, and hence $I(v) = I$ for all views v with rank 1. At such views v, the procedure above applies the iterated elimination of strictly dominated choices.

Note that the bottom-up procedure corresponds to an alternative elimination order for iterated strict dominance for unawareness where (i) we first do the eliminations only at the views with rank 1 until no further eliminations are possible, (ii) we subsequently do the eliminations only at views with rank 2 until no further eliminations are possible, and so on, until we have covered all views. By Theorem 5.3 we thus know that the bottom-up procedure will always yield the same output as the original procedure. Especially when there are many views in the game, the bottom-up procedure may be more user-friendly than the original procedure, since at every round we only have to deal with one view at the time.

On a qualitative level, the bottom-up procedure is similar to the *backwards order of elimination* (Perea, 2012, Section 8.10) for the *backward dominance procedure* (Perea, 2014) for dynamic

[5] I thank an anonymous referee for suggesting this procedure.

games. According to the backwards order of elimination, we first do all the eliminations at the last information sets in the dynamic game until no further eliminations are possible. Subsequently, we do all the eliminations at the information sets just before the last information sets, and so on, until all information sets are covered. Also this elimination order is a kind of bottom-up order, where we start at the end of the game and gradually move backwards. In the bottom-up procedure above we start at the smallest views, and gradually move to larger and larger views.

As an illustration, let us apply the bottom-up procedure to the example "A day at the beach" in Table 1. Here, the views with rank 1 are v_1' and v_2'. In fact, as v_1' and v_2' are identical, we can write $v' := v_1' = v_2'$, with $I(v') = \{1, 2\}$. Since no choices can be eliminated at v' we have that

$C_1^*(v') = \{n, c\}$ and $C_2^*(v') = \{n, c\}$.

We then turn to the views with rank 2, which are v_1 and v_2. Again, v_1 and v_2 are identical, so we can write $v := v_1 = v_2$, with $I(v) = \{1, 2\}$.

Round 1. By definition we have that $C_{-1}^1(v) = \{f, d, n, c\}$. As your choice c is strictly dominated by $(0.5) \cdot f + (0.5) \cdot d$ in $(C_1^0(v), C_{-1}^1(v))$, we have that

$C_1^1(v) = \{f, d, n\}$.

Similarly, $C_{-2}^1(v) = \{f, d, n, c\}$. Since Barbara's choice d is strictly dominated by $(0.5) \cdot n + (0.5) \cdot c$, it follows that

$C_2^1(v) = \{f, n, c\}$.

Round 2. By construction, $C_{-1}^2(v) = \{f, n, c\}$ as choice d for Barbara is not in $C_2^1(v)$ and not in $C_2^*(v')$. But then, your choice n is strictly dominated by d in $(C_1^1(v), C_{-1}^2(v))$, and thus

$C_1^2(v) = \{f, d\}$.

In turn, $C_{-2}^1(v) = \{f, d, n, c\}$, since your choices f, d and n are in $C_1^1(v)$, and your choice c is in $C_1^*(v')$. But then,

$C_2^2(v) = C_2^1(v) = \{f, n, c\}$.

Since the procedure terminates here, we conclude that the choice-view pairs for you surviving the bottom-up procedure are $(f, v), (d, v), (n, v')$ and (c, v'), whereas the choice-view pairs that survive for Barbara are $(f, v), (n, v), (c, v), (n, v')$ and (c, v'). Note that this output is exactly the same as in the original procedure.

6. Fixed beliefs on views

In the literature on games with unawareness, it is typically assumed that every player holds some *exogenously given belief hierarchy on views*. See, for instance, Rêgo and Halpern (2012), Heifetz et al. (2013b) and Feinberg (2021). Following this approach, we reconcile in this section the concept of common belief in rationality with the assumption that the *belief hierarchy on views is fixed.* One important difference with Feinberg (2021) is that we allow for *truly probabilistic* belief hierarchies on views, and not only belief hierarchies consisting of probability 1 beliefs on views. The reason is that we wish to allow for situations in which a player is *uncertain* about the precise view adopted by his opponent, and therefore assigns positive probability to various possible views for this opponent.

6.1. Common belief in rationality with fixed beliefs on views

Different from Rêgo and Halpern (2012), Heifetz et al. (2013b), and Feinberg (2021) but in accordance with, for instance, Heinsalu (2014), Heifetz et al. (2013a), Meier and Schipper (2014) and Guarino (2020), we decide to encode belief hierarchies on views by means of epistemic models with *types*.[6] The reason is that such encodings are easy to work with, and turn out to be convenient for designing proofs and an associated elimination procedure as well. Such an epistemic model may be seen as a reduced version of the one used in Section 3, since now a type only holds a belief about the opponents' types, instead of the opponents' *choices* and types.

Definition 6.1 (*Epistemic Model for Views*)**.** An epistemic model for views is a tuple $M^{view} = (R_i, \hat{w}_i, p_i)_{i\in I}$ where, for every player i,

(a) R_i is a finite set of types,
(b) the view mapping $\hat{w}_i$ assigns to every type $r_i \in R_i$ some view $\hat{w}_i(r_i) \in V_i$. For a given type r_i and a player $j \neq i$, we denote by $R_j(\hat{w}_i(r_i))$ the set of types $r_j \in R_j$ with view $\hat{w}_j(r_j) \subseteq \hat{w}_i(r_i)$, and define $R_{-i}(\hat{w}_i(r_i)) := \times_{j\neq i} R_j(\hat{w}_i(r_i))$,
(c) the belief mapping p_i assigns to every type $r_i \in R_i$ some probabilistic belief $p_i(r_i) \in \Delta(R_{-i}(\hat{w}_i(r_i)))$.

Similarly to Section 3, condition (c) guarantees that a type r_i only assigns probabilities to opponents' views that it is aware of. More than this, we assume that type r_i is *not aware* of any opponents' views and types that are outside $R_{-i}(\hat{w}_i(r_i))$. As such, the whole epistemic model may be viewed as a description from the modeler's point of view, whereas the various types r_i may only be aware of a part of the epistemic model.

We call the types in this model *view-types,* since they generate belief hierarchies on views. For every view-type $r_i \in R_i$, let $h_i(r_i)$ be the belief hierarchy on views induced by r_i. The precise construction of this belief hierarchy can be found in Appendix A.2.1 of the Appendix.

Compare this to the epistemic models we considered in Definition 3.1, used to encode belief hierarchies on *choices and views.* In such an epistemic model $M = (T_i, \hat{v}_i, b_i)_{i\in I}$, every type t_i induces a belief hierarchy on choices and views, and hence also on views alone. Let $h_i(t_i)$ be the induced belief hierarchy on views. The precise construction of $h_i(t_i)$ can be found in Appendix A.2.2 of the Appendix.

With these definitions at hand, we can now formally define what we mean by common belief in rationality with fixed beliefs on views.

Definition 6.2 (*Common Belief in Rationality with Fixed Beliefs on Views*)**.** Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$, an epistemic model $M^{view} = (R_i, \hat{w}_i, p_i)_{i\in I}$ for views, and a view-type pair $(v_i, r_i) \in V_i \times R_i$ where $v_i = \hat{w}_i(r_i)$. A choice $c_i \in C_i(v_i)$ can rationally be made under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i, if there is an epistemic model $M = (T_j, \hat{v}_j, b_j)_{j\in I}$ for choices and views, and a type $t_i^* \in T_i$ with view $\hat{v}_i(t_i^*) = v_i$, such that $h_i(t_i^*) = h_i(r_i)$, type t_i^* expresses common belief in rationality, and c_i is optimal for t_i^*.

In contrast to Section 4, we now fix the belief hierarchy on views. In other words, among all possible stories that are compatible with common belief in rationality and the view v_i, we restrict to those stories that are consistent with the given belief hierarchy h_i on views.

In the following sections we will design a procedure that yields precisely the choices that can rationally be made under this concept, and show that it is always possible to reason in accordance with this concept.

6 Here, we use the term "epistemic model" in a broad sense, describing any model that encodes any sort of belief hierarchies. In this case, the model we present encodes belief hierarchies about views alone. Hence, the belief hierarchy (and also the types in the model) do not involve beliefs about choices. This is different from how epistemic models are defined at other places, where the types are required to specify beliefs about the opponents' choices.

6.2. Recursive procedure

We will now present a recursive elimination procedure, called *iterated strict dominance with fixed beliefs on views,* that characterizes precisely those choices that can rationally be made, with every possible view, under common belief in rationality with a fixed belief hierarchy on views. Not surprisingly, the procedure is quite similar to *iterated strict dominance for unawareness* (without fixed belief hierarchies on views). There are two important differences. The first is that decision problems will now be defined for *every view-type* $r_i \in R_i$ rather than for every view. Moreover, the sets $C^k_{-i}(v_i)$ of opponents' choice combinations as defined in iterated strict dominance with unawareness, restricting the possible beliefs that player i can hold at round k, will now be replaced by sets of possible *probabilistic* beliefs $B^k_i(r_i)$, representing the possible probabilistic beliefs that player i can hold at round k if he holds view $\hat{w}_i(r_i)$ and has the belief hierarchy on views induced by r_i.

To define the procedure formally, we need some additional notation. Consider some Euclidean space $\mathbf{R}^n$, some subsets $A_1, \ldots, A_K$ of $\mathbf{R}^n$, and some numbers $x_1, \ldots, x_K \in \mathbf{R}$. Then, by

$$\sum_{k\in\{1,\ldots,K\}} x_k \cdot A_k := \left\{ \sum_{k\in\{1,\ldots,K\}} x_k \cdot a_k \mid a_k \in A_k \text{ for all } k \in \{1, \ldots, K\} \right\}$$

we define the corresponding "linear combination" of these sets $A_1, \ldots, A_K$.

