Chapter 5

Common Belief in Rationality with Incomplete Information

In this part we focus on scenarios where a player may be *uncertain* about the precise *conditional preference relations* held by his opponents. Such situations are called games with *incomplete information.* In the previous part, where we investigated *standard* games, we assumed that every player was perfectly informed about the conditional preference relations of his opponents. In this chapter we start by formalizing the central idea of *common belief in rationality* for such games with incomplete information, and show that the resulting choices can be characterized by the *generalized iterated strict dominance* procedure – a procedure similar to the *iterated elimination of strictly dominated choices* for standard games. At the end, we consider scenarios where the players hold *fixed* beliefs about the opponents' conditional preference relations. In Chapter 5 of the online appendix we study some economic applications.

5.1 Incomplete Information

In the previous two chapters we assumed that a player was always fully informed about the conditional preference relations of his opponents. That is, a player could always tell, for every belief that his opponent could hold, what the resulting opponent's preference relation over his choices would be. Since we assume that these conditional preference relations always have an expected utility representation, this is the same as saying that a player always knows, for every opponent, the specific utility function that induces the opponent's conditional preference relation.

In many situations, however, a player may not be fully informed about the opponent's utility function. Such situations will be called *games with incomplete information,* and they will be the subject of study in this part. In this section we start by illustrating the idea of *incomplete information* by means of an example, after which we offer a formal definition of a game with incomplete information.

You	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2
yellow	1	1	1	0

u_1

Barbara	*blue*	*green*	*red*	*yellow*
blue	0	2	2	2
green	1	0	1	1
red	4	4	0	4
yellow	3	3	3	0

u_2^r

Barbara	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
green	2	0	2	2
red	1	1	0	1
yellow	3	3	3	0

u_2^b

Table 5.1.1 Decision problems for "What is Barbara's favorite color?"

The main difference with standard games is that we now define, for every player, a *collection* of decision problems – one for every conditional preference relation that is deemed possible by his opponents – rather than a single decision problem. At the end of this section we informally describe the idea of *common belief in rationality* for games with incomplete information.

5.1.1 Example

To illustrate what we mean by *incomplete information,* consider the following example.

Example 5.1: What is Barbara's favorite color?

Recall the example "Going to a party" from Section 3.1, where you and Barbara have to decide which color to wear at the party. Important is that Barbara and you both dislike it when the other person wears the same color. It is now one year later, and the same party is organized again. Your preferences over colors have not changed: You still prefer *blue* to *green*, *green* to *red*, and *red* to *yellow*. However, Barbara's tastes have really changed during the last year, and therefore you are no longer sure about her preferences over the colors. You think she may either prefer *red* to *yellow*, *yellow* to *blue*, and *blue* to *green*, as before, but you think there is also a chance she may prefer *blue* to *yellow*, *yellow* to *green*, and *green* to *red*.

This situation can be represented by the three decision problems in Table 5.1.1. Note that we have specified *two* decision problems for Barbara, since you are uncertain about Barbara's conditional preference relation. More precisely, you believe that Barbara's conditional preference relation is either induced by the utility function u_2^r in the first matrix or by the utility function u_2^b in the second matrix. Here, the superindices r and b specify Barbara's favorite color under these two conditional preference relations.

On the other hand, we assume that you believe that Barbara is certain about *your* conditional preference relation. For that reason, we have only specified one decision problem for you, where the conditional preference relation is induced by the unique utility function u_1 for you.

5.1.2 Games with Incomplete Information

Such situations, where a player is uncertain about the conditional preference relation of some opponent, are called games with *incomplete information.* In the example above, you are uncertain about Barbara's conditional preference relation: You believe that her conditional preference relation is either given by the utility function u_2^r or by the utility function u_2^b.

In general, a player i may believe that opponent j's utility function u_j belongs to some finite set U_j of possible utility functions, without precisely knowing which utility function opponent j has. This gives rise to the following formal definition of games with incomplete information.

Definition 5.1.1 (Game with incomplete information) *A **game with incomplete information** $(C_i, U_i)_{i \in I}$ specifies*

(a) a finite set of players I,

(b) for every player i, a finite set of choices C_i, and

(c) for every player i, a finite set U_i of possible utility functions. Every utility function u_i in U_i assigns to every choice $c_i \in C_i$ and opponents' choice combination $c_{-i} \in C_{-i}$ some utility $u_i(c_i, c_{-i})$.

Note that every utility function u_i in U_i gives rise to a new decision problem (C_i, C_{-i}, u_i), with a new conditional preference relation induced by u_i. Of course, player i *knows* his own conditional preference relation, and therefore knows which utility function u_i belongs to him. However, player i may believe his opponents are *uncertain* about i's utility function. This is why we may still include several possible utility functions for player i if we view the game from i's perspective.

In fact, when U_i contains more than one utility function, then player i *will* believe that each of his opponents is uncertain about i's utility function. More precisely, player i believes that every opponent j will believe that i's utility function belongs to U_i, without knowing exactly which utility function in U_i is held by player i.

On the other hand, if U_j contains more than one utility function, and the game is viewed from player i's perspective, then player i will believe that j's utility function is in U_j, without knowing exactly which utility function u_j belongs to player j. In the example above, for instance, you (player 1) believe that Barbara's (player 2's) utility function is in $U_2 = \{u_2^r, u_2^b\}$, without knowing exactly which utility function Barbara holds. At the same time, $U_1 = \{u_1\}$ only contains one utility function for you, since you believe that Barbara is certain about your conditional preference relation.

In the special case where U_i only contains a *single* utility function for every player i, there would be no uncertainty about the players' conditional preference relations, and we would be back to the setting of *standard* games in Chapters 3 and 4.

5.1.3 Reasoning about Others' Decision Problems

Let us return to the example "What is Barbara's favorite color?" above. What colors would you consider wearing, and why? To answer this question, let us assume that you reason in accordance with *common belief in rationality* as introduced in Chapter 3 for standard games. As we will see, it will now be important for you to reason about *both* decision problems for Barbara.

We have seen in Chapter 3 that the color *yellow* is irrational for you, since it is strictly dominated by the randomized choice where you select *blue* and *green* with probability 0.5. Therefore, if you believe that Barbara believes in your rationality, you will believe that Barbara assigns probability 0 to your choice *yellow,* no matter whether her utility function is u_2^r or u_2^g. Or, equivalently, in both of Barbara's decision problems we may eliminate the state *yellow.*

Barbara	*blue*	*green*	*red*
red	4	4	0
yellow	3	3	3

u_2^r

Barbara	*blue*	*green*	*red*
blue	0	4	4
yellow	3	3	3

u_2^b

Table 5.1.2 Barbara's two–fold reduced decision problems in "What is Barbara's favorite color?"

You	*blue*	*red*	*yellow*
blue	0	4	4
green	3	3	3

u_1

Table 5.1.3 Your three-fold reduced decision problem in "What is Barbara's favorite color?"

But then, in Barbara's decision problem for u_2^r, her choices *blue* and *green* become irrational, because they are strictly dominated by her choice *yellow.* Thus, we may eliminate Barbara's choices *blue* and *green* at her utility function u_2^r. Similarly, at the utility function u_2^b, her choices *green* and *red* become irrational, since they are strictly dominated by her choice *yellow.* Thus, we may eliminate Barbara's choices *green* and *red* at her utility function u_2^b. This leads to Barbara's 2-fold reduced decision problems in Table 5.1.2.

Recall that you do not know whether Barbara's conditional preferences are given by u_2^r or u_2^b. However, for both conditional preference relations it will never be optimal for Barbara to choose *green* if she believes in your rationality. Thus, if you believe in Barbara's rationality, and believe that Barbara believes in your rationality, then you must believe that Barbara will definitely not choose *green* – no matter whether her utility function is u_2^r or u_2^b. We may thus eliminate the state *green* from your (unique) decision problem.

Afterwards, your choices *red* and *yellow* are both irrational, since they are both strictly dominated by your choice *green.* We can therefore eliminate your choices *red* and *yellow,* and arrive at your 3-fold reduced decision problem in Table 5.1.3. Thus, if you believe in Barbara's rationality, and believe that Barbara believes in your rationality, then the only colors that you can possibly rationally choose are *blue* and *green.*

In fact, we will see in the next few sections that you can rationally choose both of the colors *blue* and *green* under common belief in rationality. Important is that in the reasoning above, you had to reason about *both* of Barbara's decision problems. Indeed, to conclude that Barbara will definitely not choose *green* it was important to realize that, at a certain point in the reasoning process, the color *green* became irrational for Barbara at *both* of her decision problems.

5.2 Belief Hierarchies, Beliefs Diagrams and Types

The main goal of this chapter is to provide a formal definition of *common belief in rationality* for games with incomplete information, and to see how the resulting choices can be characterized by an iterated elimination procedure.

5.2.1 Belief Hierarchies

Let us first see what we need to formalize the idea that you *believe in the opponents' rationality.* Recall that in a game with incomplete information, player i may be uncertain about opponent j's utility function. Hence, for player i to believe in j's rationality means that player i believes that j chooses optimally, given what i believes that j believes about his opponents' choices, and given what i believes about j's *utility function.* Indeed, to verify whether a given choice is optimal for player j, we need not only j's belief about his opponents' choices, but also j's utility function.

To formalize the expression above, we thus need player i's belief about j's choice and j's utility function, summarized by i's *first-order* belief about player j, together with player i's belief about j's belief about his opponents' choices, which is part of i's *second-order* belief about player j.

As a next step, what do we need to formally state that player i believes that j believes in k's rationality? In words, it means that player i believes that player j believes that k chooses optimally, given what i believes that j believes about k's belief about his opponents' choices, and given what i believes that j believes about k's utility function. To state this formally, we thus need player i's belief about j's belief about k's choice and k's utility function, summarized by i's second-order belief, and i's belief about j's belief about k's belief about his opponents choices, which is part of i's third-order belief.

By continuing in this fashion we see that for a formal definition of common belief in rationality, we need to specify a full belief hierarchy for player i, where his first-order belief is a belief about the opponents' choices and utility functions, his second-order belief is a belief about what his opponents believe about the other players' choices and utility functions, and so on. We thus obtain the following definition of a belief hierarchy.

Definition 5.2.1 (Belief hierarchies) *A **belief hierarchy** for player i specifies*

*(1) a **first-order belief,** which is a belief about the choices and utility functions of i's opponents,*

*(2) a **second-order belief,** which is a belief about what every opponent j believes about the choices and utility functions of j's opponents,*

*(3) a **third-order belief,** which is a belief about what every opponent j believes about what each of his opponents k believes about the choices and utility functions of k's opponents,*

and so on.

As you can see, the crucial difference with a belief hierarchy for standard games, as discussed in Chapter 3, is that a player must now also entertain beliefs about the opponents' utility functions, beliefs about the opponents' beliefs about the other players' utility functions, and so on.

Similarly to the case of standard games, a major difficulty with these belief hierarchies is that they involve infinitely many orders of belief. Fortunately, we can use the same techniques as in Chapter 3 to encode such infinite belief hierarchies in a finite manner: We can either use *beliefs diagrams*, or epistemic models with *types*, to summarize the belief hierarchies in a convenient and finite way.

5.2.2 Beliefs Diagrams

One way to encode belief hierarchies about choices and utility functions is by means of a *beliefs diagram.* Like for the case of standard games, it acts as a *visualization* of belief hierarchies, and is particularly useful for illustrations and examples. However, it will also be used in Chapter 6 to define *symmetric* belief hierarchies, like we did in Chapter 4 for standard games.

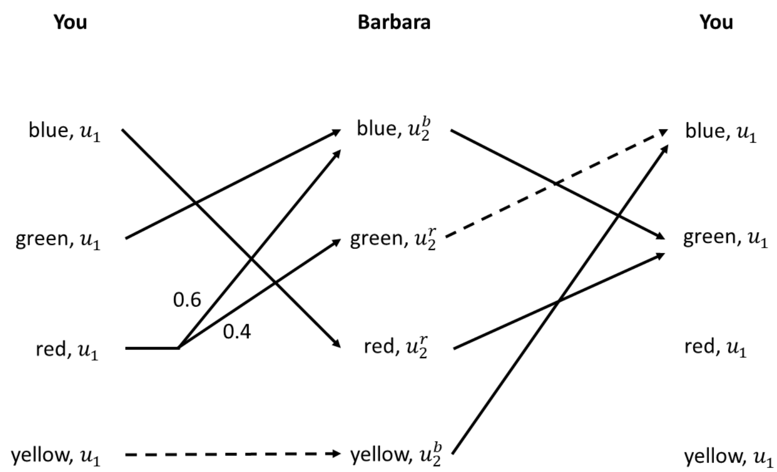

Figure 5.2.1 Beliefs diagram for "What is Barbara's favorite color?"

To see what a beliefs diagram looks like for games with incomplete information, consider the beliefs diagram in Figure 5.2.1 for the example "What is Barbara's favorite color?". The arrow from your choice-utility pair (*blue,* u_1) to Barbara's choice-utility pair (*red,* u_2^r) indicates that you believe that Barbara chooses *red* while having the utility function u_2^r. The fact that this arrow is *solid* means that under this belief, it is *optimal* for you to choose *blue* if your utility function is u_1. Indeed, if your utility function is u_1 and you believe that Barbara chooses *red,* then it is optimal for you to choose *blue.*

Similarly, the forked arrow that leaves your choice-utility pair (*red,* u_1) represents the belief where you assign probability 0.6 to the event that Barbara chooses *blue* while having the utility function u_2^b, and where you assign probability 0.4 to the event that Barbara chooses *green* while having the utility function u_2^r. This forked arrow is solid because under this belief it is optimal for you to choose *red* when you hold the utility function u_1.

In contrast, the arrow from your choice-utility pair (*yellow,* u_1) to Barbara's choice-utility pair (*yellow,* u_2^b) is *dashed* because it is *not* optimal for you to choose *yellow* under the utility function u_1 if you hold this belief.

Note that for Barbara, there is a solid arrow from her choice-utility pair (*blue,* u_2^b) to your pair (*green,* u_1), and another solid arrow from her pair (*red,* u_2^r) to the same pair (*green,* u_1) of yours. In other words, if Barbara believes that you choose *green,* then it is optimal for her to choose *blue* if her utility function is u_2^b, whereas it would be optimal for her to choose *red* if her utility function would be u_2^r. Indeed, for a given belief about you, the optimal choice for Barbara depends on the particular utility function she has.

Question 5.2.1 *In the beliefs diagram from Figure 5.2.1, consider the arrow from Barbara's pair (yellow,* u_2^b*) to your pair (blue,* u_1*). Suppose that in this arrow we replace* u_2^b *by the other utility function* u_2^r *for Barbara. Should the arrow be solid or dashed? Explain your answer.*

Similarly to Chapter 3, we can read two, or more, arrows consecutively, to obtain higher-order beliefs. Start, for instance, at your pair (*blue,* u_1), and follow the first two arrows. Then, your second-order belief would be that you believe that Barbara believes that you choose *green* while holding the utility function u_1.

Question 5.2.2 *(a) Suppose we start again at your pair (blue, u_1). What is your third-order belief?*

(b) Suppose we start at your pair (red, u_1). Describe your second- and third-order belief.

In this way, whenever we start at a choice-utility pair for you or Barbara, we can always derive a first-order belief, second-order belief, and higher-order beliefs about the players' choices and utility functions. That is, we can always derive a *full* belief hierarchy on choices and utility functions if we start at some choice-utility pair in a beliefs diagram.

5.2.3 Types

Recall that a belief hierarchy for a player in a game with incomplete information specifies (i) a first-order belief about the opponents' choices and utility functions, (ii) a second-order belief about the opponents' first-order beliefs, (iii) a third-order belief about the opponents' second-order beliefs, and so on. In other words, a belief hierarchy describes (a) a belief about the opponents' choices, (b) a belief about the opponents' utility functions, and (c) a belief about the opponents' first-order, second-order and all higher-order beliefs – that is, a belief about the opponents' belief hierarchies.

Now, let us summarize the belief hierarchy and the utility function of a player by a *type.* Then, by (a), (b) and (c) above, a type t_i for player i specifies a utility function $v_i(t_i)$ for player i, together with a belief $b_i(t_i)$ about the opponents' choices and types. This insight leads to the following definition of an *epistemic model* with *types.*

Definition 5.2.2 (Epistemic model) *Consider a game with incomplete information $(C_i, U_i)_{i\in I}$. An **epistemic model** $M = (T_i, v_i, b_i)_{i\in I}$ specifies*

(a) for every player i a finite set of types T_i,

(b) for every player i and every type $t_i \in T_i$, a utility function $v_i(t_i)$ from U_i,

(c) for every player i and every type $t_i \in T_i$, a probability distribution $b_i(t_i)$ on the opponents' choice-type combinations. This probability distribution $b_i(t_i)$ represents t_i's belief about the opponents' choices and types.

The only difference with an epistemic model from Chapter 3 is that a type now also specifies the player's *utility function.* Of course, for standard games, where there is only one possible utility function for every player, such a specification would be redundant.

Similarly as before, we can interpret an epistemic model as a mathematical translation of a beliefs diagram. To see this, consider the beliefs diagram from Figure 5.2.1. This beliefs diagram can be translated into the epistemic model from Table 5.2.1.

Suppose we start at your type t_1^{blue}. What would be the corresponding first-order belief? From the epistemic model, it can be seen that your type t_1^{blue} believes that Barbara wears *red* and is of type t_2^{red}. Since Barbara's type t_2^{red} has the utility function $v_2(t_2^{red}) = u_2^r$, the first-order belief is that you believe that Barbara wears *red* while having the utility function u_2^r.

What would be the second-order belief for your type t_1^{blue}? From the epistemic model, we see that t_1^{blue} believes that Barbara has type t_2^{red} which believes, in turn, that you choose *green* and that you have type t_1^{green}. Since your type t_1^{green} has utility function $v_1(t_1^{green}) = u_1$, the second-order belief is that you believe that Barbara believes that you wear *green* while having the utility function u_1.

Question 5.2.3 *Describe the first-order, second-order and third-order belief of your type t_1^{yellow}, based on the epistemic model above.*

Hence, belief hierarchies can be derived from types in a similar way as we have seen in Chapter 3.

