Chapter 7

Common Belief in Rationality with Unawareness

In the previous two chapters we have investigated scenarios where the decision makers are uncertain about the decision problems of others. Remember that a decision problem for you in a game consists of the following three ingredients: (i) the choices that you believe are available to your opponents, (ii) the choices that you believe are available to yourself, and (iii) your conditional preference relation, which assigns to every belief about the opponents' choices a preference relation over your own choices.

To be precise, the uncertainty we considered in the previous two chapters only concerned the third component, not the first two components. Indeed, we implicitly assumed that everybody involved knew exactly which choices were available to every decision maker, including himself.

In this chapter we turn to situations where a player may be *unaware* of some choices that are actually available to his opponents, and may even be *unaware* of certain choices that are actually available to himself. The crucial difference with *uncertainty* is that being unaware of an event precludes you to even reason about this event – the event is simply not in your dictionary, and you cannot even contemplate the possibility of this particular event.

More concretely, if you can actually make the choice a, but believe that the opponent is unaware of a, then you believe that the opponent cannot even reason about the possibility that you could ever choose a. You believe that the choice a is not in the vocabulary of the opponent.

For a given player, the choices he believes are available to his opponents, together with the choices he believes are available to himself, constitute the *view* of that player. In particular, if you hold a certain view, you may well believe that your opponents hold a view different from yours.

In this chapter we start by an example that illustrates the notion of unawareness in games, and show informally how a player can reason in accordance with common belief in rationality there. Next, we explain that belief hierarchies must satisfy the following condition: If you are unaware of a choice a, then you cannot believe that an opponent is aware of the same choice a. Similarly, if you believe that an opponent is unaware of a choice a, then you must believe that this opponent cannot believe that somebody else is aware of a, and so on. This condition, which we refer to as the *awareness principle,* makes the analysis fundamentally different from that of games with incomplete information.

Once the awareness principle is being imposed, the concept of common belief in rationality can be defined analogously to how we did it for games with incomplete information. We introduce a recursive procedure, *iterated strict dominance for unawareness,* that yields precisely those choices that can rationally be made under common belief in rationality. As you will see, the procedure is very similar to the *generalized iterated strict dominance* procedure we used for games with incomplete information.

Subsequently, we impose *fixed beliefs on views,* similarly to how we imposed fixed beliefs on utilities in games with incomplete information. More precisely, we fix a belief hierarchy that you may hold about the players' views in the games, and explore which choices you can rationally make under common belief in rationality with this particular belief hierarchy on views.

Finally, it is shown that additionally imposing *simple*, or even *symmetric*, belief hierarchies necessarily leads to trivial cases of unawareness, where you believe that it is commonly believed that everybody holds exactly the *same view* of the game. That is, we would essentially be back to the case of standard games, where all players hold the same view of the game. For that reason, we do not devote a seperate chapter to the case of correct and symmetric beliefs here.

In Chapter 7 of the online appendix we discuss some economic applications of games with unawareness.

7.1 Unawareness

We first present an example, illustrating the notion of unawareness in games. Based on this example, we then provide a general definition of a game with unawareness.

7.1.1 Example

As already announced above, we will study situations where a player may be unaware of some choices that are available to others, or even to himself. This new phenomenon will be illustrated by the following example.

Example 7.1: A day at the beach.

You and Barbara spend the holiday on a small island, where there are four beaches: *Nextdoor Beach, Closeby Beach, Faraway Beach* and *Distant Beach.* The first two beaches are close to the hotel, and both you and Barbara are aware of these beaches. Moreover, both of you know this. The other two beaches are really far away, and very difficult to find. Even your phone is not able to find these beaches.

However, yesterday, when making a long and nice walk, you discovered these two beaches by accident. The problem is that you do not know whether Barbara is aware of these two beaches or not. You will also not ask her because you had another fierce discussion yesterday, and therefore you would rather not see Barbara today. The same holds for Barbara.

At the same time, you would like to go to the beach this morning, because the weather is simply splendid. But which beach should you go to?

You have seen all four beaches, and although all of them are nice, you prefer *Faraway Beach* to *Distant Beach,* you prefer *Distant Beach* to *Nextdoor Beach,* and *Nextdoor Beach* to *Closeby Beach.* But remember that you want to avoid Barbara today.

You	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	4	4	4
Distant	3	0	3	3
Nextdoor	2	2	0	2
Closeby	1	1	1	0

v_1^{all}

You	*Nextdoor*	*Closeby*
Nextdoor	0	2
Closeby	1	0

v_1^{two}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	2	2	2
Distant	1	0	1	1
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Barbara	*Nextdoor*	*Closeby*
Nextdoor	0	4
Closeby	3	0

v_2^{two}

Table 7.1.1 Decision problems for "A day at the beach"

Yesterday, before the fight, Barbara told you that she likes *Nextdoor Beach* better than *Closeby Beach.* Moreover, you believe that Barbara prefers *Closeby Beach* to *Faraway Beach,* and prefers *Faraway Beach* to *Distant Beach,* in case she is aware of the last two beaches. But, as stated above, you do not know whether Barbara is aware of these two beaches or not. Similarly to you, also Barbara prefers to avoid your presence today.

This situation can be represented by the decision problems in Table 7.1.1. Here, v_2^{all} represents Barbara's state of mind where she is aware of all four beaches, whereas v_2^{two} is her state of mind where she is only aware of the two beaches close to the hotel. We refer to v_2^{all} and v_2^{two} as the possible *views* for Barbara. You are thus uncertain about the view that Barbara has: She could either have view v_2^{all} or view v_2^{two}.

Similarly, v_1^{all} and v_1^{two} represent your views where you are aware of all beaches, and where you are only aware of the two beaches close to the hotel, respectively. Remember from the story that your actual view is v_1^{all}. Why, then, do we include your smaller view v_1^{two} in the table?

Well, if you believe that Barbara holds the small view v_2^{two}, then you must necessarily believe that Barbara believes that your view is v_1^{two}, and not your actual view v_1^{all}. Indeed, if you believe that Barbara is only aware of the two closest beaches, then you think that Barbara is unaware of the existence of any other beaches on the island. Therefore, you must believe that Barbara believes that you are only aware of the two closest beaches as well, because the other two beaches are simply not in Barbara's vocabulary.

7.1.2 Reasoning about Others' Decision Problems

Which beaches can you rationally go to under *common belief in rationality*? To start with, note that in your decision problem with view v_1^{all}, the choice *Closeby Beach* is strictly dominated by the randomized choice where you select *Faraway Beach* and *Distant Beach* with probability 0.5. Therefore, by Theorem 2.6.1, going to *Closeby Beach* can never be optimal for you for any belief, and can thus be eliminated.

Similarly, at Barbara's view v_2^{all}, her choice *Distant Beach* can be eliminated because it is strictly dominated by the randomized choice that selects *Nextdoor Beach* and *Closeby Beach* with probability 0.5. This yields the one-fold reduced decision problems in Table 7.1.2.

You	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	4	4	4
Distant	3	0	3	3
Nextdoor	2	2	0	2

v_1^{all}

You	*Nextdoor*	*Closeby*
Nextdoor	0	2
Closeby	1	0

v_1^{two}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	2	2	2
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Barbara	*Nextdoor*	*Closeby*
Nextdoor	0	4
Closeby	3	0

v_2^{two}

Table 7.1.2 One-fold reduced decision problems for "A day at the beach"

You	*Faraway*	*Nextdoor*	*Closeby*
Faraway	0	4	4
Distant	3	3	3

v_1^{all}

You	*Nextdoor*	*Closeby*
Nextdoor	0	2
Closeby	1	0

v_1^{two}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	2	2	2
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Barbara	*Nextdoor*	*Closeby*
Nextdoor	0	4
Closeby	3	0

v_2^{two}

Table 7.1.3 Two-fold reduced decision problems for "A day at the beach"

If you believe in Barbara's rationality, then you think that Barbara will definitely not choose *Distant Beach*. Indeed, if Barbara's view is v_2^{all} then Barbara is aware of *Distant Beach,* but going there would not be rational for her. On the other hand, if Barbara's view is v_2^{two} then she would not even be aware of *Distant Beach,* and hence she could not go there in the first place. We can thus eliminate the state *Distant Beach* at your view v_1^{all}, but not at your view v_1^{two}, because with the latter view you are unaware of the state *Distant Beach.* Consequently, at your view v_1^{all} you would never choose *Nextdoor Beach* since *Distant Beach* is always better.

Now turn to Barbara's one-fold reduced decision problems. One would be tempted to say that we can eliminate the state *Closeby Beach* at her view v_2^{all}, because *Closeby Beach* is not rational for you if your view is v_1^{all}. However, this reasoning is false: If Barbara's view is v_2^{all}, then she may very well believe that you are unaware of the two more distant beaches, and hence that your view is v_1^{two}. But going to *Closeby Beach* is a rational choice for you if your view is v_1^{two}, as can be seen from the decision problem at v_1^{two}. Since Barbara cannot exclude your view v_1^{two}, she cannot exclude your choice *Closeby Beach* either, and hence the state *Closeby Beach* cannot be eliminated at v_2^{all} or v_2^{two}. As such, no additional choices can be eliminated for Barbara either. This gives rise to the two-fold reduced decision problems in Table 7.1.3.

Question 7.1.1 *Explain why, from this moment on, no states can be eliminated at Barbara's view* v_2^{all}.

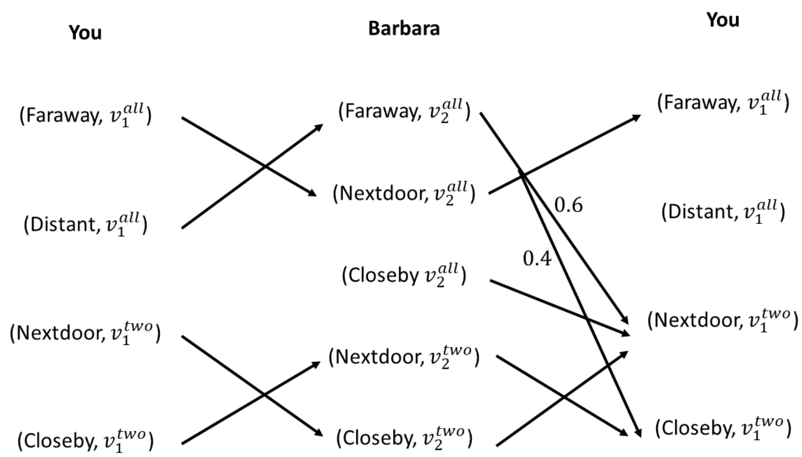

Figure 7.1.1 Beliefs diagram for "A day at the beach"

In view of the question above, no further eliminations are possible. Indeed, we will show, by means of a beliefs diagram, that all remaining choices can rationally made under common belief in rationality for the respective views. Consider the beliefs diagram in Figure 7.1.1.

It should be read as follows. Let us start at your choice-view pair (*Distant,* v_1^{all}). In your first-order belief, you believe that Barbara chooses *Faraway Beach w*hile having the view v_2^{all}. In your second-order belief you believe that Barbara believes that, with probability 0.6, you choose *Nextdoor Beach* while having the view v_1^{two}, and that with probability 0.4 you choose *Closeby Beach* while having the view v_1^{two}. Hence, in your second-order belief you believe that Barbara believes that your view is v_1^{two}, different from your actual view v_1^{all}.

Question 7.1.2 *Consider the belief hierarchy that starts at your choice-view pair* (Faraway, v_1^{all})*. Describe, in words, the first, second and third-order belief. What do you believe that Barbara believes about your view?*

It may be verified that for each choice-view pair with an outgoing arrow in the beliefs diagram, the choice is optimal for the respective view and the belief that is given by the arrow. For that reason, all arrows are solid. Consider, for instance, the choice-view pair (*Distant,* v_1^{all}), with the arrow that goes to (*Faraway,* v_2^{all}). If your view is v_1^{all} and you believe that Barbara goes to *Faraway Beach,* then it is indeed optimal to go to *Distant Beach.* Therefore, all the belief hierarchies in the beliefs diagram express common belief in rationality.

Now, consider the choices *Faraway Beach* and *Distant Beach* that survived the procedure for you at your view v_1^{all}. Since *Faraway Beach* is optimal for the belief hierarchy that starts at (*Faraway,* v_1^{all}), *Distant Beach* is optimal for the belief hierarchy that starts at (*Distant,* v_1^{all}), and both belief hierarchies express common belief in rationality, it follows that you can rationally go to *Faraway Beach* and *Distant Beach* under common belief in rationality with the view v_1^{all}.

Question 7.1.3 *What choices can you rationally make under common belief in rationality with the view* v_1^{two}*?*

From the procedure above and the beliefs diagram, it can similarly be concluded that under common belief in rationality, Barbara can rationally go to *Faraway Beach, Nextdoor Beach* and *Closeby*

Beach if her view is v_2^{all}, whereas she can rationally go to *Nextdoor Beach* and *Closeby Beach* if her view is v_2^{two}. Can you explain why?

7.1.3 Games with Unawareness

With the example at hand, we are now able to provide a general definition of a game with unawareness. Recall that for a given player, the choices he believes to be available to his opponents, together with the choices he believes to be available to himself, constitute the *view* of that player. Formally, this leads to the following definition.

Definition 7.1.1 (Views) *A **view** v_i for player i specifies for every player j (including player i himself) a set $C_j(v_i)$ of choices.*

*We say that a view v_i for player i is **contained** in a view v_k for player k, if for every player j, every choice in $C_j(v_i)$ is also in $C_j(v_k)$.*

Hence, if the view v_i is contained in the view v_k, then player k with view v_k is aware of all the choices that player i with view v_i is aware of. But not necessarily *vice versa.* Consider, as an illustration, the views $v_1^{all}, v_1^{two}, v_2^{all}, v_2^{two}$ in the example "A day at the beach". Then, Barbara's view v_2^{two} is contained in your view v_1^{all}, but not *vice versa.* Can you explain why?

Question 7.1.4 *Consider the example "A day at the beach". List all the views for you and Barbara that are contained in v_1^{all}, and all the views for you and Barbara that are contained in v_1^{two}.*

Suppose now that player i holds the view v_i. That is, player i is, for every player j, only aware of the choices in $C_j(v_i)$, and no other. But then, he must believe that every opponent k is, for every player j, only aware of the choices in $C_j(v_i)$, but possibly less. Indeed, since he is only aware of the choices in $C_j(v_i)$, he cannot even imagine an opponent reasoning about player j's choices outside $C_j(v_i)$. In other words, player i must believe that every opponent k holds a view that is contained in v_i. We call this the *awareness principle.*

Definition 7.1.2 (Awareness principle) *A player with view v must believe that every opponent holds a view that is contained in v.*

This principle plays a key role in the present chapter, as we will see. A consequence of the awareness principle is that, for every view v_i that is considered for player i in the game, we must consider for every opponent j a view v_j that is contained in v_i.

In the definition of games with unawareness that we will employ in this chapter, we assume that a player may be unaware of some of the actual choices in the game, or that he may be aware of more choices than some of his opponents, but that otherwise he will be correct about the opponents' conditional preference relations. That is, we do not allow for *incomplete information* in the game.

This may be formalized as follows: Consider two views v_i and v_i' for the same player i, with their respective conditional preference relations $\succsim_i^{v_i}$ and $\succsim_i^{v_i'}$. Then, for every belief b_i that is possible in both v_i and v_i', and for every two choices c_i, c_i' that are present in both v_i and v_i', the induced preference relation between c_i and c_i' must be the same in $\succsim_i^{v_i}$ as in $\succsim_i^{v_i'}$.

In terms of expected utility representations, this means the following: Suppose the conditional preference relations for v_i and v_i' are represented by the utility functions u_i and u_i', respectively. Then, for every opponents' choice combination c_{-i} that is present in both v_i and v_i', and for every choice c_i that is present in both v_i and v_i', we must have that $u_i(c_i, c_{-i}) = u_i'(c_i, c_{-i})$.

By gathering all the elements above, we arrive at the following general definition of a *game with unawareness.*

Definition 7.1.3 (Game with unawareness) *A* ***game with unawareness*** *specifies*

(a) a finite set of players I,

(b) for every player i a finite collection V_i of possible views, and

(c) for every view v_i in V_i a utility function $u_i^{v_i}$ that assigns to every choice c_i and every opponents' choice combination c_{-i} in the view v_i some utility $u_i^{v_i}(c_i, c_{-i})$.

Moreover, for every player i the following properties must hold:

(d) for every view v_i in V_i and every opponent j, there is a view v_j in V_j that is contained in v_i, and

(e) for two different views v_i, v_i' in V_i, it must be that

$$u_i^{v_i}(c_i, c_{-i}) = u_i^{v_i'}(c_i, c_{-i})$$

for every choice c_i and opponents' choice combination c_{-i} that is present in both v_i and v_i'.

Here, condition (d) guarantees that for every view v_i that is being considered for player i, there is for every opponent j some view v_j that player i can reason about while having the view v_i. Indeed, by the awareness principle, a player with view v_i can only reason about opponent's views v_j that are contained in v_i.

On the other hand, condition (e) states that a player may be unaware of some of the actual choices in the game, or may be aware of more choices than some of his opponents, but otherwise he will always be correct about the opponents' conditional preference relations. See our discussion above. Thus, condition (e) rules out elements of incomplete information in the game.

As an illustration of the definition, consider the example "A day at the beach". The sets of views for you and Barbara are $V_1 = \{v_1^{all}, v_1^{two}\}$ and $V_2 = \{v_2^{all}, v_2^{two}\}$, respectively. Moreover, the sets of choices that you are aware of at both of your views are given by

$$C_1(v_1^{all}) = \{\textit{Faraway, Distant, Nextdoor, Closeby}\},\ C_2(v_1^{all}) = \{\textit{Faraway, Distant, Nextdoor, Closeby}\}$$
$$C_1(v_1^{two}) = \{\textit{Nextdoor, Closeby}\},\ C_2(v_1^{two}) = \{\textit{Nextdoor, Closeby}\},$$

and similarly for Barbara. The utility functions $u_1^{v_1^{all}}, u_1^{v_1^{two}}, u_2^{v_2^{all}}$ and $u_2^{v_2^{two}}$ are given by the four decision problems in Table 7.1.1.

Question 7.1.5 *Explain why condition (d) is satisfied in this example.*

To see that condition (e) is satisfied, consider your decision problems in Table 7.1.1 for the views v_1^{all} and v_1^{two}. Note that your choices *Nextdoor* and *Closeby,* and the states – that is, Barbara's choices – *Nextdoor* and *Closeby,* are present in both views v_1^{all} and v_1^{two}. Moreover, the utilities for these choices and states are the same in the associated utility functions $u_1^{v_1^{all}}$ and $u_1^{v_1^{two}}$. As the same holds for Barbara, we conclude that condition (e) is satisfied.

7.2 Belief Hierarchies, Beliefs Diagrams and Types

To formally define the concept of *common belief in rationality* for games with unawareness, we need to talk about the *belief hierarchies* that the players have about the choices and *views* of the various players in the game. We will see that such belief hierarchies can be visualized by means of *beliefs diagrams*, and encoded mathematically by means of *epistemic models with types.* Moreover, the way to do so is very similar to what we have seen for games with incomplete information. It essentially boils down to replacing beliefs about utility functions by beliefs about views, and imposing the awareness principle that we have seen in the previous section.

7.2.1 Belief Hierarchies

If we wish to formalize the idea of common belief in rationality for games with unawareness, we must first specify what it means for player i to believe in opponent j's rationality. Intuitively, this means that player i believes that player j makes an optimal choice, given what i believes that j believes about the other players' choices, and given what i believes is player j's *view* of the game. Hence, we need (i) player i's first-order belief about j's choice, (ii) player i's first-order belief about j's view, and (iii) player i's second-order belief about player j's belief about the choices of others.

Suppose next that we want to formally define what it means for player i to believe that player j believes in some opponent k's rationality. Intuitively, it means that player i believes that j believes that k chooses optimally, given what i believes that j believes that k believes about his opponents' choices, and given what i believes that j believes is k's view. For this we thus need (iv) player i's second-order belief about j's belief about k's choice, (v) player i's second-order belief about j's belief about k's view, and (vi) player i's third-order belief about j's belief about k's belief about his opponents' choices.

If we continue like this, we arrive at the following definition of a belief hierarchy for games with unawareness.

Definition 7.2.1 (Belief hierarchies) *A* ***belief hierarchy*** *for player i specifies*

(1) a ***first-order belief,*** *which is a belief about the choices and views of i's opponents,*

(2) a ***second-order belief,*** *which is a belief about what every opponent j believes about the choices and views of j's opponents,*

(3) a ***third-order belief,*** *which is a belief about what every opponent j believes about what each of his opponents k believes about the choices and views of k's opponents,*

and so on.

Moreover, the second-order and higher-order beliefs must satisfy the ***awareness principle:***

If player i believes that player j chooses c_j and has view v_j, then c_j must be part of the view v_j, and player i must believe that j believes that every opponent has a view contained in v_j.

If player i believes that player j believes that player k chooses c_k and has view v_k, then c_k must be part of the view v_k, and i must believe that j believes that k believes that every opponent has a view contained in v_k.

And so on.

Note that the awareness principle consists of two parts: First, it requires that if you believe that the opponent chooses c_j and believe that the opponent has the view v_j, then c_j must be part of v_j. Indeed, player j can only choose c_j if he has a view v_j with which he is *aware* of his own choice c_j.

The second part states that if you believe that the opponent has view v_j, then you must believe that the opponent believes that everybody else has a view contained in v_j. This is exactly the awareness principle as discussed in the previous section.

As an illustration, consider the beliefs diagram from Figure 7.1.1. It may be verified that all belief hierarchies generated by this beliefs diagram satisfy the awareness principle above.

7.2.2 Beliefs Diagrams

For the case of incomplete information, we have seen that belief hierarchies can be visualized by means of beliefs diagrams where the arrows go from a choice-utility pair of a certain player i to choice-utility pairs for the players other than i. In games with unawareness the same is true if we replace choice-utility pairs by choice-view pairs. That is, we can visualize belief hierarchies for unawareness by beliefs diagrams where the arrows always go from a choice-view pair of a player i to opponents' choice-view pairs.

In fact, we have already seen such a beliefs diagram in Figure 7.1.1 for the example "A day at the beach". There, we have arrows from your choice-view pairs to Barbara's choice-view pairs, and *vice versa.* As to guarantee that the awareness principle holds for the induced belief hierarchies, we must make sure that an arrow from a choice-view pair (c_i, v_i) will only go to opponents' choice-view pairs (c_j, v_j) where (i) the choice c_j is part of the view v_j, and (ii) the view v_j is contained in the view v_i.

Question 7.2.1 *Suppose that the beliefs diagram satisfies the conditions (i) and (ii) above. Consider an arrow from a choice-view pair (c_i, v_i) to an opponents' choice-view pair (c_j, v_j). Explain why the opponent's choice c_j must be part of the own view v_i.*

It may be verified that all the arrows in the beliefs diagram of Figure 7.1.1 satisfy the conditions (i) and (ii) of the awareness principle.

7.2.3 Types

We have seen above that a belief hierarchy for a game with unawareness specifies (i) a first-order belief about the opponents' choice-view pairs, (ii) a second-order belief about the opponents' first-order beliefs, (iii) a third-order belief about the opponents' second-order beliefs, and so on. In other words, a belief hierarchy for player i specifies, for every opponent j, a belief about j's choice, j's belief hierarchy and j's view.

Similarly to what we have done for the case of incomplete information, the pair consisting of a belief hierarchy and a view for player j will be called a *type.* With this new terminology, a type t_i for player i will specify a view $w_i(t_i)$ and, for every opponent j, a belief $b_i(t_i)$ about j's choice and j's type. This naturally leads to the following definition of an *epistemic model* with *types*.

Definition 7.2.2 (Epistemic model) *Consider a game with unawareness with sets of views V_i for every player i. An* ***epistemic model*** *$M = (T_i, w_i, b_i)_{i \in I}$ specifies*

(a) for every player i a finite set of types T_i,

(b) for every player i and every type $t_i \in T_i$, a view $w_i(t_i)$ from V_i,

Types	$T_1 = \{t_1^{all,F}, t_1^{all,D}, t_1^{two,N}, t_1^{two,C}\}$, $T_2 = \{t_2^{all,F}, t_2^{all,N}, t_2^{all,C}, t_2^{two,N}, t_2^{two,C}\}$		
Views and beliefs for you	$w_1(t_1^{all,F}) = v_1^{all}$	$b_1(t_1^{all,F}) =$	$(Nextdoor,\ t_2^{all,N})$
	$w_1(t_1^{all,D}) = v_1^{all}$	$b_1(t_1^{all,D}) =$	$(Faraway,\ t_2^{all,F})$
	$w_1(t_1^{two,N}) = v_1^{two}$	$b_1(t_1^{two,N}) =$	$(Closeby,\ t_2^{two,C})$
	$w_1(t_1^{two,C}) = v_1^{two}$	$b_1(t_1^{two,C}) =$	$(Nextdoor,\ t_2^{two,N})$
Views and beliefs for Barbara	$w_2(t_2^{all,F}) = v_2^{all}$	$b_2(t_2^{all,F}) =$	$(0.6) \cdot (Nextdoor,\ t_1^{two,N}) + (0.4) \cdot (Closeby,\ t_1^{two,C})$
	$w_2(t_2^{all,N}) = v_2^{all}$	$b_2(t_2^{all,N}) =$	$(Faraway,\ t_1^{all,F})$
	$w_2(t_2^{all,C}) = v_2^{all}$	$b_2(t_2^{all,C}) =$	$(Nextdoor,\ t_1^{two,N})$
	$w_2(t_2^{two,N}) = v_2^{two}$	$b_2(t_2^{two,N}) =$	$(Closeby,\ t_1^{two,C})$
	$w_2(t_2^{two,C}) = v_2^{two}$	$b_2(t_2^{two,C}) =$	$(Nextdoor,\ t_1^{two,N})$

Table 7.2.1 Epistemic model for "A day at the beach"

(c) for every player i and every type $t_i \in T_i$, a probability distribution $b_i(t_i)$ on the opponents' choice-type combinations. This probability distribution $b_i(t_i)$ represents t_i's belief about the opponents' choices and types.