Definition 6.3 (*Iterated Strict Dominance with Fixed Beliefs on Views*)**.** Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$ and an epistemic model
$M^{view} = (R_i, \hat{w}_i, p_i)_{i\in I}$ for views.
(Initial step) For every player i, and every view-type $r_i \in R_i$, define

$$B^0_i(r_i) := \sum_{(r_j)_{j\neq i}\in R_{-i}(\hat{w}_i(r_i))} p_i(r_i)((r_j)_{j\neq i}) \cdot \Delta(\times_{j\neq i} C_j(\hat{w}_j(r_j))),$$

and $C^0_i(r_i) := C_i(\hat{w}_i(r_i))$.
(Inductive step) For $k \geq 1$, every player i, and every view-type $r_i \in R_i$, define

$$B^k_i(r_i) := \sum_{(r_j)_{j\neq i}\in R_{-i}(\hat{w}_i(r_i))} p_i(r_i)((r_j)_{j\neq i}) \cdot \Delta(\times_{j\neq i} C^{k-1}_j(r_j)),$$

and

$$C^k_i(r_i) := \{c_i \in C^{k-1}_i(r_i) | \quad c_i \text{ is optimal for some belief } \beta_i \in B^k_i(r_i) \text{ among choices in } C^{k-1}_i(r_i)\}.$$

A pair (c_i, r_i), consisting of a choice and view-type, is said to survive the procedure if $c_i \in C^k_i(r_i)$ for every $k \geq 0$.

More precisely, this procedure is the iterated strict dominance procedure with fixed beliefs on views *as given by* M^{view}. As a short-hand, we will refer to this procedure as the *iterated strict dominance procedure for* M^{view}.

Consider now the special case where every view-type in M^{view} assigns probability 1 to one specific view for every opponent. Then, it may be verified that the procedure above is equivalent to the *extensive-form rationalizability procedure* in Heifetz et al. (2013b), when applied to the special case of static games. The procedure in Heifetz et al. (2013b) is designed for *dynamic* games with unawareness, and hence can also be applied to *static* games.

Our procedure above is quite similar to the procedures of *interim correlated rationalizability* (Dekel et al., 2007) and *interim (independent) rationalizability* Ely and Pęski (2006) for games with *incomplete information*. Also interim correlated rationalizability assumes a fixed belief hierarchy, not on views but on *utility functions*. The interim correlated rationalizability procedure then recursively restricts, for every possible *utility function* and every belief hierarchy on *utilities*, the set of choices for the respective player. In turn, we recursively restrict the player's set of choices for every possible *view* and belief hierarchy on *views* (as encoded by a view-type r_i). Also Schipper (2016) offers an interim rationalizability concept for a context with network formation and unawareness.

Similarly to the case without fixed belief hierarchies on views, there is still an important difference between the two procedures. In the case of unawareness, not every belief hierarchy on views can be chosen, because this belief hierarchy must express common belief in smaller views for an appropriately chosen view of the respective player. A similar condition is not present in the case of incomplete information, as in principle every possible belief hierarchy on utility functions may be regarded as reasonable. The reason, again, is that in the context of incomplete information, a player with a certain utility function has mental access to *all* utility functions in the model — something that is not true for views in games with unawareness.

To conclude this subsection, we compare the case of fixed belief hierarchies on views to the case where these belief hierarchies are left free. Clearly, if for a given view v_i we look at each individual belief hierarchy on views h_i, then this is the same as putting no restrictions on the belief hierarchy on views. Consequently, if for every such belief hierarchy on views h_i we derive the choices that player i can rationally make under common belief in rationality with the view v_i and this particular belief hierarchy on views h_i (as defined in this section), then we should obtain exactly the choices that player i can rationally make under common belief in rationality with the view v_i (as defined in Section 5). We know by Theorem 6.2 that the choices that player i can rationally make under common belief in rationality with the view v_i and the fixed belief hierarchy on views h_i are given by the iterated strict dominance procedure with fixed beliefs on views. On the other hand, Theorem 5.2 guarantees that the choices that player i can rationally make under common belief in rationality with the view v_i are given by the iterated strict dominance procedure for unawareness. Consequently, if for a given view v_i and every possible belief hierarchy on views h_i that is accessible from v_i, we run the iterated strict dominance procedure with fixed beliefs on views, and collect all the delivered choices for player i at view v_i, this will deliver exactly the same output as when we would run the iterated strict dominance procedure for unawareness (without fixed beliefs on views) and look at the delivered choices for player i at v_i.

6.3. Non-empty output and characterization result

Like in Section 5, we first show that the procedure always delivers a non-empty output, and subsequently prove that the procedure yields, for every view-type, exactly those choices that can rationally be made under common belief in rationality with this particular view-type.

Theorem 6.1 (*Non-Empty Output*)**.** *Consider a static game with unawareness*
$G^u = (G^{base}, (V_i)_{i\in I})$ and an epistemic model $M^{view} = (R_i, \hat{w}_i, p_i)_{i\in I}$ for views. Then, for every player i, and every view-type $r_i \in R_i$, there is some choice $c_i \in C_i$ such that (c_i, r_i) survives the iterated strict dominance procedure for M^{view}.

The reader will note that the proof for this result is very similar to the one we gave for Theorem 5.1. We thus conclude that, no matter which belief hierarchy on views we impose, it is always possible for a player to reason in accordance with this particular belief hierarchy on views, while respecting common belief in rationality.

We next show that the procedure selects, for every view and every belief hierarchy on views encoded by M^{view}, exactly those choices that can rationally be made under common belief in rationality for this specific view and belief hierarchy on views.

Theorem 6.2 (*Characterization of Common Belief in Rationality*). *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$ and an epistemic model $M^{view} = (R_i, \hat{w}_i, p_i)_{i\in I}$ for views. Then, for every player i, every choice $c_i \in C_i$, every view $v_i \in V_i$, and every view-type $r_i \in R_i$ with $\hat{w}_i(r_i) = v_i$, player i can rationally choose c_i under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i, if and only if, (c_i, r_i) survives the iterated strict dominance procedure for M^{view}.*

Also here, the proof follows a similar structure as the one for Theorem 5.2. From Theorem 6.1 we know that the procedure always delivers a non-empty set of choices for every possible view-type in the game. The "if" direction of Theorem 6.2 therefore implies that for every view v_i and view-type r_i with $\hat{w}_i(r_i) = v_i$, we can always construct an epistemic model, and a type t_i^* within it with view v_i that expresses common belief in rationality, and which holds the belief hierarchy on views induced by r_i. The following result thus obtains.

Corollary 6.1 (*Common Belief in Rationality is Always Possible*). *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$ and an epistemic model $M^{view} = (R_i, \hat{w}_i, p_i)_{i\in I}$ for views. Then, for every player i, every view v_i, and every view-type $r_i \in R_i$ with $\hat{w}_i(r_i) = v_i$, there is an epistemic model $M = (T_j, \hat{v}_j, b_j)_{j\in I}$, and a type $t_i^* \in T_i$, such that t_i^* has view v_i, has the belief hierarchy on views induced by r_i, and expresses common belief in rationality.*

In other words, it is always possible to reason in accordance with common belief in rationality, while respecting the bounds set by a fixed view and a fixed belief hierarchy on views.

6.4. Example

To see how the procedure of *iterated strict dominance with fixed beliefs on views* works, consider the example "A day at the beach".

Example 3 (*Procedure for "A day at the Beach"*). Recall that you are unsure whether Barbara is aware of the two remote beaches or not. Assume now that you deem the event that she is aware of these two beaches equally likely as the event that she is not. In case Barbara is aware of the two remote beaches, you believe that Barbara believes that you are also aware of these two beaches. Indeed, you know by experience that Barbara believes that you are aware of everything that she is aware of herself. In case Barbara is not aware of these two beaches, she must of course believe that you are also not aware of these. This situation can be summarized by Table 3, with the fixed belief hierarchy on views induced by your view-type r_1 at the bottom of the table. This belief hierarchy on views is also graphically represented by the arrows between the various views. Indeed, if you have view v_1 and view-type r_1, then the induced belief hierarchy on views matches exactly the story above.

The *iterated strict dominance procedure for M^{view}* proceeds as follows.

Initial step. The initial sets of beliefs are given by

$$\begin{aligned} B_1^0(r_1) &= (0.5)\cdot\Delta(C_2(v_2)) + (0.5)\cdot\Delta(C_2(v_2')) \\ &= (0.5)\cdot\Delta(\{f,d,n,c\}) + (0.5)\cdot\Delta(\{n,c\}) \\ &= \{\beta_1 \in \Delta(\{f,d,n,c\}) | \beta_1(f)+\beta_1(d) \leq 0.5\} \end{aligned}$$

$$\begin{aligned} B_1^0(r_1') &= \Delta(C_2(v_2')) = \Delta(\{n,c\}), \\ B_2^0(r_2) &= \Delta(C_1(v_1)) = \Delta(\{f,d,n,c\}), \\ B_2^0(r_2') &= \Delta(C_1(v_1')) = \Delta(\{n,c\}), \end{aligned}$$

whereas the initial sets of choices are

$$\begin{aligned} &C_1^0(r_1) = \{f,d,n,c\},\ C_1^0(r_1') = \{n,c\}, \\ &C_2^0(r_2) = \{f,d,n,c\},\ C_2^0(r_2') = \{n,c\}. \end{aligned}$$

Round 1. By definition, the sets of beliefs remain the same as in the initial step. Note that choices n and c are not optimal for you at view v_1 for any belief in $B_1^1(r_1)$, and that Barbara's choice d is not optimal for her at view v_2 for any belief in $B_2^1(r_2)$. Hence, we obtain

$$\begin{aligned} &C_1^1(r_1) = \{f,d\},\ C_1^1(r_1') = \{n,c\}, \\ &C_2^1(r_2) = \{f,n,c\},\ C_2^1(r_2') = \{n,c\}. \end{aligned}$$

Round 2. The new sets of beliefs are

$$\begin{aligned} B_1^2(r_1) &= (0.5)\cdot\Delta(C_2^1(r_2)) + (0.5)\cdot\Delta(C_2^1(r_2')) \\ &= (0.5)\cdot\Delta(\{f,n,c\}) + (0.5)\cdot\Delta(\{n,c\}) \\ &= \{\beta_1 \in \Delta(\{f,n,c\}) | \beta_1(f) \leq 0.5\} \\ B_1^2(r_1') &= \Delta(C_2^1(r_2')) = \Delta(\{n,c\}), \\ B_2^2(r_2) &= \Delta(C_1^1(r_1)) = \Delta(\{f,d\}), \\ B_2^2(r_2') &= \Delta(C_1^1(r_1')) = \Delta(\{n,c\}). \end{aligned}$$