Types	$T_1 = \{t_1^{blue}, t_1^{green}, t_1^{red}, t_1^{yellow}\}$, $T_2 = \{t_2^{blue}, t_2^{green}, t_2^{red}, t_2^{yellow}\}$					
Utilities and beliefs for you	$v_1(t_1^{blue})$	$=$	u_1	$b_1(t_1^{blue})$	$=$	$(red,\ t_2^{red})$
	$v_1(t_1^{green})$	$=$	u_1	$b_1(t_1^{green})$	$=$	$(blue,\ t_2^{blue})$
	$v_1(t_1^{red})$	$=$	u_1	$b_1(t_1^{red})$	$=$	$(0.6)\cdot(blue,\ t_2^{blue}) + (0.4)\cdot(green,\ t_2^{green})$
	$v_1(t_1^{yellow})$	$=$	u_1	$b_1(t_1^{yellow})$	$=$	$(yellow,\ t_2^{yellow})$
Utilities and beliefs for Barbara	$v_2(t_2^{blue})$	$=$	u_2^b	$b_2(t_2^{blue})$	$=$	$(green,\ t_1^{green})$
	$v_2(t_2^{green})$	$=$	u_2^r	$b_2(t_2^{green})$	$=$	$(blue,\ t_1^{blue})$
	$v_2(t_2^{red})$	$=$	u_2^r	$b_2(t_2^{red})$	$=$	$(green,\ t_1^{green})$
	$v_2(t_2^{yellow})$	$=$	u_2^b	$b_2(t_2^{yellow})$	$=$	$(blue,\ t_1^{blue})$

Table 5.2.1 Epistemic model for "What is Barbara's favorite color?"

5.3 Common Belief in Rationality

In this section we will use epistemic models with types to provide a formal definition of *common belief in rationality* – the central reasoning concept in this book. As will become clear, the definition is highly similar to that of Chapter 3. We build the definition up in the same way as in Chapter 3, by first defining what it means for a choice to be optimal for a type and subsequently defining belief in the opponents' rationality. The latter definition, when used inductively, then leads to k-fold belief in rationality for every k, and common belief in rationality.

5.3.1 Optimal Choices for Types

Consider a type t_i for player i in an epistemic model. Recall that t_i specifies both a belief $b_i(t_i)$ about the opponents' choices and types, and a conditional preference relation summarized by a utility function $v_i(t_i)$. Let $b_i^1(t_i)$ be the induced first-order belief about the opponents' choices. Then, a choice c_i is said to be optimal for t_i if it is optimal for the first-order belief $b_i^1(t_i)$ in combination with the utility function $v_i(t_i)$.

Definition 5.3.1 (Optimal choice for a type) *A choice c_i is optimal for a type t_i if*

$$v_i(t_i)(c_i, b_i^1(t_i)) \geq v_i(t_i)(c_i', b_i^1(t_i))$$

for all choices $c_i' \in C_i$.

In particular, the optimality of a choice depends on the specific utility function that the type has.

Question 5.3.1 *Consider the epistemic model in Table 5.2.1. For each of Barbara's types, find the optimal choice(s).*

You will have seen that Barbara's types t_2^{blue} and t_2^{red} have different optimal choices, although they share the same belief about your choice. The reason is that these two types have different utility functions, and thus different conditional preference relations.

5.3.2 Common Belief in Rationality

The key ingredient in the definition of common belief in rationality is *belief in the opponents' rationality.* Informally, this means that you believe that each of your opponents chooses optimally, given your belief about his first-order belief, and given your belief about his *utility function.* If your type is t_i, then your belief $b_i(t_i)$ about the opponent's choice-type pairs specifies, in particular, your belief about opponent j's first-order belief, and your belief about opponent j's utility function. Indeed, you hold a belief about j's type, and j's type specifies j's first-order belief and j's utility function. To believe in j's rationality then means that your type t_i only assigns positive probability to choice-type pairs for opponent j where the choice is optimal for his type – exactly the way we defined it in Chapter 3.

Definition 5.3.2 (Belief in the opponents' rationality) *Type t_i* ***believes in the opponents' rationality*** *if the belief $b_i(t_i)$ on the opponents' choice-type combinations assigns, for every opponent j, only positive probability to choice-type pairs (c_j, t_j) where the choice c_j is optimal for the type t_j.*

The only difference with Chapter 3 is that, in order to verify whether type t_i believes in j's rationality, we must also take into account t_i's belief about j's utility function.

Question 5.3.2 *For each of your and Barbara's types in Table 5.2.1, verify whether it believes in the opponent's rationality or not.*

In exactly the same way as in Chapter 3, we can now define k-fold belief in rationality for every $k \in \{1, 2, 3, ...\}$, and common belief in rationality.

Definition 5.3.3 (Common belief in rationality) *A type t_i expresses 1-fold belief in rationality if t_i believes in the opponents' rationality.*

A type t_i expresses 2-fold belief in rationality if $b_i(t_i)$ only assigns positive probability to opponents' types that express 1-fold belief in rationality.

A type t_i expresses 3-fold belief in rationality if $b_i(t_i)$ only assigns positive probability to opponents' types that express 2-fold belief in rationality.

And so on.

A type t_i expresses ***common belief in rationality*** *if it expresses 1-fold belief in rationality, 2-fold belief in rationality, 3-fold belief in rationality, and so on, ad infinitum.*

For games with incomplete information, the definition of common belief in rationality has thus not become more difficult compared to the case of standard games: The only modification has been with the definition of optimality of a choice for a type, which now involves the specific utility function specified by that type. All other aspects of the definition of common belief in rationality have remained the same.

Question 5.3.3 *For each of your and Barbara's types in Table 5.2.1, verify whether it expresses common belief in rationality or not.*

In Theorem 3.3.1 from Chapter 3 we have seen an easy way to verify that all types in an epistemic model express common belief in rationality: If all types believe in the opponents' rationality, then also all types will express common belief in rationality. We have also seen an analogous result for beliefs diagrams in Theorem 3.3.2. These results remain true for games with incomplete information as well, for exactly the same reasons. We will therefore not repeat these arguments here.

As an illustration, consider again the epistemic model in Table 5.2.1. Concentrate on the smaller epistemic model we obtain if we leave out the type t_1^{red} for you, but include all other types. This is a well-defined epistemic model, since none of Barbara's types assigns a positive probability to your type t_1^{red}. In Question 5.3.2 you have shown that each of the types in this smaller epistemic model believes in the opponent's rationality. Therefore, by the result above, we conclude that every type in this smaller epistemic model expresses common belief in rationality. In other words, each of the types in Table 5.2.1, except your type t_1^{red}, expresses common belief in rationality.

Finally, consider a choice c_i for player i, together with a utility function u_i in the set U_i of possible utility functions. Similarly to Chapter 3, we can then define what it means that player i can rationally make the choice c_i under common belief in rationality *with the utility function* u_i.

Definition 5.3.4 (Rational choice under common belief in rationality) *Player i can* ***rationally make choice*** c_i ***under common belief in rationality with the utility function*** $u_i \in U_i$ *if there is some epistemic model $M = (T_i, v_i, b_i)_{i \in I}$, and some type $t_i \in T_i$ for player i within that model, such that (a) type t_i expresses common belief in rationality, (b) type t_i has utility function u_i and (c) choice c_i is optimal for the type t_i.*

Note that we must add the part "with the utility function $u_i \in U_i$" here, since the optimality of a choice also depends on the specific utility function $u_i \in U_i$ we consider.

5.4 Recursive Procedure

In this section we will develop a recursive elimination procedure that yields, for a given player and a given utility function, all choices he can rationally make under common belief in rationality with that utility function. The procedure is called *generalized iterated strict dominance.* The main difference with the *iterated elimination of strictly dominated choices* from Chapter 3 is that for every player, we have a decision problem for *each* of his possible *utility functions,* and that in every round we eliminate states and choices from all such decision problems. Recall that in Chapter 3 we only had one decision problem for every player. At the end, we will show that this procedure is order independent. That is, the final result does not depend on the speed and order of elimination.

5.4.1 One-fold Belief in Rationality

Similarly to what we did in Chapter 3, we first characterize the choices you can rationally make for a given utility function, without putting any restrictions on your belief. Consider a player i and a utility function u_i in U_i, which gives rise to his decision problem that corresponds to u_i. From Theorem 2.6.1 we know that the choices that are optimal for player i for some belief under this utility function u_i are precisely the choices that are not strictly dominated in this decision problem. By eliminating the strictly dominated choices at each of the decision problems for each of the possible utility functions, we obtain the *one-fold reduced decision problems.* These contain the choices that can rationally be made for each of the possible utility functions.

Now, we go one step further and focus on the consequences of 1-fold belief in rationality. What choices can you rationally make then, and how can we characterize these? If player i expresses 1-fold

You	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2

u_1

Barbara	*blue*	*green*	*red*	*yellow*
blue	0	2	2	2
red	4	4	0	4
yellow	3	3	3	0

u_2^r

Barbara	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
green	2	0	2	2
yellow	3	3	3	0

u_2^b

Table 5.4.1 One-fold reduced decision problems in "What is Barbara's favorite color?"

belief in rationality, then he believes that every opponent j chooses optimally, given j's belief and given j's utility function. In other words, for every possible utility function u_j that j may have, player i will only assign positive probability to choices that are optimal for j for some belief in the associated decision problem. By the insight above, i will thus, for each of j's utility functions u_j, only assign positive probability to choices c_j that are in the one-fold reduced decision problem at u_j. Overall, i will thus assign probability zero to opponent's choices c_j that are *not* in *any* of j's one-fold reduced decision problems, for *any* of j's utility functions u_j in U_j. Or, put differently, player i will eliminate from each of his one-fold reduced decision problems those states that involve choices c_j that are *not* in *any* of j's one-fold reduced decision problems.

Hence, for each of i's decision problems we obtain a smaller decision problem where, possibly, some states have been eliminated. By Theorem 2.6.1, the choices that are optimal for player i for some belief in this smaller decision problem are precisely the choices that are not strictly dominated within this smaller decision problem. By eliminating the strictly dominated choices, we are left with the choices that player i can rationally make if he expresses 1-fold belief in rationality.

To find these choices, we have thus implemented the following procedure: In round 1 we eliminate for every player, and each of his decision problems, the choices that are strictly dominated. This leads to the one-fold reduced decision problems. In round 2 we first eliminate at every decision problem for player i those states that involve opponents' choices c_j that are *not* in *any* of j's one-fold reduced decision problems. Subsequently, we eliminate i's choices that are strictly dominated within this smaller decision problem. This procedure is called *two-fold elimination of strictly dominated choices* and yields, for every player and every utility function, precisely those choices that can rationally be made for this utility function if the player expresses 1-fold belief in rationality.

To illustrate this procedure, let us go back to the example "What is Barbara's favorite color?", with the decision problems from Table 5.1.1.

Round 1. In your single decision problem, your choice *yellow* is strictly dominated by the randomized choice $(0.5)\cdot$ *blue* + $(0.5)\cdot$ *green,* and can thus be eliminated. In Barbara's decision problem at u_2^r, her choice *green* is strictly dominated by the randomized choice $(0.5)\cdot$ *red* + $(0.5)\cdot$ *yellow,* and can thus be eliminated. Finally, in Barbara's decision problem at u_2^b, her choice *red* is strictly dominated by the randomized choice $(0.5)\cdot$ *blue* + $(0.5)\cdot$ *yellow,* and can thus be eliminated. This yields the one-fold reduced decision problems in Table 5.4.1.

Round 2. In your one-fold reduced decision problem we cannot eliminate any state, since each of

You	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2

u_1

Barbara	*blue*	*green*	*red*
red	4	4	0
yellow	3	3	3

u_2^r

Barbara	*blue*	*green*	*red*
blue	0	4	4
yellow	3	3	3

u_2^b

Table 5.4.2 Two-fold reduced decision problems in "What is Barbara's favorite color?"

Barbara's colors is in some of Barbara's one-fold reduced decision problem.

In Barbara's decision problem at u_2^r we can eliminate the state *yellow,* as it is not in your single one-fold reduced decision problem. Subsequently, Barbara's choice *blue* becomes strictly dominated by *yellow,* and can thus be eliminated at u_2^r.

In Barbara's decision problem at u_2^b we can also eliminate the state *yellow* for the same reason as above. Subsequently, Barbara's choice *green* becomes strictly dominate by *yellow,* and can thus be eliminated at u_2^b. This leads to the two-fold reduced decision problems in Table 5.4.2.

These two-fold reduced decision problems specify, for every player and every utility function, the choices that this player can rationally make for this specific utility function if he expresses 1-fold belief in rationality. For instance, if Barbara has the utility function u_2^b, she can rationally wear *blue* or *yellow* if she expresses 1-fold belief in rationality.

5.4.2 Two-fold Belief in Rationality

Suppose now that you express 1-fold and 2-fold belief in rationality. What choices can you rationally make then for each of your utility functions, and how can we characterize these? Consider a player i who expresses 1-fold and 2-fold belief in rationality. He then assigns, for every opponent j and each of j's utility functions u_j, only positive probability to choices c_j that j can rationally make under 1-fold belief in rationality for this specific utility function u_j. By our insights above, these are precisely the choices c_j that are in j's two-fold reduced decision problem at u_j. Hence, player i assigns, for every utility function u_j, only positive probability to choices c_j that are in the two-fold reduced decision problem at u_j. Overall, player i will thus assign probability zero to all opponent's choices c_j that are *not* in *any* of j's two-fold reduced decision problems. In other words, player i will eliminate, from each of his two-fold reduced decision problems, those states that involve opponents' choices c_j which are *not* in *any* of j's two-fold reduced decision problems. This leads to smaller decision problems for player i.

By Theorem 2.6.1, the choices that are optimal for player i in these smaller decision problems are precisely the choices that are not strictly dominated within these smaller decision problems. We can therefore eliminate those choices for player i that are strictly dominated in these smaller decision problems, and obtain the three-fold reduced decision problems for every player i.

The procedure we have implemented above is the following: We start by applying the two-fold elimination of strictly dominated choices. At each of player i's decision problems, we then eliminate those states that involve opponents' choices c_j that are not in any of j's two-fold reduced decision

You	*blue*	*red*	*yellow*
blue	0	4	4
green	3	3	3

u_1

Barbara	*blue*	*green*	*red*
red	4	4	0
yellow	3	3	3

u_2^r

Barbara	*blue*	*green*	*red*
blue	0	4	4
yellow	3	3	3

u_2^b

Table 5.4.3 Three-fold reduced decision problems in "What is Barbara's favorite color?"

problems. Subsequently, we remove for player i those choices that have become strictly dominated in these smaller decision problems. This procedure is called the *three-fold elimination of strictly dominated choices* and yields, for every player and each of his utility functions, those choices that the player can rationally make for this particular utility function if he expresses 1-fold and 2-fold belief in rationality.

As an illustration, let us go back to the example "What is Barbara's favorite color?", together with the two-fold reduced decision problems in Table 5.4.2.

Round 3. From your decision problem we can remove the state *green,* since Barbara's choice *green* is not present in any of Barbara's two-fold reduced decision problems. Afterwards, your choice *red* becomes strictly dominated by *green,* and can thus be eliminated. In Barbara's decison problems, nothing can be eliminated in round 3. This yields the three-fold reduced decision problem in Table 5.4.3. Thus, if you express 1-fold and 2-fold belief in rationally, you can rationally wear *blue* or *green.*

5.4.3 Common Belief in Rationality

Above we have seen that the choices you can rationally make under 1-fold belief in rationality are those that survive the two-fold elimination of strictly dominated choices, whereas the choices you can rationally make under 1-fold and 2-fold belief in rationality are characterized by the three-fold elimination of strictly dominated choices. By continuing along these arguments we conclude more generally that for every $k \in \{1, 2, 3, ...\}$, the choices you can rationally make while expressing 1-fold up to k-fold belief in rationality are precisely those that survive the $k+1$-fold elimination of strictly dominated choices. As a consequence, the choices that are possible under *common belief in rationality* are those that survive *all* rounds of this elimination procedure, which will be called the *generalized iterated strict dominance* procedure.

Definition 5.4.1 (Generalized iterated strict dominance procedure) *Start by writing down the decision problems for every player i and every utility function u_i in U_i.*

Round 1. *From every decision problem, eliminate those choices that are strictly dominated. This leads to the 1-fold reduced decision problems.*

Round 2. *From every 1-fold reduced decision problem, eliminate those states that involve opponents' choices c_j that did not survive round 1 at any of j's utility functions u_j. Within the (possibly smaller) decision problem so obtained, eliminate all choices that are strictly dominated. This leads to the 2-fold reduced decision problems.*

You	*blue*	*red*	*yellow*
blue	0	4	4
green	3	3	3

u_1

Barbara	*blue*	*green*
red	4	4

u_2^r

Barbara	*blue*	*green*
blue	0	4
yellow	3	3

u_2^b

Table 5.4.4 Four-fold and final reduced decision problems in "What is Barbara's favorite color?"

Round 3. *From every 2-fold reduced decision problem, eliminate those states that involve opponents' choices c_j that did not survive round 2 at any of j's utility functions u_j. Within the (possibly smaller) decision problem so obtained, eliminate all choices that are strictly dominated. This leads to the 3-fold reduced decision problems.*

Continue like this until no further states and choices can be eliminated. The choices for a player i that eventually remain in his decision problem at a certain utility function u_i are said to survive the ***generalized iterated strict dominance procedure*** *at u_i.*

Clearly, if we are dealing with a *standard* game without incomplete information, where there is only one possible utility function for every player, then this procedure reduces to the *iterated elimination of strictly dominated choices* from Chapter 3.

As an illustration of the procedure, let us go back to the example "What is Barbara's favorite color?". We have seen that the first three rounds of the generalized iterated strict dominance procedure led to the three-fold reduced decision problems in Table 5.4.3.

Round 4. In your decision problem, nothing can be eliminated. In Barbara's decision problem at u_2^r we can remove the state *red* since your choice *red* did not survive round 3 at your decision problem. Subsequently, Barbara's choice *yellow* gets strictly dominated by her choice *red* and can thus be eliminated. In Barbara's decision problem at u_2^b we can eliminate the state *red* for the same reason as at u_2^r. But afterwards, no remaining choice for Barbara is strictly dominated. This leads to the four-fold reduced decision problems in Table 5.4.4. As no further states or choices can be eliminated after this round, this is where the procedure terminates.

The final decision problems at every utility function thus specify the choices that the player can rationally make under common belief in rationality with that particular utility function. That is, under common belief in rationality you can rationally wear *blue* or *green,* Barbara can only rationally wear *red* if her utility function is u_2^r, and Barbara can rationally wear *blue* or *yellow* if her utility function is u_2^b.

Based on our arguments above we arrive at the following general result.

Theorem 5.4.1 (Procedure for common belief in rationality) *(a) For every $k \in \{1, 2, 3, ...\}$, the choices that player i can rationally make with utility function $u_i \in U_i$ while expressing up to k-fold belief in rationality are precisely the choices that survive the first $k+1$ rounds of the generalized iterated strict dominance procedure at u_i.*

(b) The choices that player i can rationally make with utility function $u_i \in U_i$ under common belief in

rationality are exactly the choices that survive all rounds of the generalized iterated strict dominance procedure at u_i.