Moreover, every type t_i must satisfy the ***awareness principle:***

If $b_i(t_i)$ assigns positive probability to an opponents' choice-type pair (c_j, t_j), then the choice c_j must be part of the view $w_j(t_j)$, and the view $w_j(t_j)$ must be contained in the view $w_i(t_i)$.

Note that the awareness principle in the definition of an epistemic model is nothing more than a translation of the awareness principles we have discussed for belief hierarchies and beliefs diagrams. Indeed, the first part states that if you believe that the opponent chooses c_j and has the type t_j, then the choice c_j must be in the view held by t_j. The second part states that if you hold the view $w_i(t_i)$, then you must believe that every opponent j holds a view that is contained in your own view.

Similarly to Question 7.2.1, it can be shown that the awareness principle implies the following: If the type t_i assigns positive probability to an opponent's choice c_j, then the choice c_j must be part of the view $w_i(t_i)$ held by t_i. Can you explain why? Thus, a type will only consider opponent's choices that he is actually aware of.

This definition of an epistemic model with types is almost identical to the one we have seen for the case of incomplete information. The only difference is that utility functions have been substituted by views, and that we have imposed, in addition, the awareness principle.

As an illustration, consider again the beliefs diagram from Figure 7.1.1. This beliefs diagram can be translated into the epistemic model of Table 7.2.1. It may be verified that this epistemic model satisfies the awareness principle above.

7.3 Common Belief in Rationality

Now that we know how to encode belief hierarchies by means of epistemic models with types, we are ready to formally define the central idea of *common belief in rationality* for games with unawareness. Like for the case with incomplete information, we do so in three steps: We first define what it means for a choice to be optimal for a type, after which we formally state what it means to believe in the opponents' rationality. Finally, we use this to formalize common belief in rationality.

7.3.1 Optimal Choices for Types

Consider a type t_i for player i within an epistemic model. Recall that the type t_i specifies the view $w_i(t_i)$ that t_i has, and the belief $b_i(t_i)$ that t_i has about the opponents' choices and types. Intuitively, a choice c_i is optimal for the type t_i if it is optimal for the belief that t_i holds about the opponents' choices, within the bounds set by t_i's view of the game.

Definition 7.3.1 (Optimal choice for a type) *Consider a type t_i with the view $w_i(t_i)$, the utility function $u_i^{w_i(t_i)}$, and the first-order belief $b_i^1(t_i)$ on the opponents' choices. Take a choice c_i that is part of the view $w_i(t_i)$. Then, the choice c_i is* ***optimal*** *for the type t_i if*

$$u_i^{w_i(t_i)}(c_i, b_i^1(t_i)) \geq u_i^{w_i(t_i)}(c_i', b_i^1(t_i))$$

for all choices c_i' that are part of the view $w_i(t_i)$.

That is, given the belief about the opponent's choices, the choice c_i is at least as good as all other choices for himself that the type t_i is aware of. In the epistemic model from Table 7.2.1, it may be verified that your choices *Faraway, Distant, Nextdoor* and *Closeby* are optimal for your types $t_1^{all,F}, t_1^{all,D}, t_2^{two,N}$ and $t_2^{two,C}$, respectively. Can you explain why? Moreover, the optimal choices for Barbara's types $t_2^{all,F}, t_2^{all,N}, t_2^{all,C}, t_2^{two,N}$ and $t_2^{two,C}$ are *Faraway, Nextdoor, Closeby, Nextdoor* and *Closeby,* respectively. Again, can you explain why?

7.3.2 Common Belief in Rationality

Recall that common belief in rationality states that you believe that the opponents choose rationally, you believe that every opponent believes that every other player chooses rationally, and so on. The crucial step towards formally defining this notion is to formalize what we mean by "believing that the opponent chooses rationally". Like for the case of standard games and games with incomplete information, it means that you only assign positive probability to opponent's choice-type pairs where the choice is optimal for the type. In fact, the definition of belief in the opponent's rationality is literally the same as for standard games in Chapter 3 and games with incomplete information in Chapter 5.

Definition 7.3.2 (Belief in the opponents' rationality) *Type t_i* ***believes in the opponents' rationality*** *if the belief $b_i(t_i)$ on the opponents' choice-type combinations assigns, for every opponent j, only positive probability to choice-type pairs (c_j, t_j) where the choice c_j is optimal for the type t_j.*

With this definition, it is now easy to define common belief in rationality. In fact, the definition is exactly the same as for standard games in Chapter 3, and for games with incomplete information in Chapter 5.

Definition 7.3.3 (Common belief in rationality) *A type t_i expresses 1-fold belief in rationality if t_i believes in the opponents' rationality.*

A type t_i expresses 2-fold belief in rationality if $b_i(t_i)$ only assigns positive probability to opponents' types that express 1-fold belief in rationality.

A type t_i expresses 3-fold belief in rationality if $b_i(t_i)$ only assigns positive probability to opponents' types that express 2-fold belief in rationality.

And so on.

A type t_i expresses ***common belief in rationality*** *if it expresses 1-fold belief in rationality, 2-fold belief in rationality, 3-fold belief in rationality, and so on, ad infinitum.*

An easy way to check that all types in an epistemic model express *common* belief in rationality is to check that all types *believe in the opponents' rationality.* If this is the case, then it follows by arguments similar to those in Chapter 3 that all types will automatically express common belief in rationality as well. As an illustration, consider the epistemic model in Table 7.2.1. It may be verified that all types believe in the opponents' rationality. Can you explain why? Therefore, all types in the epistemic model express common belief in rationality.

Similarly to Chapters 3 and 5, we say that a choice c_i can rationally be made under common belief in rationality with a certain view v_i if there is a belief hierarchy that expresses common belief in rationality, such that the choice c_i is optimal for this particular belief hierarchy under the view v_i.

Definition 7.3.4 (Rational choice under common belief in rationality) *Player i can* ***rationally make choice c_i under common belief in rationality with the view v_i*** *if there is some epistemic model $M = (T_i, w_i, b_i)_{i \in I}$, and some type $t_i \in T_i$ for player i within that model, such that (a) type t_i expresses common belief in rationality, (b) type t_i has the view v_i and (c) choice c_i is optimal for the type t_i.*

Consider again the epistemic model from Table 7.2.1. Recall that your choices *Faraway, Distant, Nextdoor* and *Closeby* are optimal for your types $t_1^{all,F}, t_1^{all,D}, t_2^{two,N}$ and $t_2^{two,C}$, respectively. As all of these types express common belief in rationality, the first two types have the view v_1^{all}, and the last two types have the view v_1^{two}, we conclude that under common belief in rationality with the view v_1^{all} you can rationally go to *Faraway Beach* and *Distant Beach,* whereas under common belief in rationality with the view v_1^{two} you can rationally go to *Nextdoor Beach* and *Closeby Beach.* Moreover, as we have seen in Section 7.1, these are the *only* choices you can rationally make under common belief in rationality for each of these two views.

7.4 Recursive Procedure

In this section we will develop a recursive elimination procedure, *iterated strict dominance for unwareness,* that yields for every player, and each of his possible views, the choices that he can rationally make under common belief in rationality. We have seen that the treatments of games with unawareness and games with incomplete information are quite similar, and it should therefore not be surprising that the procedure of this section bears some resemblance with the *generalized iterated strict dominance* procedure for games with incomplete information. As we have done in Chapters 3 and 5, we build the procedure up in steps: We first characterize the choices that can rationally be made at the different views under 1-fold belief in rationality, and then characterize the choices that can rationally be made under 2-fold belief in rationality. These two steps will be sufficient to indicate how the full procedure looks like. The procedure will be illustrated by a new example. We will also show that at every view, at least one choice will survive the procedure. This, in turn, will imply that reasoning in accordance with common belief in rationality will always be possible. We finally show how to use the procedure for constructing an epistemic model where all types express common belief in rationality.

7.4.1 One-fold Belief in Rationality

We start with the most basic question: How can we characterize, for a given view v_i, the choices that player i can rationally make at this view? Recall that the view v_i corresponds to a decision problem $(C_i(v_i), C_{-i}(v_i), u_i^{v_i})$, where $C_i(v_i)$ are the choices for himself that player i is aware of, $C_{-i}(v_i)$ are the opponents' choice combinations (states) that player i is aware of, and $u_i^{v_i}$ is an expected utility representation of his conditional preference relation. By Theorem 2.6.1 we thus know that the choices that player i can rationally make with the view v_i are precisely the choices that are *not strictly dominated* in v_i's decision problem. In round 1 we can thus eliminate, for every view, those choices that are strictly dominated within that view. This leads to the one-fold reduced decision problems. As an illustration, consider the example "A day at the beach", where the one-fold reduced decision problems have been represented in Table 7.1.2.

Now suppose that player i holds the view v_i and expresses 1-fold belief in rationality. What choices can he rationally make then? Remember, by the *awareness principle*, that player i can only reason about opponents' views v_j that are contained in v_i. Hence, to express 1-fold belief in rationality means that for every opponent's view v_j contained in v_i, player i should only assign positive probability to choices c_j that are rational for player j in v_j. By the insight above, this is equivalent to saying that for every opponent's view v_j contained in v_i, player i should only assign positive probability to choices c_j that are not strictly dominated within the view v_j. Or, in other words, player i must assign probability zero to opponent's choices c_j that are strictly dominated for *every view v_j that is contained in v_i*. That is, within the view v_i we eliminate those states that involve opponents' choices that are strictly dominated within *every view that is contained in v_i*. But then, by construction of round 1, we eliminate at v_i those states that involve opponents' choices that have *not survived* round 1 at *any view that is contained in v_i*.

By eliminating these states at v_i, we obtain a reduced decision problem at v_i. The remaining states are precisely those states that you can assign positive probability to at v_i if you express 1-fold belief in rationality. But then, by Theorem 2.6.1, the choices that you can rationally make at v_i under 1-fold belief in rationality are precisely those choices that are not strictly dominated in this reduced decision problem at v_i. By eliminating those choices that *are* strictly dominated within the reduced decision problem at v_i we obtain the two-fold reduced decision problem at v_i. It contains precisely those choices for player i that he can rationally make with the view v_i if he expresses 1-fold belief in rationality.

To illustrate this second round, consider again the example "A day at the beach", and the one-fold reduced decision problems from Table 7.1.2. Consider the view v_1^{all}. Note that Barbara's choice *Distant* did not survive round 1 at any view for Barbara that is contained in v_1^{all}. Indeed, Barbara's choice *Distant* got eliminated in round 1 at her view v_2^{all}, which is contained in v_1^{all}, and was not even present from the beginning at her view v_2^{two}, contained in v_1^{all}, because Barbara is not aware of *Distant Beach* if her view is v_2^{two}. Therefore, at v_1^{all} we can eliminate the state *Distant.* In the reduced decision problem so obtained, your choice *Nextdoor* becomes strictly dominated by *Distant,* and can therefore be eliminated. This yields your two-fold reduced decision problem at view v_1^{all} as represented in Table 7.1.3.

Consider next Barbara's one-fold reduced decision problem at the view v_2^{all} in Table 7.1.2. Note that each of your choices survived round 1 at some view that is contained in v_2^{all}. Indeed, your choices *Faraway, Distant* and *Nextdoor* survived round 1 at your view v_1^{all}, which is contained in v_2^{all}, whereas your choice *Closeby* survived round 1 at your view v_1^{two}, which is also contained in v_2^{all}. Therefore, no state can be eliminated for Barbara at v_2^{all}, and hence no additional choice for Barbara can be eliminated there either in round 2. For every view, the two-fold reduced decision problems for "A day at the beach" can be found in Table 7.1.3.

7.4.2 Two-fold Belief in Rationality

Consider player i with view v_i, and suppose he expresses one-fold and two-fold belief in rationality. What choices can he rationally make then? Recall that player i can only consider opponent's views v_j that are contained in v_i. If he expresses one-fold and two-fold belief in rationality, then for each of these opponent's views v_j he must believe that player j makes a choice c_j that is rational for him under one-fold belief in rationality at that view v_j. We have just seen that these choices c_j are precisely player j's choices in his two-fold reduced decision problem at v_j. That is, if you hold the view v_i and express one-fold and two-fold belief in rationality, you must, for every opponent's view v_j contained in v_i, only assign positive probability to choices c_j that survived round 2 at v_j. In other words, you must assign probability zero to all opponent's choices c_j that did not survive round 2 at any view v_j that is contained in v_i. That is, from your decision problem at v_i you must eliminate all states that involve opponents' choices that did *not survive round 2* at *any view that is contained in* v_i. This leads to a reduced decision problem at v_i.

Therefore, by Theorem 2.6.1, the choices you can rationally make at v_i under one-fold and two-fold belief in rationality are precisely the choices that are not strictly dominated in this reduced decision problem. By eliminating the choices for player i that *are* strictly dominated at this reduced decision problem, we arrive at the three-fold reduced decision problem at v_i. It contains precisely those choices that player i can rationally make with the view v_i if he expresses one-fold and two-fold belief in rationality.

Note that in the example "A day at the beach", the three-fold reduced decision problems are the same as the two-fold reduced decision problems. Consider, for instance, Barbara's two-fold reduced decision problem at her view v_2^{all} in Table 7.1.3. Note that each of your choices survives round 2 at some view that is contained in v_2^{all}. Indeed, your choices *Faraway* and *Distant* survived round 2 at your view v_1^{all}, contained in v_2^{all}, whereas your choices *Nextdoor* and *Closeby* survived round 2 at your view v_1^{two}, which is also contained in v_2^{all}. Therefore, no state can be eliminated from the two-fold reduced decision problem at view v_2^{all}. Similarly for the other three views. In the example, the procedure thus terminates at round 2.

7.4.3 Common Belief in Rationality

The arguments above naturally lead to a procedure that yields, for every view, precisely those choices that can rationally be made under *common* belief in rationality. This procedure will be called *iterated strict dominance for unawareness.*

We have already seen above that the choices that can rationally be made under one-fold belief in rationality at a particular view are those that survive the first two rounds of eliminations at that view. Moreover, the choices that can rationally be made under one-fold and two-fold belief in rationality at a particular view are those that survive the first three rounds of eliminations at that view. By extending the arguments above, it can similarly be shown that for every view, the choices that can rationally be made if you express up to k-fold belief in rationality are those that survive the first $k+1$ rounds of eliminations at that view.

Definition 7.4.1 (Iterated strict dominance for unawareness) *Start by writing down the decision problems for every player i and every view v_i in V_i.*

Round 1. *From every decision problem, eliminate those choices that are strictly dominated. This leads to the 1-fold reduced decision problems.*

Round 2. *For every player i and every view v_i , eliminate those states that involve opponents' choices that did not survive round 1 at any view contained in v_i. Within the (possibly smaller) decision problem so obtained, eliminate all choices that are strictly dominated. This leads to the 2-fold reduced decision problems.*

Round 3. *For every player i and every view v_i , eliminate those states that involve opponents' choices that did not survive round 2 at any view contained in v_i. Within the (possibly smaller) decision problem so obtained, eliminate all choices that are strictly dominated. This leads to the 3-fold reduced decision problems.*

Continue like this until no further states and choices can be eliminated. The choices for a player i that eventually remain in his decision problem at a certain view v_i are said to survive ***iterated strict dominance for unawareness*** *at v_i.*

By extending the arguments we have been using above, we conclude that this procedure yields, for every view, precisely those choices that can rationally be made under common belief in rationality. As we have done in Chapters 3 and 5, this result can be fine-tuned by stating that for every $k \in \{1, 2, 3, ...\}$, the first $k+1$ rounds of the procedure yield precisely those choices that can rationally be made if the players express up to k-fold belief in rationality.

Theorem 7.4.1 (Procedure for common belief in rationality) *(a) For every $k \in \{1, 2, 3, ...\}$, the choices that player i can rationally make with view v_i while expressing up to k-fold belief in rationality are precisely the choices that survive the first $k+1$ rounds of iterated strict dominance for unawareness at v_i.*

(b) The choices that player i can rationally make with view v_i under common belief in rationality are exactly the choices that survive all rounds of iterated strict dominance for unawareness at v_i.

It is not difficult to see that the procedure will always terminate within finitely many rounds. Indeed, since there are finitely many views in the game with unawareness, and finitely many choices and states within every view, there must be a round after which no further choices and states can be eliminated. This is where the procedure will terminate.

Similarly to the elimination procedures in Chapters 3 and 5, the output of this elimination procedure also does not depend on the specific order by which we eliminate the choices and states at the various rounds.

Theorem 7.4.2 (Order independence) *Changing the order of elimination in iterated strict dominance for unawareness does not change the sets of choices that survive the procedure at each of the decision problems.*

In Section 7.5 we will use this order independence property to present an alternative elimination procedure, the *bottom-up procedure,* which yields exactly the same output as the procedure above, but is somewhat easier to use. In the alternative procedure we start by looking at the "minimal" views in the game, and do the eliminations there until we can go no further. Afterwards, we look at the "slightly larger" views v that only contain v itself and "minimal" views, and do the eliminations there until we can go no further. We proceed like this, by considering larger and larger views, until we have convered all the views in the game. For the details the reader will have to wait until Section 7.5.

7.4.4 Example

We will now illustrate the procedure above by means of a new example.

Example 7.2: Too much wine.

Yesterday evening Barbara and you had a party at Chris' house while Chris was away. Chris' house heavily suffered from the party because you both had too much wine. Early in the evening you both started dancing on the table, which broke the table in two. Later you played football in the living room and broke one of the windows. Afterwards you climbed on the roof and started jumping, which severly damaged the roof. Towards the end of the evening you painted the front door in the color pink.

The morning after, Chris comes home and remains in a state of shock for an hour after seeing all the damage. To find out what happened, he wakes you and Barbara up, and you both must whisper in his ear what has happened during the party.

Despite the wine you remember everything, and you can whisper five different stories into Chris' ear:

Innocent: You tell Chris that none of this was your or Barbara's fault, and that all the damage was caused by others.

Table: You tell Chris that Barbara and you danced on the table, but that you do not know what happened to the window, roof and door.

Window: You tell Chris about the dancing and the football, which broke the table and the window, but state that you do not know what happened to the roof and door.

Roof: You tell Chris about the dancing, the football and the jumping on the roof, which broke the table and the window and damaged the roof, but state that you do not know what happened to the door.

Door: You tell Chris about the dancing, the football, the jumping on the roof and painting the door, which broke the table and the window, damaged the roof and ruined the door.

However, you are not certain that Barbara remembers everything, because of the wine. You are quite confident that Barbara remembers breaking the table and the window, but you are not sure whether she remembers the events that followed. As such, you think that Barbara may have three different views of the evening: The view v_2^{window} where she only remembers breaking the table and the window, the view v_2^{roof} where she only remembers breaking the table and the window, and damaging the roof, and the view v_2^{door} where she remembers everything.

If her view is v_2^{window} she can only whisper the stories *innocent, table* and *window* into Chris' ear, since she cannot even imagine that you could ever have been jumping on the roof or painting the door

You	*innocent*	*table*	*window*
innocent	0	−550	−800
table	50	−250	−800
window	−200	−200	−500

v_1^{window}

You	*innocent*	*table*	*window*	*roof*
innocent	0	−550	−800	−1050
table	50	−250	−800	−1050
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_1^{roof}

You	*innocent*	*table*	*window*	*roof*	*door*
innocent	0	−550	−800	−1050	−1300
table	50	−250	−800	−1050	−1300
window	−200	−200	−500	−1050	−1300
roof	−450	−450	−450	−750	−1300
door	−700	−700	−700	−700	−1000

v_1^{door}

Table 7.4.1 Decision problems for "Too much wine"

in the color pink. In that case, she can only imagine your view v_1^{window} where you only remember breaking the table and the window.

Similarly, if her view is v_2^{roof} she can only whisper the stories *innocent, table, window* and *roof* into Chris' ear, and she can only imagine your views v_1^{window} and v_1^{roof}.

Finally, if her view is v_2^{door} she can whisper all five stories into Chris' ear, and she can imagine each of your views v_1^{window}, v_1^{roof} and v_1^{door}.

Now, for each of the four damages caused yesterday evening it will cost 500 euros to repair the damage, and Chris will let you and Barbara pay evenly for the damages he believes you have caused. More precisely, when you and Barbara have both whispered a story into Chris' ear, then Chris will believe the most detailed story of the two. Moreover, if you both tell different stories, then he will reward the person with the most detailed story with a bonus of 300 euros for being so honest, and punish the other person with a penalty of 300 euros for lying to him. If you both tell the same story, then there will be no bonus or penalty.

For instance, if you tell the story *roof* and Barbara tells the story *window,* then Chris will believe your story, and hence he believes that you and Barbara broke the table, the window and the roof. Since the total cost is 1500 euros you must both pay 750 euros. However, you earn a bonus of 300 euros whereas Barbara incurs a penalty of 300 euros, which makes your net payment 450 euros, and Barbara's net payment 1050 euros.

This story can be translated into the game with unawareness as shown in Table 7.4.1. In this game there are three possible views for you and three possible views for Barbara, which are v_1^{window}, v_1^{roof}, v_1^{door} and v_2^{window}, v_2^{roof}, v_2^{door}, respectively. We only wrote down the decision problems for your views, since the decision problems for Barbara's views are similar, by symmetry.

Note that we need your views v_1^{window} and v_1^{roof}, despite the fact that your true view is v_1^{door}. The reason is that you are uncertain about Barbara's view, and uncertain about what Barbara believes about your view. For instance, if you believe that Barbara's view is v_2^{roof}, then you must believe that Barbara can only imagine your views v_1^{window} and v_1^{roof}, and not your true view v_1^{door}.

The question is: Which story, or stories, can you rationally whisper into Chris' ear under common belief in rationality given your actual view v_1^{roof}? To answer this question we use the procedure *iterated*

You	*innocent*	*table*	*window*
table	50	−250	−800
window	−200	−200	−500

v_1^{window}

You	*innocent*	*table*	*window*	*roof*
table	50	−250	−800	−1050
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_1^{roof}

You	*innocent*	*table*	*window*	*roof*	*door*
table	50	−250	−800	−1050	−1300
window	−200	−200	−500	−1050	−1300
roof	−450	−450	−450	−750	−1300
door	−700	−700	−700	−700	−1000

v_1^{door}

Table 7.4.2 One-fold reduced decision problems in "Too much wine"

strict dominance for unawareness.

Round 1. At your view v_1^{window}, your choice *innocent* is strictly dominated by the randomized choice (0.9)· *table* + (0.1)· *window,* and can therefore be eliminated. Similarly, at your views v_1^{roof} and v_1^{door}, your choice *innocent* is strictly dominated by the randomized choice (0.95)· *table* + (0.05)· *roof* and the randomized choice (0.95)· *table* + (0.05)· *door,* respectively, and can therefore be eliminated at these two views also. Similarly for Barbara. This results in the one-fold reduced decision problems from Table 7.4.2.

Round 2. At your view v_1^{window} you can only imagine Barbara's view v_2^{window} at which her choice *innocent* did not survive. We can thus eliminate the state *innocent* at your view v_1^{window}. Afterwards, your choice *table* becomes strictly dominated by *window* at your view v_1^{window}, and can thus be eliminated there.

At your view v_1^{roof} you can only imagine Barbara's views v_2^{window} and v_2^{roof}, at which her choice *innocent* did not survive. We can thus eliminate the state *innocent* at your view v_1^{roof}. Afterwards, your choice *table* becomes strictly dominated by the randomized choice (0.95)· *window* + (0.05)· *roof* at your view v_1^{roof}, and can thus be eliminated there.

At your view v_1^{door} you can imagine Barbara's views v_2^{window}, v_2^{roof} and v_2^{door}, at which her choice *innocent* did not survive. We can thus eliminate the state *innocent* at your view v_1^{door}. Afterwards, your choice *table* becomes strictly dominated by the randomized choice (0.95)· *window* + (0.05)· *door* at your view v_1^{door}, and can thus be eliminated there.

Similarly for Barbara. This results in the two-fold reduced decision problems from Table 7.4.3.

Round 3. At your view v_1^{window} you can only imagine Barbara's view v_2^{window} at which her choice *table* did not survive. We can thus eliminate the state *table* at your view v_1^{window}.

At your view v_1^{roof} you can only imagine Barbara's views v_2^{window} and v_2^{roof}, at which her choice *table* did not survive. We can thus eliminate the state *table* at your view v_1^{roof}. Afterwards, your choice *window* becomes strictly dominated by *roof* at your view v_1^{roof}, and can thus be eliminated there.