Then, Barbara's choices f and c are not optimal at her view v_2 for any belief in $B_2^2(r_2)$. The new sets of choices are thus given by

$$\begin{aligned} &C_1^2(r_1) = \{f,d\},\ C_1^2(r_1') = \{n,c\}, \\ &C_2^2(r_2) = \{n\},\ C_2^2(r_2') = \{n,c\}. \end{aligned}$$

Round 3. The new sets of beliefs are

$$\begin{aligned} B_1^3(r_1) &= (0.5)\cdot\Delta(C_2^2(r_2)) + (0.5)\cdot\Delta(C_2^2(r_2')) \\ &= (0.5)\cdot\Delta(\{n\}) + (0.5)\cdot\Delta(\{\cdot n,c\}) \\ &= \{\beta_1 \in \Delta(\{n,c\}) | \beta_1(c) \leq 0.5\}, \\ B_1^3(r_1') &= \Delta(C_2^2(r_2')) = \Delta(\{n,c\}), \\ B_2^3(r_2) &= \Delta(C_1^2(r_1)) = \Delta(\{f,d\}), \\ B_2^3(r_2') &= \Delta(C_1^2(r_1')) = \Delta(\{n,c\}). \end{aligned}$$

Note that at your view v_1, your choice d is not optimal for any belief in $B_1^3(r_1)$. Hence, the new sets of choices are

$$\begin{aligned} &C_1^3(r_1) = \{f\},\ C_1^3(r_1') = \{n,c\}, \\ &C_2^3(r_2) = \{n\},\ C_2^3(r_2') = \{n,c\}. \end{aligned}$$

Round 4. The new sets of beliefs are

$$\begin{aligned} B_1^4(r_1) &= \{\beta_1 \in \Delta(\{n,c\}) | \beta_1(c) \leq 0.5\} \\ B_1^4(r_1') &= \Delta(C_2^3(r_2')) = \Delta(\{n,c\}), \\ B_2^4(r_2) &= \Delta(C_1^3(r_1)) = \Delta(\{f\}), \\ B_2^4(r_2') &= \Delta(C_1^3(r_1')) = \Delta(\{n,c\}). \end{aligned}$$

Since no further choices can be eliminated from $C_1^3(r_1)$, $C_1^3(r_1')$, $C_2^3(r_2)$ and $C_2^3(r_2')$ we have that

$$\begin{aligned} &C_1^4(r_1) = C_1^3(r_1) = \{f\},\ C_1^4(r_1') = C_1^3(r_1') = \{n,c\}, \\ &C_2^4(r_2) = C_2^3(r_2) = \{n\},\ C_2^4(r_2') = C_2^3(r_2') = \{n,c\}, \end{aligned}$$

and the procedure terminates.

We thus conclude that you can only rationally go to the *Faraway Beach* under common belief in rationality with the view v_1 and the belief hierarchy on views induced by r_1.

Table 3
"A day at the beach" with fixed beliefs on views.

	G^{base}	Faraway	Distant	Nextdoor	Closeby			
Base game	Faraway	0,0	4,1	4,4	4,3			
	Distant	3,2	0,0	3,4	3,3			
	Nextdoor	2,2	2,1	0,0	2,3			
	Closeby	1,2	1,1	1,4	0,0			
	v_1	Faraway	Distant	Nextdoor	Closeby	v'_1	Nextdoor	Closeby
Your views	Faraway	0	4	4	4	Nextdoor	0	2
	Distant	3	0	3	3	Closeby	1	0
	Nextdoor	2	2	0	2			
	Closeby	1	1	1	0			
			↓ (0.5) ↑		(0.5) ↘		↓ ↑	
	v_2	Faraway	Distant	Nextdoor	Closeby	v'_2	Nextdoor	Closeby
Barbara's views	Faraway	0	2	2	2	Nextdoor	0	4
	Distant	1	0	1	1	Closeby	3	0
	Nextdoor	4	4	0	4			
	Closeby	3	3	3	0			
Epistemic model for views M^{view}					$R_1 = \{r_1, r'_1\}$, $R_2 = \{r_2, r'_2\}$ $\hat{w}_1(r_1) = v_1$ and $p_1(r_1) = (0.5) \cdot r_2 + (0.5) \cdot r'_2$ $\hat{w}_1(r'_1) = v'_1$ and $p_1(r'_1) = r'_2$ $\hat{w}_2(r_2) = v_2$ and $p_2(r_2) = r_1$ $\hat{w}_2(r'_2) = v'_2$ and $p_2(r'_2) = r'_1$			

Table 4
Base game in Feinberg's example.

	b_1	b_2	b_3
a_1	0,2	3,3	0,2
a_2	2,2	2,1	2,1
a_3	1,0	4,0	0,1

Compare this to the case where we did not fix the belief hierarchy on views. As we saw in Section 5, you could rationally visit the *Faraway Beach* and the *Distant Beach* under common belief in rationality with the view v_1 if we allow for *any* belief hierarchy on views that is cognitively feasible for v_1. Indeed, the epistemic model from Table 2 shows that under common belief in rationality with the view v_1, you can rationally choose the *Distant Beach* if you hold the belief hierarchy induced by type t'_1. In that belief hierarchy, you believe that Barbara has view v_2, believe that Barbara believes that you have view v'_1, believe that Barbara believes that you believe that Barbara has view v'_2, and so on. Clearly, this belief hierarchy is different from the one induced by r_1.

6.5. Different elimination orders

Similarly as we have done in Section 5.4 for the procedure without fixed beliefs on views, it can be shown that the procedure in this section is also order independent. That is, no matter in which order, and with what speed, we eliminate the beliefs and choices at the various view-types, we will always end up with the same output. In order to save space, we leave the proof to the reader.

This property allows us to choose, for a specific instance of a game with unawareness, and a given epistemic model for views, an order of elimination that is most convenient for this situation. In many cases, such an order will be similar to the bottom-up order of elimination that we discussed in Section 5.5 for the case without fixed beliefs on views. To illustrate this, we consider the following classical example by Feinberg (2005), which also appears in Heinsalu (2014) and Meier and Schipper (2014).

Example 4 (*Feinberg's Example*)**.** Consider the base game in Table 4 between two players, Alice and Bob. The choices for Alice (player 1) are in the rows, whereas the choices for Bob (player 2) are in the columns. Suppose that Bob is aware of all choices in the base game, believes that Alice is aware of all choices, but believes that Alice believes that Bob is unaware of her choice a_3. What choices could Bob then rationally choose under common belief in rationality?

As a first step towards answering the question, we first model this story as a static game with unawareness, with an associated epistemic model for views. See Table 5. The story above corresponds with view-type r_2 for Bob.

We will apply iterated strict dominance with fixed beliefs on views with the "most convenient" order of elimination. This means that we will start with the view-types r'_1 and r'_2 and do all the eliminations there, since these view-types have the smallest view $v' := v'_1 = v'_2$. We first eliminate choice b_3 for Bob since it is not optimal for any belief. Since no further choices for Alice or Bob can be removed after this elimination, we are done for view v' and the associated view-types r'_1 and r'_2. Hence, for the view-types r'_1 and r'_2 all choices survive except b_3.

We then turn to the view-type r_1 for Alice, which believes that, with probability 1, Bob's view-type is r'_2. As b_3 has been eliminated for r'_2, Alice believes that Bob will not choose b_3. But then, a_1 cannot be optimal for Alice at r_1. After removing a_1, no further eliminations are possible at r_1.

Finally, we move to view-type r_2 for Bob which believes that, with probability 1, Alice has view-type r_1. As a_1 has been eliminated for Alice at r_1, Bob believes that Alice will not choose a_1. As a consequence, b_2 cannot be optimal for Bob. After removing b_2 for Bob at r_2, no further eliminations are possible.

In particular, we see that only the choices b_1 and b_3 survive for Bob at r_2. That is, given the story above Bob can only rationally choose b_1 and b_3 under common belief in rationality.

7. Related literature

Roughly speaking, the literature on unawareness can be divided into two categories. The first category explores the logical foundations of unawareness in a single agent and multi-agent setting, without an explicit reference to games, whereas the second category applies the logic of unawareness to games. For a survey of this literature we refer the reader to Schipper (2014).

Table 5
Feinberg's example as a static game with unawareness with fixed beliefs on views.

	G^{base}	b_1	b_2	b_3				
Base game	a_1	0,2	3,3	0,2				
	a_2	2,2	2,1	2,1				
	a_3	1,0	4,0	0,1				
	v_1	b_1	b_2	b_3	v_1'	b_1	b_2	b_3
Alice's views	a_1	0	3	0	a_1	0	3	0
	a_2	2	2	2	a_2	2	2	2
	a_3	1	4	0				
				↘		↓		
			↑			↑		
	v_2	a_1	a_2	a_3	v_2'	a_1	a_2	
Bob's views	b_1	2	2	0	b_1	2	2	
	b_2	3	1	0	b_2	3	1	
	b_3	2	1	1	b_3	2	1	
Epistemic model for views M^{view}						$R_1 = \{r_1, r_1'\}$, $R_2 = \{r_2, r_2'\}$ $\hat{w}_1(r_1) = v_1$ and $p_1(r_1) = r_2'$ $\hat{w}_1(r_1') = v_1'$ and $p_1(r_1') = r_2'$ $\hat{w}_2(r_2) = v_2$ and $p_2(r_2) = r_1$ $\hat{w}_2(r_2') = v_2'$ and $p_2(r_2') = r_1'$		

An important question being addressed by the first category is how unawareness can be modeled in a meaningful way, both syntactically and semantically. See, for instance, Fagin and Halpern (1988), Dekel et al. (1998), Modica and Rustichini (1999), Halpern (2001), Heifetz et al. (2006, 2008, 2013a), Halpern and Rêgo (2008) and Li (2009).

A general conclusion in this literature is that in a multi-agent setting, every agent must be endowed with his own, *subjective* state space that only contains those objects he is aware of, and which therefore may be substantially smaller than the *full* state space. This principle is also reflected in our definition of a game with unawareness, and how we set up an epistemic model to encode belief hierarchies about choices and views.