Note that the procedure is guaranteed to terminate within finitely many rounds. Indeed, since there are finitely many choices and finitely many possible utility functions for every player, there are finitely many decision problems to start with, each having a finite number of choices and states. In every "active" round, we eliminate at least one state or choice in at least one of the decision problems. As such, there can only be finitely many active rounds, after which the procedure terminates.

Moreover, like with the *iterated elimination of strictly dominated choices* for standard games, it can be shown that the order in which we eliminate choices and states at the various decision problems does not matter for the final output in the *generalized iterated strict dominance* procedure.

Theorem 5.4.2 (Order independence) *Changing the order of elimination in the generalized iterated strict dominance procedure does not change the sets of choices that survive the procedure at each of the decision problems.*

Recall that in the original formulation of the procedure, we must at every round investigate each of the decision problems, and detect at every decision problem all states and choices that can be eliminated there. The result above states that if at some rounds we only investigate some – but not all – decision problems, or at a particular decision problem only eliminate some – but not all – states or choices that could be eliminated, then we are still guaranteed to arrive at the same output eventually. Provided, of course, we do not forget to eliminate a state or choice at a given decision problem forever.

5.4.4 Example

To further illustrate the procedure and the result above, we introduce a new example.

Example 5.2: Chris' drawings.

Chris was really good at drawing when he was a teenager. But nowadays he is a bit tired of all these drawings in his room, and decides to organize an auction to get rid of these hundreds of drawings. Chris is somewhat disappointed, though, to see that only you and Barbara show up at the auction. The rules of the auction are simple: You and Barbara must independently write down a bid from $\{20, 40, 60, 80, 100\}$ on a piece of paper, and give it to Chris in a sealed envelope. Chris opens the envelopes, and gives the drawings to the person with the highest bid. This person must then pay the bid, in euros, to Chris. If both you and Barbara have written down the same bid, then Chris will toss a coin to decide who gets the drawings. This type of auction is called a *first price auction.*

As to the utilities that you and Barbara obtain, we assume that you and Barbara have a valuation in $\{30, 50, 70, 90\}$ for the drawings. If you win the auction with a bid of b_1 and your valuation is w_1, then your utility will be $w_1 - b_1$. If you do not win the auction, then your utility will be 0. Similarly for Barbara: If she wins the auction with a bid of b_2 and her valuation is w_2, then her utility is $w_2 - b_2$. If she does not win the auction, her utility is 0. That is, the valuation represents the highest price you are willing to pay for the collection of drawings.

Of course, you know what your own valuation is, but you are uncertain about Barbara's valuation. Similarly for Barbara. As the utility functions for you and Barbara depend on the specific valuations that you have, this is a game with *incomplete information.* For every possible valuation w_1 that you can have, we denote the associated utility function by $u_1^{w_1}$, and similarly for Barbara.

You	20	40	60	80	100
20	5	0	0	0	0
40	−10	−5	0	0	0
60	−30	−30	−15	0	0
80	−50	−50	−50	−25	0
100	−70	−70	−70	−70	−35

u_1^{30}

You	20	40	60	80	100
20	15	0	0	0	0
40	10	5	0	0	0
60	−10	−10	−5	0	0
80	−30	−30	−30	−15	0
100	−50	−50	−50	−50	−25

u_1^{50}

You	20	40	60	80	100
20	25	0	0	0	0
40	30	15	0	0	0
60	10	10	5	0	0
80	−10	−10	−10	−5	0
100	−30	−30	−30	−30	−15

u_1^{70}

You	20	40	60	80	100
20	35	0	0	0	0
40	50	25	0	0	0
60	30	30	15	0	0
80	10	10	10	5	0
100	−10	−10	−10	−10	−5

u_1^{90}

Table 5.4.5 Decision problems for "Chris' drawings"

The decision problems for you at each of the possible utility functions can be found in Table 5.4.5. Since the situation is symmetric between Barbara and you, Barbara's decision problems look exactly the same, and have therefore been omitted to save space. Recall that if you both submit the same bid, then you will only win the auction with probability 0.5. For instance, if your valuation is 50 and you both bid 20, then your expected utility will be $(0.5) \cdot (50 - 20) + (0.5) \cdot 0 = 15$. Similarly for the other entries where you and Barbara make the same bid.

Now suppose that your valuation is 70. What bids can you rationally make under common belief in rationality? To answer this question we rely on Theorem 5.4.1 and run the *generalized iterated strict dominance* procedure.

Round 1. At each of your decision problems, the bid 100 is strictly dominated by 80, and can therefore be eliminated. This leads to the one-fold reduced decision problems in Table 5.4.6. Barbara's one-fold reduced decision problems look exactly the same.

Round 2. In each of your decision problems we start by eliminating the state 100, since Barbara's bid 100 did not survive round 1 at any of her decision problems. Subsequently, the bid 80 becomes strictly dominated by 60 at each of your valuations, except 90. We can thus eliminate the bid 80 at your valuations 30, 50 and 70. Moreover, at your valuation 90, the bid 20 has become strictly dominated by the randomized bid $(0.9) \cdot 40 + (0.1) \cdot 80$. We can thus eliminate bid 20 at your valuation 90. Similarly for Barbara. This leads to the two-fold reduced decision problems in Table 5.4.7.

Note that after round 2, no state can be eliminated in your decision problems, as each of Barbara's bids 20, 40, 60 and 80 have survived round 2 at some of her valuations. As such, no choice can be eliminated either, and the procedure terminates here.

Thus, if your valuation is 70, then you can rationally make the bids 20, 40 and 60 under common belief in rationality. Note that, in order to reach this conclusion, we also had to reason about decision problems for you where your valuation was different from 70. The reason is that Barbara does not know your valuation, and hence you must reason about Barbara who may reason about valuations for you that are different from your actual valuation.

You	20	40	60	80	100
20	5	0	0	0	0
40	−10	−5	0	0	0
60	−30	−30	−15	0	0
80	−50	−50	−50	−25	0

u_1^{30}

You	20	40	60	80	100
20	15	0	0	0	0
40	10	5	0	0	0
60	−10	−10	−5	0	0
80	−30	−30	−30	−15	0

u_1^{50}

You	20	40	60	80	100
20	25	0	0	0	0
40	30	15	0	0	0
60	10	10	5	0	0
80	−10	−10	−10	−5	0

u_1^{70}

You	20	40	60	80	100
20	35	0	0	0	0
40	50	25	0	0	0
60	30	30	15	0	0
80	10	10	10	5	0

u_1^{90}

Table 5.4.6 One-fold reduced decision problems in "Chris' drawings"

You	20	40	60	80
20	5	0	0	0
40	−10	−5	0	0
60	−30	−30	−15	0

u_1^{30}

You	20	40	60	80
20	15	0	0	0
40	10	5	0	0
60	−10	−10	−5	0

u_1^{50}

You	20	40	60	80
20	25	0	0	0
40	30	15	0	0
60	10	10	5	0

u_1^{70}

You	20	40	60	80
40	50	25	0	0
60	30	30	15	0
80	10	10	10	5

u_1^{90}

Table 5.4.7 Two-fold reduced decision problems in "Chris' drawings"

5.4.5 Common Belief in Rationality is Always Possible

With Theorem 5.4.1 at hand it is now easy to show that common belief in rationality is always possible in every game with incomplete information, provided we have finitely many possible choices and utility functions for every player. Recall from our arguments above that the generalized iterated strict dominance procedure must always terminate within finitely many rounds for such games.

Moreover, it can never happen that at a certain round we eliminate all remaining states or all remaining choices in a given decision problem. The argument for the latter is almost identical to Chapter 3: Consider a certain round k, a player i, a utility function $u_i \in U_i$, and the associated $(k-1)$-fold reduced decision problem (D_i, D_{-i}, u_i) that has been produced before round k starts. Now, fix for every opponent j a utility function u_j and a choice c_j in j's associated $(k-1)$-fold reduced decision problem. Then, the state $(c_j)_{j \neq i}$ is in D_{-i}, and will not be eliminated from (D_i, D_{-i}, u_i) at round k. Next, take a choice c_i for player i that is optimal under the utility function u_i if he believes that, with probability 1, the state is $(c_j)_{j \neq i}$. Then, by Theorem 2.6.1, c_i will certainly survive round k at utility function u_i. Hence, we conclude that at the end of round k, there will, for every player i and every utility function $u_i \in U_i$, be at least one state and one choice left in the associated decision problem. As this holds for every round k, and the procedure terminates after finitely many rounds, there will at the end of the procedure be at least one state and one choice left in every decision problem.

Part (b) in Theorem 5.4.1 guarantees that for every choice c_i that is left at the end of the procedure at utility function u_i, there will be an epistemic model, and a type t_i with utility function u_i within it, such that t_i expresses common belief in rationality, and c_i is optimal for t_i. As we have seen that, for every player i and every utility function u_i, there *will* be at least one choice left for player i at the end, there must always be a type t_i that expresses common belief in rationality.

In fact, we can say a little more: Based on the proof of Theorem 5.4.1 we can always construct a *single* epistemic model M such that, for every player i and every utility function u_i, there is a type t_i in M that has utility function u_i and expresses common belief in rationality. We thus arrive at the following conclusion.

Theorem 5.4.3 (Common belief in rationality is always possible) *Consider a game with incomplete information $(C_i, U_i)_{i \in I}$ which, for every player i, contains finitely many choices and finitely many utility functions. Then, there is an epistemic model M such that for every player i and every utility function $u_i \in U_i$, there is a type t_i in M such that $v_i(t_i) = u_i$ and t_i expresses common belief in rationality.*

In the next subsection we will explain how the outcome of the procedure can be used to construct such an epistemic model M with these properties.

5.4.6 Using the Procedure to Construct Epistemic Models

Suppose that for a specific game with incomplete information we run the *generalized iterated strict dominance* procedure. Consider, for a given player i and utility function $u_i \in U_i$, some choice c_i that has survived at the decision problem associated with u_i. Then, by construction of the procedure, c_i is not strictly dominated in the final decision problem at u_i. By Theorem 2.6.1 we thus know that c_i is optimal, given the utility function u_i, for some belief on the surviving states in this final decision problem.

Moreover, again by construction, every surviving state only involves opponents' choices c_j that have survived at some decision problem, with some utility function $u_j \in U_j$. As such, every state in

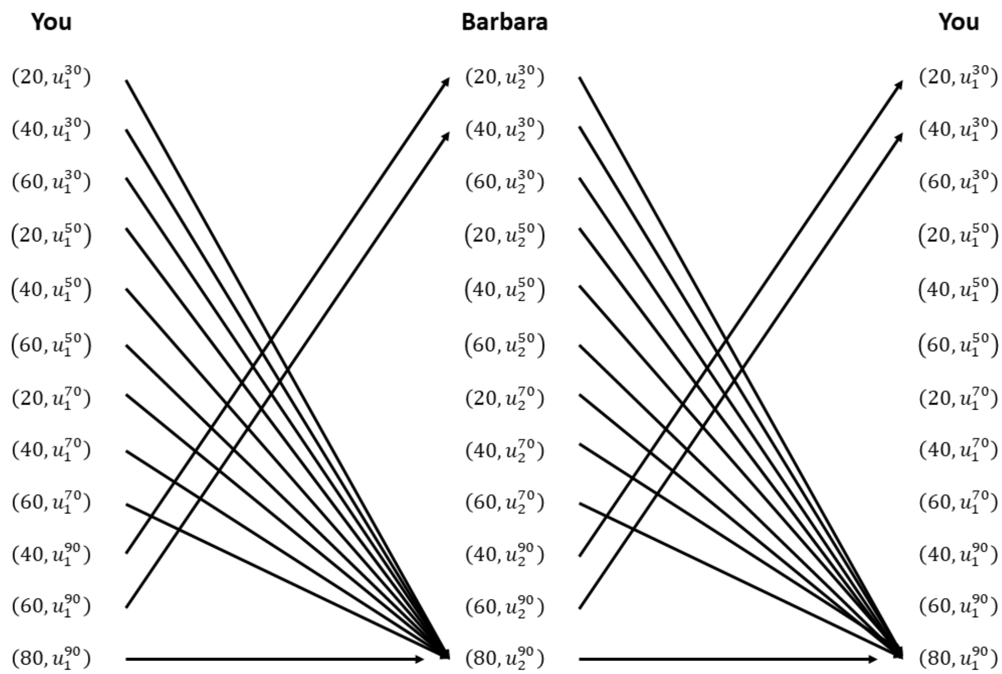

Figure 5.4.1 Beliefs diagram for "Chris' drawings"

the final decision problem at u_i only contains opponents' choices c_j that are optimal, for some utility function $u_j \in U_j$, and some belief on the surviving states in the decision problem at u_j.

These two insights thus allow us to build a beliefs diagram, with solid arrows only, where every choice that appears in some final decision problem, for some utility function, is present. But then, all belief hierarchies in this beliefs diagram will express common belief in rationality. Therefore, if we translate this beliefs diagram into an epistemic model, we obtain for every choice c_i that has survived at some decision problem, for some utility function u_i, a type t_i that (i) has utility function u_i, (ii) expresses common belief in rationality, and (iii) for which c_i is optimal.

As an illustration, consider the example "Chris' drawings", for which the final decision problems can be found in Table 5.4.7. Note that in your decision problems at u_1^{30}, u_1^{50} and u_1^{70}, each of the surviving choices is optimal, given the respective utility function, if you believe that Barbara chooses 80. In the decision problem at u_1^{90}, bid 40 is optimal if you believe that Barbara chooses 20, bid 60 is optimal if you believe that Barbara chooses 40, and bid 80 is optimal if you believe that Barbara chooses 80. Since the same can be said about Barbara, this yields the beliefs diagram in Figure 5.4.1.

Question 5.4.1 *Translate the beliefs diagram from Figure 5.4.1 into an epistemic model.*

It may be verified that each of the types in this epistemic model believes in the opponent's rationality, and thus every type expresses common belief in rationality. Note also that for every bid that survives the procedure at some decision problem, with some utility function, there is an associated type with that utility function that expresses common belief in rationality, and for which that bid is optimal.

Question 5.4.2 *In a similar fashion, construct an epistemic model for the example "What is Barbara's favorite color?", where for every choice that survives the procedure at some decision problem,* *with some utility function, there is an associated type with that utility function that expresses common belief in rationality, and for which that choice is optimal.*

Therefore, the elimination procedure is not only useful for finding the choices that are possible under common belief in rationality, but also as a basis for constructing beliefs diagrams and epistemic models that justify these choices.

5.5 *Fixed Beliefs on Utilities

In the concept of common belief in rationality as discussed above, the players were free to hold any possible belief about the opponents' utility functions. In many situations, however, some beliefs about the opponents' utilities seem more plausible than others. In this section we investigate, as an extreme case, the scenario where the players are required to hold a *fixed* belief about the opponents' utility functions, and combine this with common belief in rationality. We start with an example to illustrate this idea, after which we define the new concept formally, and characterize the resulting choices by means of a recursive procedure similar to generalized iterated strict dominance.

5.5.1 Example

Let us return to the example "What is Barbara's favorite color?". We have seen that under common belief in rationality you can rationally choose the colors *blue* and *green.* Moreover, the beliefs diagram in Figure 5.2.1 provides a justification for these choices: Under common belief in rationality it is optimal to choose *blue* if you believe that Barbara chooses *red,* has utility function u_2^r, and holds the belief hierarchy that starts at (*red,* u_2^r), whereas it is optimal to choose *green* if you believe that Barbara chooses *blue,* has utility function u_2^b, and holds the belief hierarchy that starts at (*blue,* u_2^b).

Note that these two justifications involve two different beliefs about Barbara's utility function. Suppose now that you are relatively certain that Barbara's utility function is u_2^r. In that case, the second justification seems implausible, because it reveals a state of mind in which you are confident that Barbara's utility function is u_2^b and not u_2^r. More precisely, assume that you are 80% confident that Barbara's utility function is u_2^r, and that you believe that Barbara believes this, you believe that Barbara believes that you believe that Barbara believes this, and so on. What choices could you rationally make under common belief in rationality in this particular scenario?

Recall that under common belief in rationality without restrictions on the beliefs on utilities, you could only rationally choose *blue* and *green,* Barbara could only rationally choose *red* if her utility function is u_2^r, and Barbara could only rationally choose *blue* or *yellow* if her utility function is u_2^b. Clearly, these are the only candidates for optimal choices in the new, more restrictive scenario, where we additionally impose the belief $p = (0.8) \cdot u_2^r + (0.2) \cdot u_2^b$ on Barbara's utilities.

Hence, if you hold the belief p about Barbara's utilities then, under common belief in rationality, you must assign probability 0.8 to Barbara choosing *red.* Indeed, under common belief in rationality, *red* is the only color she can rationally wear in case her utility function is u_2^r, whereas she cannot rationally wear *red* if her utility function is u_2^b. But if you assign probability 0.8 to Barbara wearing *red,* then the only optimal choice for you is to wear *blue.* We thus see that under common belief in rationality with the fixed belief p on Barbara's utility function, the only color you can rationally wear is *blue,* and not *green.*

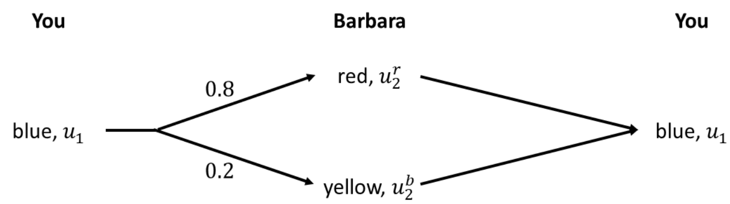

Figure 5.5.1 Common belief in rationality with fixed belief on utilities in "What is Barbara's favorite color?"

As a justification for this choice *blue* we may consider the beliefs diagram in Figure 5.5.1. Consider your belief hierarchy that starts at (*blue,* u_1). This belief hierarchy expresses common belief in rationality as it involves only solid arrows, it supports your choice *blue,* and throughout the belief hierarchy the only belief about Barbara's utility function that enters is $p = (0.8) \cdot u_2^r + (0.2) \cdot u_2^b$. That is, not only do you hold this belief p about Barbara's utilities, you also believe that Barbara believes that you hold this belief, and so on.

Question 5.5.1 *Suppose now that we impose the belief* $p = (0.5) \cdot u_2^r + (0.5) \cdot u_2^b$ *on Barbara's utilities. Show, by means of a beliefs diagram, that under common belief in rationality you can rationally choose* blue *and* green.