At your view v_1^{door} you can imagine Barbara's views v_2^{window}, v_2^{roof} and v_2^{door}, at which her choice *table* did not survive. We can thus eliminate the state *table* at your view v_1^{door}. Afterwards, your choice *window* becomes strictly dominated by the randomized choice (0.95)· *roof* + (0.05)· *door* at

You	*table*	*window*
window	−200	−500

v_1^{window}

You	*table*	*window*	*roof*
window	−200	−500	−1050
roof	−450	−450	−750

v_1^{roof}

You	*table*	*window*	*roof*	*door*
window	−200	−500	−1050	−1300
roof	−450	−450	−750	−1300
door	−700	−700	−700	−1000

v_1^{door}

Table 7.4.3 Two-fold reduced decision problems in "Too much wine"

You	*window*
window	−500

v_1^{window}

You	*window*	*roof*
roof	−450	−750

v_1^{roof}

You	*window*	*roof*	*door*
roof	−450	−750	−1300
door	−700	−700	−1000

v_1^{door}

Table 7.4.4 Three-fold reduced decision problems in "Too much wine"

your view v_1^{door}, and can thus be eliminated there.

Similarly for Barbara. This results in the three-fold reduced decision problems from Table 7.4.4.

Afterwards, no more states or choices can be eliminated, and hence the procedure terminates in Round 3. In particular, we see that at your actual view v_1^{door}, you can rationally whisper the stories *roof* and *door* into Chris' ear under common belief in rationality.

7.4.5 Common Belief in Rationality is Always Possible

It is not difficult to see that the procedure will always eventually leave at least one choice and state at each of the decision problems. To see why, consider first the full decision problems at the beginning of the procedure. At a given decision problem for player i, associated with a view v_i, fix an arbitrary belief on the states, and consider a choice that is optimal for this belief. By Theorem 2.6.1 we know that this choice will not be strictly dominated within the decision problem, and will thus survive the first round at that decision problem. Hence, for every decision problem there is at least one choice that survives the first round.

We next turn to Round 2. Consider a one-fold reduced decision problem for player i that is associated with a view v_i. For every opponent j, consider a view v_j in V_j that is contained in v_i. Note that such views v_j exist, by condition (d) in Definition 7.1.3. By our argument above, there is for every such view v_j a choice c_j that is still present in the one-fold reduced decision problem at v_j. Hence, by construction, the state $(c_j)_{j \neq i}$ survives Round 2 in the decision problem at v_i. As such, there is at every one-fold reduced decision problem at least one state that survives Round 2.

Now, consider the one-fold reduced decision problem at view v_i and eliminate the states according to the rules of Round 2. We know from above that some states must remain. Fix an arbitrary belief on the remaining states, and a choice c_i that is optimal for this belief within the remaining decision problem at v_i. Then, by Theorem 2.6.1, the choice c_i is not strictly dominated within the remaining decision problem at v_i, and thus survives Round 2 at v_i. As such, we know that for every view v_i there must be at least one state and one choice that survives Round 2 at v_i.

By repeating this argument we conclude that for every round k, and every view v_i, there must be at least one state and one choice that survives round k at v_i. Since we have seen earlier that the procedure must terminate within finitely many rounds, we know that for every view there must be at least one choice that survives the procedure at this view.

On the other hand, we know by Theorem 7.4.1 that for every view v_i and every choice c_i that survives the procedure at v_i, there is an epistemic model and a type t_i with view v_i within it, such that t_i expresses common belief in rationality and the choice c_i is optimal for t_i. In particular, for every player i and every view v_i, there is an epistemic model and a type t_i with view v_i within it, such that t_i expresses common belief in rationality.

As in earlier chapters, we can say a little more: From the proof of Theorem 7.4.1 it follows that there is a *single* epistemic model M such that for every player i and every view v_i, there is a type t_i *within* M with view v_i such that t_i expresses common belief in rationality. We have thus established the following result.

Theorem 7.4.3 (Common belief in rationality is always possible) *Consider a game with unawareness which, for every player i, contains finitely many views and finitely many choices and states per view. Then, there is an epistemic model M such that for every player i and every view $v_i \in V_i$, there is a type t_i in M such that $w_i(t_i) = v_i$ and t_i expresses common belief in rationality.*

In the following subsection we will show how we can use the output of the procedure to generate such an epistemic model M.

7.4.6 Using the Procedure to Construct Epistemic Models

Consider a game with unawareness, and suppose that after running the procedure we are left with some choices and states at every possible view. Take a view v_i, and a choice c_i that has survived at that view. Then, by construction of the procedure, the choice c_i is not strictly dominated within the final reduced decision problem at v_i. Hence, by Theorem 2.6.1, the choice c_i is optimal for a belief on the surviving states in the final reduced decision problem at v_i. Therefore, within a beliefs diagram the choice c_i at view v_i can be supported by solid outgoing arrows.

By construction, every surviving state at v_i only contains opponents' choices c_j that (a) are part of the view v_i, and (b) for which there is a view v_j contained in v_i such that c_j is optimal for some belief within the final reduced decision problem at v_j. Hence, every solid outgoing arrow that supports choice c_i at view v_i leads to a choice-view pair (c_j, v_j) where v_j is contained in v_i, and c_j is optimal for some belief within the view v_j. That is, the choice c_j at v_j can be supported by solid arrows as well.

By continuing this argument, we can build a beliefs diagram where every choice c_i that survives at a view v_i can be supported by an infinite chain of solid arrows. Recall that such an infinite chain of solid arrows gives rise to a belief hierarchy that expresses common belief in rationality. By translating this beliefs diagram into an epistemic model M, we thus obtain an epistemic model with the desired properties of Theorem 7.4.3.

As an illustration, consider the example "Too much wine" and the final reduced decision problems from Table 7.4.4. Within the final reduced decision problem at v_1^{window}, the only surviving choice

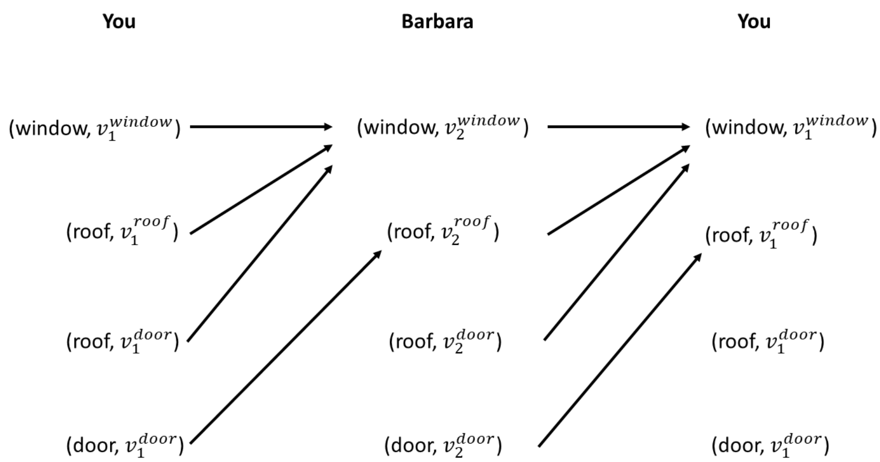

Figure 7.4.1 Beliefs diagram for "Too much wine"

window is optimal if you believe that Barbara has view v_2^{window} and chooses *window.* At v_1^{roof}, the only surviving choice *roof* is optimal if you believe, for instance, that Barbara has view v_1^{window} and chooses *window.* Finally, at the view v_1^{door}, your choice *roof* is optimal if you believe that Barbara has view v_1^{window} and chooses *window,* whereas your choice *door* is optimal if you believe that Barbara has view v_1^{roof} and chooses *roof.* Similarly for Barbara.

These insights give rise to the beliefs diagram in Figure 7.4.1. Note that in this beliefs diagram, every choice that survives at a given view is always supported by a belief hierarchy that expresses common belief in rationality, and that satisfies the *awareness principle.* Moreover, in each of your belief hierarchies present, you always believe that Barbara believes that your view is v_1^{window} – that is, that you were too drunk to remember what happened after crushing the window. Even though you actually remember everything that happened.

Question 7.4.1 *Translate the beliefs diagram from Figure 7.4.1 into an epistemic model.*

Note that each of the types in your model satisfies the awareness principle, and expresses common belief in rationality. Moreover, for every player i, every view v_i and every choice c_i that survives the procedure at v_i, there is a type t_i within the epistemic model that has the view v_i, expresses common belief in rationality, and for which the choice c_i is optimal.

7.5 Bottom-Up Procedure

In Theorem 7.4.2 we have seen that, for the eventual output of *iterated strict dominance for unawareness,* it does not matter in which order we eliminate states and choices at the various decision problems. This result will allow us to use a very convenient order of elimination, which we will call the *bottom-up procedure.*

Within that order we start with the smallest views in the game, and recursively eliminate choices and states there until we can go no further. The smallest views are said to have *rank* 1. Subsequently, we move to the views that only contain themselves or the smallest views as subviews. Such views are said to have rank 2. For all views of rank 2 we recursively eliminate choices and states until we can go no further. And so on, until we have covered all views in the game. As our examples will demonstrate, this order of elimination is very efficient. Moreover, by Theorem 7.4.2, it will deliver exactly the same output as the original procedure, and will therefore also characterize those choices that can rationally be made under common belief in rationality.

Before we define the bottom-up procedure, we first formalize the ranking of the views as discussed above. Afterwards, we introduce the bottom-up procedure, and show that it yields the same output as the original procedure.

7.5.1 Ranking of Views

We start by introducing the *ranking* of views, where views with rank 1 are the "smallest" views around, the views with rank 2 are the "smallest" amongst the views that do not have rank 1, the views with rank 3 are the "smallest" amongst the views that do not have rank 1 or 2, and so on. Before doing so, we first define what it means for a view to be *smallest* amongst a given set of views.

Definition 7.5.1 (Smallest view) *Consider a set V of views for possibly different players, which may not contain all views that are present in the game with unawareness. A view v in V is smallest amongst the views in V if v does not contain any view v' in V that has less choices than v for some player.*

As an illustration, consider the example "Too much wine", and consider the set V that contains all six views in the game. Then the views that are smallest amongst the views in V are v_1^{window} and v_2^{window}.

Question 7.5.1 *Now consider the set V' of views that contains all views except v_1^{window} and v_2^{window}. Which views are smallest amongst the views in V'? What if we consider the set V'' of views that contains the views v_1^{roof}, v_1^{door} and v_2^{door}?*

It can easily be seen that for every set of views V, there is always at least one view that is smallest amongst the views in V. Indeed, consider a view v in V where the total number of choices for all players is minimal amongst all views in V. Suppose that v contains a view v' in V. Then, by definition, $C_i(v') \subseteq C_i(v)$ for all players i. Now suppose, contrary to what we want to show, that v' contains less choices than v for some of the players i. Then, the total number of choices in v' would be less than the total number of choices in v, which cannot be. Therefore, v does not contain a view v' in V with less choices than in v, which means that the view v is smallest amongst the views in V.

Views with rank $1, 2, 3, ...$ can now be formalized as follows.

Definition 7.5.2 (Rank of a view) *Consider a game with unawareness, where V is the set of all views for all the players in that game.*

Rank 1. *A view v has rank 1 if it is smallest amongst the views in V.*

Rank 2. *A view v has rank 2 if it does not have rank 1, and it is smallest amongst the views that do not have rank 1.*

Rank 3. *A view v has rank 3 if it does not have rank 1 or 2, and it is smallest amongst the views that do not have rank 1 or 2.*

And so on.

Intuitively, the views with rank 1 are the smallest possible views in the game, the views with rank 2 are the second to smallest views, the views with rank 3 the third to smallest views, and so on. As an illustration, let us go back to the example "Too much wine", where the set of all possible views is

$$V = \{v_1^{window}, v_1^{roof}, v_1^{door}, v_2^{window}, v_2^{roof}, v_2^{door}\}.$$

We have seen above that the smallest views amongst the views in V are v_1^{window} and v_2^{window}, and hence these are the views with rank 1. Amongst the views that do not have rank 1, the smallest views are v_1^{roof} and v_2^{roof}, and hence these are the views with rank 2. Finally, amongst the views that do not have rank 1 or 2, which are only the views v_1^{door} and v_2^{door}, the smallest views are v_1^{door} and v_2^{door}. These are thus the views with rank 3. There are no views with a rank higher than 3.

Question 7.5.2 *Consider the example "A day at the beach". Classify the four possible views in terms of their rank.*

Recall that within every collection of views there is always at least one smallest view. As such, we can conclude that for every game with unawareness with finitely many views, there is always a number $K \in \{1, 2, 3, ...\}$ such that (i) for every $k \in \{1, ..., K\}$ there is at least one view with rank k, and (ii) every view has some rank $k \in \{1, ..., K\}$.

Moreover, the views with rank 1 are very special in the following sense: Every player i has at least one view v_i with rank 1, and every such view v_i can only contain views with exactly the same choices as v_i itself. In other words, if you have a view of rank 1, then you must necessarily believe that all your opponents share your view.

To see why every player i must have a view with rank 1, consider some arbitrary view v for some player j with rank 1. Fix some player $i \neq j$. By Definition 7.1.3 (d), the view v must contain a view $v_i \in V_i$ for player i. But then, v_i must necessarily have rank 1 itself. Therefore, every player i has at least one view with rank 1.

Now take some view v_i for player i with rank 1, and suppose that v_i contains some view $v_j \in V_j$ for player j. As v_i has rank 1, the view v_j cannot contain less choices than v_i. But then, v_j must contain exactly the same choices as v_i, since $C_k(v_j) \subseteq C_k(v_i)$ for every player k. Thus, every view contained in v_i must display the same choices as v_i itself.

As an illustration, consider the example "Too much wine". Note that both you and Barbara have a view with rank 1, which are v_1^{window} and v_2^{window}, respectively. Moreover, your view v_1^{window} only contains Barbara's view v_2^{window}, which contains the same choices as v_1^{window}, and similarly for v_2^{window}. Hence, with the view v_1^{window} you believe that Barbara shares your view, and similarly for Barbara's view v_2^{window}.

7.5.2 Bottom-Up Procedure

We have seen in Theorem 7.4.2 that the order of elimination does not matter for the eventual output of *iterated strict dominance for unawareness.* In particular, we can always use the following, very convenient, order of elimination: We first recursively eliminate the choices and states at all views with rank 1, according to the criteria of the original procedure. Note that for the eliminations of states at views with rank 1 it is sufficient to only concentrate on views with rank 1, and not any larger views, as a player with a view of rank 1 can only reason about opponents' views of rank 1.

Subsequently we turn to the views of rank 2, and recursively eliminate the choices and states there, taking into account the eliminations we have already performed at views of rank 1. Note that for the

You	*innocent*	*table*	*window*
innocent	0	−550	−800
table	50	−250	−800
window	−200	−200	−500

v_1^{window} ⟶

You	*innocent*	*table*	*window*
table	50	−250	−800
window	−200	−200	−500

v_1^{window} ⟶

⟶

You	*table*	*window*
window	−200	−500

v_1^{window} ⟶

You	*window*
window	−500

v_1^{window}

Table 7.5.1 Bottom-up procedure for "Too much wine for Barbara" at views of rank 1'

eliminations of states at views of rank 2 it is sufficient to only concentrate on views of rank 1 and 2, as a player with a view of rank 2 can only reason about views that have rank 2 or 1.

Afterwards we turn to the views of rank 3, and so on, until we have covered all views in the game. This procedure will be called the *bottom-up procedure.* To see how the bottom-up procedure works, let us apply it to a variation of the example "Too much wine".

Example 7.3: Too much wine for Barbara.

The events are the same as in the original example "Too much wine". However, now you are convinced that Barbara was too drunk to remember what happened to the door, and you are convinced that Barbara is convinced that you were too drunk to remember what happened after breaking the window. In terms of views, this means that only Barbara's views v_2^{window} and v_2^{roof} are present, but not v_2^{door}, because you believe that Barbara had too much wine to hold the view v_2^{door}. On the other hand, for our story it is sufficient to only consider your views v_1^{window} and v_1^{door} . The reason is that v_1^{door} is your actual view, with which you remember everything. At the same time, according to the story, you believe with the view v_1^{door} that Barbara believes that your view is v_1^{window}.

Hence, the only views present are $v_1^{window}, v_1^{door}, v_2^{window}$ and v_2^{roof}. The views of rank 1 are v_1^{window} and v_2^{window}, the only view of rank 2 is Barbara's view v_2^{roof}, and the only view of rank 3 is your view v_1^{door}.

We will now apply the bottom-up procedure to this scenario. We start with the analysis of the views of rank 1, which are v_1^{window} and v_2^{window}. The full decision problem for your view v_1^{window} is the first matrix of Table 7.5.1, and the full decision problem for Barbara's view v_2^{window} is similar.

At v_1^{window}, your choice *innocent* is strictly dominated by the randomized choice that assigns probability 0.9 to *table* and probability 0.1 to *window,* and can thus be eliminated. This yields the one-fold reduced decision problem at v_1^{window} as represented by the second matrix in Table 7.5.1, and similarly for Barbara.

In the one-fold reduced decision problem at v_1^{window}, you can only reason about Barbara's view v_2^{window} at which her choice *innocent* is no longer present. We can thus eliminate the state *innocent* at v_1^{window}. In the resulting reduced decision problem at v_1^{window}, your choice *table* is strictly dominated by *window,* and can thus be eliminated. This leads to the two-fold reduced decision problem at v_1^{window} as represented by the third matrix in Table 7.5.1, and similarly for Barbara.

In the two-fold reduced decision problem at v_1^{window}, you can only reason about Barbara's view v_2^{window} at which her choice *table* is no longer present. We can thus eliminate the state *table* at v_1^{window}, which leads to the three-fold reduced decision problem at v_1^{window} as represented by the fourth matrix in Table 7.5.1. Similarly for Barbara. This completes the analysis of the views with rank 1 in the

Barbara	*innocent*	*table*	*window*	*roof*
innocent	0	−550	−800	−1050
table	50	−250	−800	−1050
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_2^{roof}

$\longrightarrow$

Barbara	*window*
roof	−450

v_2^{roof}

Table 7.5.2 Bottom-up procedure for "Too much wine for Barbara" at view of rank 2

You	*innocent*	*table*	*window*	*roof*	*door*
innocent	0	−550	−800	−1050	−1300
table	50	−250	−800	−1050	−1300
window	−200	−200	−500	−1050	−1300
roof	−450	−450	−450	−750	−1300
door	−700	−700	−700	−700	−1000

v_1^{door}

$\longrightarrow$

You	*window*	*roof*
roof	−450	−750
door	−700	−700

v_1^{door}

Table 7.5.3 Bottom-up procedure for "Too much wine for Barbara" at view of rank 3

bottom-up procedure.

We next turn to the only view of rank 2, which is Barbara's view v_2^{roof}. The full decision problem at v_2^{roof} is the first matrix in Table 7.5.2. Note that Barbara, with view v_2^{roof}, can only reason about your view v_1^{window} of rank 1, at which only your choice *window* is left. We can therefore eliminate the states *innocent, table* and *roof* from v_2^{roof}. In fact, the states *innocent* and *table* can be eliminated because the associated choices have been eliminated at v_1^{window}, whereas the state *roof* can be eliminated because the associated choice is not even present at v_1^{window}. Afterwards, Barbara's choices *innocent, table* and *window* are strictly dominated by *roof,* and can thus be eliminated. This yields the reduced decision problem represented by the second matrix in Table 7.5.2. The analysis of the view v_2^{roof} of rank 2 is hereby complete.

We finally turn to the only view of rank 3, which is your view v_1^{door}. The full decision problem at v_1^{door} is the first matrix in Table 7.5.3. With the view v_1^{door} you can only reason about Barbara's views v_2^{window} and v_2^{roof}, at which only her choices *window* and *roof* are left. We can thus eliminate the states *innocent, table* and *door* at v_1^{door}. In fact, we can eliminate the states *innocent* and *table* because the associated choices have been eliminated at both v_2^{window} and v_2^{roof}, whereas we can eliminate the state *door* because the associated choice is not even present at the views v_2^{window} and v_2^{roof}. Afterwards, the choices *innocent, table* and *window* are strictly dominated by the choice *roof* and can thus be eliminated. The resulting reduced decision problem is the second matrix in Table 7.5.3. This completes the analysis of the view with rank 3.

The bottom-up procedure terminates here, because we have covered all the views in the game. The choices that are left for you at the view v_1^{door} are *roof* and *door.* As you will show in the next question, these are precisely the stories you can rationally whisper into Chris' ear under common belief in rationality when your actual view is v_1^{door}.

Question 7.5.3 *Consider the example "Too much wine for Barbara". Apply the original procedure, iterated strict dominance for unawareness, to this example. What choices can you rationally make* *under common belief in rationality if your view is* v_1^{door}*? Which procedure is easier to use: The bottom-up procedure, or the original procedure?*

You have probably noted that the bottom-up procedure was easier to use, and shorter, in this case. This will typically be the case when all views of rank 2 or higher only contain views that have a strictly lower rank, as is the case in this example. In such situations, the analysis of a view of rank $k \geq 2$ becomes relatively easy in the bottom-up procedure, because a player with a view of rank k only deems possible views of lower ranks, for which the surviving choices have already been determined by the previous rounds of the bottom-up procedure.

We are now ready to formally introduce the bottom-up procedure.

Definition 7.5.3 (Bottom-up procedure) *Start by writing down the decision problem for every view.*

For all views with ***rank 1*** *we apply the following procedure:*

Round 1. *From every view with rank 1, eliminate those choices that are strictly dominated. This leads to the* 1*-fold reduced decision problems.*

Round 2. *From every view* v *with rank 1, eliminate those states that involve opponents' choices that did not survive round 1 at any view contained in* v*. Within the (possibly smaller) decision problem so obtained, eliminate all choices that are strictly dominated. This leads to the* 2*-fold reduced decision problems.*

Continue until no further eliminations are possible at views with rank 1.

Subsequently, for all views with ***rank 2*** *we apply the following procedure:*

Round 1. *From every view* v *with rank 2 that only contains opponents' views of rank 1, eliminate those states that involve opponents' choices that did not survive the previous rounds at any rank 1 view contained in* v*. Subsequently, from every view with rank 2, eliminate those choices that are strictly dominated. This leads to the* 1*-fold reduced decision problems.*

Round 2. *From every view* v *with rank 2, eliminate those states that involve opponents' choices that did not survive the previous rounds at any view contained in* v*. Within the (possibly smaller) decision problem so obtained, eliminate all choices that are strictly dominated. This leads to the* 2*-fold reduced decision problems.*

Continue until no further eliminations are possible at views with rank 2.

In the same way we go over the views with rank $3, 4, ...$ *until all views have been covered.*

To further illustrate this procedure, let us go back to the example "A day at the beach". We start with the views of rank 1, which are v_1^{two} and v_2^{two}, with their full decision problems as depicted in Table 7.1.1. Since no choice is strictly dominated at v_1^{two} or v_2^{two}, there is nothing that can be eliminated at these two views.

We then turn to the views of rank 2, which are v_1^{all} and v_2^{all}. The associated decision problems can be found in Table 7.1.1. Recall that at v_1^{all}, your choice *Closeby* is strictly dominated by the randomized choice where you select *Faraway* and *Distant* with probability 0.5. We can thus eliminate your choice *Closeby* at v_1^{all} in Round 1. Similarly, at v_2^{all}, Barbara's choice *Distant* is strictly dominated by the randomized choice that selects *Nextdoor* and *Closeby* with probability 0.5. We can thus eliminate Barbara's choice *Distant* at view v_2^{all} in Round 1. This yields the one-fold reduced decision problems at v_1^{all} and v_2^{all} as depicted in Table 7.5.4.

You	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	4	4	4
Distant	3	0	3	3
Nextdoor	2	2	0	2

v_1^{all}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	2	2	2
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Table 7.5.4 Bottom-up procedure for "A day at the beach", Round 1 at views of rank 2

You	*Faraway*	*Nextdoor*	*Closeby*
Faraway	0	4	4
Distant	3	3	3

v_1^{all}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	2	2	2
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Table 7.5.5 Bottom-up procedure for "A day at the beach", Round 2 at views of rank 2

We now turn to Round 2 at v_1^{all} and v_2^{all}. At v_1^{all} you can only reason about Barbara's views v_2^{all} and v_2^{two}, at which her choice *Distant* does not appear. Indeed, we have eliminated Barbara's choice *Distant* at her view v_2^{all} in Round 1, whereas her choice *Distant* was not even present from the beginning at her view v_2^{two}. Therefore, we can eliminate the state *Distant* at your view v_1^{all}. Afterwards, your choice *Nextdoor* becomes strictly dominated by *Distant* at v_1^{all}, and can thus be eliminated there. This leads to the two-fold reduced decision problem at v_1^{all}, as depicted in Table 7.5.5. At Barbara's view v_2^{all} she can reason about your views v_1^{all} and v_1^{two}. As each of your choices is still present, either at v_1^{all} or v_1^{two}, we cannot eliminate any state at her view v_2^{all}.

After this round, no more states or choices can be eliminated at the views of rank 2, which are v_1^{all} and v_2^{all}. Thus, the bottom-up procedure ends here.

7.5.3 Equivalence with Original Procedure

The bottom-up procedure may be viewed as a particular order of elimination within the original procedure, *iterated strict dominance for unawareness.* Indeed, at the beginning we would only perform the required eliminations at the views of rank 1, until we can go no further. Afterwards we would turn to the views of rank 2 and perform the required eliminations there until we can go no further. And so on, until we have covered all the views in the game.