To model a game with unawareness, we assume for every player a finite collection of possible views on the game. The implicit understanding is that a player with a certain view only has mental access to those choices that are part of his view, and to those views in the model that are smaller than his own. In other words, the subjective state space for a player with view v_i only contains the choices inside v_i, and the views for the opponents and himself that are contained in v_i.

Papers in the second category deal specifically with static or dynamic games with unawareness, and can thus be seen as applications of the logic of unawareness. See, for instance, Feinberg (2004), Feinberg (2021), Čopič and Galeotti (2006), Rêgo and Halpern (2012), Heifetz et al. (2013b), Grant and Quiggin (2013), Halpern and Rêgo (2014), Meier and Schipper (2014) and Schipper (2021). Our paper clearly falls within this category as well.

As we already mentioned in Section 2, an important difference between our way of modeling games with unawareness and that of most other papers is that we do not exogenously specify a unique belief hierarchy on views for every player. In fact, of the abovementioned papers only Čopič and Galeotti (2006) and Meier and Schipper (2014) do not fix the belief hierarchies on views in their model. Moreover, we allow for *probabilistic* belief hierarchies on views, whereas most papers above – exceptions being Feinberg (2004), Rêgo and Halpern (2012), Halpern and Rêgo (2014), Heifetz et al. (2013a) and Meier and Schipper (2014) – restrict to deterministic belief hierarchies on views. We find such probabilistic beliefs on views important, as they allow for cases where a player is truly uncertain about the precise view held by an opponent.

In terms of the approach adopted, this paper is one of the few to provide an *epistemic* analysis of the players' reasoning in games with unawareness, through the epistemic conditions of common belief in rationality. Another example is Guarino (2020), who offers an epistemic characterization of *extensive-form rationalizability* (Pearce, 1984; Battigalli, 1997; Heifetz et al., 2013b) for dynamic games with unawareness.

Like our paper, also Heifetz et al. (2013b) and Feinberg (2021) investigate the implications of common (strong) belief in rationality by studying the concepts of *rationalizability* and *extensive-form rationalizability*, respectively. One difference with our approach is that the latter papers do not investigate these concepts on an epistemic basis.

8. Concluding remarks

The goal of this paper has been to investigate the reasoning of players in static games with unawareness through the basic concept of *common belief in rationality*. Our approach has been primarily *epistemic*, as we started by formulating the epistemic conditions that constitute common belief in rationality, and subsequently designed a recursive elimination procedure that characterizes exactly those choices that can rationally be made, for every possible view, under this epistemic concept. We did so for two scenarios: one in which we only restrict the possible views that may enter the players' belief hierarchies, and one in which we fix the players' belief hierarchies on views.

An interesting open question is how one can epistemically characterize various *equilibrium* concepts that have been proposed for games with unawareness, such as *action-awareness equilibrium* (Čopič and Galeotti, 2006), *extended Nash equilibrium* (Feinberg, 2021), *generalized Nash equilibrium* (Halpern and Rêgo, 2014), *generalized sequential equilibrium* (Rêgo and Halpern, 2012), *sequential equilibrium* (Grant and Quiggin, 2013), *equilibrium of Bayesian game with unawareness* (Meier and Schipper, 2014) and *self-confirming equilibrium* (Schipper, 2021).

Another problem that could be addressed in the future is how one could formulate the backward induction concept of *common belief in future rationality* (Perea, 2014) for dynamic games with unawareness. Moreover, it could be explored how this concept would relate to the forward induction concept of *extensive-form rationalizability* as defined by Heifetz et al. (2013b) for dynamic games with unawareness. These, and other, open problems are left for future research.

Appendix

A.1. Proofs of Section 5

For the proofs of Section 5 we heavily rely on Lemma 3 in Pearce (1984). We will present this result below within the framework of decision problems, because we can then readily apply it for our specific purposes. Consider a decision problem (D_i, D_{-i}), a choice $c_i \in D_i$ and a probabilistic belief $\beta_i \in \Delta(D_{-i})$ about the opponents' choice combinations. Choice c_i is said to be *optimal* for β_i within the decision problem (D_i, D_{-i}) if

$$\sum_{c_{-i}\in D_{-i}} \beta_i(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in D_{-i}} \beta_i(c_{-i}) \cdot u_i(c_i', c_{-i}) \text{ for all } c_i' \in D_i.$$

Lemma 3 in Pearce (1984) states that a choice is optimal for at least one belief, if and only if, the choice is not strictly dominated.

Lemma A.1 (*Pearce, 1984*)**.** *Consider a decision problem (D_i, D_{-i}) and an available choice $c_i \in D_i$. Then, c_i is optimal for some probabilistic belief within the decision problem (D_i, D_{-i}), if and only if, c_i is not strictly dominated within the decision problem (D_i, D_{-i}).*

As we will see, this result is the cornerstone to the proofs of Section 5.

Proof of Theorem 5.1. Note that in the iterated strict dominance procedure for unawareness, $C_i^{k+1}(v_i) \subseteq C_i^k(v_i)$ for every player i, every view $v_i \in V_i$ and every round $k \geq 0$. Since there are only finitely many choices and views in the game, the procedure must terminate after finitely many rounds. That is, there is some $K \geq 0$ such that $C_i^k(v_i) = C_i^K(v_i)$ and $C_{-i}^k(v_i) = C_{-i}^K(v_i)$ for every player i, view $v_i \in V_i$ and every $k \geq K$. As such, it is sufficient to show that $C_i^k(v_i)$ is always non-empty for every player i, every view $v_i \in V_i$ and every $k \geq 0$. We prove so by induction on k.

For $k = 0$ this is clear since $C_i^0(v_i) = C_i(v_i)$, which is non-empty.

Take now some $k \geq 1$ and assume that $C_j^{k-1}(v_j)$ is non-empty for every player j and every view $v_j \in V_j$. Consider some player i and some view $v_i \in V_i$. We show that $C_i^k(v_i)$ is non-empty.

For every opponent $j \neq i$, take some view $v_j \in V_j$ that is contained in v_i. Note that such view v_j exists by Definition 2.1. For every opponent $j \neq i$, take a choice $c_j \in C_j^{k-1}(v_j)$, which is possible because $C_j^{k-1}(v_j)$ is non-empty by the induction assumption. Then, by construction, the choice combination $(c_j)_{j\neq i}$ is in $C_{-i}^k(v_i)$. Let the choice $c_i \in C_i(v_i)$ be optimal, among all choices in $C_i(v_i)$, for the belief β_i that assigns probability 1 to $(c_j)_{j\neq i}$. Hence, $\beta_i \in \Delta(C_{-i}^k(v_i))$. By Lemma A.1 it then follows that c_i is not strictly dominated within the decision problem $(C_i(v_i), C_{-i}^k(v_i))$. In particular, c_i is not strictly dominated within the decision problem $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$, and hence $c_i \in C_i^k(v_i)$. We thus conclude that $C_i^k(v_i)$ is non-empty.

By induction, it follows that $C_i^k(v_i)$ is always non-empty for every player i, every view $v_i \in V_i$ and every round $k \geq 0$. As we have seen, this completes the proof. ■

Proof of Theorem 5.2. (a) Suppose that choice c_m^* can rationally be made under common belief in rationality with view v_m^*. We show that (c_m^*, v_m^*) will survive the procedure.

For every player i and every view $v_i \in V_i$, let $C_i^{cbr}(v_i)$ be the set of choices in $C_i(v_i)$ that player i can rationally make under common belief in rationality with the view v_i. We show, by induction on k, that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$ for every $k \geq 0$, every player i and every view $v_i \in V_i$.

For $k = 0$ this is obviously true since $C_i^0(v_i) = C_i(v_i)$.

Now, consider some $k \geq 1$ and assume that $C_i^{cbr}(v_i) \subseteq C_i^{k-1}(v_i)$ for every player i and every view $v_i \in V_i$. Consider some player i, some view v_i, and assume that $c_i \in C_i^{cbr}(v_i)$. By the induction assumption we know that $c_i \in C_i^{k-1}(v_i)$. As $c_i \in C_i^{cbr}(v_i)$, there is some epistemic model $M = (T_j, \hat{v}_j, b_j)_{j\in I}$, and some type $t_i \in T_i$ with $\hat{v}_i(t_i) = v_i$, such that t_i expresses common belief in rationality and c_i is optimal for t_i. Let $b_i^C(t_i)$ be the marginal of the belief $b_i(t_i)$ on C_{-i}. Then, in light of the above,

$$\sum_{c_{-i}\in C_{-i}(v_i)} b_i^C(t_i)(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}(v_i)} b_i^C(t_i)(c_{-i}) \cdot u_i(c_i', c_{-i}) \text{ for all } c_i' \in C_i(v_i). \tag{A.1}$$

Since t_i expresses common belief in rationality, we conclude that $b_i^C(t_i)((c_j)_{j\neq i}) > 0$ only if, for every $j \neq i$, choice c_j is in $C_j^{cbr}(v_j)$ for some view $v_j \subseteq v_i$. Since by the induction assumption we have that $C_j^{cbr}(v_j) \subseteq C_j^{k-1}(v_j)$, we conclude that $b_i^C(t_i)((c_j)_{j\neq i}) > 0$ only if, for every $j \neq i$, choice c_j is in $C_j^{k-1}(v_j)$ for some view $v_j \subseteq v_i$. Hence, by definition of the procedure, $b_i^C(t_i) \in \Delta(C_{-i}^k(v_i))$.

In view of (A.1) we thus conclude that $c_i \in C_i^{k-1}(v_i)$ is optimal for the belief $b_i^C(t_i) \in \Delta(C_{-i}^k(v_i))$ within the reduced decision problem $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$. By Lemma A.1 it then follows that c_i is not strictly dominated for the reduced decision problem $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$, and hence $c_i \in C_i^k(v_i)$, by definition of the procedure. As this holds for every $c_i \in C_i^{cbr}(v_i)$, we conclude that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$, which was to show. By induction on k we conclude that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$ for every k, every player i and every view $v_i \in V_i$.