Hence, we see that the particular belief we impose on Barbara's utilities has important consequences for the choices you can rationally make under common belief in rationality.

5.5.2 Definition

We will now formally define, within an epistemic model with types, what we mean by imposing fixed beliefs on utility functions. As you will see, the structure of the definition is similar to that of common belief in rationality.

Definition 5.5.1 (Fixed beliefs on utilities) *For every player* i, *let* $p_i \in \Delta(U_i)$ *be a fixed probability distribution on* i*'s possible utility functions, and let* $p = (p_i)_{i \in I}$ *be the collection of these probability distributions. Consider an epistemic model* $(T_i, v_i, b_i)_{i \in I}$.

A type t_i *expresses* 1*-fold belief in* p *if* t_i *assigns to every profile* $(u_j)_{j \neq i}$ *of opponents' utility functions the probability* $\prod_{j \neq i} p_j(u_j)$.

A type t_i *expresses* 2*-fold belief in* p *if* t_i *only assigns positive probability to opponents' types* t_j *that express* 1*-fold belief in* p.

A type t_i *expresses* 3*-fold belief in* p *if* t_i *only assigns positive probability to opponents' types* t_j *that express* 2*-fold belief in* p.

And so on.

A type t_i *expresses* ***common belief in*** p *if it expresses* k*-fold belief in* p *for every* $k \in \{1, 2, 3, ...\}$.

Thus, imposing a fixed belief on utilities is formalized by stating that a type t_i expresses common belief in a fixed profile $p = (p_i)_{i \in I}$ of beliefs on utility functions. Or, equivalently, t_i's belief hierarchy on utility functions is the *simple* belief hierarchy induced by the profile $p = (p_i)_{i \in I}$ of beliefs.

We can now naturally define what it means that a choice can rationally be made, for a given utility function, under common belief in rationality with fixed beliefs on utility functions.

Definition 5.5.2 (Rational choice with fixed beliefs on utilities) *Let $p = (p_i)_{i \in I}$ be a profile of beliefs on utility functions, and $u_i \in U_i$ a utility function. Then, player i can* ***rationally make the choice c_i with utility function u_i under common belief in rationality and common belief in*** *p if there is an epistemic model $(T_i, v_i, b_i)_{i \in I}$ and a type $t_i \in T_i$ such that (a) t_i expresses common belief in rationality, (b) t_i expresses common belief in p, (c) t_i has utility function u_i, and (d) c_i is optimal for t_i.*

Note the similarity with the definition without restrictions on the beliefs about utilities: The only difference is that we added condition (b).

5.5.3 Recursive Procedure

Fix a profile $p = (p_i)_{i \in I}$ of beliefs on utility functions. Can we design a recursive procedure that computes, for every player and each of his utility functions, the choices he can rationally make under common belief in rationality and common belief in p? The anwer is "yes", and the procedure, as we will see, is rather similar to the generalized iterated strict dominance procedure.

We first ask which choices are optimal for player i, at utility function u_i, for some belief about the opponents' choices and utility functions, without yet imposing any restrictions on this belief. The answer is simple: By Theorem 2.6.1, these are precisely the choices that are not strictly dominated in the decision problem at u_i. By eliminating the strictly dominated choices at u_i we obtain the 1-fold reduced decision problem at u_i.

We then ask: What choices can player i rationally make if he expresses 1-fold belief in rationality and 1-fold belief in p? Consider i's first-order belief b_i^1 on the opponents' utility functions and choices. As i expresses 1-fold belief in p, the induced belief on the opponents' utility functions must be in accordance with p. We say that the first-order belief *respects* p. Moreover, if b_i^1 assigns positive probability to a choice-utility pair (c_j, u_j) for opponent j, then c_j must be in the 1-fold reduced decision problem at u_j, as i expresses 1-fold belief in rationality. Thus, the choices that player i can rationally make if he expresses 1-fold belief in rationality and 1-fold belief in p are precisely those choices c_i that are optimal for a first-order belief b_i^1 on the opponents choices and utility functions where (i) b_i^1 respects p, and (ii) b_i^1 only assigns positive probability to pairs (c_j, u_j) where c_j is in the 1-fold reduced decision problem at u_j. These are precisely the choices we keep at each of i's decision problems. This leads, for every player i and every utility function u_i, to the 2-fold reduced decision problem at u_i.

In the next round we wish to single out those choices that player i can rationally make if he expresses 1-fold and 2-fold belief in rationality, and 1-fold and 2-fold belief in p. By a similar argument as above, these are precisely those choices c_i that are optimal for a first-order belief b_i^1 on the opponents' choices and utility functions where (i) b_i^1 respects p, and (ii) b_i^1 only assigns positive probability to pairs (c_j, u_j) where c_j is in the 2-fold reduced decision problem at u_j. By keeping only those choices, this leads to the 3-fold reduced decision problems. And so on.

These arguments will lead to the procedure below. To formally define this procedure, we need the following two additional definitions: For a first-order belief b_i^1 on opponents' choices and utility functions, we say that choice c_i is *optimal* for b_i^1 with the utility function u_i if c_i is optimal for the induced belief on the opponents' choices. Also, we say that b_i^1 *respects* the profile $p = (p_i)_{i \in I}$ of beliefs on utility functions if the induced belief on the opponents' utility functions assigns probability $\prod_{j \neq i} p_j(u_j)$ to every opponents' combination of utility functions $(u_j)_{j \neq i}$.

Definition 5.5.3 (Generalized iterated strict dominance with fixed beliefs on utilities) *Fix a profile $p = (p_i)_{i \in I}$ of beliefs on utility functions, where $p_i \in \Delta(U_i)$ for every player i. Start by writing* *down the decision problems for every player i and every utility function u_i in U_i.*

Round 1. *From every decision problem, eliminate those choices that are strictly dominated. This leads to the 1-fold reduced decision problems.*

Round 2. *At every 1-fold reduced decision problem, only keep those choices c_i which are optimal for a first-order belief on opponents' choices and utility functions that (i) respects p, and (ii) only assigns positive probability to pairs (c_j, u_j) where c_j is in the 1-fold reduced decision problem at u_j. This leads to the 2-fold reduced decision problems.*

Round 3. *At every 2-fold reduced decision problem, only keep those choices c_i which are optimal for a first-order belief on opponents' choices and utility functions that (i) respects p, and (ii) only assigns positive probability to pairs (c_j, u_j) where c_j is in the 2-fold reduced decision problem at u_j. This leads to the 3-fold reduced decision problems.*

Continue like this until no further choices can be eliminated. The choices for a player i that eventually remain in his decision problem at a certain utility function u_i are said to survive the ***generalized iterated strict dominance procedure with fixed beliefs** p **on utility functions*** *at u_i.*

By the arguments above, we conclude that this procedure will yield precisely those choices that can rationally be made under common belief in rationality and common belief in p.

Theorem 5.5.1 (Procedure with fixed beliefs on utilities) *Fix a profile $p = (p_i)_{i \in I}$ of beliefs on utility functions, where $p_i \in \Delta(U_i)$ for every player i.*

(a) For every $k \in \{1, 2, 3, ...\}$, the choices that player i can rationally make with utility function $u_i \in U_i$ while expressing up to k-fold belief in rationality and up to k-fold belief in p are precisely the choices that survive the first $k+1$ rounds of the generalized iterated strict dominance procedure with fixed beliefs p on utility functions at u_i.

(b) The choices that player i can rationally make with utility function $u_i \in U_i$ under common belief in rationality and common belief in p are exactly the choices that survive all rounds of the generalized iterated strict dominance procedure with fixed beliefs p on utility functions at u_i.

In the same way as for the generalized iterated strict dominance procedure without restrictions on the beliefs on utility functions, it can be shown that this procedure terminates within finitely many steps, and that for every player i and every utility function $u_i \in U_i$ at least one choice for player i survives the procedure at u_i. Together with Theorem 5.5.1, we thus arrive at the following conclusion.

Theorem 5.5.2 (Possibility of common belief in rationality with fixed beliefs on utilities) *Consider a game with incomplete information $(C_i, U_i)_{i \in I}$ which, for every player i, contains finitely many choices and finitely many utility functions. Fix a profile $p = (p_i)_{i \in I}$ of beliefs on utility functions. Then, there is an epistemic model M such that for every player i and every utility function $u_i \in U_i$, there is a type t_i in M such that $v_i(t_i) = u_i$ and t_i expresses common belief in rationality and common belief in p.*

Moreover, as we will see in the next subsection, we can use the procedure to construct such an epistemic model with the properties above.

Like we have seen with the *generalized iterated strict dominance* procedure without fixed beliefs on utilities, also the procedure above is order independent. That is, the order by which we eliminate choices at the various decision problems, and the order by which we investigate the different decision problems, does not matter for the eventual output of the procedure.

You	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
green	3	0	3	3

u_1

Barbara	*blue*	*green*	*red*	*yellow*
red	4	4	0	4
yellow	3	3	3	0

u_2^r

Barbara	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
yellow	3	3	3	0

u_2^b

Table 5.5.1 Two-fold reduced decision problems in "What is Barbara's favorite color?"

Theorem 5.5.3 (Order independence) *Changing the order of elimination in the generalized iterated strict dominance procedure with fixed beliefs on utilities does not change the sets of choices that survive the procedure at each of the decision problems.*

In particular, if at some round we forget to eliminate a choice at a certain decision problem, then this will not affect the final result. Provided, of course, we do not forget to eliminate this choice forever.

5.5.4 Illustration of the Procedure

We will now illustrate the procedure by means of the two examples we have seen so far in this chapter. Let us first return to the example "What is Barbara's favorite color?", and fix the profile $p = (p_1, p_2)$ of beliefs on utility functions, where p_1 assigns probability 1 to your unique utility function u_1, and p_2 assigns probability 0.8 to Barbara's utility function u_2^r and probability 0.2 to Barbara's utility function u_2^b . Then, the procedure would go as follows.

Round 1. The first round is the same as in Section 5.4, and leads to the 1-fold reduced decision problems in Table 5.4.1.

Round 2. Consider a first-order belief b_1^1 for you that respects p and that only assigns positive probability to pairs (c_2, u_2) where c_2 has survived the first round at u_2. Since p must assign probability 0.8 to u_2^r and Barbara's choice *green* did not survive round 1 at u_2^r, the belief b_1^1 can assign probability at most 0.2 to Barbara choosing *green.* But then, the expected utility of choosing *green* yourself is at least $(0.8) \cdot 3 = 2.4$, which means that choosing *red* cannot be optimal for b_1^1. We thus eliminate your choice *red.*

In Barbara's decision problems at u_2^r and u_2^b, Barbara must assign probability zero to you choosing *yellow,* since your choice *yellow* did not survive round 1 at your unique decision problem. But then, it cannot be optimal for Barbara to choose *blue* at u_2^r or to choose *green* at u_2^b. We thus eliminate Barbara's choices *blue* at u_2^r and *green* at u_2^b, leading to the 2-fold reduced decision problems in Table 5.5.1.

Round 3. Consider a first-order belief b_1^1 for you that respects p and that only assigns positive probability to pairs (c_2, u_2) where c_2 has survived the second round at u_2. As b_1^1 assigns probability 0.8 to u_2^r, and Barbara's choice *blue* did not survive round 2 at u_2^r, the belief b_1^1 assigns probability at most 0.2 to Barbara choosing *blue.* But then, the expected utility of choosing *blue* yourself is at least $(0.8) \cdot 4 = 3.2$, and hence it cannot be optimal to choose *green.* We thus eliminate your choice *green.*

You	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4

u_1

Barbara	*blue*	*green*	*red*	*yellow*
red	4	4	0	4

u_2^r

Barbara	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4
yellow	3	3	3	0

u_2^b

Table 5.5.2 Three-fold reduced decision problems in "What is Barbara's favorite color?"

You	*blue*	*green*	*red*	*yellow*
blue	0	4	4	4

u_1

Barbara	*blue*	*green*	*red*	*yellow*
red	4	4	0	4

u_2^r

Barbara	*blue*	*green*	*red*	*yellow*
yellow	3	3	3	0

u_2^b

Table 5.5.3 Final decision problems in "What is Barbara's favorite color?"

On the other hand, Barbara must assign probability zero to your choices *red* and *yellow* at both of her decision problems, since these choices did not survive round 2 at your unique decision problem. But then, *yellow* cannot be optimal for Barbara at u_2^r, and will thus be eliminated at u_2^r. This leads to the 3-fold reduced decision problems in Table 5.5.2.

Round 4. At u_2^b, Barbara must assign probability 1 to your choice *blue,* as this is the only choice that has survived round 3 at your unique decision problem. But then, Barbara can only optimally choose *yellow* at u_2^b, and thus we eliminate her choice *blue* there. This leads to the final decision problems in Table 5.5.3. By Theorem 5.5.1 we thus conclude that under common belief in rationality and common belief in p, you can only rationally wear *blue,* Barbara can only rationally wear *red* if her utility function is u_2^r, and Barbara can only rationally wear *yellow* if her utility function is u_2^b.

Like in Section 5.4, we can use the output of the procedure to construct an epistemic model that satisfies the properties from Theorem 5.5.2. In this case this is particularly easy as there is only one choice left at every decision problem. We first construct the beliefs diagram which we have already seen in Figure 5.5.1, which can then be translated into the epistemic model from Table 5.5.4.

Question 5.5.2 *In the example "What is Barbara's favorite color?", consider the profile $p = (p_1, p_2)$ of beliefs on utilities where p_1 assigns probability 1 to u_1, and p_2 assigns probability 0.5 to u_2^r and u_2^b. Apply the procedure with this particular p to the game, and find for every player and utility function the choices that can rationally be made under common belief in rationality and common belief in p.*

We finally illustrate the procedure by the example "Chris' drawings". Fix the profile $p = (p_1, p_2)$ of beliefs on utilities where p_1 assigns equal probability to each of the utility functions $u_1^{30}, u_1^{50}, u_1^{70}$ and u_1^{90}, and similarly for p_2. This represents a situation where you express common belief in the event that both you and Barbara deem each of the four possible opponent's valuations equally likely.

Round 1. The eliminations are the same as in Section 5.4, which leads to the 1-fold reduced decision

Types	$T_1 = \{t_1^{blue}\}, \quad T_2 = \{t_2^{red}, t_2^{yellow}\}$					
Utilities and beliefs for you	$v_1(t_1^{blue})$	$=$	u_1	$b_1(t_1^{blue})$	$=$	$(0.8) \cdot (red,\ t_2^{red})$ $+(0.2) \cdot (yellow,\ t_2^{yellow})$
Utilities and beliefs for Barbara	$v_2(t_2^{red})$	$=$	u_2^r	$b_2(t_2^{red})$	$=$	$(blue,\ t_1^{blue})$
	$v_2(t_2^{yellow})$	$=$	u_2^b	$b_2(t_2^{yellow})$	$=$	$(blue,\ t_1^{blue})$

Table 5.5.4 Epistemic model for "What is Barbara's favorite color?" with fixed beliefs on utilities

You	20	40	60	80	100
20	5	0	0	0	0
40	−10	−5	0	0	0
60	−30	−30	−15	0	0

u_1^{30}

You	20	40	60	80	100
20	15	0	0	0	0
40	10	5	0	0	0
60	−10	−10	−5	0	0

u_1^{50}

You	20	40	60	80	100
20	25	0	0	0	0
40	30	15	0	0	0
60	10	10	5	0	0

u_1^{70}

You	20	40	60	80	100
40	50	25	0	0	0
60	30	30	15	0	0
80	10	10	10	5	0

u_1^{90}

Table 5.5.5 Two-fold reduced decision problems in "Chris' drawings"

problems for you in Table 5.4.6. The 1-fold reduced decision problems for Barbara look exactly the same.

Round 2. You must assign probability zero to Barbara's bid 100, since 100 did not survive round 1 at any of Barbara's decision problems. But then, bidding 80 is always worse than bidding 60 at your utility functions u_1^{30}, u_1^{50} and u_1^{70}. Moreover, at your utility function u_1^{90}, bidding 20 cannot be optimal, since either 40 or 80 will be better. To see this, distinguish two cases: If you assign positive probability to Barbara choosing 20 or 40, then 40 is better than 20 for you. If you assign probability zero to Barbara choosing 20 and 40, then you must assign positive probability to Barbara choosing 60 or 80, and then 80 will better than 20 for you. Thus, we can eliminate the bid 80 at your utility functions u_1^{30}, u_1^{50} and u_1^{70}, and the bid 20 at u_1^{90}. Similarly for Barbara. This leads to the 2-fold reduced decision problems for you in Table 5.5.5. The 2-fold reduced decision problems for Barbara look exactly the same.

Round 3. In your first-order belief you must assign probability at most 0.25 to Barbara choosing 80, as 80 only survives round 2 at u_2^{90} to which you must assign probability 0.25. Since you assign probability zero to Barbara choosing 100 and probability at most 0.25 to Barbara choosing 80, bidding 40 is better than bidding 60 at u_1^{30} and u_1^{50}. Moreover, at u_1^{70} either 40 or 60 is better than 20. To see this, distinguish two cases: If you assign positive probability to Barbara choosing 20 or 40, then

You	20	40	60	80	100
20	5	0	0	0	0
40	−10	−5	0	0	0

u_1^{30}

You	20	40	60	80	100
20	15	0	0	0	0
40	10	5	0	0	0

u_1^{50}

You	20	40	60	80	100
40	30	15	0	0	0
60	10	10	5	0	0

u_1^{70}

You	20	40	60	80	100
40	50	25	0	0	0
60	30	30	15	0	0

u_1^{90}

Table 5.5.6 Three-fold reduced decision problems in "Chris' drawings"

You	20	40	60	80	100
20	5	0	0	0	0

u_1^{30}

You	20	40	60	80	100
20	15	0	0	0	0
40	10	5	0	0	0

u_1^{50}

You	20	40	60	80	100
40	30	15	0	0	0
60	10	10	5	0	0

u_1^{70}

You	20	40	60	80	100
40	50	25	0	0	0
60	30	30	15	0	0

u_1^{90}

Table 5.5.7 Four–fold reduced decision problems in "Chris' drawings"

40 is better than 20 for you. If you assign probability zero to Barbara choosing 20 and 40, then you must assign positive probability to Barbara choosing 60, as you can assign at most probability 0.25 to Barbara choosing 80. But then, 60 is better than 20 for you. Also, at u_1^{90} the expected utility difference between bidding 60 and bidding 80 is at least

$$(0.75) \cdot 5 + (0.25) \cdot (-5) > 0,$$

and hence bidding 60 is better than bidding 80 at u_1^{90}. We can thus eliminate 60 at u_1^{30} and u_1^{50}, eliminate 20 at u_1^{70} and eliminate 80 at u_1^{90}. Similarly for Barbara. This leads to the 3-fold reduced decision problems in Table 5.5.6.