Note than when we do the eliminations at views of rank k, we can solely concentrate on views of rank k or less, since a player with a view of rank k can only reason about opponents' views of rank k or less. In that sense, the bottom-up procedure is well-defined, and corresponds to a specific order of elimination in *iterated strict dominance for unawareness.*

Since we have seen in Theorem 7.4.2 that the specific order of elimination does not matter for the output of *iterated strict dominance for unawareness,* we conclude that the *bottom-up procedure* must always deliver the same output at the end as *iterated strict dominance for unawareness.*

Theorem 7.5.1 (Equivalence with original procedure) *The bottom-up procedure always yields the same final output as iterated strict dominance for unawareness.*

In particular, it follows from Theorems 7.4.1 and 7.5.1 that the *bottom-up procedure*, for every view v_i, eventually selects exactly those choices that player i can rationally make under common belief in rationality with view v_i.

However, the bottom-up procedure must be handled with care, for the following reason: In Theorem 7.4.1 (a) we have seen that for every view v_i and every number k, the choices that player i can rationally make with view v_i while expressing up to k-fold belief in rationality are precisely those choices that survive the first $k+1$ rounds of *iterated strict dominance for unwareness* at v_i. This result, however, is not true for the *bottum-up procedure.*

To see this, let us go back to the example "Too much wine for Barbara" and the bottom-up procedure we applied to this example. Recall that during the first three rounds, we only performed eliminations at the views of rank 1, which are v_1^{window} and v_2^{window}. In particular, your choice *innocent* is still present at the view v_1^{door} after the third round of the procedure. However, your choice *innocent* is not optimal for any belief at the view v_1^{door} and hence, in particular, you cannot rationally choose *innocent* with the view v_1^{door} while expressing up to 2-fold belief in rationality. In turn, your choice *innocent* would already be eliminated in the *first* round of *iterated strict dominance for unawareness* at the view v_1^{door}.

7.6 *Fixed Beliefs on Views

Recall that a player with view v can only reason about views that are contained in v. In particular, in his belief this player can only assign positive probability to opponent's views that are contained in v. But apart from this condition, we have not yet put any restrictions on the particular probabilities that this player can assign to the various opponents' views. However, within a given story some beliefs about the opponents' views may be much more reasonable than others. In this section we implement this idea in a rather extreme way, by imposing for every player, and each of his views, a *fixed belief on the opponents' views.* That is, we think that these beliefs stand out as the most plausible beliefs, and we require the players to adhere to these specific beliefs.

This is very similar to how we modelled *fixed* beliefs on *utilities* in games with *incomplete information* – see Section 5.5. There is one crucial difference, however: For games with incomplete information we imposed, for a given player i, the *same* belief on the opponents' utility functions, irrespective of the utility function u_i that player i has himself. If we translate this to games with unawareness, then we would be imposing on player i the same belief on the opponents' views, irrespective of the view v_i that player i has himself. The problem is that this cannot work. To see this, consider, for instance, the example "Too much wine", and suppose we would impose on you the belief that assigns probability

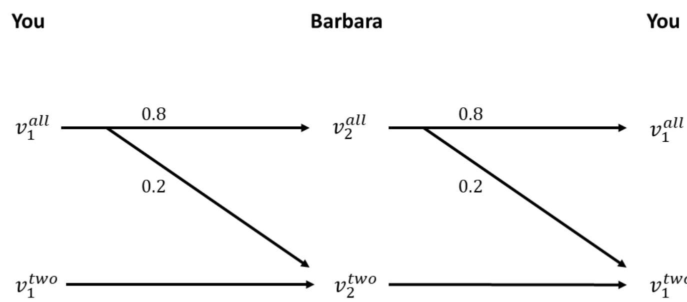

Figure 7.6.1 Fixed beliefs on views in "A day at the beach"

0.5 to Barbara's views v_2^{roof} and v_2^{door}, irrespective of your own view. This belief, however, will only be possible if your own view is v_1^{door}, since otherwise it would violate the awareness principle. As such, the belief on the opponents' views we impose on a player must necessarily depend on his own view.

In this section we start with an example which illustrates what we mean by fixed beliefs on views, and how we can combine this with the concept of common belief in rationality. Afterwards, we give a formal definition of common belief in rationality with fixed beliefs on views, and present a recursive elimination procedure, similar to *iterated strict dominance for unawareness,* that characterizes the choices that are possible under this concept. We finally present a bottom-up version of this procedure, and argue why it yields exactly the same output.

7.6.1 Example

Let us go back to the example "A day at the beach", with the views and decision problems as depicted in Table 7.1.1. Clearly, if your view is v_1^{two}, then you can only hold one possible belief about Barbara's view, which is to believe with probability 1 that Barbara's view is v_2^{two}. Similarly for Barbara.

But suppose now that your view is v_1^{all}, that is, that you are aware also of the two remote beaches on the island. Since you have discovered these beaches by making a nice, long walk, and you know that Barbara likes walking too, you deem it likely that also Barbara will be aware of these two beaches. A reasonable belief in this case would be to assign probability 0.8 to the event that Barbara is also aware of these two beaches, and to assign probability 0.2 to the event that she is not. Or, in terms of views, we would be imposing the belief that assigns probability 0.8 to Barbara's view v_2^{all} and probability 0.2 to Barbara's view v_2^{two}. If we impose the same belief for Barbara when her view is v_2^{all}, the imposed beliefs can be visualized by the beliefs diagram in Figure 7.6.1.

In fact, for each of your views we are not only imposing a fixed belief of views, but a fixed belief *hierarchy* on views. For instance, if your view is v_1^{all}, we are imposing the belief hierarchy in which (i) you assign probability 0.8 to Barbara's view v_2^{all} and probabability 0.2 to Barbara's view v_2^{two}, (ii) you assign probability 0.8 to the event that Barbara assigns probability 0.8 to your view v_1^{all} and probability 0.2 to your view v_1^{two}, and you assign probability 0.2 to the event that Barbara assigns probability 1 to your view v_1^{two}, and so on.

Question 7.6.1 *Consider the belief hierarchy on views we impose when your view is v_1^{two}. Describe the first- and second-order belief of this belief hierarchy.*

You	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	4	4	4
Distant	3	0	3	3
Nextdoor	2	2	0	2
Closeby	1	1	1	0

v_1^{all}

You	*Nextdoor*	*Closeby*
Nextdoor	0	2
Closeby	1	0

v_1^{two}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	2	2	2
Distant	1	0	1	1
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Barbara	*Nextdoor*	*Closeby*
Nextdoor	0	4
Closeby	3	0

v_2^{two}

Table 7.6.1 Decision problems for "A day at the beach"

You	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	4	4	4
Distant	3	0	3	3
Nextdoor	2	2	0	2

v_1^{all}

You	*Nextdoor*	*Closeby*
Nextdoor	0	2
Closeby	1	0

v_1^{two}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	2	2	2
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Barbara	*Nextdoor*	*Closeby*
Nextdoor	0	4
Closeby	3	0

v_2^{two}

Table 7.6.2 One-fold reduced decision problems for "A day at the beach" with fixed beliefs on utilities

Which beaches can you rationally go to under common belief in rationality with these fixed beliefs on views, if your actual view is v_1^{all}? To answer this question we start with the decision problems at the various views, as reproduced in Table 7.6.1.

Round 1. Recall that at the view v_1^{all}, your choice *Closeby* is never optimal for any belief, and can thus be eliminated there. Similarly, at the view v_2^{all} Barbara's choice *Distant* is never optimal for any belief, and can thus be eliminated there. This leads to the one-fold reduced decision problems in Table 7.6.2.

Round 2. At your view v_1^{all} you will assign probability 0 to Barbara choosing *Distant,* because *Distant* was eliminated at her view v_2^{all} in Round 1, whereas she is not even aware of this choice at her view v_2^{two}. But then, your choice *Nextdoor* can no longer be optimal at v_1^{all} because *Distant* will always be better. Hence, we can eliminate your choice *Nextdoor* at your view v_1^{all}.

Consider next Barbara's view v_2^{all}. Recall that we impose the belief where Barbara assigns probability 0.8 to your view v_1^{all} and probability 0.2 to your view v_1^{two}. As your choice *Closeby* only survived Round 1 at your view v_1^{two}, Barbara must assign probability at most 0.2 to you choosing *Closeby.* But then, Barbara's expected utility by choosing *Closeby* is at least $(0.8) \cdot 3 = 2.4$, which means that it can no longer be optimal for Barbara to choose *Faraway.* We can therefore eliminate Barbara's choice

You	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	4	4	4
Distant	3	0	3	3

v_1^{all}

You	*Nextdoor*	*Closeby*
Nextdoor	0	2
Closeby	1	0

v_1^{two}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Nextdoor	4	4	0	4
Closeby	3	3	3	0

v_2^{all}

Barbara	*Nextdoor*	*Closeby*
Nextdoor	0	4
Closeby	3	0

v_2^{two}

Table 7.6.3 Two-fold reduced decision problems for "A day at the beach" with fixed beliefs on utilities

You	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Faraway	0	4	4	4

v_1^{all}

You	*Nextdoor*	*Closeby*
Nextdoor	0	2
Closeby	1	0

v_1^{two}

Barbara	*Faraway*	*Distant*	*Nextdoor*	*Closeby*
Nextdoor	4	4	0	4

v_2^{all}

Barbara	*Nextdoor*	*Closeby*
Nextdoor	0	4
Closeby	3	0

v_2^{two}

Table 7.6.4 Three-fold reduced decision problems for "A day at the beach" with fixed beliefs on utilities

Faraway at her view v_2^{all}. This leads to the two-fold reduced decision problems in Table 7.6.3.

Round 3. At your view v_1^{all} you must now assign probability 0 to Barbara choosing *Faraway* or *Distant,* as both of these choices did not survive Round 2 at Barbara's view v_2^{all}, and Barbara is not even aware of these choices at her view v_2^{two}. But then, your choice *Distant* can no longer be optimal at v_1^{all}, because *Faraway* will always be better. We can thus eliminate your choice *Distant* at v_1^{all}.

At her view v_2^{all}, Barbara must assign probability 0.8 to your view v_1^{all} and probability 0.2 to your view v_1^{two}. As only your choices *Faraway* and *Distant* survived round 2 at your view v_1^{all}, Barbara must assign probability 0.8 to your choices *Faraway* and *Distant* together. But then, Barbara's expected utility from choosing *Nextdoor* is at least $(0.8) \cdot 4 = 3.2$, which means that her choice *Closeby* can no longer be optimal. We thus eliminate Barbara's choice *Closeby* at her view v_2^{all}. This leads to the three-fold reduced decision problems in Table 7.6.4.

Since only your choice *Faraway* survives at your view v_1^{all}, we conclude that under common belief in rationality with these fixed beliefs on views, you can only rationally go to *Faraway Beach* if your view is v_1^{all}. Recall that without any restrictions on the beliefs on views, you could rationally go to either *Faraway Beach* or *Distant Beach* under common belief in rationality with this view.

To support this finding, consider the beliefs diagram in Figure 7.6.2. It may be verified that all belief hierarchies on choices and views express common belief in rationality, and respect the fixed beliefs on views as stated above. As the choice *Faraway* is optimal for the view v_1^{all} under the belief hierarchy that starts at (*Faraway,* v_1^{all}), we indeed conclude that you can rationally choose *Faraway* at the view v_1^{all} under common belief in rationality with the fixed beliefs on views above.

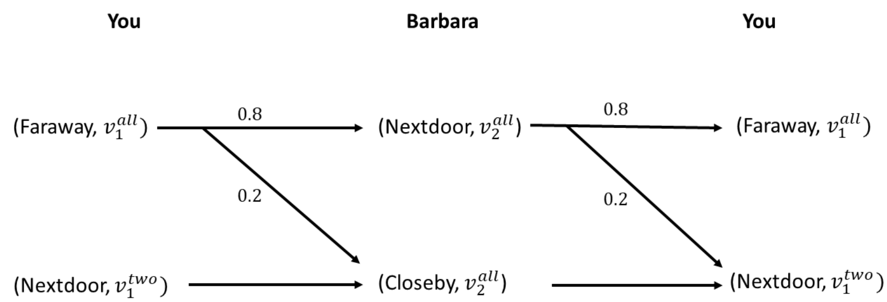

Figure 7.6.2 Beliefs diagram for "A day at the beach" with fixed beliefs on views

7.6.2 Definition

We will now provide a formal definition of fixed beliefs on views, and show how this can be combined with the conditions of common belief in rationality.

Definition 7.6.1 (Fixed beliefs on views) *A* ***fixed belief combination on views*** $p = (p_i(v_i))_{i \in I, v_i \in V_i}$ *assigns to every player* i *and every view* $v_i \in V_i$ *a probabilistic belief* $p_i(v_i)$ *on the opponents' view combinations, where* $p_i(v_i)$ *only assigns positive probability to opponents' views* $v_j \in V_j$ *that are contained in* v_i*.*

For instance, the fixed belief combination p on views considered above in the example "A day at the beach" contains the beliefs

$$p_1(v_1^{two}) = v_2^{two} \text{ and } p_1(v_1^{all}) = (0.8) \cdot v_2^{all} + (0.2) \cdot v_2^{two}$$

for you, and the beliefs

$$p_2(v_2^{two}) = v_1^{two} \text{ and } p_2(v_2^{all}) = (0.8) \cdot v_1^{all} + (0.2) \cdot v_1^{two}$$

for Barbara.

Such a fixed belief combination on views can always be visualized by a beliefs diagram on views, like we did in Figure 7.6.1. Moreover, we have seen above that if we start from a view v_i in this beliefs diagram and follow the arrows, we obtain a belief *hierarchy* on views. In that sense, a fixed belief combination on views p induces, for every player i and every view $v_i \in V_i$, a belief hierarchy on views.

Now, consider a type within an epistemic model, prescribing a view and generating a belief hierarchy on choices and views. We say that this type expresses *common belief* in the fixed belief combination p on views if its belief hierarchy on views is exactly the one prescribed by p. But we can also define up to k-fold belief in p, for every k.

Definition 7.6.2 (Type respecting fixed beliefs on views) *Consider a fixed belief combination on views* p*, and an epistemic model* $(T_i, w_i, b_i)_{i \in I}$*.*

A type t_i *with view* v_i *expresses* 1*-fold belief in* p *if* t_i*'s belief about the opponents' views is given by* $p_i(v_i)$*.*

A type t_i *expresses* 2*-fold belief in* p *if* t_i *only assigns positive probability to opponents' types* t_j *that* *express* 1*-fold belief in* p.

A type t_i *expresses* 3*-fold belief in* p *if* t_i *only assigns positive probability to opponents' types* t_j *that express* 2*-fold belief in* p.

And so on.

A type t_i *expresses* ***common belief in*** p *if it expresses* k*-fold belief in* p *for every* $k \in \{1, 2, 3, ...\}$.

In a similar way as in Section 5.5.2, we can now define what it means that you can rationally make a choice under common belief in rationality and common belief in a fixed belief combination on views.

Definition 7.6.3 (Rational choice with fixed beliefs on views) *Let* p *be a fixed belief combination on views, and* $v_i \in V_i$ *a view. Then, player* i *can* ***rationally make the choice*** c_i ***with view*** v_i ***under common belief in rationality and common belief in*** p, *if there is an epistemic model* $(T_i, w_i, b_i)_{i \in I}$ *and a type* $t_i \in T_i$ *such that (a)* t_i *expresses common belief in rationality, (b)* t_i *expresses common belief in* p, *(c)* t_i *has view* v_i, *and (d)* c_i *is optimal for* t_i.

In the following subsection we will present an elimination procedure, similar to *iterated strict dominance for unawareness,* that yields for every view precisely those choices you can rationally make under common belief in rationality and common belief in a fixed belief combination p on views.

7.6.3 Recursive Procedure

Consider a fixed belief combination on views p, which prescribes for every player i and every view $v_i \in V_i$ a probabilistic belief $p_i(v_i)$ about the opponents' views. Can we design a recursive elimination procedure, similar to *iterated strict dominance for unawarenss,* that selects for every player and view exactly those choices he can rationally make with this view under common belief in rationality and common belief in p?

As before, we start with a more basic question: For a given player i and view v_i, which choices can this player rationally make with *some* belief that satisfies the awareness principle, but without yet imposing any other restrictions on the belief? We know from the first round of the *iterated strict dominance procedure for unawareness* that these are precisely the choices that are not strictly dominated within the decision problem at v_i. This yields the one-fold reduced decision problem at view v_i.

Next, we ask: At a given view v_i, what choices can player i rationally make if he expresses 1-fold belief in p and 1-fold belief in rationality? That is, at the view v_i player i's first-order belief about the opponents' choice-view pairs must be such that (i) the induced belief about the opponents' views is $p_i(v_i)$, and (ii) it only assigns positive probability to opponent's choice-view pairs (c_j, v_j) where the choice c_j is in the one-fold reduced decision problem at view v_j. Thus, we only keep those choices for player i at view v_i that are optimal for some first-order belief b_i^1 about the opponents' choices and views that satisfy the properties (i) and (ii) above. This yields the two-fold reduced decision problem at view v_i.

Afterwards, we wish to identify those choices that player i can rationally make with view v_i if he expresses up to 2-fold belief in p and up to 2-fold belief in rationality? That is, at the view v_i player i's first-order belief about the opponents' choice-view pairs must be such that (i) the induced belief about the opponents' views is $p_i(v_i)$, and (ii) it only assigns positive probability to opponent's choice-view pairs (c_j, v_j) where the choice c_j is in the two-fold reduced decision problem at view v_j. Thus, we only keep those choices for player i at view v_i that are optimal for some first-order belief b_i^1

about the opponents' choices and views that satisfy the properties (i) and (ii) above. This yields the three-fold reduced decision problem at view v_i.

By continuing in this way, we arrive at the recursive elimination procedure stated below. Like in Section 5.5.3, we say that a choice c_i is *optimal* at a view v_i for a first-order belief b_i^1 about the opponents' choices and views if it is optimal for the induced belief about the opponents' choices.

Definition 7.6.4 (Procedure with fixed beliefs on views) *Let p be a fixed belief combination on views. Start by writing down the decision problems for every player i and every view v_i in V_i.*

Round 1. *At every view v_i, eliminate from the associated decision problem those choices that are strictly dominated. This leads to the 1-fold reduced decision problems.*

Round 2. *At every view v_i, keep at the associated 1-fold reduced decision problem only those choices c_i which are optimal for a first-order belief b_i^1 on opponents' choices and views where (i) b_i^1's belief about the opponents' views is $p_i(v_i)$, and (ii) b_i^1 only assigns positive probability to pairs (c_j, v_j) where c_j is in the 1-fold reduced decision problem at v_j. This leads to the 2-fold reduced decision problems.*

Round 3. *At every view v_i, keep at the associated 2-fold reduced decision problem only those choices c_i which are optimal for a first-order belief b_i^1 on opponents' choices and views where (i) b_i^1's belief about the opponents' views is $p_i(v_i)$, and (ii) b_i^1 only assigns positive probability to pairs (c_j, v_j) where c_j is in the 2-fold reduced decision problem at v_j. This leads to the 3-fold reduced decision problems.*

Continue like this until no further choices can be eliminated. The choices for a player i that eventually remain in his decision problem at a certain view v_i are said to survive the ***iterated strict dominance procedure for unawareness with fixed beliefs p on the views****.*

In view of our arguments above, we can conclude that this procedure will always yield precisely those choices that can rationally be made under common belief in rationality with fixed beliefs on p.

Theorem 7.6.1 (Procedure for common belief in rationality with fixed beliefs on views) *Consider a fixed belief combination p on views.*

(a) For every $k \in \{1, 2, 3, ...\}$, the choices that player i can rationally make with a view $v_i \in V_i$ while expressing up to k-fold belief in rationality and up to k-fold belief in p are precisely the choices that survive the first $k+1$ rounds of the iterated strict dominance procedure for unawareness with fixed beliefs p on views at v_i.

(b) The choices that player i can rationally make with view $v_i \in V_i$ under common belief in rationality and common belief in p are exactly the choices that survive all rounds of the iterated strict dominance procedure for unawareness with fixed beliefs p on views at v_i.

Similarly to what we saw for other procedures so far, also this procedure terminates within finitely many rounds, and for every view v_i there is at least one choice for player i that survives the procedure at this view. If we combine this insight with Theorem 7.6.1 it follows that there will always be, for every player, a belief hierarchy that expresses common belief in rationality and common belief in a fixed belief combination p on views.

Theorem 7.6.2 (Common belief in rationality with fixed beliefs on views is possible) *Consider a game with unawareness which, for every player i, contains finitely many views and finitely many choices per view. Consider a fixed belief combination p on views. Then, there is an epistemic model M such that for every player i and every view $v_i \in V_i$, there is a type t_i in M such that $w_i(t_i) = v_i$ and t_i expresses common belief in rationality and common belief in p.*

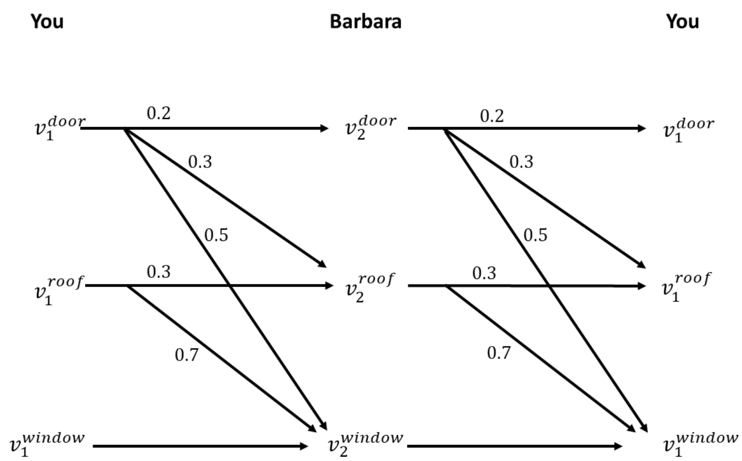

Figure 7.6.3 Fixed beliefs on views for "Too much wine"

We have seen that for all the procedures discussed so far the order of elimination did not matter for the eventual output. The same is true for the procedure discussed in this section.

Theorem 7.6.3 (Order independence) *Changing the order of elimination in the iterated strict dominance procedure for unawareness with fixed beliefs on views does not change the sets of choices that survive the procedure at each of the decision problems.*

That is, even if we do not eliminate, at some of the rounds and some of the views, all the choices that we can, we are still guaranteed to eventually end up with the choices that can rationally be made under common belief in rationality and common belief in the fixed belief combination on views.

7.6.4 Illustration of the Procedure

To illustrate the procedure with fixed beliefs on views, let us return to the original story of "Too much wine" in Example 7.2. The decision problems can be found in Table 7.4.1.

Now suppose that, whenever you remember what happened to the roof or door, you are quite confident that Barbara was too drunk to remember this. More precisely, if your view is v_1^{roof}, then you believe that with probability 0.7 Barbara's view is v_2^{window} and that with probability 0.3 her view is v_2^{roof}. If your view is v_1^{door}, then you assign probability 0.5 to Barbara holding the view v_2^{window}, probability 0.3 to her holding the view v_2^{roof} and probability 0.2 to her holding the view v_2^{door}. Similarly for Barbara. This fixed combination p of beliefs can be visualized by the beliefs diagram in Figure 7.6.3. With these fixed beliefs on views, what stories can you rationally tell to Chris if you remember everything that happened? To answer this question we use the iterated strict dominance procedure for unawareness with the fixed beliefs p on views.

Round 1. We have seen in Section 7.4.4 that your choice *innocent* is strictly dominated in the decision problems for your views v_1^{window}, v_1^{roof} and v_1^{door}. We can thus eliminate your choice *innocent* at these three views, and similarly for Barbara. This yields the one-fold reduced decision problems in Table 7.6.5.

You	*innocent*	*table*	*window*
table	50	−250	−800
window	−200	−200	−500

v_1^{window}

You	*innocent*	*table*	*window*	*roof*
table	50	−250	−800	−1050
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_1^{roof}

You	*innocent*	*table*	*window*	*roof*	*door*
table	50	−250	−800	−1050	−1300
window	−200	−200	−500	−1050	−1300
roof	−450	−450	−450	−750	−1300
door	−700	−700	−700	−700	−1000

v_1^{door}

Table 7.6.5 One-fold reduced decision problems in "Too much wine"

Round 2. At each of your views you believe that Barbara will not choose *innocent.* We have seen in Section 7.4.4 that under these circumstances, it can no longer be optimal to choose *table* at any of your three views. Hence, we can eliminate the choice *table* at each of your three views.

However, we will see that at your view v_1^{door} we can also eliminate your choice *door.* Compare your choices *roof* and *door* at this view v_1^{door}. Then, the difference in expected utility between the two choices is

$$\begin{aligned} u_1(roof) - u_1(door) &= (b_1(table) + b_1(window)) \cdot (-450 - (-700)) + b_1(roof) \cdot (-750 - (-700)) + \\ &\quad + b_1(door) \cdot (-1300 - (-1000)) \\ &= (b_1(table) + b_1(window)) \cdot 250 - b_1(roof) \cdot 50 - b_1(door) \cdot 300, \qquad (7.6.1) \end{aligned}$$

where $b_1(table)$, $b_1(window)$, $b_1(roof)$ and $b_1(door)$ denote the probabilities you assign to these four choices.