Now, let us return to the choice c_m^* and the view v_m^* such that c_m^* can rationally be made under common belief in rationality with the view v_m^*. Then, $c_m^* \in C_m^{cbr}(v_m^*)$ and hence, by the analysis above, $c_m^* \in C_m^k(v_m^*)$ for every $k \geq 0$. Hence, (c_m^*, v_m^*) survives the procedure, which completes the proof of part (a).

(b) Suppose that (c_m^*, v_m^*) survives the procedure. We show that c_m^* can rationally be made under common belief in rationality with the view v_m^*.

For every player i, and every view $v_i \in V_i$, let $C_i^\infty(v_i) := \cap_{k\geq 0} C_i^k(v_i)$ be the set of choices that survive the procedure for view v_i, and let $C_{-i}^\infty(v_i) := \cap_{k\geq 0} C_{-i}^k(v_i)$ be the set of opponents' choice combinations that survive the procedure at v_i. By Theorem 5.1 we know that all these sets $C_i^\infty(v_i)$ and $C_{-i}^\infty(v_i)$ are non-empty.

By construction, every choice $c_i \in C_i^\infty(v_i)$ is not strictly dominated within the decision problem $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$. Hence, by Lemma A.1, there is for every choice $c_i \in C_i^\infty(v_i)$ some belief $\beta_i^{c_i,v_i} \in \Delta(C_{-i}^\infty(v_i))$ such that c_i is optimal for $\beta_i^{c_i,v_i}$ within the decision problem $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$. We will show that, in fact, c_i is optimal for $\beta_i^{c_i,v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$. Let $c_i^* \in C_i(v_i)$ be optimal for $\beta_i^{c_i,v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$. Then, by Lemma A.1, c_i^* is not strictly dominated within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$, and hence c_i^* must be in $C_i^\infty(v_i)$. As c_i is optimal for $\beta_i^{c_i,v_i}$ within the decision problem $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$, it follows that

$$\sum_{c_{-i}\in C_{-i}^\infty(v_i)} \beta_i^{c_i,v_i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}^\infty(v_i)} \beta_i^{c_i,v_i}(c_{-i}) \cdot u_i(c_i^*, c_{-i}).$$

As c_i^* is optimal for $\beta_i^{c_i,v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$, it follows that c_i is optimal for $\beta_i^{c_i,v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$ as well. Moreover, since $\beta_i^{c_i,v_i} \in \Delta(C_{-i}^\infty(v_i))$ we know, by construction of the procedure, that $\beta_i^{c_i,v_i}$ only assigns positive probability to opponents' choices c_j where $c_j \in C_j^\infty(v_j[c_i, v_i, c_j])$ for some view $v_j[c_i, v_i, c_j] \subseteq v_i$.

Let us return to the pair (c_m^*, v_m^*) that survives the procedure. On the basis of the beliefs $\beta_i^{c_i,v_i}$ and views $v_j[c_i, v_i, c_j]$ above, we will construct an epistemic model $M = (T_i, \hat{v}_i, b_i)_{i\in I}$ such that there is some type t_m^* in M with view $\hat{v}_m(t_m^*) = v_m^*$ for which c_m^* is optimal.

In M, let the set of types for every player i be given by

$$T_i := \{t_i^{c_i,v_i} \mid v_i \in V_i(v_m^*) \text{ and } c_i \in C_i^\infty(v_i)\}.$$

Moreover, for every type $t_i^{c_i,v_i} \in T_i$, let the view be $\hat{v}_i(t_i^{c_i,v_i}) := v_i$, and let the belief $b_i(t_i^{c_i,v_i})$ on $\Delta(C_{-i}(\hat{v}_i(t_i^{c_i,v_i})) \times T_{-i}(\hat{v}_i(t_i^{c_i,v_i})))$ be given by

$$b_i(t_i^{c_i,v_i})((c_j,t_j)_{j\neq i}) := \begin{cases} \beta_i^{c_i,v_i}((c_j)_{j\neq i}), & \text{if } t_j = t_j^{c_j,v_j[c_i,v_i,c_j]} \\ & \text{for all } j \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it may be verified that conditions (c) and (d) in Definition 3.1 are satisfied, and hence $M = (T_i, \hat{v}_i, b_i)_{i\in I}$ is an epistemic model.

Note that every type $t_i^{c_i,v_i}$ has the belief $\beta_i^{c_i,v_i}$ about the opponents' choices. Since we have seen above that c_i is optimal for $\beta_i^{c_i,v_i}$ among all choices in $C_i(v_i)$, it follows that c_i is optimal for $t_i^{c_i,v_i}$ as well. By construction, every type $t_i^{c_i,v_i}$ only assigns positive probability to pairs $(c_j, t_j^{c_j,v_j})$ for every opponent $j \neq i$, where $c_j \in C_j^\infty(v_j)$. Since we have seen that c_j is optimal for $t_j^{c_j,v_j}$, it follows that every type $t_i^{c_i,v_i}$ in the epistemic model believes in the opponents' rationality. As a consequence, every type in the epistemic model expresses *common* belief in rationality.

Now, consider the pair (c_m^*, v_m^*) that we started with. As $v_m^* \in V_m(v_m^*)$ and $c_m^* \in C_m^\infty(v_m^*)$, it follows that $t_m^{c_m^*,v_m^*} \in T_m$. We have seen above that c_m^* is optimal for $t_m^{c_m^*,v_m^*}$, and that $t_m^{c_m^*,v_m^*}$ expresses common belief in rationality. As $\hat{v}_m(t_m^{c_m^*,v_m^*}) = v_m^*$, it thus follows that c_m^* can rationally be chosen under common belief in rationality with the view v_m^*. This completes the proof. ■

Proof of Theorem 5.3. Let $(D^0, \ldots, D^K)$ be the "full speed" elimination order, where $D^{k+1} = r(D^k)$ for all k, and let $(E^0, \ldots, E^L)$ be a different elimination order for iterated strict dominance for unawareness. We will show that $D^K = E^L$.

To show this, we first prove, by induction on k, that $D^k \subseteq E^k$ for all k.

For $k = 0$ this is true, since D^0 and E^0 are both the collection of full decision problems. Suppose now that $k \geq 1$, and make the induction assumption that $D^{k-1} \subseteq E^{k-1}$. Then, $D^k = r(D^{k-1})$ and

$$r(E^{k-1}) \subseteq E^k \subseteq E^{k-1}. \tag{A.2}$$

We show that $D^k \subseteq E^k$. Let $D^k = (D_i^k(v_i), D_{-i}^k(v_i))_{i\in I, v_i\in V_i}$ and $E^k = (E_i^k(v_i), E_{-i}^k(v_i))_{i\in I, v_i\in V_i}$. Take some $(c_j)_{j\neq i} \in D_{-i}^k(v_i)$. As $D^k = r(D^{k-1})$ we know, by definition, for every $j \neq i$ that $c_j \in D_j^{k-1}(v_j)$ for some $v_j \subseteq v_i$. By the induction assumption, it follows for every $j \neq i$ that $c_j \in E_j^{k-1}(v_j)$ for some $v_j \subseteq v_i$. That is, $(c_j)_{j\neq i}$ belongs to the set of opponents' choice combinations of $r(E^{k-1})$ at view v_i. Together with (A.2) we conclude that $(c_j)_{j\neq i} \in E_{-i}^k(v_i)$. We thus have shown that $D_{-i}^k(v_i) \subseteq E_{-i}^k(v_i)$.

Next, take some $c_i \in D_i^k(v_i)$. As $D^k = r(D^{k-1})$ we know, by definition, that c_i is not strictly dominated within $(D_i^{k-1}(v_i), D_{-i}^k(v_i))$. By Lemma A.1, there is a probabilistic belief $\beta_i \in \Delta(D_{-i}^k(v))$ such that

$$u_i(c_i, \beta_i) \geq u_i(c_i', \beta_i) \text{ for all } c_i' \in D_i^{k-1}(v_i). \tag{A.3}$$

We will show that, in fact, (A.3) holds for all $c_i' \in C_i(v_i)$. Suppose not. Then, there is some $c_i' \in C_i(v_i)$ with

$$u(c_i, \beta_i) < u_i(c_i', \beta_i). \tag{A.4}$$

Now, let c_i'' be the choice in $C_i(v_i)$ that is optimal for the belief β_i amongst all choices in $C_i(v_i)$. Then, by (A.4),

$$u_i(c_i, \beta_i) < u_i(c_i', \beta_i) \leq u_i(c_i'', \beta_i). \tag{A.5}$$

Recall that $\beta_i \in \Delta(D_{-i}^k(v_i))$. Since c_i'' is optimal for $\beta_i \in \Delta(D_{-i}^k(v))$ amongst all choices in $C_i(v_i)$, it follows by Lemma A.1 that c_i'' is not strictly dominated within $(C_i(v_i), D_{-i}^k(v_i))$. Hence, it must be that $c_i'' \in D_i^k(v_i)$ and hence, in particular, $c_i'' \in D_i^{k-1}(v_i)$. However, this insight together with (A.5) would contradict (A.3). We thus conclude that (A.3) holds for all $c_i' \in C_i(v_i)$.

By Lemma A.1 we may thus conclude that c_i is not strictly dominated in $(C_i(v_i), D_{-i}^k(v_i))$. As $D_{-i}^k(v_i) \subseteq E_{-i}^k(v_i)$, it follows that c_i is not strictly dominated in $(C_i(v_i), E_{-i}^k(v_i))$ and hence, in particular, c_i is not strictly dominated in $(E_i^{k-1}(v_i), E_{-i}^k(v_i))$. Thus, by definition, c_i is in $r(E^{k-1})$ at view v_i. This, together with (A.2), implies that $c_i \in E_i^k(v_i)$. Hence, we have shown that $D_i^k(v_i) \subseteq E_i^k(v_i)$.

Since we have already shown that $D_{-i}^k(v_i) \subseteq E_{-i}^k(v_i)$, it follows that $D^k \subseteq E^k$. By induction on k, this holds for all k. In particular, it follows that $D^K \subseteq E^L$.

We next show, by induction on k, that $E^L \subseteq D^k$ for every k.