Round 4. In your first-order belief you must assign probability zero to Barbara choosing 80 or 100, since both bids did not survive round 3 at any of Barbara's decision problems. Moreover, you must assign probability at most 0.5 to Barbara bidding 60, since Barbara's choice 60 only survives round 3 at u_1^{70} and u_1^{90} to which you assign probability 0.25 each. But then, bidding 20 is better than bidding 40 at u_1^{30}. We can thus eliminate 40 at u_1^{30}. This leads to the 4-fold reduced decision problems in Table 5.5.7.

Round 5. You can assign probability at most 0.5 to Barbara choosing 60, since Barbara's choice 60 only survived round 4 at u_2^{70} and u_2^{90}, to which you both assign probability 0.25. Moreover, you must assign probability at least 0.25 to Barbara choosing 20, as 20 is the only choice for Barbara that survived round 4 at u_2^{30}, to which you assign probability 0.25. But then, at u_1^{70} the expected utility difference between choosing 40 and 60 for you is at least

$$(0.25) \cdot (30 - 10) + (0.25) \cdot (15 - 10) + (0.5) \cdot (0 - 5) > 0,$$

You	20	40	60	80	100
20	5	0	0	0	0

u_1^{30}

You	20	40	60	80	100
20	15	0	0	0	0
40	10	5	0	0	0

u_1^{50}

You	20	40	60	80	100
40	30	15	0	0	0

u_1^{70}

You	20	40	60	80	100
40	50	25	0	0	0
60	30	30	15	0	0

u_1^{90}

Table 5.5.8 Five–fold reduced decision problems in "Chris' drawings"

and hence 40 is better than 60 for you at u_1^{70}. We can therefore eliminate your choice 60 at u_1^{70}, which leads to the 5-fold reduced decision problems in Table 5.5.8.

It may be verified that after this round no further eliminations are possible. On the basis of Theorem 5.5.1 we thus conclude that under common belief in rationality and common belief in p, you can only rationally bid 20 if your valuation is 30, you can only rationally bid 20 or 40 if your valuation is 50, you can only rationally bid 40 if your valuation is 70, and you can only rationally bid 40 or 60 if your valuation is 90.

Moreover, we can use the output of the procedure to derive the epistemic model in Table 5.5.9. Here, we assume that Barbara's types hold similar beliefs as your types, since the situation is symmetric between you and Barbara. For the sake of brevity, we did not write down the beliefs of Barbara's types.

The superindices 30 and 20 in the type $t_1^{30,20}$ indicate that this type has the utility function u_1^{30}, and that the bid 20 is optimal for this type. The superindices of the other types should be interpreted in a similar fashion.

It may be verified that all types believe in the opponent's rationality, and therefore all types in the epistemic model express *common* belief in rationality. Moreover, every type believes in p, which implies that all types express *common* belief in p. Thus, for each of your possible valuations w_1 and every bid c_1 that survives the procedure at w_1, there is a type $t_1^{w_1,c_1}$ that expresses common belief in rationality, expresses common belief in p, holds the utility function $u_1^{w_1}$ such that the bid c_1 is optimal for $t_1^{w_1,c_1}$. Similarly for Barbara.

Let us conclude with an intuitive analysis of the results for this example. We observe at least two important phenomena: Under common belief in rationality and common belief in p, a player with valuation w_i will always bid lower than w_i. Moreover, the higher the valuation w_i, the bigger the gap between the highest bid he will consider and his valuation.

To explain the first phenomenon, observe first that bidding 100 can never be optimal for any valuation, since it would yield a positive probability of winning the auction, in which case you would have to pay more than your valuation. But then, for valuations below 90 it can never be optimal to bid 80 either, since the probability of winning would again be positive, in which case you would have to pay more than your valuation. As both players i hold the belief p_j on the opponent's utility function, a player believes that, with positive probability, the opponent will have a valuation lower than 90, and will thus bid lower than 80. This implies, in turn, that bidding 60 will not be optimal for valuations lower than 60 because it will induce a positive probability of winning, in which case you would have to pay more than your valuation. For similar reasons it can finally be concluded that bidding 40 will not be optimal for a player with valuation 20. Thus, indeed, a player will never bid

Types	$T_1 = \{t_1^{30,20}, t_1^{50,20}, t_1^{50,40}, t_1^{70,40}, t_1^{90,40}, t_1^{90,60}\}$		
	$T_2 = \{t_2^{30,20}, t_2^{50,20}, t_2^{50,40}, t_2^{70,40}, t_2^{90,40}, t_2^{90,60}\}$		
Utilities and beliefs for you	$v_1(t_1^{30,20}) = u_1^{30}$	$b_1(t_1^{30,20}) =$	$(0.25)\cdot(20, t_2^{30,20}) + (0.25)\cdot(20, t_2^{50,20})$ $+(0.25)\cdot(40, t_2^{70,40}) + (0.25)\cdot(60, t_2^{90,60})$
	$v_1(t_1^{50,20}) = u_1^{50}$	$b_1(t_1^{50,20}) =$	$(0.25)\cdot(20, t_2^{30,20}) + (0.25)\cdot(20, t_2^{50,20})$ $+(0.25)\cdot(40, t_2^{70,40}) + (0.25)\cdot(60, t_2^{90,60})$
	$v_1(t_1^{50,40}) = u_1^{50}$	$b_1(t_1^{50,40}) =$	$(0.25)\cdot(20, t_2^{30,20}) + (0.25)\cdot(40, t_2^{50,40})$ $+(0.25)\cdot(40, t_2^{70,40}) + (0.25)\cdot(60, t_2^{90,60})$
	$v_1(t_1^{70,40}) = u_1^{70}$	$b_1(t_1^{70,40}) =$	$(0.25)\cdot(20, t_2^{30,20}) + (0.25)\cdot(20, t_2^{50,20})$ $+(0.25)\cdot(40, t_2^{70,40}) + (0.25)\cdot(60, t_2^{90,60})$
	$v_1(t_1^{90,40}) = u_1^{90}$	$b_1(t_1^{90,40}) =$	$(0.25)\cdot(20, t_2^{30,20}) + (0.25)\cdot(20, t_2^{50,20})$ $+(0.25)\cdot(40, t_2^{70,40}) + (0.25)\cdot(60, t_2^{90,60})$
	$v_1(t_1^{90,60}) = u_1^{90}$	$b_1(t_1^{90,60}) =$	$(0.25)\cdot(20, t_2^{30,20}) + (0.25)\cdot(40, t_2^{50,40})$ $+(0.25)\cdot(40, t_2^{70,40}) + (0.25)\cdot(60, t_2^{90,60})$

Table 5.5.9 Epistemic model for "Chris' drawings"

more than his valuation.

To understand the second phenomenon, note that the expected utility of making a certain bid under a certain belief consists of the product of two parts: The probability of winning, and the expected utility conditional on winning the auction. Moreover, by increasing your bid you increase the first part but decrease the second part, as you would have to pay a higher price in case you win. As such, a player has to find the "right balance" between the probability of winning and the expected utility conditional on winning.

If your valuation is low then it is best to bid just below your valuation, as to ensure an acceptable probability of winning. By bidding lower, the probability of winning would become so low that it can no longer be compensated by the higher expected utility conditional on winning. If, on the other hand, your valuation is medium or high, then bidding just below your valuation will no longer be optimal. In that case, the probability of winning will be relatively high or really high, depending on the valuation. But then, the overall expected utility could be increased by lowering your bid, which would increase the expected utility conditional on winning while still guaranteeing an acceptable probability of winning. Moreover, the margin by which you could lower your bid in this case would be higher if your valuation is higher.

Question 5.5.3 *In the auction above, assume that the person who wins the auction is still the person with the highest bid, but that person will now pay the bid of the other person. This is called a second-price auction. Set up the decision problems for the different valuations, and explain why under common belief in rationality and common belief in* p *(for any* p*), every bid can rationally be made. Yet, show that for every valuation there are only two bids that are not weakly dominated by another bid. Which two bids?*

5.6 Proofs

5.6.1 Proofs of Section 5.4

To prove Theorem 5.4.1 we need the following optimality property, similar to the one from the proof section of Chapter 3.

Lemma 5.6.1 (Optimality property) *For every player i, every utility function $u_i \in U_i$ and every round $k \geq 0$, let $C_i^k(u_i)$ be the set of choices for player i that survive the first k rounds of the generalized iterated strict dominance procedure at u_i, and let $C_i^*(u_i)$ be the set of choices that survive all rounds there. Similarly, let $C_{-i}^k(u_i)$ be the set of states that survive the first k rounds, and let $C_{-i}^*(u_i)$ be the set of states that survive all rounds, at u_i.*

(a) For every $k \geq 1$, a choice c_i is in $C_i^k(u_i)$ if and only if c_i is optimal for some belief in $(C_i, C_{-i}^k(u_i), u_i)$.

(b) A choice c_i is in $C_i^*(u_i)$ if and only if c_i is optimal for some belief in $(C_i, C_{-i}^*(u_i), u_i)$.

The proof of this lemma is essentially identical to the one for Lemma 3.6.1 and is therefore omitted.

Proof of Theorem 5.4.1. (a) For every player i and utility function $u_i \in U_i$, let $BR_i^k(u_i)$ denote the set of choices that player i can rationally make while expressing up to k-fold belief in rationality with utility function u_i. Recall from above that $C_i^k(u_i)$ and $C_{-i}^k(u_i)$ denote the set of choices and set of states, respectively, that survive the first k rounds of the generalized iterated strict dominance procedure at u_i. We will show that $BR_i^k(u_i) = C_i^{k+1}(u_i)$ for every player i, every utility function $u_i \in U_i$ and every $k \geq 1$. We show this in two steps: (i) prove that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$ for all $k \geq 1$, and (ii) prove that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$.

(i) Show that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$ for all $k \geq 1$.

We prove this by induction on k. For $k = 1$, take some $c_i \in BR_i^1(u_i)$. Then, there is some epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses 1-fold belief in rationality, $v_i(t_i) = u_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j). Since t_i expresses 1-fold belief in rationality, c_j must be optimal for t_j. Hence, c_j is optimal for t_j's first-order belief in the full decision problem $(C_j, C_{-j}, v_j(t_j))$ which, by Lemma 5.6.1, implies that $c_j \in C_j^1(v_j(t_j))$. Hence, t_i's first-order belief only assigns positive probability to opponents' choices c_j which are in $C_j^1(u_j)$ for some u_j, and thus only assigns positive probability to states in $C_{-i}^2(u_i)$. As c_i is optimal for t_i, we conclude that c_i is optimal for t_i's first-order belief in $(C_i, C_{-i}^2(u_i), u_i)$ which implies, by Lemma 5.6.1, that c_i is in $C_i^2(u_i)$. We thus have shown that every choice $c_i \in BR_i^1(u_i)$ must be in $C_i^2(u_i)$, and hence $BR_i^1(u_i) \subseteq C_i^2(u_i)$.

Now suppose that $k \geq 2$ and that, by the induction assumption, $BR_i^{k-1}(u_i) \subseteq C_i^k(u_i)$ for all players i and all utility functions u_i. Consider some player i and some $c_i \in BR_i^k(u_i)$. Then, there is some epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses up to k-fold belief in rationality, $v_i(t_i) = u_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j). Since t_i expresses up to k-fold belief in rationality, the choice c_j must be optimal for t_j and t_j must express up to $(k-1)$-fold belief in rationality. Hence, $c_j \in BR_j^{k-1}(v_j(t_j))$. Since, by the induction assumption, $BR_j^{k-1}(v_j(t_j)) \subseteq C_j^k(v_j(t_j))$, we know that $c_j \in C_j^k(v_j(t_j))$. We thus conclude that t_i's first-order belief only assigns positive probability to opponents' choices c_j that are in $C_j^k(u_j)$ for some utility function u_j, and hence only assigns positive probability to states in $C_{-i}^{k+1}(u_i)$. As c_i is optimal for t_i, we conclude that c_i is optimal for t_i's first-order belief in $(C_i, C_{-i}^{k+1}(u_i), u_i)$ which implies, by Lemma 5.6.1, that c_i is in $C_i^{k+1}(u_i)$. We thus have shown that every choice $c_i \in BR_i^k(u_i)$ must be in $C_i^{k+1}(u_i)$, and hence $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$. By induction on k, we conclude that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$ for all players i, all utility functions $u_i \in U_i$, and all $k \geq 1$. This completes the proof of (i).

(ii) Show that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$.

Hence, for every choice $c_i \in C_i^{k+1}(u_i)$ we must show that there is some epistemic model, and some type $t_i^{u_i,c_i}$ in it, such that $t_i^{u_i,c_i}$ expresses up to k-fold belief in rationality, $v_i(t_i^{u_i,c_i}) = u_i$, and c_i is optimal for $t_i^{u_i,c_i}$. We will now construct a *single* epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ that contains *all* such types. For every player i, define the set of types

$$T_i = \{t_i^{u_i,c_i} \mid u_i \in U_i,\ c_i \in C_i^1(u_i)\}$$

where $v_i(t_i^{u_i,c_i}) = u_i$. To define the beliefs of these types about the opponents' choice-type combinations we distinguish the following three cases, assuming that the procedure terminates at the end of round K.

Case 1. Suppose that $c_i \in C_i^1(u_i) \backslash C_i^2(u_i)$. Then, by Lemma 5.6.1, c_i is optimal for some belief $b_i^{u_i,c_i} \in \Delta(C_{-i})$ within (C_i, C_{-i}, u_i). For every opponent j choose some arbitrary type $\hat{t}_j \in T_j$, and define

$$b_i(t_i^{u_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{u_i,c_i}((c_j)_{j\neq i}), & \text{if } t_j = \hat{t}_j \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{5.6.1}$$

for all $(c_j, t_j)_{j\neq i}$ in $C_{-i} \times T_{-i}$.

Case 2. Suppose that $c_i \in C_i^k(u_i) \backslash C_i^{k+1}(u_i)$ for some $k \in \{2, ..., K-1\}$. Then, by Lemma 5.6.1, c_i is optimal for some belief $b_i^{u_i,c_i} \in \Delta(C_{-i}^k)$ within (C_i, C_{-i}^k, u_i). By construction of the procedure, for every $(c_j)_{j\neq i} \in C_{-i}^k$ and every $j \neq i$, there is some utility function $u_j^{k-1}[c_j] \in U_j$ such that $c_j \in C_j^{k-1}(u_j^{k-1}[c_j])$. Define

$$b_i(t_i^{u_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{u_i,c_i}((c_j)_{j\neq i}), & \text{if } c_j \in C_j^{k-1} \text{ and } t_j = t_j^{u_j^{k-1}[c_j],c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{5.6.2}$$

for all $(c_j, t_j)_{j\neq i}$ in $C_{-i} \times T_{-i}$.

Case 3. Suppose that $c_i \in C_i^K(u_i)$. As the procedure terminates at round K we have that $c_i \in C_i^*(u_i)$. Hence, by Lemma 5.6.1, c_i is optimal for some belief $b_i^{u_i,c_i} \in \Delta(C_{-i}^*)$ within (C_i, C_{-i}^*, u_i). By construction of the procedure, for every $(c_j)_{j\neq i} \in C_{-i}^*$ and every $j \neq i$, there is some utility function $u_j^*[c_j]$ such that $c_j \in C_j^*(u_j^*[c_j])$. Define

$$b_i(t_i^{u_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{u_i,c_i}((c_j)_{j\neq i}), & \text{if } c_j \in C_j^* \text{ and } t_j = t_j^{u_j^*[c_j],c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{5.6.3}$$

for all $(c_j, t_j)_{j\neq i}$ in $C_{-i} \times T_{-i}$. This completes the construction of the epistemic model $M = (T_i, v_i, b_i)_{i\in I}$.

Note that in this epistemic model, every type $t_i^{u_i,c_i}$ holds the first-order belief $b_i^{u_i,c_i}$ on choices. As, by definition, c_i is optimal for $b_i^{u_i,c_i}$ within (C_i, C_{-i}, u_i), we conclude that c_i is optimal for $t_i^{u_i,c_i}$, for every player i and every $c_i \in C_i^1(u_i)$.

We now show that for every $k \geq 2$ and every choice $c_i \in C_i^k(u_i)$, the associated type $t_i^{u_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality. We show this by induction on k.

For $k = 2$, consider some choice $c_i \in C_i^2(u_i)$ and the associated type $t_i^{u_i,c_i}$ with the belief given by (5.6.2) or (5.6.3). By (5.6.2) and (5.6.3), the belief $b_i(t_i^{u_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where $c_j \in C_j^1(u_j)$. As c_j is optimal for $t_j^{u_j,c_j}$, the type $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where c_j is optimal for $t_j^{u_j,c_j}$. Hence, $t_i^{u_i,c_i}$ expresses 1-fold belief in rationality. This holds for every type $t_i^{u_i,c_i}$ where $c_i \in C_i^2(u_i)$.

Suppose now that $k \geq 3$ and that, by the induction assumption, $t_i^{u_i,c_i}$ expresses up to $(k-2)$-fold belief in rationality for every $c_i \in C_i^{k-1}(u_i)$ and every player i. Consider some choice $c_i \in C_i^k(u_i)$ and the associated type $t_i^{u_i,c_i}$ with the belief given by (5.6.2) or (5.6.3). By (5.6.2) and (5.6.3) it follows that $b_i(t_i^{u_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where $c_j \in C_j^{k-1}(u_j)$. By the induction assumption we know that $t_j^{u_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality. As c_j is optimal for $t_j^{u_j,c_j}$, we conclude that $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where c_j is optimal for $t_j^{u_j,c_j}$, and $t_j^{u_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality. Hence, $t_i^{u_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality. This holds for every type $t_i^{u_i,c_i}$ where $c_i \in C_i^k(u_i)$.

By induction on k, we conclude that for every $k \geq 2$ and every choice $c_i \in C_i^k(u_i)$, the associated type $t_i^{u_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality.

We next show that for every $c_i \in C_i^K(u_i)$, the associated type $t_i^{u_i,c_i}$ expresses *common* belief in rationality. Consider the smaller epistemic model $M^* = (T_i^*, v_i, b_i)_{i \in I}$ where the set of types for player i is

$$T_i^* := \{t_i^{u_i,c_i} \mid u_i \in U_i \text{ and } c_i \in C_i^*(u_i)\},$$

and the beliefs of the types are given by (5.6.3). Note that this is a well-defined epistemic model, since by (5.6.3) every type $t_i^{u_i,c_i} \in T_i^*$ with $c_i \in C_i^*(u_i)$ only assigns positive probability to opponent's types $t_j^{u_j,c_j} \in T_j^*$ where $c_j \in C_j^*(u_j)$. We show that every type in M^* believes in the opponents' rationality.