Note that Barbara is only able to choose *roof* if her view is either v_2^{roof} or v_2^{door}, and that she is only able to choose *door* if her view is v_2^{door}. Also recall that at the view v_1^{door} you only assign probability 0.3 to Barbara having the view v_2^{roof} and probability 0.2 to Barbara holding the view v_2^{door}. Therefore, you can assign at most probability 0.2 to Barbara choosing *door.* Moreover, if you assign probability 0.2 to Barbara choosing *door,* you can assign at most probability 0.3 to Barbara choosing *roof.*

Thus, in view of (7.6.1), the beliefs that are most favorable for your choice *door* compared to the choice *roof* are the beliefs where $b_1(door) = 0.2$, $b_1(roof) = 0.3$ and $b_1(table) + b_1(window) = 0.5$. But even for these most favorable beliefs we have that

$$u_1(roof) - u_1(door) = (0.5) \cdot 250 - 0.3 \cdot 50 - 0.2 \cdot 300 > 0,$$

which means that *roof* is still better than *door.* Hence, with the fixed beliefs on views it can no longer be optimal to choose *door* when your view is v_1^{door}. We thus eliminate your choice *door* at v_1^{door}. This leads to the two-fold reduced decision problems in Table 7.6.6.

Round 3. At your views v_1^{roof} and v_1^{door} you believe that Barbara will not choose *innocent* or *table.* But then, at the view v_1^{roof} it can no longer be optimal to choose *window* since *roof* will always be better. We can thus eliminate *window* at your view v_1^{roof}.

At the view v_1^{door} you will assign probability 0.2 to Barbara having the view v_2^{door}. Since Barbara is only able to choose *door* if her view is v_2^{door}, you can assign probability at most 0.2 to Barbara

You	*innocent*	*table*	*window*
window	−200	−200	−500

v_1^{window}

You	*innocent*	*table*	*window*	*roof*
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_1^{roof}

You	*innocent*	*table*	*window*	*roof*	*door*
window	−200	−200	−500	−1050	−1300
roof	−450	−450	−450	−750	−1300

v_1^{door}

Table 7.6.6 Two-fold reduced decision problems in "Too much wine"

You	*innocent*	*table*	*window*
window	−200	−200	−500

v_1^{window}

You	*innocent*	*table*	*window*	*roof*
roof	−450	−450	−450	−750

v_1^{roof}

You	*innocent*	*table*	*window*	*roof*	*door*
roof	−450	−450	−450	−750	−1300

v_1^{door}

Table 7.6.7 Three-fold reduced decision problems in "Too much wine"

choosing *door.* As *roof* is better than *window* for you when Barbara chooses *window* or *roof,* and both choices are equally good when Barbara chooses *door,* it follows that for all the beliefs you can hold at Round 3, choosing *roof* is better than choosing *window.* We can thus eliminate your choice *window* at your view v_1^{door} as well. This yields the three-fold reduced decision problems in Table 7.6.7.

Clearly, these decision problems cannot be reduced any further, and thus the procedure terminates here. In particular, we see that under common belief in rationality with the fixed beliefs on views from Figure 7.6.3, you can only rationally tell the story *roof* if your view is v_1^{door}.

That is, even when you remember everything, it is optimal not to reveal what happened with the door. The reason is that you deem it quite likely that Barbara does not remember what happened to the roof or the door. But then, it is better for you to only reveal what happened to the roof, but not what happened to the door.

Recall from Section 7.4.4 that without any restrictions on the beliefs on the views, you could rationally tell the whole truth if you remember everything. The reason is clear: Without any further restrictions on the beliefs on the views, you could possibly deem it very likely that Barbara also remembers what happened to the door, making it optimal for you to tell the whole truth.

7.6.5 Bottom-Up Procedure

In Theorem 7.6.3 we have seen that the order of elimination is not important for the output of the iterated strict dominance procedure for unawareness with fixed beliefs on views. Similarly to the case without restrictions on the beliefs on views, we could thus go for a "bottom-up" version of the procedure without changing the final outcome. That is, we could start by analyzing the smallest views in the game, followed by the second to smallest views, and so on, until we have covered all the

You	*innocent*	*table*	*window*
innocent	0	−550	−800
table	50	−250	−800
window	−200	−200	−500

v_1^{window} ⟶

You	*innocent*	*table*	*window*
table	50	−250	−800
window	−200	−200	−500

v_1^{window} ⟶

You	*innocent*	*table*	*window*
window	−200	−200	−500

v_1^{window}

Table 7.6.8 Bottom-up procedure with fixed beliefs on views for "Too much wine" at views of rank 1

You	*innocent*	*table*	*window*	*roof*
innocent	0	−550	−800	−1050
table	50	−250	−800	−1050
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_1^{roof} ⟶

You	*innocent*	*table*	*window*	*roof*
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_1^{roof} ⟶

You	*innocent*	*table*	*window*	*roof*
roof	−450	−450	−450	−750

v_1^{roof}

Table 7.6.9 Bottom-up procedure with fixed beliefs on views for "Too much wine" at views of rank 2

views in the game. In many cases, this specific order of elimination will be easier than the "standard" order prescribed by the original procedure. Instead of providing a formal definition of the bottom-up procedure, we illustrate it by means of an example. This will probably be enough for the reader to understand how the bottom-up version would work in general.

Let us return to the example "Too much wine" with the fixed beliefs on views from Figure 7.6.3. We start with the views of rank 1 which – remember – are the smallest views in the game. That is, we start with v_1^{window} and v_2^{window}. The full decision problem that corresponds to your view v_1^{view} is represented by the first matrix in Table 7.6.8. Note that at your view v_1^{window} you must believe that Barbara's view is v_2^{window}. Your choice *innocent* is strictly dominated by the randomized choice where you choose *table* and *window* with probabilities 0.9 and 0.1, respectively. We can therefore eliminate your choice *innocent,* leading to the second matrix in Table 7.6.8. The same applies to Barbara.

Since you believe that Barbara's view is v_2^{window}, you will believe that Barbara does not choose *innocent.* But then, choosing *window* will always be better than choosing *table,* and we can thus eliminate your choice *table.* This leads to the third matrix in Table 7.6.8, after which the analysis of your view v_1^{window} stops. The same holds for Barbara.

We then move to the views with rank 2 – the second to smallest views – which are v_1^{roof} and v_2^{roof}. The full decision problem for you at the view v_1^{roof} is represented by the first matrix in Table 7.6.9. Recall that with the view v_1^{roof} you believe that, with probability 0.7, Barbara has the view v_1^{window} and that with probability 0.3 she holds the view v_2^{roof}. Since we have seen that Barbara must choose *window* if her view is v_2^{window}, you must assign probability at least 0.7 to Barbara choosing *window.* But then, it can no longer be rational for you to choose *innocent* or *table.*

You	*innocent*	*table*	*window*	*roof*	*door*
innocent	0	−550	−800	−1050	−1300
table	50	−250	−800	−1050	−1300
window	−200	−200	−500	−1050	−1300 ⟶
roof	−450	−450	−450	−750	−1300
door	−700	−700	−700	−700	−1000

v_1^{door}

You	*innocent*	*table*	*window*	*roof*	*door*
⟶ *roof*	−450	−450	−450	−750	−1300

v_1^{door}

Table 7.6.10 Bottom-up procedure with fixed beliefs on views for "Too much wine" at views of rank 3

To see this, recall first that your choice *innocent* cannot be optimal for any belief. To see why *table* cannot be optimal for any of the beliefs above, compare the choices *table* and *window,* and the expected utilities induced by these two choices. Then,

$$\begin{aligned} u_1(window) - u_1(table) &= b_1(innocent)\cdot(-200-50) + b_1(table)\cdot(-200-(-250)) + \\ &\quad + b_1(window)\cdot(-500-(-800)) + b_1(roof)\cdot(-1050-(-1050)) \\ &= -250\cdot b_1(innocent) + 50\cdot b_1(table) + 300\cdot b_1(window). \end{aligned}$$

where $b_1(innocent)$, $b_1(table)$, $b_1(window)$ and $b_1(roof)$ denote the probabilities you assign to these choices of Barbara. As $b_1(window) \geq 0.7$ we conclude that $u_1(window) - u_1(table) > 0$, and hence choosing *window* will always better than choosing *table.* By eliminating the choices *innocent* and *table* we arrive at the second matrix in Table 7.6.9. Similarly for Barbara.

Hence, you believe that Barbara will choose *window* if her view is v_2^{window} and you believe that she will choose *window* or *roof* if her view is v_2^{roof}. Since you only deem possible Barbara's views v_2^{window} and v_2^{roof} at your view v_1^{roof}, you will believe that Barbara only chooses *window* or *roof.* But then, your choice *window* can no longer be optimal as choosing *roof* is always better. Eliminating your choice *window* leads to the last matrix in Table 7.6.9. Similarly for Barbara. This concludes the analysis of the views of rank 2.

We finally move to the views of rank 3 – the largest views – which are v_1^{door} and v_2^{door}. The full decision problem at your view v_1^{door} is represented by the first matrix in Table 7.6.10. Recall that with the view v_1^{door} you assign probability 0.5 to Barbara having the view v_2^{window}, probability 0.3 to her holding the view v_2^{roof} and probability 0.2 to her having the view v_2^{door}. Moreover, based on the eliminations above, you believe that Barbara will choose *window* if her view is v_2^{window} and that she will choose *roof* if her view is v_2^{roof}. As a consequence, you assign probability at least 0.5 to Barbara choosing *window* and probability at least 0.3 to her choosing *roof.* But then, your only optimal choice is *roof.*

To see this, recall first that your choice *innocent* cannot be optimal for any belief. To see why *table* cannot be optimal for any of the beliefs above, compare the choices *table* and *roof.* Then,

$$\begin{aligned} u_1(roof) - u_1(table) &= b_1(innocent)\cdot(-450-50) + b_1(table)\cdot(-450-(-250)) + \\ &\quad + b_1(window)\cdot(-450-(-800)) + b_1(roof)\cdot(-750-(-1050)) \\ &\quad + b_1(door)\cdot(-1300-(-1300)) \\ &= -500\cdot b_1(innocent) - 200\cdot b_1(table) + 350\cdot b_1(window) + 300\cdot b_1(roof). \end{aligned}$$

As $b_1(window) \geq 0.5$ and $b_1(roof) \geq 0.3$, it follows that $u_1(window) - u_1(table) > 0$, and hence choosing *roof* is always better than choosing *table.*

To see why *window* cannot be optimal for any of the beliefs above, compare the choices *window* and *roof.* Then,

$$\begin{aligned} u_1(roof) - u_1(window) &= b_1(innocent) \cdot (-450 - (-200)) + b_1(table) \cdot (-450 - (-200)) + \\ &\quad + b_1(window) \cdot (-450 - (-500)) + b_1(roof) \cdot (-750 - (-1050)) \\ &\quad + b_1(door) \cdot (-1300 - (-1300)) \\ &= -250 \cdot b_1(innocent) - 250 \cdot b_1(table) + 50 \cdot b_1(window) + 300 \cdot b_1(roof). \end{aligned}$$

Since $b_1(window) \geq 0.5$ and $b_1(roof) \geq 0.3$ we have that $b_1(innocent) + b_1(table) \leq 0.2$. But then, $u_1(roof) - u_1(window) > 0$, which means that *roof* is always better than *window.*

Finally, to verify that *door* cannot be optimal for any of the beliefs above, compare the choices *door* and *roof.* Then,

$$\begin{aligned} u_1(roof) - u_1(door) &= b_1(innocent) \cdot (-450 - (-700)) + b_1(table) \cdot (-450 - (-700)) + \\ &\quad + b_1(window) \cdot (-450 - (-700)) + b_1(roof) \cdot (-750 - (-700)) \\ &\quad + b_1(door) \cdot (-1300 - (-1000)) \\ &= 250 \cdot b_1(innocent) + 250 \cdot b_1(table) + 250 \cdot b_1(window) \\ &\quad - 50 \cdot b_1(roof) - 300 \cdot b_1(door). \end{aligned}$$

Since $b_1(window) \geq 0.5$ and $b_1(roof) \geq 0.3$ we have that $b_1(door) \leq 0.2$. But then, the least favorable belief for choosing *roof* compared to choosing *door* is the belief b_1 that assigns probability 0.5 to Barbara choosing *window,* probability 0.3 to her choosing *roof* and probability 0.2 to her choosing *door.* Even under this least favorable belief, we have that $u_1(roof) - u_1(door) > 0$. Hence, choosing *roof* is always better than choosing *door.*

By eliminating the choices *innocent, table, window* and *door* we arrive at the second matrix in Table 7.6.7. Similarly for Barbara. This completes the analysis of the largest views v_1^{door} and v_2^{door}. The bottom-up procedure thereby terminates.

According to the outcome of the procedure, you can only rationally tell the story *window* if your view is v_1^{window}, and you can only rationally tell the story *roof* to Chris if your view if v_1^{roof} or v_1^{door}. This is precisely the conclusion we drew based on the original procedure from Section 7.6.4.

This must be the case: The bottom-up procedure can be viewed as a special order of elimination in the original procedure. Moreover, by Theorem 7.6.3, the outcome of the original procedure is independent of the specific order of elimination chosen. This then implies that the bottom-up procedure must always yield the same outcome as the original procedure, also in the case of fixed beliefs on views.

7.7 Correct and Symmetric Beliefs

In Parts II and III of this book, where we discussed standard games and games with incomplete information, respectively, we have combined the restrictions of common belief in rationality with those of a simple, or symmetric, belief hierarchy. In principle we could do this for games with unawareness as well. However, it turns out that imposing symmetric belief hierarchies necessarily leads to *trivial* cases of unawareness, where you believe that everybody else shares your view of the game, believe that every opponent believes that everybody else shares his view of the game, and so on. That is, we would be back to a *standard* game where every player holds the same view of the game. Since every simple belief hierarchy is symmetric, the same holds if we would impose a *simple* belief hierarchy.

To see why a symmetric belief hierarchy leads to trivial cases of unawareness, consider a symmetric belief hierarchy for a player. By definition, such a symmetric belief hierarchy would be induced by a symmetric weighted beliefs diagram, containing arrows from choice-view pairs to opponents' choice-view combinations. Consider two different players, i and j, and suppose that in this symmetric weighted beliefs diagram there would be an arrow from player i's choice-view pair (c_i, v_i) to the opponents' choice-view combinations, containing player j's choice-view pair (c_j, v_j). By symmetry of the weighted beliefs diagram, there must also be an arrow from the choice-view pair (c_j, v_j) to (c_i, v_i).

Hence, in the belief hierarchy starting at (c_i, v_i), player i believes that, with some positive probability, player j has the view v_j while believing, with some positive probability, that player i has the view v_i. By the awareness principle, the view v_i must thus be contained in v_j, since otherwise player j with view v_j could not reason about the view v_i.

At the same time, player i believes in this belief hierarchy that, with some positive probability, player j believes that, with some positive probability, that player i has the view v_i while believing, with some positive probability, that j's view is v_j. Again, by the awareness principle, it would follow that the view v_j must be contained in the view v_i, since otherwise player i with view v_i could not reason about the view v_j.

Hence, we conclude that v_i must be included in v_j and that v_j must be included in v_i. This, however, is only possible if the views v_i and v_j are equal. As such, we see that whenever there is an arrow from a view v_i to an opponent's view v_j in a symmetric weighted beliefs diagram, then the two views must equal.

But then, all views that enter a symmetric belief hierarchy induced by this symmetric weighted beliefs diagram must be equal as well. We thus conclude that for every symmetric belief hierarchy there is a *single* view v such that the player believes, with probability 1, that (i) all opponents have the view v, (ii) that all opponents believe with probability 1 that all other players have the view v, and so on. But then, we are back to the situation of a *standard* game with a unique view v shared by all the players. Since every simple belief hierarchy is symmetric, the same would hold if we impose a *simple* belief hierarchy instead. For this reason, we do not treat correct and symmetric beliefs in a separate chapter in the part on unawareness, because it would bring us back to the analysis of standard games.

7.8 Proofs

7.8.1 Proofs of Section 7.4

To prove Theorem 7.4.1 we need the following optimality property, similar to the one from the proof sections of Chapters 3 and 5. In the statement of this lemma we denote by $C_{-i}(v_i)$ the set of opponents' choice-combinations in the view v_i.

Lemma 7.8.1 (Optimality property) *For every player i, every view $v_i \in V_i$ and every round $k \geq 0$, let $C_i^k(v_i)$ be the set of choices for player i that survive the first k rounds of the iterated strict dominance procedure for unawareness at v_i, and let $C_i^*(v_i)$ be the set of choices that survive all rounds there. Similarly, let $C_{-i}^k(v_i)$ be the set of states that survive the first k rounds, and let $C_{-i}^*(v_i)$ be the set of states that survive all rounds, at v_i.*

(a) For every $k \geq 1$, a choice c_i is in $C_i^k(v_i)$ if and only if c_i is optimal for some belief in $(C_i(v_i), C_{-i}^k(v_i), u_i)$.

(b) A choice c_i is in $C_i^*(v_i)$ if and only if c_i is optimal for some belief in $(C_i(v_i), C_{-i}^*(v_i), u_i)$.

The proof of this lemma is essentially identical to the one for Lemma 3.6.1 and is therefore omitted.

Proof of Theorem 7.4.1. (a) For every player i and view $v_i \in V_i$, let $BR_i^k(v_i)$ denote the set of choices that player i can rationally make while expressing up to k-fold belief in rationality with view v_i. Recall from above that $C_i^k(v_i)$ and $C_{-i}^k(v_i)$ denote the set of choices and set of states, respectively, that survive the first k rounds of the procedure at view v_i. We will show that $BR_i^k(v_i) = C_i^{k+1}(v_i)$ for every player i, every view $v_i \in V_i$ and every $k \geq 1$. We show this in two steps: (i) prove that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$ for all $k \geq 1$, and (ii) prove that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$.

(i) Show that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$ for all $k \geq 1$.

We prove this by induction on k. For $k = 1$, take some $c_i \in BR_i^1(v_i)$. Then, there is some epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses 1-fold belief in rationality, $w_i(t_i) = v_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j) with $w_j(t_j) = v_j$. Then, v_j is contained in v_i. Moreover, since t_i expresses 1-fold belief in rationality, c_j must be optimal for t_j. Hence, c_j is optimal for t_j's first-order belief in the full decision problem $(C_j(v_j), C_{-j}(v_j), u_j)$ which, by Lemma 7.8.1, implies that $c_j \in C_j^1(v_j)$. Hence, t_i's first-order belief only assigns positive probability to opponents' choices c_j which are in $C_j^1(v_j)$ for some v_j contained in v_i, and thus only assigns positive probability to states in $C_{-i}^2(v_i)$. As c_i is optimal for t_i, we conclude that c_i is optimal for t_i's first-order belief in $(C_i(v_i), C_{-i}^2(v_i), u_i)$ which implies, by Lemma 7.8.1, that c_i is in $C_i^2(v_i)$. We thus have shown that every choice $c_i \in BR_i^1(v_i)$ must be in $C_i^2(v_i)$, and hence $BR_i^1(v_i) \subseteq C_i^2(v_i)$.

Now suppose that $k \geq 2$ and that, by the induction assumption, $BR_i^{k-1}(v_i) \subseteq C_i^k(v_i)$ for all players i and all views v_i. Consider some player i and some $c_i \in BR_i^k(v_i)$. Then, there is some epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses up to k-fold belief in rationality, $w_i(t_i) = v_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j) with $w_j(t_j) = v_j$. Then, v_j is contained in v_i. Moreover, since t_i expresses up to k-fold belief in rationality, the choice c_j must be optimal for t_j and t_j must express up to $(k-1)$-fold belief in rationality. Hence, $c_j \in BR_j^{k-1}(v_j)$. Since, by the induction assumption, $BR_j^{k-1}(v_j) \subseteq C_j^k(v_j)$, we know that $c_j \in C_j^k(v_j)$. We thus conclude that t_i's first-order belief only assigns positive probability to opponents' choices c_j that are in $C_j^k(v_j)$ for some view v_j that is contained in v_i, and hence only assigns positive probability to states in $C_{-i}^{k+1}(v_i)$. As c_i is optimal for t_i, we conclude that c_i is optimal for t_i's first-order belief in $(C_i(v_i), C_{-i}^{k+1}(v_i), u_i)$, which implies, by Lemma 7.8.1, that c_i is in $C_i^{k+1}(v_i)$. We thus have shown that every choice $c_i \in BR_i^k(v_i)$ must be in $C_i^{k+1}(v_i)$, and hence $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$. By induction on k, we conclude that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$ for all players i, all views $v_i \in V_i$, and all $k \geq 1$. This completes the proof of (i).

(ii) Show that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$.

Hence, for every choice $c_i \in C_i^{k+1}(v_i)$ we must show that there is some epistemic model, and some type $t_i^{v_i,c_i}$ in it, such that $t_i^{v_i,c_i}$ expresses up to k-fold belief in rationality, $w_i(t_i^{v_i,c_i}) = v_i$, and c_i is optimal for $t_i^{v_i,c_i}$. We will now construct a *single* epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ that contains *all* such types. For every player i, define the set of types

$$T_i = \{t_i^{v_i,c_i} \mid v_i \in V_i,\ c_i \in C_i^1(v_i)\}$$

where $w_i(t_i^{v_i,c_i}) = v_i$. To define the beliefs of these types about the opponents' choice-type combinations we distinguish the following three cases, assuming that the procedure terminates at the end of round K.

Case 1. Suppose that $c_i \in C_i^1(v_i) \backslash C_i^2(v_i)$. Then, by Lemma 7.8.1, c_i is optimal for some belief $b_i^{v_i,c_i} \in \Delta(C_{-i}(v_i))$ within $(C_i(v_i), C_{-i}(v_i), u_i)$. For every opponent j choose some arbitrary type $\hat{t}_j \in T_j$ with a view $w_j(\hat{t}_j)$ contained in v_i, and define

$$b_i(t_i^{v_i,c_i})((c_j,t_j)_{j \neq i}) := \begin{cases} b_i^{v_i,c_i}((c_j)_{j \neq i}), & \text{if } t_j = \hat{t}_j \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{7.8.1}$$

for all $(c_j, t_j)_{j \neq i}$ in $C_{-i} \times T_{-i}$.

Case 2. Suppose that $c_i \in C_i^k(v_i) \backslash C_i^{k+1}(v_i)$ for some $k \in \{2, ..., K-1\}$. Then, by Lemma 7.8.1, c_i is optimal for some belief $b_i^{v_i,c_i} \in \Delta(C_{-i}^k(v_i))$ within $(C_i(v_i), C_{-i}^k(v_i), u_i)$. By construction of the procedure, for every $(c_j)_{j \neq i} \in C_{-i}^k(v_i)$ and every $j \neq i$, there is some view $v_j^{k-1}[c_j] \in V_j$ contained in v_i such that $c_j \in C_j^{k-1}(v_j^{k-1}[c_j])$. Define

$$b_i(t_i^{v_i,c_i})((c_j,t_j)_{j \neq i}) := \begin{cases} b_i^{v_i,c_i}((c_j)_{j \neq i}), & \text{if } c_j \in C_j^{k-1}(v_i) \text{ and } t_j = t_j^{v_j^{k-1}[c_j],c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{7.8.2}$$

for all $(c_j, t_j)_{j \neq i}$ in $C_{-i} \times T_{-i}$.

Case 3. Suppose that $c_i \in C_i^K(v_i)$. As the procedure terminates at round K we have that $c_i \in C_i^*(v_i)$. Hence, by Lemma 7.8.1, c_i is optimal for some belief $b_i^{v_i,c_i} \in \Delta(C_{-i}^*(v_i))$ within $(C_i(v_i), C_{-i}^*(v_i), u_i)$. By construction of the procedure, for every $(c_j)_{j \neq i} \in C_{-i}^*$ and every $j \neq i$, there is some view $v_j^*[c_j]$ contained in v_i such that $c_j \in C_j^*(v_j^*[c_j])$. Define

$$b_i(t_i^{v_i,c_i})((c_j,t_j)_{j \neq i}) := \begin{cases} b_i^{v_i,c_i}((c_j)_{j \neq i}), & \text{if } c_j \in C_j^*(v_i) \text{ and } t_j = t_j^{v_j^*[c_j],c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{7.8.3}$$

for all $(c_j, t_j)_{j \neq i}$ in $C_{-i} \times T_{-i}$.

By (7.8.1), (7.8.2) and (7.8.3) it follows that every type satisfies the awareness principle. This completes the construction of the epistemic model $M = (T_i, w_i, b_i)_{i \in I}$.

Note that in this epistemic model, every type $t_i^{v_i,c_i}$ holds the first-order belief $b_i^{v_i,c_i}$ on choices. As, by definition, c_i is optimal for $b_i^{v_i,c_i}$ within $(C_i(v_i), C_{-i}(v_i), u_i)$, we conclude that c_i is optimal for $t_i^{v_i,c_i}$, for every player i and every $c_i \in C_i^1(v_i)$.

We now show that for every $k \geq 2$ and every choice $c_i \in C_i^k(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality. We show this by induction on k.