For $k = 0$ this is obviously true, since D^0 is the collection of full decision problems. Let $k \geq 1$ and make the induction assumption that $E^L \subseteq D^{k-1}$.

Take first some $(c_j)_{j\neq i} \in E_{-i}^L(v_i)$. As $r(E^L) = E^L$, it must be, for every $j \neq i$, that $c_j \in E_j^L(v_j)$ for some $v_j \subseteq v_i$. By the induction assumption, we thus know, for every $j \neq i$, that $c_j \in D_j^{k-1}(v_j)$ for some $v_j \subseteq v_i$. Hence, $(c_j)_{j\neq i} \in D_{-i}^k(v_i)$. We have thus shown that $E_{-i}^L(v_i) \subseteq D_{-i}^k(v_i)$.

Next, take some $c_i \in E_i^L(v_i)$. As $r(E^L) = E^L$, it must be that c_i is not strictly dominated within $(E_i^L(v_i), E_{-i}^L(v_i))$. In the same way as above, it can be shown that, in fact, c_i is not strictly dominated within $(C_i(v_i), E_{-i}^L(v_i))$. As $E_{-i}^L(v_i) \subseteq D_{-i}^k(v_i)$, it follows that c_i is not strictly dominated within $(C_i(v_i), D_{-i}^k(v_i))$. This, in turn, implies that $c_i \in D_i^k(v_i)$. Hence, we have shown that $E_i^L(v_i) \subseteq D_i^k(v_i)$.

Since we have already shown that $E_{-i}^L(v_i) \subseteq D_{-i}^k(v_i)$, it follows that $E^L \subseteq D^k$. By induction on k, this holds for every k. In particular, $E^L \subseteq D^K$. Together with the insight above that $D^K \subseteq E^L$ we conclude that $D^K = E^L$.

So far, we have shown that for the "full speed" elimination order $(D^0, \ldots, D^K)$ and any other elimination order $(E^0, \ldots, E^L)$ we have that $D^K = E^L$. But then, every two elimination orders must yield the same output, which completes the proof. ■

A.2. Belief hierarchies on views induced by types

A.2.1. Epistemic models for views

Consider an epistemic model for views $M^{view} = (R_i, \hat{w}_i, p_i)_{i\in I}$. We show how, for every player i and every view-type $r_i \in R_i$, we can derive the induced belief hierarchy $h_i(r_i)$ on views. Formally, this belief hierarchy can be written as an infinite sequence of beliefs $h_i(r_i) = (h_i^1(r_i), h_i^2(r_i), \ldots)$, where $h_i^1(r_i)$ is the induced first-order belief, $h_i^2(r_i)$ is the induced second-order belief, and so on.

We will inductively define, for every n, the nth order beliefs induced by types r_i in M^{view}, building upon the $(n-1)$th order beliefs that have been defined in the preceding step. We start by defining the first-order beliefs.

For a given view v_i, let $R_i[v_i] := \{r_i \in R_i \mid \hat{w}_i(r_i) = v_i\}$. If $v_{-i} = (v_j)_{j \neq i}$, then we define $R_{-i}[v_{-i}] := \times_{j \neq i} R_j[v_j]$.

For every player i, and every type $r_i \in R_i$, define the first-order belief $h_i^1(r_i) \in \Delta(V_{-i})$ by

$$h_i^1(r_i)(v_{-i}) := p_i(r_i)(R_{-i}[v_{-i}]) \text{ for all } v_{-i} \in V_{-i}.$$

Now, suppose that $n \geq 2$, and assume that the $(n-1)$th order beliefs $h_i^{n-1}(r_i)$ have been defined for all players i, and every type $r_i \in R_i$. Let

$$h_i^{n-1}(R_i) := \{h_i^{n-1}(r_i) \mid r_i \in R_i\}$$

be the finite set of $(n-1)$th order beliefs for player i induced by types in R_i. For every $h_i^{n-1} \in h_i^{n-1}(R_i)$, let

$$R_i[h_i^{n-1}] := \{r_i \in R_i \mid h_i^{n-1}(r_i) = h_i^{n-1}\}$$

be the set of types in R_i that have the $(n-1)$th order belief h_i^{n-1}.

Let $h_{-i}^{n-1}(R_{-i}) := \times_{j \neq i} h_j^{n-1}(R_j)$, and for a given $h_{-i}^{n-1} = (h_j^{n-1})_{j \neq i}$ in $h_{-i}^{n-1}(R_{-i})$ let

$R_{-i}[h_{-i}^{n-1}] := \times_{j \neq i} R_j[h_j^{n-1}]$.

For every type $r_i \in R_i$, let the nth order belief $h_i^n(r_i) \in \Delta(V_{-i} \times h_{-i}^{n-1}(R_{-i}))$ be given by

$$h_i^n(r_i)(v_{-i}, h_{-i}^{n-1}) := p_i(r_i)(R_{-i}[v_{-i}] \cap R_{-i}[h_{-i}^{n-1}])$$

for every $v_{-i} \in V_{-i}$ and every $h_{-i}^{n-1} \in h_{-i}^{n-1}(R_{-i})$.

Finally, for every type $r_i \in R_i$, we denote by

$$h_i(r_i) := (h_i^n(r_i))_{n \in \mathbb{N}}$$

the *belief hierarchy on views* induced by r_i.

A.2.2. Epistemic models for choices and views

Consider an epistemic model for choices and views $M = (T_i, \hat{v}_i, b_i)_{i \in I}$. We show how, for every player i and every type $t_i \in T_i$, we can derive the induced belief hierarchy $h_i(t_i)$ on views. Formally, this belief hierarchy can be written as an infinite sequence of beliefs $h_i(t_i) = (h_i^1(t_i), h_i^2(t_i), \ldots)$, where $h_i^1(t_i)$ is the induced first-order belief on views, $h_i^2(t_i)$ is the induced second-order belief on views, and so on.

We will inductively define, for every n, the nth order beliefs on views induced by types t_i in M, building upon the $(n-1)$th order beliefs on views that have been defined in the preceding step. We start by defining the first-order beliefs.

For a given view v_i, let $T_i[v_i] := \{t_i \in T_i \mid \hat{v}_i(t_i) = v_i\}$. If $v_{-i} = (v_j)_{j \neq i}$, then we define $T_{-i}[v_{-i}] := \times_{j \neq i} T_j[v_j]$.

For every player i, and every type $t_i \in T_i$, define the first-order belief on views $h_i^1(t_i) \in \Delta(V_{-i})$ by

$$h_i^1(t_i)(v_{-i}) := b_i(t_i)(C_{-i} \times T_{-i}[v_{-i}]) \text{ for all } v_{-i} \in V_{-i}.$$

Now, suppose that $n \geq 2$, and assume that the $(n-1)$th order beliefs on views $h_i^{n-1}(t_i)$ have been defined for all players i, and every type $t_i \in T_i$. Let

$$h_i^{n-1}(T_i) := \{h_i^{n-1}(t_i) \mid t_i \in T_i\}$$

be the finite set of $(n-1)$th order beliefs for player i induced by types in T_i. For every $h_i^{n-1} \in h_i^{n-1}(T_i)$, let

$$T_i[h_i^{n-1}] := \{t_i \in T_i \mid h_i^{n-1}(t_i) = h_i^{n-1}\}$$

be the set of types in T_i that have the $(n-1)$th order belief h_i^{n-1}.

Let $h_{-i}^{n-1}(T_{-i}) := \times_{j \neq i} h_j^{n-1}(T_j)$, and for a given $h_{-i}^{n-1} = (h_j^{n-1})_{j \neq i}$ in $h_{-i}^{n-1}(T_{-i})$ let

$T_{-i}[h_{-i}^{n-1}] := \times_{j \neq i} T_j[h_j^{n-1}]$.

For every type $t_i \in T_i$, let the nth order belief on views $h_i^n(t_i) \in \Delta(V_{-i} \times h_{-i}^{n-1}(T_{-i}))$ be given by

$$h_i^n(t_i)(v_{-i}, h_{-i}^{n-1}) := b_i(t_i)(C_{-i} \times (T_{-i}[v_{-i}] \cap T_{-i}[h_{-i}^{n-1}]))$$

for every $v_{-i} \in V_{-i}$ and every $h_{-i}^{n-1} \in h_{-i}^{n-1}(T_{-i})$.

Finally, for every type $t_i \in T_i$, we denote by

$$h_i(t_i) := (h_i^n(t_i))_{n \in \mathbb{N}}$$

the *belief hierarchy on views* induced by t_i.

A.3. Proofs of Section 6

Proof of Theorem 6.1. Note that $C_i^{k+1}(r_i) \subseteq C_i^k(r_i)$ for every player i, every view-type $r_i \in R_i$, and every round $k \geq 0$. Since there are only finitely many choices and view-types, the procedure must terminate after finitely many rounds. That is, there is some $K \geq 0$ such that for every $k \geq K$, $C_i^k(r_i) = C_i^K(r_i)$ for every player i and every view-type $r_i \in R_i$. As such, it is sufficient to show that $C_i^k(r_i)$ is always non-empty for every player i, every view-type $r_i \in R_i$, and every $k \geq 0$. We prove so by induction on k.

For $k = 0$ this is clear since $C_i^0(r_i) = C_i(\hat{w}_i(r_i))$, which is non-empty.

Take now some $k \geq 1$ and assume that $C_j^{k-1}(r_j)$ is non-empty for every player j and every view-type $r_j \in R_j$. Consider some player i, and some view-type $r_i \in R_i$. Then, $B_i^k(r_i)$ is non-empty, since the choice sets $C_j^{k-1}(r_j)$ are non-empty for every $j \neq i$ and every $r_j \in R_j$.

Now, take some $b_i \in B_i^k(r_i)$ and some choice $c_i \in C_i(\hat{w}_i(r_i))$ that is optimal for b_i among choices in $C_i(\hat{w}_i(r_i))$. Then, c_i will also be optimal for b_i among choices in $C_i^{k-1}(r_i)$, and hence $c_i \in C_i^k(r_i)$. We thus conclude that $C_i^k(r_i)$ is non-empty.