Consider a type $t_i^{u_i,c_i} \in T_i^*$ where $c_i \in C_i^*(u_i)$. By (5.6.3), type $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where $c_j \in C_j^*(u_j)$. Since c_j is optimal for $t_j^{u_j,c_j}$, the type $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where c_j is optimal for $t_j^{u_j,c_j}$. Hence, $t_i^{u_i,c_i} \in T_i^*$ believes in the opponents' rationality. Since this holds for every type $t_i^{u_i,c_i} \in T_i^*$, all types in M^* believe in the opponents' rationality. Hence, it follows that all types in M^* express common belief in rationality. Note that the types in M^* are exactly the types $t_i^{u_i,c_i}$ with $c_i \in C_i^K(u_i)$. Hence, for every $c_i \in C_i^K(u_i)$, the associated type $t_i^{u_i,c_i}$ expresses common belief in rationality.

We can now prove that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$. Take some $c_i \in C_i^{k+1}(u_i)$ where $k \geq 1$. Then we know from above that c_i is optimal for the associated type $t_i^{u_i,c_i}$, and that the type $t_i^{u_i,c_i}$ expresses up to k-fold belief in rationality. Hence, by definition, $c_i \in BR_i^k(u_i)$. As this holds for every $c_i \in C_i^{k+1}(u_i)$, we conclude that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$.

Since in part (i) we have already seen that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$, we may conclude that $BR_i^k(u_i) = C_i^{k+1}(u_i)$ for all $k \geq 1$. That is, a choice can rationally be made while expressing up to k-fold belief in rationality with utility function u_i precisely when the choice survives $k+1$ elimination rounds at u_i. This establishes part (a) of Theorem 5.4.1.

(b) We finally prove part (b) of Theorem 5.4.1. Suppose first that choice c_i can rationally be made under common belief in rationality with utility function u_i. Then, in particular, for every $k \geq 1$, the choice c_i can rationally be made while expressing up to k-fold belief in rationality with utility function u_i. By part (a) we then know that c_i survives $k+1$ rounds of elimination at u_i. Since this holds for every $k \geq 1$, we conclude that c_i survives all rounds of elimination at u_i.

Suppose next that the choice c_i survives all rounds of elimination at u_i. Then, $c_i \in C_i^K(u_i)$, where K is the round at which the *generalized iterated strict dominance* procedure terminates. From the construction of the epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ above we know that the choice c_i is optimal for the type $t_i^{u_i, c_i}$ and that the type $t_i^{u_i, c_i}$ expresses common belief in rationality. Hence, c_i can rationally be made under common belief in rationality with utility function u_i. We thus conclude that a choice c_i can rationally be made under common belief in rationality with utility function u_i precisely when the choice c_i survives all rounds of elimination at u_i. This completes the proof of part (b), and thereby the proof of this theorem. ■

Proof of Theorem 5.4.2. Recall the definitions and results for reduction operators from Sections 3.6.3.1 and 3.6.3.2. We first show that the *generalized iterated strict dominance* procedure can be characterized by the iterated application of a *reduction operator gsd*, and subsequently prove that this reduction operator *gsd* is *monotone*. By Lemma 3.6.2 it would then follow that *gsd*, and thereby the procedure, is *order independent*.

Let $A = (C_i(u_i), C_{-i}(u_i), u_i)_{i \in I, u_i \in U_i}$ be the set that assigns to every player i and utility function $u_i \in U_i$ the (full) decision problem $(C_i(u_i), C_{-i}(u_i), u_i)$ where $C_i(u_i) = C_i$ and $C_{-i}(u_i) = C_{-i}$. The subsets of A we are interested in have the form $D = (D_i(u_i), D_{-i}(u_i), u_i)_{i \in I, u_i \in U_i}$, where $D_i(u_i) \subseteq C_i$ and $D_{-i}(u_i) \subseteq C_{-i}$ for every player i and every $u_i \in U_i$. For two such subsets $D = (D_i(u_i), D_{-i}(u_i), u_i)_{i \in I, u_i \in U_i}$ and $E = (E_i(u_i), E_{-i}(u_i), u_i)_{i \in I, u_i \in U_i}$ we write that $D \subseteq E$ if $D_i(u_i) \subseteq E_i(u_i)$ and $D_{-i}(u_i) \subseteq E_{-i}(u_i)$ for every player i and $u_i \in U_i$.

Let *gsd* be the reduction operator that assigns to every set $E = (E_i(u_i), E_{-i}(u_i), u_i)_{i \in I, u_i \in U_i}$ the subset $D = (D_i(u_i), D_{-i}(u_i), u_i)_{i \in I, u_i \in U_i}$ where, for every player i and $u_i \in U_i$,

$$D_{-i}(u_i) := \{(c_j)_{j \neq i} \in E_{-i}(u_i) \mid \text{for every } j \neq i, \ c_j \in E_j(u_j) \text{ for some } u_j \in U_j\}$$

and

$$D_i(u_i) := \{c_i \in E_i(u_i) \mid c_i \text{ not strictly dominated in } (E_i(u_i), D_{-i}(u_i), u_i)\}.$$

Then, by construction,

$$gsd^k(A) = (C_i^k(u_i), C_{-i}^k(u_i), u_i)_{i \in I, u_i \in U_i}$$

for every $k \in \{1, 2, 3, ...\}$, and hence the generalized iterated strict dominance procedure can be characterized by the iterated application of the reduction operator *gsd*. We call *gsd* the *generalized strict dominance* operator.

We next show that *gsd* is monotone. Take some sets D, E of the form above with $gsd(E) \subseteq D \subseteq E$. We show that $gsd(D) \subseteq gsd(E)$.

Let $gsd(D) = (D_i'(u_i), D_{-i}'(u_i), u_i)_{i \in I, u_i \in U_i}$ and $gsd(E) = (E_i'(u_i), E_{-i}'(u_i), u_i)_{i \in I, u_i \in U_i}$. Take some player i and utility function u_i. We start by showing that $D_{-i}'(u_i) \subseteq E_{-i}'(u_i)$. Take some $(c_j)_{j \neq i} \in D_{-i}'(u_i)$. Then, for every player j, we have that $c_j \in D_j(u_j)$ for some u_j. Since $D_j(u_j) \subseteq E_j(u_j)$, we conclude that $c_j \in E_j(u_j)$ for some u_j. As this applies to every $j \neq i$, we conclude that $(c_j)_{j \neq i} \in E_{-i}'(u_i)$. Thus, we see that $D_{-i}'(u_i) \subseteq E_{-i}'(u_i)$.

Next, we show that $D_i'(u_i) \subseteq E_i'(u_i)$. Take some $c_i \in D_i'(u_i)$. Then, c_i is not strictly dominated in $(D_i(u_i), D_{-i}'(u_i), u_i)$. By Theorem 2.6.1 it follows that there is some belief $b_i \in \Delta(D_{-i}'(u_i))$ such that

$$u_i(c_i, b_i) \geq u_i(c_i', b_i) \text{ for all } c_i' \in D_i(u_i). \tag{5.6.4}$$

Note that $b_i \in \Delta(E_{-i}'(u_i))$ since we have seen that $D_{-i}'(u_i) \subseteq E_{-i}'(u_i)$. Now, let $c_i^* \in E_i(u_i)$ be such that

$$u_i(c_i^*, b_i) \geq u_i(c_i', b_i) \text{ for all } c_i' \in E_i(u_i). \tag{5.6.5}$$

By Theorem 2.6.1 we conclude that c_i^* is not strictly dominated in $(E_i(u_i), E'_{-i}(u_i), u_i)$, and hence $c_i^* \in E'_i(u_i)$ by definition of the *gsd* operator. Since $gsd(E) \subseteq D$ we know, in particular, that $E'_i(u_i) \subseteq D_i(u_i)$, and thus we see that $c_i^* \in D_i(u_i)$. By combining (5.6.4) and (5.6.5), and using the fact that $c_i^* \in D_i(u_i)$, we conclude that

$$u_i(c_i, b_i) \geq u_i(c_i^*, b_i) \geq u_i(c'_i, b_i) \text{ for all } c'_i \in E_i(u_i).$$

By Theorem 2.6.1 it then follows that c_i is not strictly dominated in $(E_i(u_i), E'_{-i}(u_i), u_i)$, and hence c_i is in $E'_i(u_i)$. This shows that $D'_i(u_i) \subseteq E'_i(u_i)$.

Altogether, we conclude that $gsd(D) \subseteq gsd(E)$. Hence, *gsd* is monotone. By Lemma 3.6.2 it then follows that the reduction operator *gsd* is order independent. As the generalized iterated strict dominance procedure coincides with the iterated application of *gsd*, we conclude that the procedure is order independent. This completes the proof. ■

Proof of Theorem 5.4.3. Follows from the arguments in Section 5.4.5. ■

5.6.2 Proofs of Section 5.5

To prove Theorem 5.5.1 we need the following optimality property, similar to the one from the proof section of Chapter 3.

Lemma 5.6.2 (Optimality property) *For every player i, every utility function $u_i \in U_i$ and every round $k \geq 0$, let $C_i^k(u_i)$ be the set of choices for player i that survive the first k rounds of the generalized iterated strict dominance procedure with fixed beliefs p on utility functions at u_i, and let $C_i^*(u_i)$ be the set of choices that survive all rounds there.*

(a) For every $k \geq 1$, a choice c_i is in $C_i^k(u_i)$ if and only if c_i is optimal in (C_i, C_{-i}, u_i) for some first-order belief on opponents' choices and utility functions that respects p and which only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^{k-1}(u_j)$.

(b) A choice c_i is in $C_i^*(u_i)$ if and only if c_i is optimal in (C_i, C_{-i}, u_i) for some first-order belief on opponents' choices and utility functions that respects p and which only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^*(u_j)$.

Proof. (a) We prove the statement by induction on k. For $k = 1$ the statement is true by construction of the procedure.

Suppose now that $k \geq 2$ and that the statement is true for $k-1$. To show the "only if" direction for k, consider some choice $c_i \in C_i^k(u_i)$. Then, by definition, there is a first-order belief b_i on opponents' choices and utility functions such that (i) b_i respects p, (ii) b_i only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^{k-1}(u_j)$, and

$$u_i(c_i, b_i) \geq u_i(c'_i, b_i) \text{ for all } c'_i \in C_i^{k-1}(u_i). \tag{5.6.6}$$

Let $c_i^* \in C_i$ be optimal for the belief b_i within (C_i, C_{-i}, u_i). That is,

$$u_i(c_i^*, b_i) \geq u_i(c'_i, b_i) \text{ for all } c'_i \in C_i(u_i). \tag{5.6.7}$$

As $C_j^{k-1}(u_j) \subseteq C_j^{k-2}(u_j)$ for all u_j, we conclude that b_i only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^{k-2}(u_j)$. But then, by the induction assumption, $c_i^* \in C_i^{k-1}(u_i)$. By (5.6.6) we thus conclude that

$$u_i(c_i, b_i) \geq u_i(c_i^*, b_i). \tag{5.6.8}$$

By combining (5.6.8) and (5.6.7) we see that

$$u_i(c_i, b_i) \geq u_i(c_i^*, b_i) \geq u_i(c_i', b_i) \text{ for all } c_i' \in C_i,$$

and hence c_i is optimal for the belief b_i in (C_i, C_{-i}, u_i). This establishes the "only if" part.

To show the "if" part, consider some choice c_i that is optimal in (C_i, C_{-i}, u_i) for some first-order belief b_i on opponents' choices and utilities which respects p and only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^{k-1}(u_j)$. Then, in particular, c_i is optimal for this belief in $(C_i^{k-1}(u_i), C_{-i}, u_i)$, and hence $c_i \in C_i^k(u_i)$. This establishes the "if" 'direction.

By combining the "only if" and "if" direction, the statement in (a) follows for k. By induction on k, statement (a) holds for every $k \geq 1$.

(b) Suppose that the procedure terminates at the end of round K. That is, $C_i^*(u_i) = C_i^K(u_i) = C_i^{K+1}(u_i)$ for every player i and utility function u_i. Then, c_i is in $C_i^*(u_i)$ precisely when $c_i \in C_i^{K+1}(u_i)$. By applying (a) to $k = K+1$, we know that c_i is in $C_i^{K+1}(u_i)$ precisely when c_i is optimal for some first-order belief b_i on opponents' choices and utilities which respects p and only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^K(u_j)$. As $C_j^K(u_j) = C_j^*(u_j)$, this completes the proof. ■

Proof of Theorem 5.5.1. **(a)** For every player i and utility function $u_i \in U_i$, let $BR_i^k(u_i)$ denote the set of choices that player i can rationally make while expressing up to k-fold belief in rationality and up to k-fold belief in p with utility function u_i. Recall from above that $C_i^k(u_i)$ denotes the set of choices that survive the first k rounds at u_i. We will show that $BR_i^k(u_i) = C_i^{k+1}(u_i)$ for every player i, every utility function $u_i \in U_i$ and every $k \geq 1$. We show this in two steps: (i) prove that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$ for all $k \geq 1$, and (ii) prove that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$.

(i) Show that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$ for all $k \geq 1$.

We prove this by induction on k. For $k = 1$, take some $c_i \in BR_i^1(u_i)$. Then, there is some epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses 1-fold belief in rationality and 1-fold belief in p, where $v_i(t_i) = u_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j). Since t_i expresses 1-fold belief in rationality, c_j must be optimal for t_j. Hence, c_j is optimal for t_j's first-order belief in the full decision problem $(C_j, C_{-j}, v_j(t_j))$ which, by Lemma 5.6.2, implies that $c_j \in C_j^1(v_j(t_j))$. Thus, t_i's first-order belief $b_i^1(t_i)$ only assigns positive probability to pairs (c_j, u_j) with $c_j \in C_j^1(u_j)$. Moreover, as t_i expresses 1-fold belief in p, the first-order belief $b_i^1(t_i)$ respects p. Finally, as c_i is optimal for t_i, we conclude that c_i is optimal for $b_i^1(t_i)$ in (C_i, C_{-i}, u_i). This implies, by Lemma 5.6.2, that c_i is in $C_i^2(u_i)$. We thus have shown that every choice $c_i \in BR_i^1(u_i)$ must be in $C_i^2(u_i)$, and hence $BR_i^1(u_i) \subseteq C_i^2(u_i)$.

Now suppose that $k \geq 2$ and that, by the induction assumption, $BR_i^{k-1}(u_i) \subseteq C_i^k(u_i)$ for all players i and all utility functions u_i. Consider some player i and some $c_i \in BR_i^k(u_i)$. Then, there is some epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i expresses up to k-fold belief in rationality, t_i expresses up to k-fold belief in p, where $v_i(t_i) = u_i$ and c_i is optimal for t_i. Suppose that $b_i(t_i)$ assigns positive probability to some opponent's choice-type pair (c_j, t_j). Since t_i expresses up to k-fold belief in rationality and up to k-fold belief in p, the choice c_j must be optimal for t_j and t_j must express up to $(k-1)$-fold belief in rationality and up to $(k-1)$-fold belief in p. Hence, $c_j \in BR_j^{k-1}(v_j(t_j))$. Since, by the induction assumption, $BR_j^{k-1}(v_j(t_j)) \subseteq C_j^k(v_j(t_j))$, we know that $c_j \in C_j^k(v_j(t_j))$. We thus conclude that t_i's first-order belief $b_i^1(t_i)$ only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^k(u_j)$. Morever, as t_i expresses 1-fold belief in p, the first-order belief $b_i^1(t_i)$ respects p. Finally, as c_i is optimal for t_i, we conclude that c_i is optimal for t_i's first-order belief

$b_i^1(t_i)$ in (C_i, C_{-i}, u_i). This implies, by Lemma 5.6.2, that c_i is in $C_i^{k+1}(u_i)$. We thus have shown that every choice $c_i \in BR_i^k(u_i)$ must be in $C_i^{k+1}(u_i)$, and hence $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$. By induction on k, we conclude that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$ for all players i, all utility functions $u_i \in U_i$, and all $k \geq 1$. This completes the proof of (i).

(ii) Show that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$.

Hence, for every choice $c_i \in C_i^{k+1}(u_i)$ we must show that there is some epistemic model, and some type $t_i^{u_i,c_i}$ in it, such that $t_i^{u_i,c_i}$ expresses up to k-fold belief in rationality, expresses up to k-fold belief in p, that $v_i(t_i^{u_i,c_i}) = u_i$, and c_i is optimal for $t_i^{u_i,c_i}$. We will now construct a *single* epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ that contains *all* such types. For every player i, define the set of types

$$T_i = \{t_i^{u_i,c_i} \mid u_i \in U_i,\ c_i \in C_i^1(u_i)\}$$

where $v_i(t_i^{u_i,c_i}) = u_i$. To define the beliefs of these types about the opponents' choice-type combinations we distinguish the following three cases, assuming that the procedure terminates at the end of round K.

Case 1. Suppose that $c_i \in C_i^1(u_i) \backslash C_i^2(u_i)$. Then, by Lemma 5.6.2, c_i is optimal for some belief $b_i^{u_i,c_i} \in \Delta(C_{-i})$ within (C_i, C_{-i}, u_i). For every opponent j choose some arbitrary type $\hat{t}_j \in T_j$, and define

$$b_i(t_i^{u_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{u_i,c_i}((c_j)_{j\neq i}), & \text{if } t_j = \hat{t}_j \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{5.6.9}$$

for all $(c_j, t_j)_{j\neq i}$ in $C_{-i} \times T_{-i}$.

Case 2. Suppose that $c_i \in C_i^k(u_i) \backslash C_i^{k+1}(u_i)$ for some $k \in \{2, ..., K-1\}$. Then, by Lemma 5.6.2, c_i is optimal within (C_i, C_{-i}, u_i) for some first-order belief $b_i^{u_i,c_i} \in \Delta(C_{-i} \times U_{-i})$ that respects p, and only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^{k-1}(u_j)$. Define

$$b_i(t_i^{u_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{u_i,c_i}((c_j,u_j)_{j\neq i}), & \text{if } c_j \in C_j^{k-1}(u_j) \text{ and } t_j = t_j^{u_j,c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{5.6.10}$$

for all $(c_j, t_j)_{j\neq i}$ in $C_{-i} \times T_{-i}$.