For $k = 2$, consider some choice $c_i \in C_i^2(v_i)$ and the associated type $t_i^{v_i,c_i}$ with the belief given by (7.8.2) or (7.8.3). By (7.8.2) and (7.8.3), the belief $b_i(t_i^{v_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j^1[c_j],c_j})$ where $c_j \in C_j^1(v_j^1[c_j])$. In particular, $b_i(t_i^{v_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where $c_j \in C_j^1(v_j)$. As c_j is optimal for $t_j^{v_j,c_j}$, the type $t_i^{v_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where c_j is optimal for $t_j^{v_j,c_j}$. Hence, $t_i^{v_i,c_i}$ expresses 1-fold belief in rationality. This holds for every type $t_i^{v_i,c_i}$ where $c_i \in C_i^2(v_i)$.

Suppose now that $k \geq 3$ and that, by the induction assumption, $t_i^{v_i,c_i}$ expresses up to $(k-2)$-fold belief in rationality for every $c_i \in C_i^{k-1}(v_i)$ and every player i. Consider some choice $c_i \in C_i^k(v_i)$ and the associated type $t_i^{v_i,c_i}$ with the belief given by (7.8.2) or (7.8.3). By (7.8.2) and (7.8.3) it follows that $b_i(t_i^{v_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j^{k-1}[c_j],c_j})$ where $c_j \in C_j^{k-1}(v_j^{k-1}[c_j])$. In particular, $b_i(t_i^{v_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where $c_j \in C_j^{k-1}(v_j)$. By the induction assumption we know that $t_j^{v_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality. As c_j is optimal for $t_j^{v_j,c_j}$, we conclude that $t_i^{v_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where c_j is optimal for $t_j^{v_j,c_j}$, and $t_j^{v_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality. Hence, $t_i^{v_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality. This holds for every type $t_i^{v_i,c_i}$ where $c_i \in C_i^k(v_i)$.

By induction on k, we conclude that for every $k \geq 2$ and every choice $c_i \in C_i^k(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality.

We next show that for every $c_i \in C_i^K(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses *common* belief in rationality. Consider the smaller epistemic model $M^* = (T_i^*, w_i, b_i)_{i \in I}$ where the set of types for player i is

$$T_i^* := \{t_i^{v_i,c_i} \mid v_i \in V_i \text{ and } c_i \in C_i^*(v_i)\},$$

and the beliefs of the types are given by (7.8.3). Note that this is a well-defined epistemic model, since by (7.8.3) every type $t_i^{v_i,c_i} \in T_i^*$ with $c_i \in C_i^*(v_i)$ only assigns positive probability to opponent's types $t_j^{v_j^*[c_j],c_j} \in T_j^*$ where $c_j \in C_j^*(v_j^*[c_j])$. We show that every type in M^* believes in the opponents' rationality.

Consider a type $t_i^{v_i,c_i} \in T_i^*$ where $c_i \in C_i^*(v_i)$. By (7.8.3), type $t_i^{v_i,c_i}$ only assigns positive probability to opponent's types $t_j^{v_j^*[c_j],c_j} \in T_j^*$ where $c_j \in C_j^*(v_j^*[c_j])$. In particular, $t_i^{v_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where $c_j \in C_j^*(v_j)$. Since c_j is optimal for $t_j^{v_j,c_j}$, the type $t_i^{v_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where c_j is optimal for $t_j^{v_j,c_j}$. Hence, $t_i^{v_i,c_i} \in T_i^*$ believes in the opponents' rationality. Since this holds for every type $t_i^{v_i,c_i} \in T_i^*$, all types in M^* believe in the opponents' rationality. Hence, it follows that all types in M^* express common belief in rationality. Note that the types in M^* are exactly the types $t_i^{v_i,c_i}$ with $c_i \in C_i^K(v_i)$. Hence, for every $c_i \in C_i^K(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses common belief in rationality.

We can now prove that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$. Take some $c_i \in C_i^{k+1}(v_i)$ where $k \geq 1$. Then we know from above that c_i is optimal for the associated type $t_i^{v_i,c_i}$, and that the type $t_i^{v_i,c_i}$ expresses up to k-fold belief in rationality. Hence, by definition, $c_i \in BR_i^k(v_i)$. As this holds for every $c_i \in C_i^{k+1}(v_i)$, we conclude that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$.

Since in part (i) we have already seen that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$, we may conclude that $BR_i^k(v_i) = C_i^{k+1}(v_i)$ for all $k \geq 1$. That is, a choice can rationally be made while expressing up to k-fold belief in rationality with view v_i precisely when the choice survives $k+1$ elimination rounds at v_i. This establishes part (a) of Theorem 7.4.1.

(b) We finally prove part (b) of Theorem 7.4.1. Suppose first that choice c_i can rationally be made under common belief in rationality with view v_i. Then, in particular, for every $k \geq 1$, the choice c_i can rationally be made while expressing up to k-fold belief in rationality with view v_i. By part (a) we then know that c_i survives $k+1$ rounds of elimination at v_i. Since this holds for every $k \geq 1$, we conclude that c_i survives all rounds of elimination at v_i.

Suppose next that the choice c_i survives all rounds of elimination at v_i. Then, $c_i \in C_i^K(v_i)$, where K is the round at which the *iterated strict dominance procedure for unawareness* terminates. From the construction of the epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ above we know that the choice c_i is optimal for the type $t_i^{v_i,c_i}$ and that the type $t_i^{v_i,c_i}$ expresses common belief in rationality. Hence, c_i can rationally be made under common belief in rationality with view v_i.

We thus conclude that a choice c_i can rationally be made under common belief in rationality with view v_i precisely when the choice c_i survives all rounds of elimination at v_i. This completes the proof of part (b), and thereby the proof of this theorem. ■

Proof of Theorem 7.4.2. Recall the definitions and results for reduction operators from Sections 3.6.3.1 and 3.6.3.2. We first show that the *iterated strict dominance procedure for unawareness* can be characterized by the iterated application of a *reduction operator sdu*, and subsequently prove that this reduction operator *sdu* is *monotone*. By Lemma 3.6.2 it would then follow that *sdu*, and thereby the procedure, is *order independent*.

Let $A = (C_i(v_i), C_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$ be the set that assigns to every player i and view $v_i \in V_i$ the (full) decision problem $(C_i(v_i), C_{-i}(v_i), u_i)$. The subsets of A we are interested in have the form $D = (D_i(v_i), D_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$, where $D_i(v_i) \subseteq C_i(v_i)$ and $D_{-i}(v_i) \subseteq C_{-i}(v_i)$ for every player i and every $v_i \in V_i$. For two such subsets $D = (D_i(v_i), D_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$ and $E = (E_i(v_i), E_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$ we write that $D \subseteq E$ if $D_i(v_i) \subseteq E_i(v_i)$ and $D_{-i}(v_i) \subseteq E_{-i}(v_i)$ for every player i and $v_i \in V_i$.

Let *sdu* be the reduction operator that assigns to every set $E = (E_i(v_i), E_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$ the subset $D = (D_i(v_i), D_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$ where, for every player i and $v_i \in V_i$,

$$D_{-i}(v_i) := \{(c_j)_{j \neq i} \in E_{-i}(v_i) \mid \text{for every } j \neq i, \ c_j \in E_j(v_j) \text{ for some } v_j \in V_j \text{ contained in } v_i\}$$

and

$$D_i(v_i) := \{c_i \in E_i(v_i) \mid c_i \text{ not strictly dominated in } (E_i(v_i), D_{-i}(v_i), u_i)\}.$$

Then, by construction,

$$sdu^k(A) = (C_i^k(v_i), C_{-i}^k(v_i), u_i)_{i \in I, v_i \in V_i}$$

for every $k \in \{1, 2, 3, ...\}$, and hence the iterated strict dominance procedure for unawareness can be characterized by the iterated application of the reduction operator *sdu*. We call *sdu* the *strict dominance* operator for *unawareness*.

We next show that *sdu* is monotone. Take some sets D, E of the form above with $sdu(E) \subseteq D \subseteq E$. We show that $sdu(D) \subseteq sdu(E)$.

Let $sdu(D) = (D'_i(v_i), D'_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$ and $sdu(E) = (E'_i(v_i), E'_{-i}(v_i), u_i)_{i \in I, v_i \in V_i}$. Take some player i and view v_i. We start by showing that $D'_{-i}(v_i) \subseteq E'_{-i}(v_i)$. Take some $(c_j)_{j \neq i} \in D'_{-i}(v_i)$. Then, for every player j, we have that $c_j \in D_j(v_j)$ for some v_j contained in v_i. Since $D_j(v_j) \subseteq E_j(v_j)$, we conclude that $c_j \in E_j(v_j)$ for some v_j contained in v_i. As this applies to every $j \neq i$, we conclude that $(c_j)_{j \neq i} \in E'_{-i}(v_i)$. Thus, we see that $D'_{-i}(v_i) \subseteq E'_{-i}(v_i)$.

Next, we show that $D'_i(v_i) \subseteq E'_i(v_i)$. Take some $c_i \in D'_i(v_i)$. Then, c_i is not strictly dominated in $(D_i(v_i), D'_{-i}(v_i), u_i)$. By Theorem 2.6.1 it follows that there is some belief $b_i \in \Delta(D'_{-i}(v_i))$ such that

$$u_i(c_i, b_i) \geq u_i(c'_i, b_i) \text{ for all } c'_i \in D_i(v_i). \tag{7.8.4}$$

Note that $b_i \in \Delta(E'_{-i}(v_i))$ since we have seen that $D'_{-i}(v_i) \subseteq E'_{-i}(v_i)$. Now, let $c_i^* \in E_i(v_i)$ be such that

$$u_i(c_i^*, b_i) \geq u_i(c'_i, b_i) \text{ for all } c'_i \in E_i(v_i). \tag{7.8.5}$$

By Theorem 2.6.1, we conclude that c_i^* is not strictly dominated in $(E_i(v_i), E'_{-i}(v_i), u_i)$, and hence $c_i^* \in E'_i(v_i)$ by definition of the *sdu* operator. Since $sdu(E) \subseteq D$ we know, in particular, that $E'_i(v_i) \subseteq D_i(v_i)$, and thus we see that $c_i^* \in D_i(v_i)$. By combining (7.8.4) and (7.8.5), and using the fact that $c_i^* \in D_i(v_i)$, we conclude that

$$u_i(c_i, b_i) \geq u_i(c_i^*, b_i) \geq u_i(c'_i, b_i) \text{ for all } c'_i \in E_i(v_i).$$

By Theorem 2.6.1 it then follows that c_i is not strictly dominated in $(E_i(v_i), E'_{-i}(v_i), u_i)$, and hence c_i is in $E'_i(v_i)$. This shows that $D'_i(v_i) \subseteq E'_i(v_i)$.

Altogether, we conclude that $sdu(D) \subseteq sdu(E)$. Hence, *sdu* is monotone. By Lemma 3.6.2 it then follows that the reduction operator *sdu* is order independent. As the iterated strict dominance procedure for unawareness coincides with the iterated application of *sdu*, we conclude that the procedure is order independent. This completes the proof. ■

Proof of Theorem 7.4.3. We first show that the *iterated strict dominance procedure for unawareness* leaves, for every player i and every view $v_i \in V_i$, at least one choice and one state in the associated decision problem after the procedure has terminated. To show this, we prove, by induction on k, that $C_i^k(v_i)$ and $C_{-i}^k(v_i)$ are always non-empty for every $k \in \{1, 2, 3, ...\}$.

For $k = 1$ we know, by construction, that $C_{-i}^1(v_i) = C_{-i}(v_i)$, which is non-empty. Now, take some belief $b_i \in \Delta(C_{-i}^1(v_i))$ and some choice c_i that is optimal for b_i in $(C_i(v_i), C_{-i}^1(v_i), u_i)$. Then, by Theorem 2.6.1, c_i is not strictly dominated in $(C_i(v_i), C_{-i}^1(v_i), u_i)$, which means that $c_i \in C_i^1(v_i)$. Thus, $C_i^1(v_i)$ is non-empty.

Now, take some $k \geq 2$, and assume that $C_{-i}^{k-1}(v_i)$ and $C_i^{k-1}(v_i)$ is non-empty for every player i and every v_i. Consider a player i and a view v_i. Let $(c_j)_{j \neq i}$ be such that, for every player j, the choice c_j is in $C_j^{k-1}(v_j)$ for some v_j contained in v_i. Then, by construction, $(c_j)_{j \neq i} \in C_{-i}^k(v_i)$, and thus $C_{-i}^k(v_i)$ is non-empty.

Next, take some belief $b_i \in \Delta(C_{-i}^k(v_i))$ and let c_i be optimal for b_i in $(C_i(v_i), C_{-i}^k(v_i), u_i)$. Then, it follows by Lemma 7.8.1 that $c_i \in C_i^k(v_i)$, and hence $C_i^k(v_i)$ is non-empty.

By induction on k it follows that $C_{-i}^k(v_i)$ and $C_i^k(v_i)$ are non-empty for all k. As the procedure terminates within K rounds, the sets $C_{-i}^K(v_i)$ and $C_i^K(v_i)$ that remain at the end must all be non-empty.

But then, we can construct an epistemic model M^* as in the proof of Theorem 7.4.1. Since this epistemic model has all the properties stated in Theorem 7.4.3, the proof is complete. ■

7.8.2 Proof of Section 7.5

Proof of Theorem 7.5.1. From Theorem 7.4.2 and its proof we know that the *iterated strict dominance procedure for unawareness* is obtained by the iterated application of the reduction operator *sdu*, and that the operator *sdu* is order independent. To prove Theorem 7.5.1 it is therefore sufficient to show that the bottom-up procedure corresponds to a specific elimination order of *sdu*.

As in the proof of Theorem 7.4.2, let $A = (C_i(v_i), C_{-i}(v_i), u_i)_{i\in I, v_i\in V_i}$ be the set that assigns to every player i and view $v_i \in V_i$ the (full) decision problem $(C_i(v_i), C_{-i}(v_i), u_i)$. Suppose that M is the highest rank that a view can achieve. Let

$$(D^0, D^{1.1}, ..., D^{1.K_1}, D^{2.1}, ..., D^{2.K_2}, ..., D^{M.1}, ..., D^{M.K_M})$$

be the sequence of nested subsets of A induced by the bottom-up procedure, where $D^0 = A$, $sdu(D^{M.K_M}) = D^{M.K_M}$, where $D^{1.1}, ..., D^{1.K_1}$ correspond to the elimination rounds for the views of rank 1, $D^{2.1}, ..., D^{2.K_2}$ correspond to the elimination rounds for the views of rank 2, and so on.

We will now show that this sequence of nested subsets is an elimination order for *sdu*. Since $D^0 = A$ and $sdu(D^{M.K_M}) = D^{M.K_M}$, the properties (a) and (c) in the definition of an elimination order (see Section 3.6.3.1) are satisfied. It remains to prove property (b) there. That is, for two subsequent rounds $m.k$ and $m'.k'$ we must show that

$$sdu(D^{m.k}) \subseteq D^{m'.k'} \subseteq D^{m.k}. \tag{7.8.6}$$

As, by construction, $D^{m'.k'} \subseteq D^{m.k}$, it only remains to show that

$$sdu(D^{m.k}) \subseteq D^{m'.k'}.$$

We distinguish two cases: (1) $m'.k' = m.k + 1$, and (2) $m'.k' = m + 1.1$.

Case 1. Suppose that $m'.k' = m.k + 1$. By definition, we have that

$$D^{m.k+1} = (D_i^{m.k+1}(v_i), D_{-i}^{m.k+1}(v_i), u_i)_{i\in I, v_i\in V_i}$$

where, for every player i and every $v_i \in V_i$ with rank m

$$D_{-i}^{m.k+1}(v_i) := \{(c_j)_{j\neq i} \in D_{-i}^{m.k}(v_i) \mid \text{for every } j \neq i,\ c_j \in D_j^{m.k}(v_j) \text{ for some } v_j \in V_j \text{ contained in } v_i\},$$

and

$$D_i^{m.k+1}(v_i) := \{c_i \in D_i^{m.k}(v_i) \mid c_i \text{ not strictly dominated in } (D_i^{m.k}(v_i), D_{-i}^{m.k+1}(v_i), u_i)\}.$$

For every view v_i that does not have rank m we have

$$D_{-i}^{m.k+1}(v_i) = D_{-i}^{m.k}(v_i) \text{ and } D_i^{m.k+1}(v_i) = D_i^{m.k}(v_i).$$

Moreover, by definition, $sdu(D^{m.k}) = (E_i^{m.k+1}(v_i), E_{-i}^{m.k+1}(v_i), u_i)_{i\in I, v_i\in V_i}$ where

$$E_{-i}^{m.k+1}(v_i) := \{(c_j)_{j\neq i} \in D_{-i}^{m.k}(v_i) \mid \text{for every } j \neq i,\ c_j \in D_j^{m.k}(v_j) \text{ for some } v_j \in V_j \text{ contained in } v_i\}$$

and

$$E_i^{m.k+1}(v_i) := \{c_i \in D_i^{m.k}(v_i) \mid c_i \text{ not strictly dominated in } (D_i^{m.k}(v_i), E_{-i}^{m.k+1}(v_i), u_i)\}.$$

By construction, it holds that $E_{-i}^{m.k+1}(v_i) \subseteq D_{-i}^{m.k+1}(v_i)$ for every player i and every $v_i \in V_i$. We now show that $E_i^{m.k+1}(v_i) \subseteq D_i^{m.k+1}(v_i)$ for every player i and every $v_i \in V_i$. Take some $c_i \in E_i^{m.k+1}(v_i)$. Then, $c_i \in D_i^{m.k}(v_i)$ and c_i is not strictly dominated in $(D_i^{m.k}(v_i), E_{-i}^{m.k+1}(v_i), u_i)$. Hence, by Theorem 2.6.1, c_i is optimal in $(D_i^{m.k}(v_i), E_{-i}^{m.k+1}(v_i), u_i)$ for some belief $b_i \in \Delta(E_{-i}^{m.k+1}(v_i))$. As $E_{-i}^{m.k+1}(v_i) \subseteq D_{-i}^{m.k+1}(v_i)$, it follows that $b_i \in \Delta(D_{-i}^{m.k+1}(v_i)$ also. Thus, the choice c_i is optimal in $(D_i^{m.k}(v_i), D_{-i}^{m.k+1}(v_i), u_i)$ for some belief $b_i \in \Delta(D_{-i}^{m.k+1}(v_i)$. But then, by Theorem 2.6.1, the choice c_i is not strictly dominated in $(D_i^{m.k}(v_i), D_{-i}^{m.k+1}(v_i), u_i)$, and hence $c_i \in D_i^{m.k+1}(v_i)$ by definition. As this holds for every $c_i \in E_i^{m.k+1}(v_i)$, we conclude that $E_i^{m.k+1}(v_i) \subseteq D_i^{m.k+1}(v_i)$ for every player i and every $v_i \in V_i$.

As we have already seen that $E_{-i}^{m.k+1}(v_i) \subseteq D_{-i}^{m.k+1}(v_i)$ for every player i and every $v_i \in V_i$ it follows that $sdu(D^{m.k}) \subseteq D^{m.k+1}$. This, in turn, establishes (7.8.6).

Case 2. Suppose that $m'.k' = m+1.1$. Then, we have that $m.k = m.K_m$. By definition, we have that

$$D^{m+1.1} = (D_i^{m+1.1}(v_i), D_{-i}^{m+1.1}(v_i), u_i)_{i \in I, v_i \in V_i}$$

where, for every player i and every $v_i \in V_i$ with rank $m+1$,

$$D_{-i}^{m+1.1}(v_i) := \{(c_j)_{j \neq i} \in D_{-i}^{m.K_m}(v_i) \mid \text{for every } j \neq i, \\ c_j \in D_j^{m.K_m}(v_j) \text{ for some } v_j \in V_j \text{ contained in } v_i\},$$

and

$$D_i^{m+1.1}(v_i) := \{c_i \in D_i^{m.K_m}(v_i) \mid c_i \text{ not strictly dominated in } (D_i^{m.K_m}(v_i), D_{-i}^{m+1.1}(v_i), u_i)\}.$$

For every view v_i that does not have rank $m+1$ we have that

$$D_{-i}^{m+1.1}(v_i) = D_{-i}^{m.K_m}(v_i) \text{ and } D_i^{m+1.1}(v_i) = D_i^{m.K_m}(v_i).$$

Moreover, by definition, $sdu(D^{m.k}) = (E_i^{m+1.1}(v_i), E_{-i}^{m+1.1}(v_i), u_i)_{i \in I, v_i \in V_i}$ where

$$E_{-i}^{m+1.1}(v_i) := \{(c_j)_{j \neq i} \in D_{-i}^{m.K_m}(v_i) \mid \text{for every } j \neq i, \\ c_j \in D_j^{m.K_m}(v_j) \text{ for some } v_j \in V_j \text{ contained in } v_i\}$$

and

$$E_i^{m+1.1}(v_i) := \{c_i \in D_i^{m.K_m}(v_i) \mid c_i \text{ not strictly dominated in } (D_i^{m.K_m}(v_i), E_{-i}^{m+1.1}(v_i), u_i)\}.$$

In a similar way as for Case 1 it can be shown that $sdu(D^{m.K_m}) \subseteq D^{m+1.1}$. This, in turn, implies (7.8.6). This completes Case 2.

By (7.8.6) we thus conclude that the sequence of nested subsets above is an elimination order for *sdu*. As this sequence of nested subsets is induced by the bottom-up procedure, we conclude that the bottom-up procedure corresponds to a specific elimination procedure for *sdu*. Since we know, from Theorem 7.4.2, that the reduction operator *sdu* is order independent, we conclude that the bottom-up procedure yields the same output as the original procedure. This completes the proof. ■

7.8.3 Proofs of Section 7.6

To prove Theorem 7.6.1 we need the following optimality property, similar to Lemma 5.6.2 in the proof section of Chapter 5.

Lemma 7.8.2 (Optimality property) *For every player i, every view $v_i \in V_i$ and every round $k \geq 0$, let $C_i^k(v_i)$ be the set of choices for player i that survive the first k rounds of the iterated strict dominance procedure for unawareness with fixed beliefs p on views at v_i, and let $C_i^*(v_i)$ be the set of choices that survive all rounds there.*

(a) A choice c_i is in $C_i^1(v_i)$, if and only if, c_i is optimal in $(C_i(v_i), C_{-i}(v_i), u_i)$ for some first-order belief b_i^1 on opponents' choices and views.

(b) For every $k \geq 2$, a choice c_i is in $C_i^k(v_i)$, if and only if, c_i is optimal in $(C_i(v_i), C_{-i}(v_i), u_i)$ for some first-order belief b_i^1 on opponents' choices and views where (i) b_i^1's belief about the opponents' views is $p_i(v_i)$ and (ii) b_i^1 only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^{k-1}(v_j)$.

(c) A choice c_i is in $C_i^*(v_i)$, if and only if, c_i is optimal in $(C_i(v_i), C_{-i}(v_i), u_i)$ for some first-order belief b_i^1 on opponents' choices and views where (i) b_i^1's belief about the opponents' views is $p_i(v_i)$ and (ii) b_i^1 only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^*(v_j)$.

Proof. (a) and (b). We prove the statements (a) and (b) by induction on k. We start by showing the statement in (a) for $k = 1$. Recall that $C_i^1(v_i)$ contains precisely those choices in $C_i(v)$ that are not strictly dominated in $(C_i(v_i), C_{-i}(v_i), u_i)$. By Theorem 2.6.1 these are precisely the choices that are optimal in $(C_i(v_i), C_{-i}(v_i), u_i)$ for some first-order belief b_i^1 on opponents' choices and views. Hence, the statement in (a) follows.

Suppose now that $k \geq 2$ and that the statement in (a) or (b) is true for $k - 1$. To show the "only if" direction for k, consider some choice $c_i \in C_i^k(v_i)$. Then, by definition, there is a first-order belief b_i^1 on opponents' choices and views such that (i) b_i^1's belief on views is $p_i(v_i)$, (ii) b_i^1 only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^{k-1}(v_j)$, and

$$u_i(c_i, b_i^1) \geq u_i(c_i', b_i^1) \text{ for all } c_i' \in C_i^{k-1}(v_i). \tag{7.8.7}$$

Let $c_i^* \in C_i$ be optimal for the belief b_i^1 within $(C_i(v_i), C_{-i}(v_i), u_i)$. That is,

$$u_i(c_i^*, b_i^1) \geq u_i(c_i', b_i^1) \text{ for all } c_i' \in C_i(v_i). \tag{7.8.8}$$

As $C_j^{k-1}(v_j) \subseteq C_j^{k-2}(v_j)$ for all v_j, we conclude that b_i^1 only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^{k-2}(v_j)$. But then, by the induction assumption, $c_i^* \in C_i^{k-1}(v_i)$. By (7.8.7) we thus conclude that

$$u_i(c_i, b_i^1) \geq u_i(c_i^*, b_i^1). \tag{7.8.9}$$

By combining (7.8.9) and (7.8.8) we see that

$$u_i(c_i, b_i^1) \geq u_i(c_i^*, b_i^1) \geq u_i(c_i', b_i^1) \text{ for all } c_i' \in C_i,$$

and hence c_i is optimal for the belief b_i^1 in $(C_i(v_i), C_{-i}(v_i), u_i)$. This establishes the "only if" part.

To show the "if" part, consider some choice c_i that is optimal in $(C_i(v_i), C_{-i}(v_i), u_i)$ for some first-order belief b_i^1 on opponents' choices and views where b_i^1's belief on views is $p_i(v_i)$ and b_i^1 only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^{k-1}(v_j)$. Then, in particular, c_i is optimal for this belief in $(C_i^{k-1}(v_i), C_{-i}(v_i), u_i)$, and hence $c_i \in C_i^k(v_i)$. This establishes the "if" 'direction.