By induction, it follows that $C_i^k(r_i)$ is always non-empty for every player i, every view-type $r_i \in R_i$, and every round $k \geq 0$. As we have seen, this completes the proof. ■

Proof of Theorem 6.2. (a) Take $r_m^* \in R_m$ with $\hat{w}_m(r_m^*) = v_m^*$, and suppose that c_m^* can rationally be chosen under common belief in rationality with view v_m^* and the belief hierarchy on views induced by r_m^*. We show that (c_m^*, r_m^*) survives the procedure.

Assume, without loss of generality, that different types in M^{view} induce different belief hierarchies on views. For every player i and every view-type $r_i \in R_i$, let $C_i^{cbr}(r_i)$ be the set of choices in $C_i(\hat{w}_i(r_i))$ that player i can rationally make under common belief in rationality with the view $\hat{w}_i(r_i)$ and the belief hierarchy on views induced by r_i. We show, by induction on k, that $C_i^{cbr}(r_i) \subseteq C_i^k(r_i)$ for every $k \geq 0$, every player i, and every view-type $r_i \in R_i$.

For $k = 0$ this is obviously true since $C_i^0(r_i) = C_i(\hat{w}_i(r_i))$.

Now, consider some $k \geq 1$ and assume that $C_i^{cbr}(r_i) \subseteq C_i^{k-1}(r_i)$ for every player i and every view-type $r_i \in R_i$. Consider some player i and some view-type $r_i \in R_i$, and assume that $c_i \in C_i^{cbr}(r_i)$. Then, there is some epistemic model $M = (T_j, \hat{v}_j, b_j)_{j \in I}$, and some type $t_i \in T_i$, such that $\hat{v}_i(t_i) = \hat{w}_i(r_i)$, $h_i(t_i) = h_i(r_i)$, type t_i expresses common belief in rationality, and where c_i is optimal for t_i.

Let $b_i^C(t_i)$ be the marginal of the belief $b_i(t_i)$ on C_{-i}. Later, we will show that $b_i^C(t_i) \in B_i^k(r_i)$. In order to do so, we need two preliminary observations.

First, since $h_i(t_i) = h_i(r_i)$, there is for every opponent j, and every view-type r_j that receives positive probability under $p_i(r_i)$, some set of types $T_j(r_j)$ such that

$$h_j(t_j) = h_j(r_j) \text{ for all } t_j \in T_j(r_j), \tag{A.6}$$

and

$$b_i(t_i)(\times_{j\neq i}(C_j \times T_j(r_j))) = p_i(r_i)((r_j)_{j\neq i}) \tag{A.7}$$

for all $(r_j)_{j\neq i}$ in R_{-i} with $p_i(r_i)((r_j)_{j\neq i}) > 0$. Here, we use the assumption above that different types in M^{view} induce different belief hierarchies on views.

Second, since t_i expresses common belief in rationality, we have that $b_i(t_i)((c_j, t_j)_{j\neq i}) > 0$ only if for every opponent $j \neq i$, type t_j expresses common belief in rationality, and c_j is optimal for t_j. Note that in this case, there must be some $r_j \in R_j$ with $t_j \in T_j(r_j)$, in view of (A.7). Hence, by (A.6), we know that $h_j(t_j) = h_j(r_j)$. Together with the facts that c_j is optimal for t_j, and t_j expresses common belief in rationality, it follows that $c_j \in C_j^{cbr}(r_j)$ in this case. By the induction assumption, $C_j^{cbr}(r_j) \subseteq C_j^{k-1}(r_j)$. We thus conclude that

$$b_i(t_i)((c_j, t_j)_{j\neq i}) > 0 \text{ only if } t_j \in T_j(r_j) \text{ and } c_j \in C_j^{k-1}(r_j) \tag{A.8}$$

for all opponents $j \neq i$.

We will now use (A.7) and (A.8) to prove that $b_i^C(t_i) \in B_i^k(r_i)$. That is, we must show that

$$b_i^C(t_i) = \sum_{(r_j)_{j\neq i}\in R_{-i}} p_i(r_i)((r_j)_{j\neq i}) \cdot \beta_i^{(r_j)_{j\neq i}}, \tag{A.9}$$

where $\beta_i^{(r_j)_{j\neq i}} \in \Delta(\times_{j\neq i} C_j^{k-1}(r_j))$ for all $(r_j)_{j\neq i}$ with $p_i(r_i)((r_j)_{j\neq i}) > 0$.

Let

$$R_{-i}^* := \{(r_j)_{j\neq i} \in R_{-i} \mid p_i(r_i)((r_j)_{j\neq i}) > 0\}.$$

For every $(r_j)_{j\neq i} \in R_{-i}^*$, define $\beta_i^{(r_j)_{j\neq i}}$ by

$$\beta_i^{(r_j)_{j\neq i}}((c_j)_{j\neq i}) := \frac{b_i(t_i)(\times_{j\neq i}\{c_j\} \times T_j(r_j))}{p_i(r_i)((r_j)_{j\neq i})}. \tag{A.10}$$

Then, it may be verified that $\beta_i^{(r_j)_{j\neq i}}$ is a probability distribution on C_{-i}, since

$$\beta_i^{(r_j)_{j\neq i}}(C_{-i}) = \frac{b_i(t_i)(\times_{j\neq i}(C_j \times T_j(r_j)))}{p_i(r_i)((r_j)_{j\neq i})} = 1$$

because of (A.7).

We next show that $\beta_i^{(r_j)_{j\neq i}}$ only assigns positive probability to $(c_j)_{j\neq i} \in \times_{j\neq i} C_j^{k-1}(r_j)$. Indeed, suppose that $\beta_i^{(r_j)_{j\neq i}}((c_j)_{j\neq i}) > 0$. Then, by (A.10), $b_i(t_i)(\times_{j\neq i}\{c_j\}\times T_j(r_j)) > 0$, and hence we conclude by (A.8) that $c_j \in C_j^{k-1}(r_j)$ for every $j \neq i$. Hence, $(c_j)_{j\neq i} \in \times_{j\neq i} C_j^{k-1}(r_j)$. We may thus conclude that

$$\beta_i^{(r_j)_{j\neq i}} \in \Delta(\times_{j\neq i} C_j^{k-1}(r_j)) \text{ for every } (r_j)_{j\neq i} \in R_{-i}^*. \tag{A.11}$$

We finally show (A.9). By definition, for every $(c_j)_{j\neq i}$ in C_{-i}, we have that

$$\begin{aligned} b_i^C(t_i)((c_j)_{j\neq i}) &= \sum_{(t_j)_{j\neq i}\in T_{-i}} b_i(t_i)((c_j, t_j)_{j\neq i}) \\ &= \sum_{(r_j)_{j\neq i}\in R_{-i}^*} b_i(t_i)(\times_{j\neq i}\{c_j\} \times T_j(r_j)) \\ &= \sum_{(r_j)_{j\neq i}\in R_{-i}^*} p_i(r_i)((r_j)_{j\neq i}) \cdot \frac{b_i(t_i)(\times_{j\neq i}\{c_j\} \times T_j(r_j))}{p_i(r_i)((r_j)_{j\neq i})} \\ &= \sum_{(r_j)_{j\neq i}\in R_{-i}^*} p_i(r_i)((r_j)_{j\neq i}) \cdot \beta_i^{(r_j)_{j\neq i}}((c_j)_{j\neq i}), \end{aligned}$$

which implies (A.9). Here, the second equality follows from (A.7), whereas the fourth equality follows from (A.10). But then, we conclude from (A.9) and (A.11) that $b_i^C(t_i) \in B_i^k(r_i)$.

Remember from above that c_i is optimal for t_i. Hence, c_i is optimal for the marginal belief $b_i^C(t_i) \in B_i^k(r_i)$ among choices in $C_i(\hat{w}_i(r_i))$, which implies that $c_i \in C_i^k(r_i)$. As this holds for every $c_i \in C_i^{cbr}(r_i)$, we conclude that $C_i^{cbr}(r_i) \subseteq C_i^k(r_i)$. By induction on k, we may then conclude that $C_i^{cbr}(r_i) \subseteq C_i^k(r_i)$ for every k.

Now, return to the triple (c_m^*, v_m^*, r_m^*) where $\hat{w}_m(r_m^*) = v_m^*$, and such that c_m^* can rationally be chosen under common belief in rationality with the view v_m^* and the belief hierarchy on views induced by r_m^*. Then, by definition, $c_m^* \in C_m^{cbr}(r_m^*)$. By the conclusion above that $C_m^{cbr}(r_m^*) \subseteq C_m^k(r_m^*)$ for every k, it follows that $c_m^* \in C_m^k(r_m^*)$ for every k. Hence, (c_m^*, r_m^*) survives the procedure. This completes the proof of part (a).

(b) Suppose that (c_m^*, r_m^*) survives the procedure, with $\hat{w}_m(r_m^*) = v_m^*$. We show that c_m^* can rationally be chosen under common belief in rationality with view v_m^* and belief hierarchy on views induced by r_m^*. To that purpose, we will construct an epistemic model $M = (T_i, \hat{v}_i, b_i)_{i\in I}$ with a type $t_m^* \in T_m$ that has the view $\hat{v}_m(t_m^*) = v_m^*$ and where $h_m(t_m^*) = h_m(r_m^*)$, such that t_m^* expresses common belief in rationality, and c_m^* is optimal for t_m^*.

For every player i and every view-type $r_i \in R_i$, let $C_i^\infty(r_i) := \cap_{k\geq 0} C_i^k(r_i)$ be the set of choices that survive the procedure for view-type r_i, and let $B_i^\infty(r_i) := \cap_{k\geq 0} B_i^k(r_i)$ be the set of beliefs that survive the procedure at r_i. By Theorem 6.1 we know that all these sets $C_i^\infty(r_i)$ and $B_i^\infty(r_i)$ are non-empty.