Case 3. Suppose that $c_i \in C_i^K(u_i)$. As the procedure terminates at round K we have that $c_i \in C_i^*(u_i)$. Hence, by Lemma 5.6.2, c_i is optimal within (C_i, C_{-i}, u_i) for some first-order belief $b_i^{u_i,c_i} \in \Delta(C_{-i} \times U_{-i})$ that respects p, and only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^*(u_j)$. Define

$$b_i(t_i^{u_i,c_i})((c_j,t_j)_{j\neq i}) := \begin{cases} b_i^{u_i,c_i}((c_j,u_j)_{j\neq i}), & \text{if } c_j \in C_j^*(u_j) \text{ and } t_j = t_j^{u_j,c_j} \text{ for all } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{5.6.11}$$

for all $(c_j, t_j)_{j\neq i}$ in $C_{-i} \times T_{-i}$. This completes the construction of the epistemic model $M = (T_i, v_i, b_i)_{i \in I}$.

Note that in this epistemic model every type $t_i^{u_i,c_i}$ holds the first-order belief $b_i^{u_i,c_i}$. As, by definition, c_i is optimal for $b_i^{u_i,c_i}$ within (C_i, C_{-i}, u_i), we conclude that c_i is optimal for $t_i^{u_i,c_i}$, for every player i and every $c_i \in C_i^1(u_i)$.

We now show that for every $k \geq 2$ and every choice $c_i \in C_i^k(u_i)$, the associated type $t_i^{u_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality and up to $(k-1)$fold belief in p. We show this by induction on k.

For $k = 2$, consider some choice $c_i \in C_i^2(u_i)$ and the associated type $t_i^{u_i,c_i}$ with the belief given by (5.6.10) or (5.6.11). By (5.6.10) and (5.6.11), the belief $b_i(t_i^{u_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where $c_j \in C_j^1(u_j)$. As c_j is optimal for $t_j^{u_j,c_j}$, the type $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where c_j is optimal for $t_j^{u_j,c_j}$. Hence, $t_i^{u_i,c_i}$ expresses 1-fold belief in rationality. This holds for every type $t_i^{u_i,c_i}$ where $c_i \in C_i^2(u_i)$. Moreover, as $t_i^{u_i,c_i}$ holds the first-order belief $b_i^{u_i,c_i}$ on opponents' choices and utilities, which respects p, it follows that $t_i^{u_i,c_i}$ expresses 1-fold belief in p.

Suppose now that $k \geq 3$ and that, by the induction assumption, $t_i^{u_i,c_i}$ expresses up to $(k-2)$-fold belief in rationality and up to $(k-2)$-fold belief in p for every $c_i \in C_i^{k-1}(u_i)$ and every player i. Consider some choice $c_i \in C_i^k(u_i)$ and the associated type $t_i^{u_i,c_i}$ with the belief given by (5.6.10) or (5.6.11). By (5.6.10) and (5.6.11) it follows that $b_i(t_i^{u_i,c_i})$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where $c_j \in C_j^{k-1}(u_j)$. By the induction assumption we know that $t_j^{u_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality and up to $(k-2)$-fold belief in p. As c_j is optimal for $t_j^{u_j,c_j}$, we conclude that $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where c_j is optimal for $t_j^{u_j,c_j}$, and $t_j^{u_j,c_j}$ expresses up to $(k-2)$-fold belief in rationality and up to $(k-2)$-fold belief in p. Hence, $t_i^{u_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality and up to $(k-1)$-fold belief in p. This holds for every type $t_i^{u_i,c_i}$ where $c_i \in C_i^k(u_i)$.

By induction on k, we conclude that for every $k \geq 2$ and every choice $c_i \in C_i^k(u_i)$, the associated type $t_i^{u_i,c_i}$ expresses up to $(k-1)$-fold belief in rationality and up to $(k-1)$-fold belief in p.

We next show that for every $c_i \in C_i^K(u_i)$ the associated type $t_i^{u_i,c_i}$ expresses *common* belief in rationality and *common* belief in p. Consider the smaller epistemic model $M^* = (T_i^*, v_i, b_i)_{i \in I}$ where the set of types for player i is

$$T_i^* := \{t_i^{u_i,c_i} \mid u_i \in U_i \text{ and } c_i \in C_i^*(u_i)\},$$

and the beliefs of the types are given by (5.6.11). Note that this is a well-defined epistemic model, since by (5.6.11) every type $t_i^{u_i,c_i} \in T_i^*$ with $c_i \in C_i^*(u_i)$ only assigns positive probability to opponent's types $t_j^{u_j,c_j} \in T_j^*$ where $c_j \in C_j^*(u_j)$. We show that every type in M^* believes in the opponents' rationality.

Consider a type $t_i^{u_i,c_i} \in T_i^*$ where $c_i \in C_i^*(u_i)$. By (5.6.11), type $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where $c_j \in C_j^*(u_j)$. Since c_j is optimal for $t_j^{u_j,c_j}$, the type $t_i^{u_i,c_i}$ only assigns positive probability to opponent's choice-type pairs $(c_j, t_j^{u_j,c_j})$ where c_j is optimal for $t_j^{u_j,c_j}$. Hence, $t_i^{u_i,c_i} \in T_i^*$ believes in the opponents' rationality. Moreover, we have seen that $t_i^{u_i,c_i}$ expresses 1-fold belief in p.

Since this holds for every type $t_i^{u_i,c_i} \in T_i^*$, all types in M^* believe in the opponents' rationality and express 1-fold belief in p. Hence, it follows that all types in M^* express common belief in rationality and common belief in p. Note that the types in M^* are exactly the types $t_i^{u_i,c_i}$ with $c_i \in C_i^K(u_i)$. Hence, for every $c_i \in C_i^K(u_i)$, the associated type $t_i^{u_i,c_i}$ expresses common belief in rationality and common belief in p.

We can now prove that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$. Take some $c_i \in C_i^{k+1}(u_i)$ where $k \geq 1$. Then we know from above that c_i is optimal for the associated type $t_i^{u_i,c_i}$, and that the type $t_i^{u_i,c_i}$ expresses up to k-fold belief in rationality and up to k-fold belief in p. Hence, by definition, $c_i \in BR_i^k(u_i)$. As this holds for every $c_i \in C_i^{k+1}(u_i)$, we conclude that $C_i^{k+1}(u_i) \subseteq BR_i^k(u_i)$ for all $k \geq 1$.

Since in part (i) we have already seen that $BR_i^k(u_i) \subseteq C_i^{k+1}(u_i)$, we may conclude that $BR_i^k(u_i) = C_i^{k+1}(u_i)$ for all $k \geq 1$. That is, a choice can rationally be made while expressing up to k-fold belief in rationality and up to k-fold belief in p with utility function u_i precisely when the choice survives $k+1$ elimination rounds at u_i. This establishes part (a) of Theorem 5.5.1.

(b) We finally prove part (b) of Theorem 5.5.1. Suppose first that choice c_i can rationally be made under common belief in rationality and common belief in p with utility function u_i. Then, in particular, for every $k \geq 1$, the choice c_i can rationally be made while expressing up to k-fold belief in rationality and up to k-fold belief in p with utility function u_i. By part (a) we then know that c_i survives $k+1$ rounds of elimination at u_i. Since this holds for every $k \geq 1$, we conclude that c_i survives all rounds of elimination at u_i.

Suppose next that the choice c_i survives all rounds of elimination at u_i. Then, $c_i \in C_i^K(u_i)$, where K is the round at which the *generalized iterated strict dominance procedure with fixed belief p on utility functions* terminates. From the construction of the epistemic model $M = (T_i, v_i, b_i)_{i \in I}$ above we know that the choice c_i is optimal for the type $t_i^{u_i,c_i}$ and that the type $t_i^{u_i,c_i}$ expresses common belief in rationality and common belief in p. Hence, c_i can rationally be made under common belief in rationality and common belief in p with utility function u_i. We thus conclude that a choice c_i can rationally be made under common belief in rationality and common belief in p with utility function u_i precisely when the choice c_i survives all rounds of elimination at u_i. This completes the proof of part (b), and thereby the proof of this theorem. ■

Proof of Theorem 5.5.2. We first show that the *generalized iterated strict dominance procedure with fixed beliefs p on utility functions* leaves, for every player i and every utility function $u_i \in U_i$, at least one choice in the associated decision problem after the procedure has terminated. To show this, we prove, by induction on k, that $C_i^k(u_i)$ is always non-empty for every $k \in \{1, 2, 3, ...\}$.

Start with $k = 1$. Take a player i, a utility function u_i, and take a first-order belief b_i on opponents' choices and utilities. Select a choice c_i that is optimal for b_i in (C_i, C_{-i}, u_i). Then, by Theorem 2.6.1, c_i is not strictly dominated in (C_i, C_{-i}, u_i), and hence $c_i \in C_i^1(u_i)$. In particular, $C_i^1(u_i)$ is non-empty.

Now, take some $k \geq 2$, and assume that $C_i^{k-1}(u_i)$ is non-empty for every player i and every u_i. Consider a player i and a utility function u_i. Take a first-order belief $b_i \in \Delta(C_{-i} \times U_{-i})$ that respects p and only assigns positive probability to pairs (c_j, u_j) where $c_j \in C_j^{k-1}(u_j)$. Clearly, such a belief can be found since these sets $C_j^{k-1}(u_j)$ are all non-empty. Let c_i be optimal for b_i in (C_i, C_{-i}, u_i). Then, it follows by Lemma 5.6.2 that $c_i \in C_i^k(u_i)$, and hence $C_i^k(u_i)$ is non-empty.

By induction on k it follows that $C_i^k(u_i)$ is non-empty for all k. As the procedure terminates within K rounds, the sets $C_i^K(u_i)$ that remain at the end must all be non-empty.

But then, we can construct an epistemic model M^* as in the proof of Theorem 5.5.1. Since this epistemic model has all the properties stated in Theorem 5.5.2, the proof is complete. ■

Proof of Theorem 5.5.3. Recall again the definitions and results for reduction operators from Sections 3.6.3.1 and 3.6.3.2. We first show that the *generalized iterated strict dominance procedure with fixed beliefs p on utility functions* can be characterized by the iterated application of a *reduction operator gsdp*, and subsequently prove that this reduction operator *gsdp* is *monotone*. By Lemma 3.6.2 it would then follow that *gsdp*, and thereby the procedure, is *order independent*.

Let $A = (C_i(u_i))_{i \in I, u_i \in U_i}$ be the set that assigns to every player i and utility function $u_i \in U_i$ the (full) set of choices $C_i(u_i) = C_i$. The subsets of A we are interested in have the form $D = (D_i(u_i))_{i \in I, u_i \in U_i}$, where $D_i(u_i) \subseteq C_i$ for every player i and every $u_i \in U_i$. For two such subsets $D = (D_i(u_i))_{i \in I, u_i \in U_i}$ and $E = (E_i(u_i))_{i \in I, u_i \in U_i}$ we write that $D \subseteq E$ if $D_i(u_i) \subseteq E_i(u_i)$ for every player i and $u_i \in U_i$.

Let *gsdp* be the reduction operator that assigns to every set $E = (E_i(u_i))_{i \in I, u_i \in U_i}$ the subset $D = (D_i(u_i))_{i \in I, u_i \in U_i}$ where, for every player i and $u_i \in U_i$,

$$D_i(u_i) = \{c_i \in E_i(u_i) \mid c_i \text{ optimal in } (E_i(u_i), C_{-i}, u_i) \text{ for a first-order belief } b_i \in \Delta(C_{-i} \times U_{-i}) \\ \text{that respects } p \text{ and only assigns positive probability to pairs } (c_j, u_j) \text{ where } c_j \in E_j(u_j)\}.$$

Then, by construction,

$$gsdp^k(A) = (C_i^k(u_i))_{i \in I, u_i \in U_i}$$

for every $k \in \{1, 2, 3, ...\}$, and hence the generalized iterated strict dominance procedure with fixed beliefs p on utilities can be characterized by the iterated application of the reduction operator *gsdp*. We call *gsdp* the *generalized strict dominance* operator with *fixed beliefs p on utilities*.

We next show that *gsdp* is monotone. Take some sets D, E of the form above with $gsdp(E) \subseteq D \subseteq E$. We show that $gsdp(D) \subseteq gsdp(E)$.

Let $gsdp(D) = (D_i'(u_i))_{i \in I, u_i \in U_i}$ and $gsdp(E) = (E_i'(u_i))_{i \in I, u_i \in U_i}$. Take some player i and utility function u_i. We show that $D_i'(u_i) \subseteq E_i'(u_i)$. Take some $c_i \in D_i'(u_i)$. Then, c_i is optimal in $(D_i(u_i), C_{-i}, u_i)$ for a first-order belief $b_i \in \Delta(C_{-i} \times U_{-i})$ that respects p and only assigns positive probability to pairs (c_j, u_j) where $c_j \in D_j(u_j)$. That is,

$$u_i(c_i, b_i) \geq u_i(c_i', b_i) \text{ for all } c_i' \in D_i(u_i). \tag{5.6.12}$$

Since $D_j(u_j) \subseteq E_j(u_j)$ for all opponents j and utility functions u_j, we conclude that p only assigns positive probability to pairs (c_j, u_j) where $c_j \in E_j(u_j)$. Now, let $c_i^* \in E_i(u_i)$ be such that

$$u_i(c_i^*, b_i) \geq u_i(c_i', b_i) \text{ for all } c_i' \in E_i(u_i). \tag{5.6.13}$$

Then, by definition of the *gsdp* operator, we have that $c_i^* \in E_i'(u_i)$. Since $gsdp(E) \subseteq D$ we know, in particular, that $E_i'(u_i) \subseteq D_i(u_i)$, and thus we see that $c_i^* \in D_i(u_i)$. By combining (5.6.12) and (5.6.13), and using the fact that $c_i^* \in D_i(u_i)$, we conclude that

$$u_i(c_i, b_i) \geq u_i(c_i^*, b_i) \geq u_i(c_i', b_i) \text{ for all } c_i' \in E_i(u_i).$$

Hence, it follows that c_i is in $E_i'(u_i)$. This shows that $D_i'(u_i) \subseteq E_i'(u_i)$.

Altogether, we conclude that $gsdp(D) \subseteq gsdp(E)$. Hence, *gsdp* is monotone. By Lemma 3.6.2 it then follows that the reduction operator *gsdp* is order independent. As the generalized iterated strict dominance procedure with fixed beliefs p on utilities coincides with the iterated application of *gsdp*, we conclude that the procedure is order independent. This completes the proof. ■

Solutions to In-Chapter Questions

Question 5.2.1. If Barbara has the utility function u_2^r and believes that you choose *blue,* then the optimal choice for Barbara is *red* and not *yellow.* Therefore, the arrow must be dashed.

Question 5.2.2. (a) The third-order belief is that you believe that Barbara believes that you believe that Barbara chooses *blue* while having the utility function u_2^b.

(b) The second-order belief is that you assign probability 0.6 to the event that Barbara believes that you wear *green* while having the utility function u_1, and that you assign probability 0.4 to the event that Barbara believes that you wear *blue* while having the utility function u_1. The third-order belief is that you assign probability 0.6 to the event that Barbara believes that you believe that Barbara wears *blue* while having the utility function u_2^b, and that you assign probability 0.4 to the event that Barbara believes that you believe that Barbara wear *red* while having the utility function u_2^r.

Question 5.2.3. In your first-order belief, you believe that Barbara wears *yellow* while having the utility function u_2^b. In your second-order belief, you believe that Barbara believes that you wear *blue* while having the utility function u_1. In your third-order belief, you believe that Barbara believes that you believe that Barbara wears *red* while having the utility function u_2^r.

Question 5.3.1. For Barbara, *blue* is optimal for type t_2^{blue}, *red* is optimal for type t_2^{green}, *red* is optimal for type t_2^{red}, and *yellow* is optimal for type t_2^{yellow}.

Question 5.3.2. For you, your type t_1^{red} does not believe in Barbara's rationality, since it assigns positive probability to Barbara's choice-type pair (*green,* t_2^{green}), where *green* is not optimal for t_2^{green}. Each of your other types believes in Barbara's rationality. For Barbara, every type believes in your rationality.

Question 5.3.3. Recall from the previous question that all types in the epistemic model believe in the opponent's rationality, except type t_1^{red} for you. Moreover, all these types express k-fold belief in rationality for every k, and thus express common belief in rationality.

Question 5.4.1. Your set of types is

$$T_1 = \{t_1^{20,30}, t_1^{40,30}, t_1^{60,30}, t_1^{20,50}, t_1^{40,50}, t_1^{60,50}, t_1^{20,70}, t_1^{40,70}, t_1^{60,70}, t_1^{40,90}, t_1^{60,90}, t_1^{80,90}\}$$

and similarly for Barbara. The utility functions and beliefs of these types are

$$\begin{gathered}
v_1(t_1^{20,30}) = v_1(t_1^{40,30}) = v_1(t_1^{60,30}) = u_1^{30}, \\
v_1(t_1^{20,50}) = v_1(t_1^{40,50}) = v_1(t_1^{60,50}) = u_1^{50}, \\
v_1(t_1^{20,70}) = v_1(t_1^{40,70}) = v_1(t_1^{60,70}) = u_1^{70} \text{ and} \\
v_1(t_1^{40,90}) = v_1(t_1^{60,90}) = v_1(t_1^{80,90}) = u_1^{90}, \\
b_1(t_1^{20,30}) = ... = b_1(t_1^{60,70}) = (80, t_2^{80,90}) \\
b_1(t_1^{40,90}) = (20, t_2^{20,30}),\ b_1(t_1^{60,90}) = (40, t_2^{40,30}) \text{ and } b_1(t_1^{80,90}) = (80, t_2^{80,90}),
\end{gathered}$$

and similarly for Barbara.