By combining the "only if" and "if" direction, the statement in (b) follows for k. By induction on k, statements (a) and (b) hold for every $k \geq 1$.

(c) Suppose that the procedure terminates at the end of round K. That is, $C_i^*(v_i) = C_i^K(v_i) = C_i^{K+1}(v_i)$ for every player i and view v_i. Then, c_i is in $C_i^*(v_i)$ precisely when $c_i \in C_i^{K+1}(v_i)$. By applying (b) to $k = K+1$, we know that c_i is in $C_i^{K+1}(v_i)$ precisely when c_i is optimal with the view v_i for some first-order belief b_i^1 on opponents' choices and views where b_i^1's belief on views is $p_i(v_i)$ and b_i^1 only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^K(v_j)$. As $C_j^K(v_j) = C_j^*(v_j)$, this completes the proof. ■

Proof of Theorem 7.6.1. (a) For every player i and view $v_i \in V_i$, let $BR_i^k(v_i)$ denote the set of choices that player i can rationally make while expressing up to k-fold belief in rationality and up to k-fold belief in p with view v_i. Recall from above that $C_i^k(v_i)$ denotes the set of choices that survive the first k rounds at v_i. We will show that $BR_i^k(v_i) = C_i^{k+1}(v_i)$ for every player i, every view $v_i \in V_i$ and every $k \geq 1$. We show this in two steps: (i) prove that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$ for all $k \geq 1$, and (ii) prove that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$.

(i) Show that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$ for all $k \geq 1$.

We prove this by induction on k. For $k = 1$, take some $c_i \in BR_i^1(v_i)$. Then, there is some epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses 1-fold belief in rationality and 1-fold belief in p, where $w_i(t_i) = v_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j) with $w_j(t_j) = v_j$. Since t_i expresses 1-fold belief in rationality, c_j must be optimal for t_j. Hence, c_j is optimal for t_j's first-order belief in the full decision problem $(C_j(v_j), C_{-j}(v_j), u_j)$ which, by Lemma 7.8.2, implies that $c_j \in C_j^1(v_j)$. Thus, t_i's first-order belief $b_i^1(t_i)$ only assigns positive probability to pairs (c_j, v_j) with $c_j \in C_j^1(v_j)$. Moreover, as t_i expresses 1-fold belief in p, we know that $b_i^1(t_i)$'s belief about the views is $p_i(v_i)$. Finally, as c_i is optimal for t_i, we conclude that c_i is optimal for $b_i^1(t_i)$ in $(C_i(v_i), C_{-i}(v_i), u_i)$. This implies, by Lemma 7.8.2, that c_i is in $C_i^2(v_i)$. We thus have shown that every choice $c_i \in BR_i^1(v_i)$ must be in $C_i^2(v_i)$, and hence $BR_i^1(v_i) \subseteq C_i^2(v_i)$.

Now suppose that $k \geq 2$ and that, by the induction assumption, $BR_i^{k-1}(v_i) \subseteq C_i^k(v_i)$ for all players i and all views v_i. Consider some player i and some $c_i \in BR_i^k(v_i)$. Then, there is some epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses up to k-fold belief in rationality, t_i expresses up to k-fold belief in p, where $w_i(t_i) = v_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j) with $w_j(t_j) = v_j$. Since t_i expresses up to k-fold belief in rationality and up to k-fold belief in p, the choice c_j must be optimal for t_j and t_j must express up to $(k-1)$-fold belief in rationality and up to $(k-1)$-fold belief in p. Hence, $c_j \in BR_j^{k-1}(v_j)$. Since, by the induction assumption, $BR_j^{k-1}(v_j) \subseteq C_j^k(v_j)$, we know that $c_j \in C_j^k(v_j)$. We thus conclude that t_i's first-order belief b_i^1 only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^k(v_j)$. Morever, as t_i expresses 1-fold belief in p, the belief that b_i^1 has about the views is $p_i(v_i)$. Finally, as c_i is optimal for t_i, we conclude that c_i is optimal for t_i's first-order belief b_i^1 in $(C_i(v_i), C_{-i}(v_i), u_i)$. This implies, by Lemma 7.8.2, that c_i is in $C_i^{k+1}(v_i)$. We thus have shown that every choice $c_i \in BR_i^k(v_i)$ must be in $C_i^{k+1}(v_i)$, and hence $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$. By induction on k, we conclude that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$ for all players i, all views $v_i \in V_i$, and all $k \geq 1$. This completes the proof of (i).

(ii) Show that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$.

Hence, for every choice $c_i \in C_i^{k+1}(v_i)$ we must show that there is some epistemic model, and some type $t_i^{v_i,c_i}$ in it, such that $t_i^{v_i,c_i}$ expresses up to k-fold belief in rationality, expresses up to k-fold belief in p, that $w_i(t_i^{v_i,c_i}) = v_i$, and c_i is optimal for $t_i^{v_i,c_i}$. We will now construct a *single* epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ that contains *all* such types. For every player i, define the set of types

$$T_i = \{t_i^{v_i,c_i} \mid v_i \in V_i,\ c_i \in C_i^1(v_i)\}$$

where $w_i(t_i^{v_i,c_i}) = v_i$. To define the beliefs of these types about the opponents' choice-type combinations we distinguish the following three cases, assuming that the procedure terminates at the end of round K.

Case 1. Suppose that $c_i \in C_i^1(v_i) \backslash C_i^2(v_i)$. Then, by Lemma 7.8.2 (a), c_i is optimal for some belief $b_i^{v_i,c_i} \in \Delta(C_{-i}(v_i))$ within $(C_i(v_i), C_{-i}(v_i), u_i)$. For every opponent j choose some arbitrary type $\hat{t}_j \in T_j$ such that its view $w_j(\hat{t}_j)$ is contained in v_i, and define

$$b_i(t_i^{v_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{v_i,c_i}((c_j)_{j\neq i}), & \text{if } t_j = \hat{t}_j \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \quad (7.8.10)$$

for all $(c_j, t_j)_{j \neq i}$ in $C_{-i} \times T_{-i}$.

Case 2. Suppose that $c_i \in C_i^k(v_i) \backslash C_i^{k+1}(v_i)$ for some $k \in \{2, ..., K-1\}$. Then, by Lemma 7.8.2 (b), c_i is optimal within $(C_i(v_i), C_{-i}(v_i), u_i)$ for some first-order belief $b_i^{v_i,c_i} \in \Delta(C_{-i} \times V_{-i})$ which has the belief $p_i(v_i)$ on views, and only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^{k-1}(v_j)$ and v_j is contained in v_i. Define

$$b_i(t_i^{v_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{v_i,c_i}((c_j,v_j)_{j\neq i}), & \text{if } c_j \in C_j^{k-1}(v_j) \text{ and } t_j = t_j^{v_j,c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \quad (7.8.11)$$

for all $(c_j, t_j)_{j \neq i}$ in $C_{-i} \times T_{-i}$.

Case 3. Suppose that $c_i \in C_i^K(v_i)$. As the procedure terminates at round K we have that $c_i \in C_i^*(v_i)$. Hence, by Lemma 7.8.2 (c), c_i is optimal within $(C_i(v_i), C_{-i}(v_i), u_i)$ for some first-order belief $b_i^{v_i,c_i} \in \Delta(C_{-i} \times V_{-i})$ that has the belief $p_i(v_i)$ on views, and only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^*(v_j)$ and v_j is contained in v_i. Define

$$b_i(t_i^{v_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{v_i,c_i}((c_j,v_j)_{j\neq i}), & \text{if } c_j \in C_j^*(v_j) \text{ and } t_j = t_j^{v_j,c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \quad (7.8.12)$$

for all $(c_j, t_j)_{j \neq i}$ in $C_{-i} \times T_{-i}$. By construction, all types satisfy the awareness principle. This completes the construction of the epistemic model $M = (T_i, w_i, b_i)_{i \in I}$.

Note that in this epistemic model, every type $t_i^{v_i,c_i}$ holds the first-order belief $b_i^{v_i,c_i}$. As, by definition, c_i is optimal for $b_i^{v_i,c_i}$ within $(C_i(v_i), C_{-i}(v_i), u_i)$, we conclude that c_i is optimal for $t_i^{v_i,c_i}$, for every player i, every view v_i and every choice $c_i \in C_i^1(v_i)$.

We now show that for every $k \geq 2$ and every choice $c_i \in C_i^k(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality and up to $(k-1)$-fold belief in p. We show this by induction on k.

For $k = 2$, consider some choice $c_i \in C_i^2(v_i)$ and the associated type $t_i^{v_i,c_i}$ with the belief given by (7.8.11) or (7.8.12). By (7.8.11) and (7.8.12), the belief $b_i(t_i^{v_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where $c_j \in C_j^1(v_j)$. As c_j is optimal for $t_j^{v_j,c_j}$, the type $t_i^{v_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where c_j is optimal for $t_j^{v_j,c_j}$. Hence, $t_i^{v_i,c_i}$ expresses 1-fold belief in rationality. This holds for every type $t_i^{v_i,c_i}$ where $c_i \in C_i^2(v_i)$. Moreover, as $t_i^{v_i,c_i}$ holds the first-order belief $b_i^{v_i,c_i}$ on opponents' choices and views, which induces the belief $p_i(v_i)$ on views, it follows that $t_i^{v_i,c_i}$ expresses 1-fold belief in p.

Suppose now that $k \geq 3$ and that, by the induction assumption, $t_i^{v_i,c_i}$ expresses up to $(k-2)$-fold belief in rationality and up to $(k-2)$-fold belief in p for every player i, every $v_i \in V_i$ and every $c_i \in C_i^{k-1}(v_i)$. Consider some choice $c_i \in C_i^k(v_i)$ and the associated type $t_i^{v_i,c_i}$ with the belief given by (7.8.11) or (7.8.12). By (7.8.11) and (7.8.12) it follows that $b_i(t_i^{v_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where $c_j \in C_j^{k-1}(v_j)$. By the induction assumption we know that $t_j^{v_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality and up to $(k-2)$-fold belief in p. As c_j is optimal for $t_j^{v_j,c_j}$, we conclude that $t_i^{v_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where c_j is optimal for $t_j^{v_j,c_j}$, and $t_j^{v_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality and up to $(k-2)$-fold belief in p. Hence, $t_i^{v_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality and up to $(k-1)$-fold belief in p. This holds for every type $t_i^{v_i,c_i}$ where $c_i \in C_i^k(v_i)$.

By induction on k, we conclude that for every $k \geq 2$ and every choice $c_i \in C_i^k(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality and up to $(k-1)$-fold belief in p.

We next show that for every $c_i \in C_i^K(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses *common* belief in rationality and *common* belief in p. Consider the smaller epistemic model $M^* = (T_i^*, w_i, b_i)_{i \in I}$ where the set of types for player i is

$$T_i^* := \{t_i^{v_i,c_i} \mid v_i \in V_i \text{ and } c_i \in C_i^*(v_i)\},$$

and the beliefs of the types are given by (7.8.12). Note that this is a well-defined epistemic model, since by (7.8.12) every type $t_i^{v_i,c_i} \in T_i^*$ with $c_i \in C_i^*(v_i)$ only assigns positive probability to opponent's types $t_j^{v_j,c_j} \in T_j^*$ where $c_j \in C_j^*(v_j)$. We show that every type in M^* believes in the opponents' rationality.

Consider a type $t_i^{v_i,c_i} \in T_i^*$ where $c_i \in C_i^*(v_i)$. By (7.8.12), type $t_i^{v_i,c_i}$ only assigns positive probability to opponents ' choice-type pairs $(c_j, t_j^{v_j,c_j})$ where $c_j \in C_j^*(v_j)$. Since c_j is optimal for $t_j^{v_j,c_j}$, the type $t_i^{v_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{v_j,c_j})$ where c_j is optimal for $t_j^{v_j,c_j}$. Hence, $t_i^{v_i,c_i} \in T_i^*$ believes in the opponents' rationality. Moreover, we have seen that $t_i^{v_i,c_i}$ expresses 1-fold belief in p.

Since this holds for every type $t_i^{v_i,c_i} \in T_i^*$, all types in M^* believe in the opponents' rationality and express 1-fold belief in p. Hence, it follows that all types in M^* express common belief in rationality and common belief in p. Note that the types in M^* are exactly the types $t_i^{v_i,c_i}$ with $c_i \in C_i^K(v_i)$. Hence, for every $c_i \in C_i^K(v_i)$, the associated type $t_i^{v_i,c_i}$ expresses common belief in rationality and common belief in p.

We can now prove that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$. Take some $c_i \in C_i^{k+1}(v_i)$ where $k \geq 1$. Then we know from above that c_i is optimal for the associated type $t_i^{v_i,c_i}$, and that the type $t_i^{v_i,c_i}$ expresses up to k-fold belief in rationality and up to k-fold belief in p. Hence, by definition, $c_i \in BR_i^k(v_i)$. As this holds for every $c_i \in C_i^{k+1}(v_i)$, we conclude that $C_i^{k+1}(v_i) \subseteq BR_i^k(v_i)$ for all $k \geq 1$.

Since in part (i) we have already seen that $BR_i^k(v_i) \subseteq C_i^{k+1}(v_i)$, we may conclude that $BR_i^k(v_i) = C_i^{k+1}(v_i)$ for all $k \geq 1$. That is, a choice can rationally be made while expressing up to k-fold belief in rationality and up to k-fold belief in p with view v_i precisely when the choice survives $k+1$ elimination rounds at v_i. This establishes part (a) of Theorem 7.6.1.

(b) We finally prove part (b) of Theorem 7.6.1. Suppose first that choice c_i can rationally be made under common belief in rationality and common belief in p with view v_i. Then, in particular, for every

$k \geq 1$, the choice c_i can rationally be made while expressing up to k-fold belief in rationality and up to k-fold belief in p with view v_i. By part (a) we then know that c_i survives $k+1$ rounds of elimination at v_i. Since this holds for every $k \geq 1$, we conclude that c_i survives all rounds of elimination at v_i.

Suppose next that the choice c_i survives all rounds of elimination at v_i. Then, $c_i \in C_i^K(v_i)$, where K is the round at which the *iterated strict dominance procedure for unawareness with fixed beliefs p on views* terminates. From the construction of the epistemic model $M = (T_i, w_i, b_i)_{i \in I}$ above we know that the choice c_i is optimal for the type $t_i^{v_i, c_i}$ and that the type $t_i^{v_i, c_i}$ expresses common belief in rationality and common belief in p. Hence, c_i can rationally be made under common belief in rationality and common belief in p with view v_i. We thus conclude that a choice c_i can rationally be made under common belief in rationality and common belief in p with view v_i precisely when the choice c_i survives all rounds of elimination at v_i. This completes the proof of part (b), and thereby the proof of this theorem. ■

Proof of Theorem 7.6.2. We first show that the *iterated strict dominance procedure for unawareness with fixed beliefs p on views* leaves, for every player i and every view $v_i \in V_i$, at least one choice in the associated decision problem after the procedure has terminated. To show this, we prove, by induction on k, that $C_i^k(v_i)$ is always non-empty for every $k \in \{1, 2, 3, ...\}$.

Start with $k = 1$. Take a player i, a view v_i, and take a first-order belief b_i^1 on opponents' choices and views. Select a choice c_i that is optimal for b_i^1 in $(C_i(v_i), C_{-i}(v_i), u_i)$. Then, by Lemma 7.8.2 (a), $c_i \in C_i^1(v_i)$. In particular, $C_i^1(v_i)$ is non-empty.

Now, take some $k \geq 2$, and assume that $C_i^{k-1}(v_i)$ is non-empty for every player i and every v_i. Consider a player i and a view v_i. Take a first-order belief $b_i^1 \in \Delta(C_{-i} \times V_{-i})$ that has the belief $p_i(v_i)$ on views, and only assigns positive probability to pairs (c_j, v_j) where $c_j \in C_j^{k-1}(v_j)$. Clearly, such a belief can be found since these sets $C_j^{k-1}(v_j)$ are all non-empty. Let c_i be optimal for b_i^1 in $(C_i(v_i), C_{-i}(v_i), u_i)$. Then, it follows by Lemma 7.8.2 (b) that $c_i \in C_i^k(v_i)$, and hence $C_i^k(v_i)$ is non-empty.

By induction on k it follows that $C_i^k(v_i)$ is non-empty for all k. As the procedure terminates within K rounds, the sets $C_i^K(v_i)$ that remain at the end must all be non-empty.

But then, we can construct an epistemic model M^* as in the proof of Theorem 7.6.1. Since this epistemic model has all the properties stated in Theorem 7.6.2, the proof is complete. ■

Proof of Theorem 7.6.3. Recall again the definitions and results for reduction operators from Sections 3.6.3.1 and 3.6.3.2. We first show that the *iterated strict dominance procedure for unawareness with fixed beliefs p on views* can be characterized by the iterated application of a *reduction operator sdup*, and subsequently prove that this reduction operator *sdup* is *monotone*. By Lemma 3.6.2 it would then follow that *sdup*, and thereby the procedure, is *order independent*.

Let $A = (C_i(v_i))_{i \in I, v_i \in V_i}$ be the set that assigns to every player i and view $v_i \in V_i$ the (full) set of choices $C_i(v_i) = C_i$. The subsets of A we are interested in have the form $D = (D_i(v_i))_{i \in I, v_i \in V_i}$, where $D_i(v_i) \subseteq C_i$ for every player i and every $v_i \in V_i$. For two such subsets $D = (D_i(v_i))_{i \in I, v_i \in V_i}$ and $E = (E_i(v_i))_{i \in I, v_i \in V_i}$ we write that $D \subseteq E$ if $D_i(v_i) \subseteq E_i(v_i)$ for every player i and $v_i \in V_i$.

Let *sdup* be the reduction operator that assigns to the full set $(C_i(v_i))_{i \in I, v_i \in V_i}$ the subset $D = (D_i(v_i))_{i \in I, v_i \in V_i}$ where, for every player i and $v_i \in V_i$,

$$D_i(v_i) = \{c_i \in C_i(v_i) \mid c_i \text{ optimal in } (C_i(v_i), C_{-i}(v_i), u_i) \text{ for a first-order belief } b_i^1 \in \Delta(C_{-i} \times V_{-i})\},$$

and let *sdup* assign to every other set $E = (E_i(v_i))_{i \in I, v_i \in V_i} \neq (C_i(v_i\ ))_{i \in I, v_i \in V_i}$ the subset $D = (D_i(v_i))_{i \in I, v_i \in V_i}$ where, for every player i and $v_i \in V_i$,

$$D_i(v_i) = \{c_i \in E_i(v_i) \mid c_i \text{ optimal in } (E_i(v_i), C_{-i}(v_i), u_i) \text{ for a first-order belief } b_i^1 \in \Delta(C_{-i} \times V_{-i}) \text{ that has belief } p_i(v_i) \text{ on views and only assigns positive probability to pairs } (c_j, v_j) \text{ where } c_j \in E_j(v_j)\}.$$

Recall from Lemma 7.8.2 (a) that $C_i^1(v_i)$ contains precisely those choices in $C_i(v)$ that are optimal in $(C_i(v_i), C_{-i}(v_i), u_i)$ for a first-order belief $b_i^1 \in \Delta(C_{-i} \times V_{-i})$.

Then, we have that

$$sdup^k(A) = (C_i^k(v_i))_{i \in I, v_i \in V_i}$$

for every $k \in \{1, 2, 3, ...\}$, and hence the iterated strict dominance procedure for unawareness with fixed beliefs p on views corresponds to the iterated application of the reduction operator *sdup*. We call *sdup* the *strict dominance* operator for *unawareness* with *fixed beliefs p on views*.

We next show that *sdup* is monotone. Take some sets D, E of the form above with $sdup(E) \subseteq D \subseteq E$. We show that $sdup(D) \subseteq sdup(E)$.

Suppose first that $D = E$. Then, $sdup(D) = sdup(E)$, and hence it trivially holds that $sdup(D) \subseteq sdup(E)$.

Assume next that $D \neq E$ which implies, in particular, that $D \neq (C_i(v_i\ ))_{i \in I, v_i \in V_i}$. Let $sdup(D) = (D_i'(v_i))_{i \in I, v_i \in V_i}$ and $sdup(E) = (E_i'(v_i))_{i \in I, v_i \in V_i}$. Take some player i and view v_i. We show that $D_i'(v_i) \subseteq E_i'(v_i)$. Take some $c_i \in D_i'(v_i)$. Then, c_i is optimal in $(D_i(v_i), C_{-i}(v_i), u_i)$ for a first-order belief $b_i^1 \in \Delta(C_{-i} \times V_{-i})$ that has the belief $p_i(v_i)$ on views and only assigns positive probability to pairs (c_j, v_j) where $c_j \in D_j(v_j)$. That is,

$$u_i(c_i, b_i^1) \geq u_i(c_i', b_i^1) \text{ for all } c_i' \in D_i(v_i). \tag{7.8.13}$$

Since $D_j(v_j) \subseteq E_j(v_j)$ for all opponents j and views v_j, we conclude that p only assigns positive probability to pairs (c_j, v_j) where $c_j \in E_j(v_j)$. Now, let $c_i^* \in E_i(v_i)$ be such that

$$u_i(c_i^*, b_i^1) \geq u_i(c_i', b_i^1) \text{ for all } c_i' \in E_i(v_i). \tag{7.8.14}$$

Then, by definition of the *sdup* operator, we have that $c_i^* \in E_i'(v_i)$. Since $sdup(E) \subseteq D$ we know, in particular, that $E_i'(v_i) \subseteq D_i(v_i)$, and thus we see that $c_i^* \in D_i(v_i)$. By combining (7.8.13) and (7.8.14), and using the fact that $c_i^* \in D_i(v_i)$, we conclude that

$$u_i(c_i, b_i^1) \geq u_i(c_i^*, b_i^1) \geq u_i(c_i', b_i^1) \text{ for all } c_i' \in E_i(v_i).$$

Hence, it follows that c_i is in $E_i'(v_i)$. This shows that $D_i'(v_i) \subseteq E_i'(v_i)$.

Altogether, we conclude that $sdup(D) \subseteq sdup(E)$. Hence, *sdup* is monotone. By Lemma 3.6.2 it then follows that the reduction operator *sdup* is order independent. As the iterated strict dominance procedure for unawareness with fixed beliefs p on views corresponds to the iterated application of *sdup*, we conclude that the procedure is order independent. This completes the proof. ■

Solutions to In-Chapter Questions

Question 7.1.1. If Barbara believes that you have the view v_1^{all}, then she believes that you could choose *Faraway Beach* and *Distant Beach.* If Barbara believes that you have the view v_1^{two}, then she believes that you could choose *Nextdoor Beach* and *Closeby Beach.*

Question 7.1.2. In your first-order belief, you believe that Barbara chooses *Nextdoor Beach* while having the view v_2^{all}. In your second-order belief, you believe that Barbara believes that you choose *Faraway Beach w*hile having the view v_1^{all}. In your third-order belief, you believe that Barbara believes that you believe that Barbara chooses *Nextdoor Beach* while having the view v_2^{all}. In particular, you believe that Barbara believes that your view is v_1^{all} – your actual view.

Question 7.1.3. Consider the beliefs diagram from Figure 7.1.1. Note that your choice *Nextdoor Beach* is optimal for the belief hierarchy that starts at (*Nextdoor,* v_1^{two}), that *Closeby Beach* is optimal for the belief hierarchy that starts at (*Closeby,* v_1^{two}), and that both belief hierarchies express common belief in rationality. Hence, with the view v_1^{two} you can rationally go to *Nextdoor Beach* and *Closeby Beach* under common belief in rationality.

Question 7.1.4. All the views for you and Barbara are contained in v_1^{all}, whereas only the views v_1^{two} and v_2^{two} are contained in v_1^{two}.

Question 7.1.5. Your view v_1^{all} contains Barbara's views v_2^{all} and v_2^{two}, whereas your view v_1^{two} contains Barbara's view v_2^{two}. Similarly for Barbara.

Question 7.2.1. By the conditions (i) and (ii), the choice c_j must be part of the view v_j, and the view v_j must be contained in v_i. Hence, the choice c_j must be contained in v_i as well.

Question 7.4.1. Your set of types is

$$T_1 = \{t_1^{window,window},\ t_1^{roof,roof},\ t_1^{roof,door},\ t_1^{door,door}\}$$

and similarly for Barbara. The views and beliefs of the types are

$$\begin{gathered} w_1(t_1^{window,window}) = \ window,\ w_1(t_1^{roof,roof}) = \ roof, \\ w_1(t_1^{roof,door}) = w_1(\ t_1^{door,door}) = \ door, \\ b_1(t_1^{window,window}) = (window,\ t_2^{window,window}), \\ b_1(t_1^{roof,roof}) = (window,\ t_2^{window,window}), \\ b_1(\ t_1^{roof,door}) = (window,\ t_2^{window,window}), \\ b_1(t_1^{door,door}) = (roof,\ t_2^{roof,roof}), \end{gathered}$$

and similarly for Barbara.

Question 7.5.1. The views that are smallest amongst the views in V' are v_1^{roof} and v_2^{roof}. In turn, the only view that is smallest amongst the views in V'' is v_1^{roof}. Note that v_2^{door} is not smallest amongst the views in V'', since it contains the view v_1^{roof} which contains less choices for you and Barbara than v_2^{door}.