By construction, every choice $c_i \in C_i^\infty(r_i)$ is optimal for some belief $\beta_i^{c_i,r_i} \in B_i^\infty(r_i)$ among choices in $C_i^\infty(r_i)$. We will show that, in fact, c_i is optimal for $\beta_i^{c_i,r_i}$ among choices in $C_i(\hat{w}_i(r_i))$. Let $c_i^* \in C_i(\hat{w}_i(r_i))$ be optimal for $\beta_i^{c_i,r_i}$ among choices in $C_i(\hat{w}_i(r_i))$. Then, c_i^* is in $C_i^\infty(r_i)$. As c_i is optimal for $\beta_i^{c_i,r_i}$ among choices in $C_i^\infty(r_i)$, it follows that

$$\sum_{c_{-i}\in C_{-i}(\hat{w}_i(r_i))} \beta_i^{c_i,r_i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}(\hat{w}_i(r_i))} \beta_i^{c_i,r_i}(c_{-i}) \cdot u_i(c_i^*, c_{-i}).$$

As c_i^* is optimal for $\beta_i^{c_i,r_i}$ among choices in $C_i(\hat{w}_i(r_i))$, it follows that c_i is optimal for $\beta_i^{c_i,r_i}$ among choices in $C_i(\hat{w}_i(r_i))$ as well.

Moreover, since

$$\beta_i^{c_i,r_i} \in B_i^\infty(r_i) = \sum_{(r_j)_{j\neq i}\in R_{-i}} p_i(r_i)((r_j)_{j\neq i}) \cdot \Delta(\times_{j\neq i} C_j^\infty(r_j)),$$

there is for every $(r_j)_{j\neq i} \in R_{-i}$ that receives positive probability under $p_i(r_i)$, some belief $\gamma_i^{c_i,r_i}[(r_j)_{j\neq i}] \in \Delta(\times_{j\neq i} C_j^\infty(r_j))$ such that

$$\beta_i^{c_i,r_i} = \sum_{(r_j)_{j\neq i}\in R_{-i}} p_i(r_i)((r_j)_{j\neq i}) \cdot \gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]. \tag{A.12}$$

Recall the triple (c_m^*, v_m^*, r_m^*) above, where (c_m^*, r_m^*) survives the procedure and $\hat{w}_m(r_m^*) = v_m^*$. On the basis of the beliefs $\beta_i^{c_i,r_i}$ above we now construct the following epistemic model $M = (T_i, \hat{v}_i, b_i)_{i\in I}$. Let the set of types for every player i be given by

$$T_i = \{t_i^{c_i,r_i} \mid r_i \in R_i,\ \hat{w}_i(r_i) \in V_i(v_m^*) \text{ and } c_i \in C_i^\infty(r_i)\}.$$

Moreover, for every type $t_i^{c_i,r_i} \in T_i$, let the view be $\hat{v}_i(t_i^{c_i,r_i}) := \hat{w}_i(r_i)$, and let the belief $b_i(t_i^{c_i,r_i})$ on $C_{-i} \times T_{-i}$ be given by

$$b_i(t_i^{c_i,r_i})((c_j, t_j)_{j\neq i}) := \begin{cases} p_i(r_i)((r_j)_{j\neq i}) \cdot \gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]((c_j)_{j\neq i}), & \text{if } t_j = t_j^{c_j,r_j} \text{ for all } j \neq i, \\ 0, & \text{otherwise} \end{cases}. \tag{A.13}$$

To show that the epistemic model M satisfies conditions (c) and (d) in Definition 3.1, suppose that type $t_i^{c_i,r_i}$ assigns positive probability to some choice-type pair $(c_j, t_j^{c_j,r_j})$. Then, by (A.13), $p_i(r_i)$ assigns positive probability to r_j. By condition (c) in Definition 6.1, this is only possible when $\hat{w}_j(r_j) \subseteq \hat{w}_i(r_i)$. As $\hat{v}_i(t_i^{c_i,r_i}) =$

$\hat{w}_i(r_i)$ and $\hat{v}_j(t_j^{c_j,r_j}) = \hat{w}_j(r_j)$, it thus follows that $\hat{v}_j(t_j^{c_j,r_j}) \subseteq \hat{v}_i(t_i^{c_i,r_i})$. Hence, $t_j^{c_j,r_j} \in T_j(\hat{v}_i(t_i^{c_i,r_i}))$. This establishes condition (c) in Definition 3.1.

Moreover, if type $t_i^{c_i,r_i}$ assigns positive probability to some choice-type pair $(c_j, t_j^{c_j,r_j})$, then by (A.13) the belief $\gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]$ assigns positive probability to c_j. Since $\gamma_i^{c_i,r_i}[(r_j)_{j\neq i}] \in \Delta(\times_{j\neq i} C_j^\infty(r_j))$, it follows that $c_j \in C_j^\infty(r_j)$, and hence $c_j \in C_j(\hat{w}_j(r_j))$. Since $\hat{v}_j(t_j^{c_j,r_j}) = \hat{w}_j(r_j)$, we thus conclude that $c_j \in C_j(\hat{v}_j(t_j^{c_j,r_j}))$. Hence, condition (d) in Definition 3.1 is satisfied.

Altogether, we see that M satisfies (c) and (d) in Definition 3.1, and hence M is an epistemic model.

We next show that every type $t_i^{c_i,r_i}$ holds the belief $\beta_i^{c_i,r_i}$ about the opponents' choices. Let $b_i^C(t_i^{c_i,r_i})$ be the marginal belief of type $t_i^{c_i,r_i}$ on C_{-i}. Then, for every $(c_j)_{j\neq i} \in C_{-i}$ we have that

$$\begin{aligned} b_i^C(t_i^{c_i,r_i})((c_j)_{j\neq i}) &= \sum_{(t_j)_{j\neq i}\in T_{-i}} b_i(t_i^{c_i,r_i})((c_j, t_j)_{j\neq i}) \\ &= \sum_{(r_j)_{j\neq i}\in R_{-i}} b_i(t_i^{c_i,r_i})((c_j, t_j^{c_j,r_j})_{j\neq i}) \\ &= \sum_{(r_j)_{j\neq i}\in R_{-i}} p_i(r_i)((r_j)_{j\neq i}) \cdot \gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]((c_j)_{j\neq i}) \\ &= \beta_i^{c_i,r_i}((c_j)_{j\neq i}), \end{aligned}$$

where the second and third equality follow from (A.13), and the last equality follows from (A.12). Hence, we conclude that $t_i^{c_i,r_i}$ holds the belief $\beta_i^{c_i,r_i}$ about the opponents' choices.

Note that, by construction, $c_i \in C_i^\infty(r_i)$ for every type $t_i^{c_i,r_i} \in T_i$. Since we have seen above that c_i is optimal for $\beta_i^{c_i,r_i}$ among choices in $C_i(\hat{w}_i(r_i))$, and that $t_i^{c_i,r_i}$ holds the belief $\beta_i^{c_i,r_i}$ about the opponents' choices, it follows that c_i is optimal for $t_i^{c_i,r_i}$.

We use this to show that every type $t_i^{c_i,r_i}$ believes in the opponents' rationality. Suppose that $b_i(t_i^{c_i,r_i})((c_j, t_j^{c_j,r_j})_{j\neq i}) > 0$. Since $t_j^{c_j,r_j}$ holds the belief $\beta_j^{c_j,r_j}$ about the opponents' choices, and c_j is optimal for $\beta_j^{c_j,r_j}$ among the choices in $C_j(\hat{w}_j(r_j))$, it follows that c_j is optimal for $t_j^{c_j,r_j}$ for every $j \neq i$. Hence, $t_i^{c_i,r_i}$ indeed believes in the opponents' rationality. As this holds for all types in the epistemic model M, we conclude that all types in M with view v_m^* express *common* belief in rationality.

We finally show that every type $t_i^{c_i,r_i}$ has the belief hierarchy on views induced by the view-type r_i. For every $(r_j)_{j\neq i} \in R_{-i}$ we have that

$$\begin{aligned} &\sum_{(c_j)_{j\neq i}\in C_{-i}} b_i(t_i^{c_i,r_i})((c_j, t_j^{c_j,r_j})_{j\neq i}) \\ &= \sum_{(c_j)_{j\neq i}\in C_{-i}} p_i(r_i)((r_j)_{j\neq i}) \cdot \gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]((c_j)_{j\neq i}) \\ &= p_i(r_i)((r_j)_{j\neq i}) \cdot \sum_{(c_j)_{j\neq i}\in C_{-i}} \gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]((c_j)_{j\neq i}) \\ &= p_i(r_i)((r_j)_{j\neq i}), \end{aligned} \tag{A.14}$$

where the first equality follows from (A.13), and the last equality follows from the fact that $\gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]$ is a probability distribution on C_{-i}, and hence

$$\sum_{(c_j)_{j\neq i}\in C_{-i}} \gamma_i^{c_i,r_i}[(r_j)_{j\neq i}]((c_j)_{j\neq i}) = 1.$$

Eq. (A.14) thus states that the probability that type $t_i^{c_i,r_i}$ assigns to the set of tuples $\{(t_j^{c_j,r_j})_{j\neq i} \mid (c_j)_{j\neq i} \in C_{-i}\}$ is the same as the probability that view-type r_i assigns to the tuple $(r_j)_{j\neq i}$. Since this holds for every type $t_i^{c_i,r_i}$ in the epistemic model M, we conclude that every type $t_i^{c_i,r_i}$ in M has the belief hierarchy on views induced by r_i. That is, $h_i(t_i^{c_i,r_i}) = h_i(r_i)$ for every type $t_i^{c_i,r_i}$ in M.

Let us return to the triple (c_m^*, v_m^*, r_m^*) above, where (c_m^*, r_m^*) survives the procedure and $\hat{w}_m(r_m^*) = v_m^*$. Then, $c_m^* \in C_m^\infty(r_m^*)$. As $\hat{w}_m(r_m^*) = v_m^* \in V(v_m^*)$, it follows that $t_m^{c_m^*,r_m^*} \in T_m$. We have seen above that c_m^* is optimal for $t_m^{c_m^*,r_m^*}$, that $t_m^{c_m^*,r_m^*}$ expresses common belief in rationality, that $h_m(t_m^{c_m^*,r_m^*}) = h_m(r_m^*)$ and that r_m^* has the view v_m^*. It thus follows that c_m^* can rationally be chosen under common belief in rationality with the view v_m^* and the belief hierarchy on views induced by r_m^*. This completes the proof. ■

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