Question 7.5.2. The set of all views is $V = \{v_1^{all}, v_1^{two}, v_2^{all}, v_2^{two}\}$. The smallest views amongst the views in V are v_1^{two} and v_2^{two}, and these are thus the views with rank 1. Amongst the views which do not have rank 1, the smallest views are v_1^{all} and v_2^{all}, and these are therefore the views of rank 2.

Question 7.5.3. The full decision problems at the four different views are given by

You	*innocent*	*table*	*window*
innocent	0	−550	−800
table	50	−250	−800
window	−200	−200	−500

v_1^{window}

Barbara	*innocent*	*table*	*window*
innocent	0	−550	−800
table	50	−250	−800
window	−200	−200	−500

v_2^{window}

Barbara	*innocent*	*table*	*window*	*roof*
innocent	0	−550	−800	−1050
table	50	−250	−800	−1050
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_2^{roof}

You	*innocent*	*table*	*window*	*roof*	*door*
innocent	0	−550	−800	−1050	−1300
table	50	−250	−800	−1050	−1300
window	−200	−200	−500	−1050	−1300
roof	−450	−450	−450	−750	−1300
door	−700	−700	−700	−700	−1000

v_1^{door}.

Round 1. At your view v_1^{window}, your choice *innocent* is strictly dominated by the randomized choice (0.9)$\cdot$ *table* + (0.1)$\cdot$ *window,* and can therefore be eliminated. Similarly for Barbara's view v_2^{window}. At Barbara's view v_2^{roof}, her choice *innocent* is strictly dominated by the randomized choice (0.95)$\cdot$ *table* + (0.05)$\cdot$ *roof* and can therefore be eliminated. Finally, at your view v_1^{door}, your choice *innocent* is strictly dominated by the randomized choice (0.95)$\cdot$ *table* + (0.05)$\cdot$ *door,* and can therefore be eliminated. This leads to the following one-fold reduced decision problems:

You	*innocent*	*table*	*window*
table	50	−250	−800
window	−200	−200	−500

v_1^{window}

Barbara	*innocent*	*table*	*window*
table	50	−250	−800
window	−200	−200	−500

v_2^{window}

Barbara	*innocent*	*table*	*window*	*roof*
table	50	−250	−800	−1050
window	−200	−200	−500	−1050
roof	−450	−450	−450	−750

v_2^{roof}

You	*innocent*	*table*	*window*	*roof*	*door*
table	50	−250	−800	−1050	−1300
window	−200	−200	−500	−1050	−1300
roof	−450	−450	−450	−750	−1300
door	−700	−700	−700	−700	−1000

v_1^{door}.

Round 2. At your view v_1^{window} you can only reason about Barbara's view v_2^{window} at which her choice *innocent* is no longer present. We can therefore eliminate the state *innocent* from your view v_1^{window}. Afterwards, your choice *table* becomes strictly dominated by *window* at v_1^{window} and can thus be eliminated there. The same applies to Barbara's view v_2^{window}.

At the view v_2^{roof} Barbara can only reason about your view v_1^{window} at which your choice *innocent* is no longer present and the choice *roof* was not present from the beginning. We can therefore eliminate the states *innocent* and *roof* at view v_2^{roof}. Afterwards, her choice *table* becomes strictly dominated by the choice *window*, and can thus be eliminated there.

At your view v_1^{door} you can imagine Barbara's views v_2^{window} and v_2^{roof}, at which her choice *innocent* did not survive and her choice *door* was not present from the beginning. We can thus eliminate the states *innocent* and *door* at your view v_1^{door}. Afterwards, your choice *table* becomes strictly dominated by the randomized choice (0.95)· *window* + (0.05)· *door* at your view v_1^{door}, and can thus be eliminated there.

This leads to the following two-fold reduced decision problems:

You	*table*	*window*
window	−200	−500

v_1^{window}

Barbara	*table*	*window*
window	−200	−500

v_2^{window}

Barbara	*table*	*window*
window	−200	−500
roof	−450	−450

v_2^{roof}

You	*table*	*window*	*roof*
window	−200	−500	−1050
roof	−450	−450	−750
door	−700	−700	−700

v_1^{door}

Round 3. At your view v_1^{window} you can only imagine Barbara's view v_2^{window} at which her choice *table* did not survive. We can thus eliminate the state *table* at your view v_1^{window}. Similarly for Barbara's view v_2^{window}.

At her view v_2^{roof} Barbara only deems possible your view v_1^{window} at which your choice *table* did not survive. We can thus eliminate the state *table* at her view v_2^{roof}. Afterwards, her choice *window* becomes strictly dominated by *roof* at her view v_2^{roof}, and can thus be eliminated there.

At your view v_1^{door} you only deem possible Barbara's views v_2^{window} and v_2^{roof}, at which her choice *table* did not survive. We can thus eliminate the state *table* at your view v_1^{door}. Afterwards, your choice *window* becomes strictly dominated by the choice *roof*, and can thus be eliminated there.

This leads to the following three-fold reduced decision problems:

You	*window*
window	−500

v_1^{window}

Barbara	*window*
window	−500

v_2^{window}

Barbara	*window*
roof	−450

v_2^{roof}

You	*window*	*roof*
roof	−450	−750
door	−700	−700

v_1^{door}

This is where the procedure terminates. Hence, with the view v_1^{door} you can rationally whisper the stories *roof* and *door* into Chris' ear under common belief in rationality. But note that the bottom-up procedure was easier to use in this case, and more time efficient.

Question 7.6.1. In your first-order belief, you believe that Barbara's view is v_2^{two}. In your second-order belief, you believe that Barbara believes that your view is v_1^{two}.

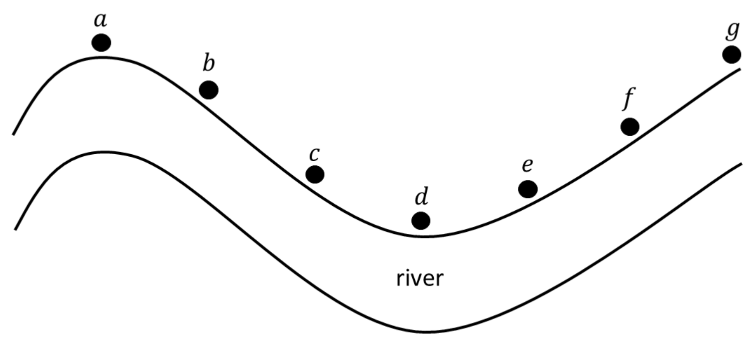

Figure 7.8.1 Map for Problem 7.1

Problems

Problem 7.1: The big discovery.

You and Barbara spend an adventurous holiday in a rain forest, somewhere in South America. By accident you discover the village of a previously unknown indigenous tribe, which you call village a. You decide to walk further, along the river, to see whether there are more villages of this tribe. During this walk you enter into a fierce discussion with Barbara, who claims to have seen the village a first, but you strongly disagree. You get angry at each other, and decide to go your separate ways. During your walk you discover six other villages of the tribe, which you call b, c, d, e, f and g, and you decide to spend the night at village g. The locations of the villages are depicted in Figure 7.8.1.

As a result of your discoveries, you are aware of these seven villages. At the same time, you are uncertain whether Barbara is also aware of these villages, because she may have taken a shorter walk, and may be spending the night at some of the villages before g. During you walk you have noticed that the people were very hostile at villages b, d and f, and for that reason you deem it unlikely that Barbara would spend the night there. But she could spend the night at village c, in which case she would only be aware of the villages a, b or c. But she could also spend the night at village e, in which case she would be aware of the villages a, b, c, d and e, or at village g, in which case she would be aware of all seven villages. Since you are convinced that Barbara will at least make it to village c, you deem it impossible that she spends the night at village a. Hence, there are three possible views for Barbara.

Barbara reasons similarly about you: If Barbara spends the night at village c, then she will definitely believe that you also spend the night at c. If she spends the night at e, she believes that you either spend the night at c or at e. If she spends the night at g, she believes that you either spend the night at c, e or g. Hence, there are three possible views for you.

Before the walk, you agreed with Barbara that you would spend the next day in one of the villages you would discover, to learn about the habits of the indigenous tribe. The question is: Which village do you go to? And which village do you believe Barbara will go to?

Because of the fight you had yesterday, Barbara wants to be as far away from you as possible. More precisely, the distance between every two neighbouring villages on the map is one kilometer, and

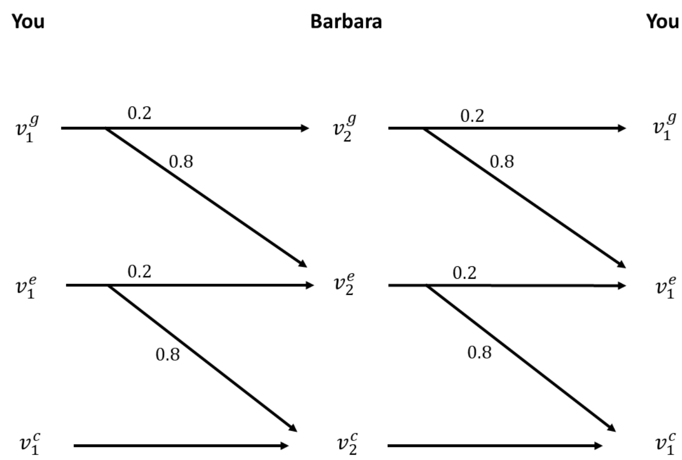

Figure 7.8.2 Fixed belief combination on views in Problem 7.1

the utility for Barbara is given by

$$u_2 = (\text{distance between Barbara and you})^2.$$

That is, if Barbara believes that you will be at village a, then the intensity by which she prefers b to a is lower than the intensity by which she prefers c to b, and so on.

For you, things are different. You would really like to make things up with Barbara, and therefore you would like to be as close to her as possible. More precisely, your utility is given by

$$u_1 = -\sqrt{\text{distance between Barbara and you}}.$$

That is, if you believe that Barbara is at village a, then the intensity by which you prefer a to b is higher than the intensity by which you prefer b to c, and so on.

(a) Translate this story into a game with unawareness, by specifying the possible views for you and Barbara, and by writing down the decision problem for each of the possible views.

(b) Find the villages that you can rationally go to under common belief in rationality. Which procedure do you use?

(c) Create a beliefs diagram with solid arrows only, that uses for each of the possible views all the choices that survive for that view in the procedure of part (b).

(d) Translate this beliefs diagram into an epistemic model where every type expresses common belief in rationality.

Now suppose that you and Barbara believe that, with a high probability, the other person has made a shorter walk whenever your own walk passed beyond village c. More precisely, assume that your belief hierarchy about the views is given by the fixed belief combination on views p in Figure 7.8.2. Here, v_1^g represents the view where you are aware of all seven villages, v_1^e is the view where you are aware of the villages a, b, c, d and e, whereas v_1^c is the view where you are only aware of the villages a, b and c. Similarly for Barbara.

***(e)** What villages can you rationally go to under common belief in rationality and common belief in p?

***(f)** Create an epistemic model that contains, for every player i, every view v_i and every choice c_i you found for view v_i in (e), a type $t_i^{v_i,c_i}$ that (i) has this view v_i, (ii) expresses common belief in rationality and common belief in p, and (iii) for which c_i is optimal. In order to do so, try to make a beliefs diagram first, and translate it into an epistemic model.

Problem 7.2: Learning a new language.

Recall the story from Problem 7.1. Barbara and you have now been staying with the indigenous tribe for a few days, and you are both trying to learn their language. As a start, you both try to learn pronouncing the numbers. So far you have learned how to pronounce the numbers 1 until 40. Since you are not aware of the pronunciation of numbers above 40, you simply cannot imagine Barbara pronouncing these numbers either.

On the other hand, you are free to believe that Barbara has learned less numbers than you have. To keep things easy, suppose you either believe that Barbara has learned the numbers 1 to 20, or the numbers 1 to 30, or the numbers 1 to 40. If you believe that Barbara has only learned the numbers 1 until k then, like you, she cannot imagine that you have learned how to pronounce any number above k. But she is free to believe that you have learned less numbers than k.

Barbara and you have agreed to compete with each other this evening, to see who is able to pronounce most numbers. The rules are as follows: The person who is able to correctly pronounce the highest amount of numbers wins 70 euros. In case of a tie, there will be a toin coss to decide who gets the 70 euros. However, to correctly pronounce the numbers 1 until k, you must first have learned these numbers during the last few days, and you must practice the pronunciation of these numbers today. Assume that the mental cost of practicing the numbers 1 to k, translated in terms of euros, is simply k.

The question is: How many numbers will you practice today? To keep things easy, suppose that you can choose between practicing the numbers 1 to 10, the numbers 1 to 20, the numbers 1 to 30, or the numbers 1 to 40. Similarly, if Barbara has learned the numbers 1 to $k \cdot 10$, where $k \in \{2, 3, 4\}$, then for every $m \in \{1, ..., k\}$ she can choose to practice the numbers 1 to $m \cdot 10$.

(a) Translate this story into a game with unawareness, by specifying the possible views for you and Barbara, and by writing down the decision problem for each of the possible views.

(b) Specify for every view its rank. Afterwards, use the bottom-up procedure to find the amounts of numbers you can rationally practice today under common belief in rationality.

(c) Create a beliefs diagram with solid arrows only, that uses for each of the possible views all the choices that survive for that view in the procedure of part (b).

(d) Translate this beliefs diagram into an epistemic model where every type expresses common belief in rationality.

It is now two days later, and you have learned, in addition, the numbers 41 to 50. You are free to believe that Barbara has learned these new numbers as well, but you cannot imagine that she has learned how to pronounce any number above 50.

(e) Use the bottom-up procedure to find the amounts of numbers you can rationally practice today under common belief in rationality.

Suppose now that if you have learned $10 \cdot k$ numbers, where $k \geq 3$, then you believe that there is a 50% chance that Barbara has learned these numbers as well, and there is a 50% chance that Barbara has learned 10 numbers less. That is, we assume the fixed belief hierarchy on views p given by the

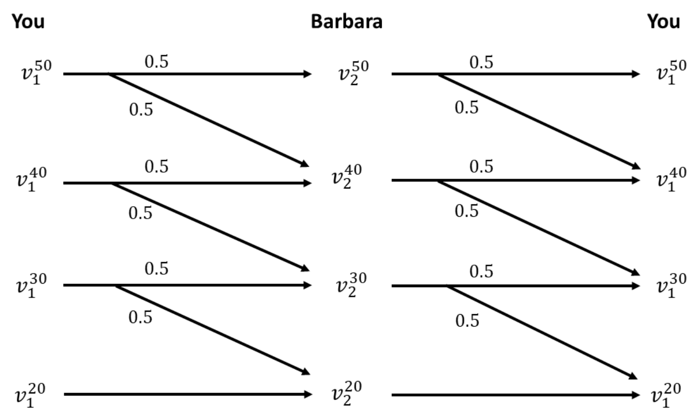

Figure 7.8.3 Fixed beliefs on views in Problem 7.2

beliefs diagram in Figure 7.8.3. Here, v_1^{50} is the view you have when you have learned the numbers 1 until 50. Similarly for the other views.

***(f)** Use the bottom-up procedure to find the amounts of numbers you can rationally practice today under common belief in rationality and common belief in p.

***(g)** Create an epistemic model that contains, for every player i, every view v_i and every choice c_i you found for view v_i in (f), a type $t_i^{v_i,c_i}$ that (i) has this view v_i, (ii) expresses common belief in rationality and common belief in p, and (iii) for which c_i is optimal. In order to do so, try to make a beliefs diagram first, and translate it into an epistemic model.

*__Problem 7.3: The temple.__

Recall the stories from Problems 7.1 and 7.2. After a few weeks of studying the language and the habits of the indigenous tribe, you and Barbara continue your journey through the forest. Within two days you discover a beautiful, old temple, completely covered by trees and plants. Indeed, you and Barbara are the first to see this temple since many centuries. Of course, you and Barbara cannot resist the temptation to enter the temple. There is only one, small entrance, and the corridor is so tiny that only person can get in at the time. Moreover, it is extremely difficult to walk, or should we say crawl, through the corridor, because it is completely dark, and full of bats who constantly attack your head. Luckily you have your smartphone with you to shine a light.

While crawling through the corridor you discover the most amazing treasures: Some beautifully decorated vases, a splendid carriage, a gorgeous silver altar, and an astonishing golden tomb. Of course you leave the treasures where they are, since you have the utmost respect for ancient cultures. However, after discovering the tomb your phone went out of battery, and you had to crawl back to the entrance in the dark. In Figure 7.8.4 you find a map of the corridor and the treasures you found.

Barbara, who was desperately waiting outside for you, immediately jumps in after you come back. Apparently, the news of the ancient temple spread fast, because two journalists arrive at the moment Barbara returns from her journey in the temple. One journalist comes to you, whereas the other journalist turns to Barbara for an interview. Of course, they both ask what you have seen in the temple, and the question is: How many of your discoveries do you reveal to the journalist? And how many discoveries do you believe that Barbara reveals to the other journalist?

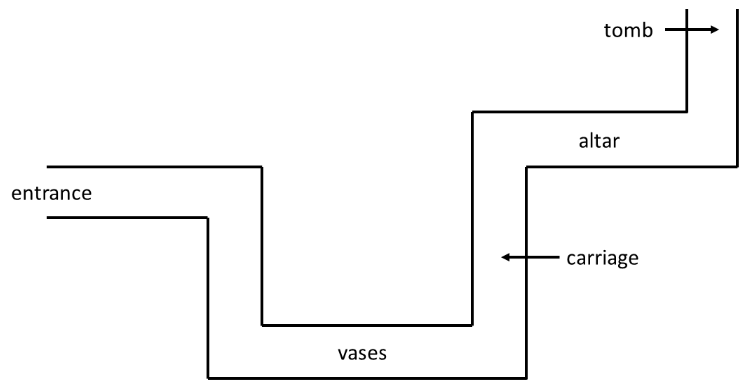

Figure 7.8.4 Map of the temple in Problem 7.3

You face a dilemma here: On the one hand you would like to reveal as little as possible about your discoveries, because more information will attract more robbers. On the other hand, you do not want to reveal less discoveries than Barbara, because you would feel embarrassed. More precisely, if you reveal at least as many discoveries as Barbara, then your utility will be the number of discoveries you believe *not* to reveal, given the total number of treasures you discovered. If, on the other hand, you reveal less discoveries than Barbara, then your utility will be the number of discoveries you believe not to reveal minus a disutility of 3.5 for feeling embarrassed. The same applies to Barbara. Note that both you and Barbara can also decide not to reveal any discoveries to the journalists.

Of course, you can only reveal the treasures you truly discovered, and the same applies to Barbara. However, you are uncertain about the treasures that Barbara discovered, because she could have returned earlier than you did. That is, you either believe that Barbara returned (i) after discovering the vases, (ii) after discovering the carriage, (iii) after discovering the altar, or (iv) after discovering the tomb, like you did.

(a) Translate this story into a game with unawareness, by specifying the possible views for you and Barbara, and by writing down the decision problem for each of the possible views.

Suppose that you and Barbara have little faith in the crawling capabilities of the other person, especially after discovering the altar or the tomb. More precisely, we assume the fixed beliefs on views p given by the beliefs diagram in Figure 7.8.5. Here, v_1^{tomb} is the view you hold when you have discovered the *tomb.* Similarly for the other views.

(b) Find the numbers of discoveries you can rationally reveal to the journalist under common belief in rationality and common belief in p.

(c) Create an epistemic model that contains, for every player i, every view v_i and every choice c_i you found for view v_i in (b), a type $t_i^{v_i,c_i}$ that (i) has this view v_i, (ii) expresses common belief in rationality and common belief in p, and (iii) for which c_i is optimal.

Suppose now that you and Barbara start to have more faith in the crawling capabilities of the other person. This results in the fixed beliefs on views p' given by the beliefs diagram in Figure 7.8.6.

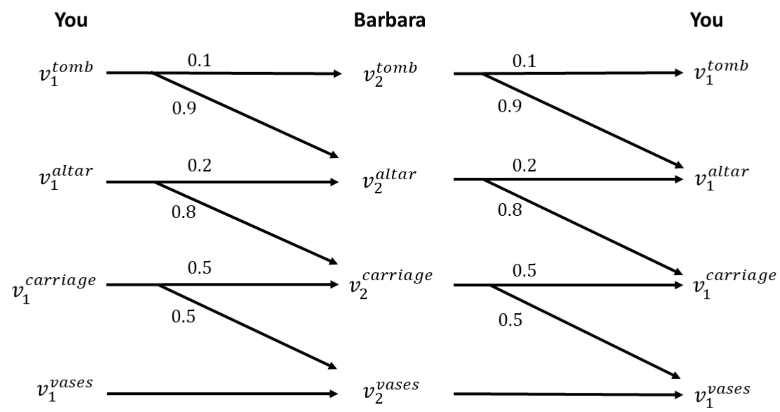

Figure 7.8.5 Fixed beliefs on views in Problem 7.3 (b)

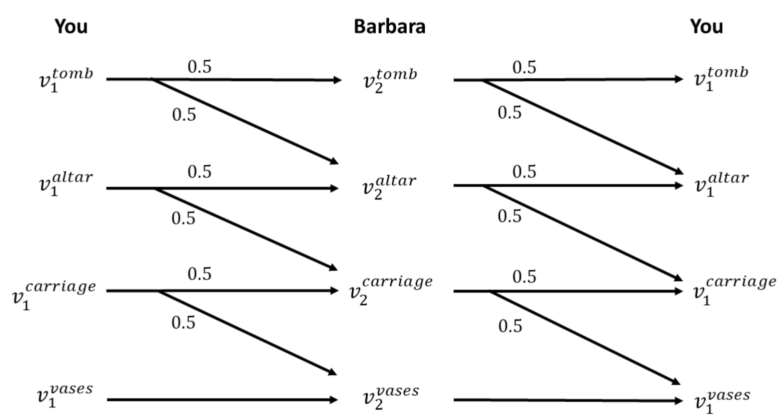

Figure 7.8.6 Fixed beliefs on views in Problem 7.3 (d)

(d) Find the numbers of discoveries you can rationally reveal to the journalist under common belief in rationality and common belief in p.

(e) Can you intuitively explain the difference in your answers to parts (b) and (d)?

Literature

Unawareness in logic. The first papers on unawareness explored its logical foundations, without an explicit reference to games. See, for instance, Fagin and Halpern (1988), Dekel, Lipman and Rustichini (1998), Modica and Rustichini (1999), Halpern (2001), Heifetz, Meier and Schipper (2006, 2008, 2013a), Halpern and Rêgo (2008) and Li (2009). An important question being addressed by these papers is how unawareness can be modeled in a meaningful way, both syntactically and semantically. A general conclusion in this literature is that in a multi-agent setting, every agent must be endowed with his own, *subjective* state space that only contains those objects he is aware of, and which therefore may be substantially smaller than the *full* state space. This principle is also reflected in our definition of a game with unawareness, and how we set up an epistemic model to encode belief hierarchies about choices and views.

To model a game with unawareness, we assume for every player a finite collection of possible views on the game. The implicit understanding is that a player with a certain view only has mental access to those choices that are part of his view, and to those views in the model that are smaller than his own. In other words, the subjective state space for a player with view v_i only contains the choices inside v_i, and the views for the opponents and himself that are contained in v_i.

Unawareness in games. The models of unawareness proposed by the papers above, and especially those that involve more than one agent, can in particular be applied to games. See, for instance, Feinberg (2004, 2021), Čopič and Galeotti (2006), Rêgo and Halpern (2012), Heifetz, Meier and Schipper (2013b), Grant and Quiggin (2013), Halpern and Rêgo (2014), Meier and Schipper (2014), Schipper (2021) and Perea (2022). See Schipper (2014) for an overview of this literature.

Most of these papers, except Čopič and Galeotti (2006) and Meier and Schipper (2014), impose a unique belief hierarchy on views, and thus follow the approach that we have called "fixed beliefs on views" in this chapter. Moreover, the papers Feinberg (2021), Čopič and Galeotti (2006), Heifetz, Meier and Schipper (2013b), Grant and Quiggin (2013) and Schipper (2021) restrict to deterministic beliefs (that is, probability 1 beliefs) on views, whereas we allow for truly probabilistic beliefs on views and choices in this chapter. We find such probabilistic beliefs on views important, as they allow for cases where a player is truly uncertain about the precise view held by an opponent.

Epistemic analysis of games with unawareness. Up until now, there are not so many papers that offer an epistemic analysis of games with unawareness. Among these few papers are Perea (2022) and Guarino (2020). Whereas Perea (2022) focuses on the epistemic concept of *common belief in rationality,* Guarino (2020) concentrates on the concept of *extensive-form rationalizability* (Pearce (1984), Battigalli (1997), Heifetz, Meier and Schipper (2013b)) for dynamic games with unawareness.

Also Heifetz, Meier and Schipper (2013b) and Feinberg (2021) investigate the implications of common (strong) belief in rationality by studying the concepts of *rationalizability* and *extensive-form rationalizability*, respectively. One difference with our approach is that the latter papers do not investigate these concepts on an epistemic basis.

Recursive elimination procedures. The recursive elimination procedures presented in this chapter – that is, *iterated strict dominance for unawareness* with and without fixed beliefs on views, and the *bottom-up procedure* – appear in Perea (2022). Also the various theorems in this chapter, which show that these procedures characterize precisely those choices that the players can rationally make under common belief in rationality (with and without fixed beliefs on views) have been proven in Perea (2022).