Question 5.4.2. Based on the final decision problems in Table 5.4.4 we can make the beliefs diagram in Figure 5.6.1. This beliefs diagram can be translated into the epistemic model where

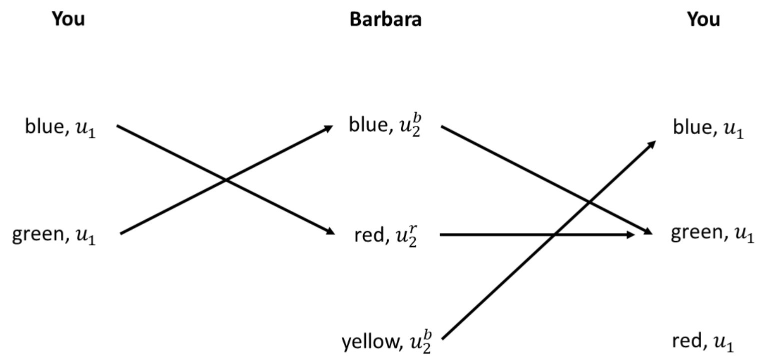

Figure 5.6.1 Beliefs diagram for Question 5.4.2

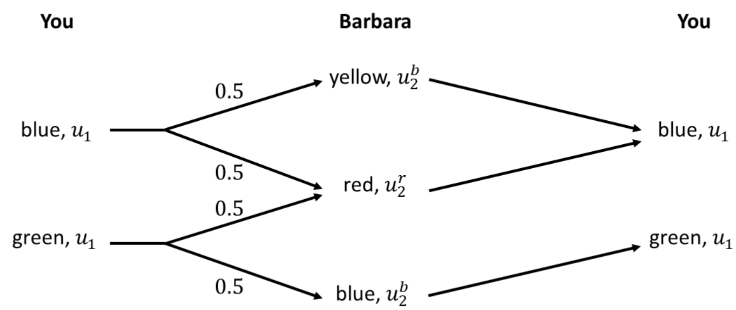

Figure 5.6.2 Beliefs diagram for Question 5.5.1

$T_1 = \{t_1^{blue}, t_1^{green}\}$ and $T_2 = \{t_2^{blue}, t_2^{red}, t_2^{yellow}\}$, and where the utility functions and beliefs are given by

$$v_1(t_1^{blue}) = v_1(t_1^{green}) = u_1,$$
$$b_1(t_1^{blue}) = (red,\ t_2^{red}),\ b_1(t_1^{green}) = (blue,\ t_2^{blue}),$$
$$v_2(t_2^{blue}) = u_2^b,\ v_2(t_2^{red}) = u_2^r,\ v_2(t_2^{yellow}) = u_2^b,$$
$$b_2(t_2^{blue}) = (green,\ t_1^{green}),\ b_2(t_2^{red}) = (green,\ t_1^{green}) \text{ and } b_2(t_2^{yellow}) = (blue,\ t_1^{blue}).$$

Question 5.5.1. Consider the beliefs diagram in Figure 5.6.2. It may be verified that your two belief hierarchies starting at (*blue,* u_1) and (*green,* u_1) both express common belief in rationality, and support the choices *blue* and *green* respectively. Moreover, in both belief hierarchies the only belief about Barbara's utilities that enters is $p = (0.5) \cdot u_2^r + (0.5) \cdot u_2^b$.

Question 5.5.2. In round 1, we can eliminate your choice *yellow,* Barbara's choice *green* at u_2^r and Barbara's choice *red* at u_2^b.

Round 2. In your decision problem, no choice can be eliminated. In particular, your choice *red* is optimal for the first-order belief $b_1^1 = (0.5) \cdot (blue,\ u_2^r) + (0.1) \cdot (blue,\ u_2^b) + (0.4) \cdot (green,\ u_2^b)$ which respects p, and which only assigns positive probability to pairs (c_2, u_2) where c_2 survives round 1 at u_2.

Barbara must assign probability zero to your choice *yellow,* and hence we can eliminate Barbara's choice *blue* at u_2^r and Barbara's choice *green* at u_2^b.

Round 3. You must assign probability zero to Barbara choosing *green,* since Barbara's choice *green* did not survive round 2 at any of Barbara's decision problems. But then, we can eliminate your choice *red.*

Round 4. Barbara must assign probability zero to your choices *yellow* and *red.* But then, *yellow* can be eliminated for Barbara at u_2^r.

Then, the procedure terminates. In particular, your choice *green* is optimal for the first-order belief $b_1^1 = (0.5) \cdot (red,\ u_2^r) + (0.5) \cdot (blue,\ u_2^b)$ which respects p, and only assigns positive probability to pairs (c_2, u_2) where c_2 survives round 4 at u_2. Similarly, your choice *blue* is optimal for the first-order belief $b_1^1 = (0.5) \cdot (red,\ u_2^r) + (0.5) \cdot (yellow,\ u_2^b)$ which respects p, and only assigns positive probability to pairs (c_2, u_2) where c_2 survives round 4 at u_2. Thus, under common belief in rationality and common belief in p, you can rationally wear *blue* and *green,* Barbara can only rationally wear *red* if her utility function is u_2^r, and Barbara can rationally wear *blue* or *yellow* if her utility function is u_2^b.

Question 5.5.3. The decision problems at the various valuations are as follows:

You	20	40	60	80	100
20	5	0	0	0	0
40	10	−5	0	0	0
60	10	−10	−15	0	0
80	10	−10	−30	−25	0
100	10	−10	−30	−50	−35

u_1^{30}

You	20	40	60	80	100
20	15	0	0	0	0
40	30	5	0	0	0
60	30	10	−5	0	0
80	30	10	−10	−15	0
100	30	10	−10	−30	−25

u_1^{50}

You	20	40	60	80	100
20	25	0	0	0	0
40	50	15	0	0	0
60	50	30	5	0	0
80	50	30	10	−5	0
100	50	30	10	−10	−15

u_1^{70}

You	20	40	60	80	100
20	35	0	0	0	0
40	70	25	0	0	0
60	70	50	15	0	0
80	70	50	30	5	0
100	70	50	30	10	−5

u_1^{90}

For every valuation, the bid 20 is optimal if you believe that Barbara bids 100, whereas the bids $40, 60, 80$ and 100 are all optimal if you believe that Barbara bids 20. Hence, for every profile of beliefs p over utility functions, the procedure *generalized iterated strict dominance with fixed beliefs p on utility functions* does not eliminate any choice at any valuation. Therefore, for every valuation, all bids can rationally be made under common belief in rationality and common belief in p.

At the same time, at valuation 30, the bids $60, 80$ and 100 are weakly dominated by 40. At valuation 50, the bid 20 is weakly dominated by 40, whereas the bids 80 and 100 are weakly dominated by 60. At valuation 70, the bids 20 and 40 are weakly dominated by 60, whereas bid 100 is weakly dominated by 80. Finally, at valuation 90, the bids $20, 40$ and 60 are weakly dominated by 80. Hence, for every valuation w_i, the only bids that are not weakly dominated are $w_i - 10$ and $w_i + 10$.

Problems

Problem 5.1: Fixing the boat.

During many years, Barbara and you have been sailing the lakes with a lovely boat called *Odysseus.* But lately you are both just too busy with other things, and you are planning to sell the boat. However, before doing so the boat must be fixed. Today you both meet to make a schedule for working on the boat. As a first step, you must both decide how many weeks of work you are willing to put in. Suppose both you and Barbara can choose between $1, 2, 3, 4$ or 5 weeks. If you work for w_1 weeks and Barbara works for w_2 weeks, then it is expected that the boat can be sold at $2000 \cdot w_1 \cdot w_2$ euros. That is, your contribution will be more effective if Barbara puts in more days of work, and *vice versa.*

If the boat is sold at V euros, and you have put in w_1 weeks of work, then you expect to receive half of the income from selling the boat, but you will incur some mental and psysical costs because of working on the boat for w_1 weeks. More precisely, your utility will be $\frac{1}{2}V - d_1 \cdot w_1^2$, where d_1 reflects the degree by which you dislike working on the boat. We refer to d_1 as the disutility degree. Similarly for Barbara, whose disutility degree is d_2. The problem, however, is that you do not know Barbara's disutility degree, and similarly for Barbara. To keep things tractable, assume that the possible disutility degrees d_1 and d_2 for you and Barbara are either 120 or 250.

(a) Formulate this situation as a game with incomplete information, by specifying for every player, and every possible utility function, the associated decision problem.

(b) Under common belief in rationality, which numbers of weeks can you rationally offer to work for both of your possible disutility degrees? Which procedure do you use?

Suppose now that the possible disutility degrees for you and Barbara are 120 and 1200, instead of 120 and 250.

(c) Under common belief in rationality, which numbers of weeks can you rationally offer to work for both of your possible disutility degrees?

(d) Make a beliefs diagram with solid arrows only that uses, for every utility function, precisely those choices you found in (c).

(e) Translate this beliefs diagram into an epistemic model where every type expresses common belief in rationality.

Consider the profile $p = (p_1, p_2)$ of beliefs on utility functions, where p_1 assigns probability 0.5 to you having the disutility degrees 120 and 1200, respectively, and similarly for p_2.

***(f)** Under common belief in rationality and common belief in p, which numbers of weeks can you rationally choose for both of your possible disutility degrees?

***(g)** Create an epistemic model that contains, for every player i, every utility function u_i, and every choice c_i you found for that utility function in (f), a type $t_i^{u_i,c_i}$ that (i) has this utility function u_i, (ii) expresses common belief in rationality and common belief in p, and (iii) for which c_i is optimal.

Problem 5.2: Dividing the revenue.

Recall the story from Problem 5.1. Since Barbara and you both dedicated five weeks to fixing the boat, you were able to sell it at a price of $50,000$ euros. The original idea was to divide this amount evenly between Barbara and you, but Chris suggested the following procedure to decide how much money each person gets: Barbara and you must independently write down a price between 10, 000 and 40, 000 on a piece of paper, put it in a sealed envelope, and give it to Chris. To make matters easy, Chris decides that you can only write down multiples of 10, 000 euros. If the sum of the two amounts is equal to 50, 000 then each person gets exactly the amount he or she wrote down. If the sum if less than 50, 000, then each person gets the amount he or she wrote down, plus one half of the amount that is left. However, if the sum is more than 50, 000, then Chris will keep all the money, and you will both go home, disappointed and empty handed.

Since you and Barbara care about each other, you would feel guilty if you receive more than Barbara, and similarly for Barbara. More precisely, if the amount a_1 you receive is bigger than the amount a_2 that Barbara receives, then your utility would be $a_1 - g_1 \cdot (a_1 - a_2)^2$, where g_1 is your degree of guilt aversion. Here, we assume that a_1 and a_2 are expressed in thousands of euros. That is, your guilt increases if the difference between your amount and Barbara's amount increases. If a_1 is less than, or equal to, a_2 then your utility will simply be a_1. Similarly for Barbara.

The problem is that you and Barbara do not know the degree of guilt aversion of the other person. To keep things tractable, suppose that the degree of guilt aversion for you and Barbara is either 0.04 or 0.1.

(a) Model this situation as a game with incomplete information by specifying, for both players and each of the possible utility functions, the associated decision problem. Remember that the utilities are based on expressing the monetary amounts in thousands of euros. Hence, if you choose 20, 000 euros and Barbara chooses 30, 000 euros, your utility is 20. Moreover, if you choose 30, 000 euros and Barbara chooses 20, 000 euros, then your utility is $30 - g_1 \cdot (30 - 20)^2$.

(b) For each of your possible degrees of guilt aversion, find the amounts you can rationally write down under common belief in rationality.

(c) Find a beliefs diagram with solid arrows only, that uses for every player and every utility function solely those choices that survived the procedure in (b).

(d) Translate this beliefs diagram into an epistemic model where all types express common belief in rationality.

Consider the profile $p = (p_1, p_2)$ of beliefs on utility functions, where p_1 assigns probabilities 0.7 and 0.3 to you having degrees of guilt aversion 0.04 and 0.1, respectively. Similarly for p_2.

***(e)** For both players, and both utility functions, find the amounts that this person can rationally write down under common belief in rationality and common belief in p.

***(f)** Create an epistemic model that contains, for every player i, every utility function u_i, and every choice c_i you found for that utility function in (e), a type $t_i^{u_i,c_i}$ that (i) has this utility function u_i, (ii) expresses common belief in rationality and common belief in p, and (iii) for which c_i is optimal.

Problem 5.3: Celebrating the sale.

Recall the stories from Problems 5.1 and 5.2. The boat has been sold at 50, 000 euros, but since Barbara and you both wrote down 30, 000 when trying to divide the revenue, Chris went home with the full revenue whereas Barbara and you were left behind empty handed and disappointed.

Now, Chris wants to celebrate the sale with Barbara and you, either by spending a long weekend in Madrid, or by having a short drink at the local pub. Of course, Chris will pay for everything. To decide what to do, Chris proposes the following procedure: You, Barbara and Chris must independently write down one of these two activities on a piece of paper. Moreover, as Chris fears that you and Barbara

You and Barbara	*Madrid*	*drink*	*home*
angry	2	6	4
not angry	6	4	2

Chris	*Madrid*	*drink*	*home*
	6	4	2

Table 5.6.1 Utilities in Problem 5.3

may still be angry at him for not sharing in the revenue, you and Barbara also have the option to write down "stay at home". Chris, on the other hand, does not have this option.

If you write down "stay at home", then you will not go anywhere, no matter how the others vote. If you vote for one of the two activities, then you will participate in the activity that receives at least two votes, if such activity exists. Note that this may not be the activity you voted for, if Barbara and Chris both vote for the other activity. If no activity received at least two votes, because Barbara voted for staying at home and Chris voted for the other activity, then you will stay at home. For Barbara, things are similar. Chris will participate in the activity that receives at least two votes, if it exists. If such activity does not exist, then he will stay at home. As a consequence, Chris will always participate if an activity takes place. However, if both you and Barbara voted for staying at home, then Chris will be very angry and will go to the pub alone in case he voted for the drink. If he voted for Madrid in this case, he will stay at home at have some drinks to ease his anger.

The problem is that you and Chris do not know whether Barbara is still angry at Chris, and Barbara and Chris do not know whether you are still angry at Chris. Moreover, the conditional preference relation for you and Barbara depend on whether you are angry or not. If you are angry at Chris, then you would rather stay at home than spending a full weekend with Chris in Madrid, but you would still prefer to have a short drink rather than staying at home, because you like the pub and you know it will not last long. If you are not angry at Chris, then you would definitely prefer a weekend in Madrid to having a short drink, which would still be better than staying at home.

More precisely, Table 5.6.1 depicts the utilities that you and Barbara derive from the three possible outcomes in case you are angry and in case you are not. The last row in the table depicts the utilities for Chris. However, if you vote for an activity, and Barbara and Chris vote for the other activity, then you will participate in an activity you did not vote for. In such a case, you will be disappointed and the utility in the matrix above will be reduced by 1. Hence, if you are not angry, vote for Madrid, and Barbara and Chris both vote for a drink, then your will go for a drink, but your utility will be $4-1=3$. The same applies to Barbara and Chris.

(a) Model this situation as a game with incomplete information and three players. That is, for every player, and each of the possible utility functions, display the associated decision problem.

(b) Under common belief in rationality, what options can you rationally vote for if you are angry and if you are not? What about Barbara and Chris?

(c) Design an epistemic model such that, for every player i, every utility function u_i for player i, and every choice c_i that survived the procedure in (b) at u_i, there is a type $t_i^{u_i,c_i}$ such that (i) $t_i^{u_i,c_i}$ has utility function u_i, (ii) $t_i^{c_i,u_i}$ expresses common belief in rationality, and (iii) c_i is optimal for $t_i^{u_i,c_i}$.

Let $p=(p_1,p_2,p_3)$ be a profile of beliefs on utility functions, where p_1 assigns probabilities 0.9 and 0.1 to you being angry at Chris and you not being angry, respectively, and similary for p_2. Clearly, p_3 assigns probability 1 to the unique utility function for Chris. This describes a scenario in which Chris believes that, with high probability, you and Barbara are angry at him.

*(d) Under common belief in rationality and common belief in p, what options can you rationally vote for if you are angry and if you are not? What about Barbara and Chris?

*(e) Design an epistemic model such that, for every player i, every utility function u_i for player i, and every choice c_i that survived the procedure in (d) at u_i, there is a type $t_i^{u_i,c_i}$ such that (i) $t_i^{u_i,c_i}$ has utility function u_i, (ii) $t_i^{c_i,u_i}$ expresses common belief in rationality and common belief in p, and (iii) c_i is optimal for $t_i^{u_i,c_i}$.

Literature

Games with incomplete information. To the best of my knowledge, Harsanyi (1962) is the first paper that systematically studies incomplete information in games. It does so in the context of bargaining between two parties, where both parties have uncertainty about the best terms that the other party is willing to accept. Since these terms characterize the party's utility function, both parties face uncertainty about the opponent's utility function.

In that context, it was important for Harsanyi to reason about the *belief hierarchies* that both parties have about the utility functions, specifying a first-order belief about the other party's utility function, a second-order belief about the belief that the other party has about his own utility function, and so on. This leads to infinite strings of beliefs, which are difficult to work with. Hence, how can such infinite belief hierarchies be encoded in an easy way?

This question led Harsanyi to his trilogy of papers Harsanyi (1967–1968), which offers a general model for games with incomplete information, and shows how infinite belief hierarchies about choices and utility functions can be encoded in a finite way by means of *types.* The main difference between the types used by Harsanyi and the types we use is that Harsanyi types also specify a randomization over choices, whereas our types do not. More precisely, every Harsanyi type specifies a utility function, a randomization over choices and a probabilistic belief about the opponents' Harsanyi types. For every Harsanyi type we can then derive an infinite belief hierarchy about the players' choices and utility functions, similarly to how types in our framework induce such belief hierarchies. But our model with types is really based on Harsanyi's seminal work.

Common belief in rationality. In this chapter, the central idea of *common belief in rationality* has been extended from standard games to games with incomplete information. Formal definitions of common belief in rationality for games with incomplete information can be found, for instance, in Battigalli and Siniscalchi (1999, 2002, 2007), Battigalli, Di Tillio, Grillo and Penta (2011), Battigalli and Prestipino (2013) and Bach and Perea (2021).

Recursive procedures. The *generalized iterated strict dominance* procedure is taken from Bach and Perea (2021). This procedure may be viewed as an extension of the iterated elimination of strictly dominated choices to games with incomplete information. In particular, at every round it explores whether a given choice is strictly dominated in a reduced decision problem. There are other, yet similar, elimination procedures for games with incomplete information which do not use strict dominance, but also characterize the choices that can rationally be made under common belief in rationality. One such procedure is Δ-rationalizability, which has been developed by Battigalli (2003), and which is also discussed in Battigalli and Siniscalchi (2003a, 2007), Battigalli, Di Tillio, Grillo and Penta (2011), Battigalli and Prestipino (2013) and Dekel and Siniscalchi (2015). This procedure starts by imposing some exogenous restrictions on the players' first-order beliefs, and then recursively eliminates in every round choice-utility pairs for the players based on optimality criteria. If no exogenous restrictions are imposed, this procedure is equivalent, in terms of output, to the generalized iterated strict dominance procedure. Cappelletti (2010) and Battigalli, Di Tillio, Grillo and Penta (2011) show how Δ-rationalizability can be characterized by procedures that rely on strict dominance arguments. A procedure similar to the latter two has been developed by Bergemann and Morris (2003) in the context of mechanism design.

Fixed beliefs on utilities. Towards the end of the chapter we have combined common belief in rationality with the idea of fixed beliefs on utilities. That is, we have studied scenarios where you believe that the players' beliefs about the opponents' utilities are transparent to everyone. This is closely related to the concepts of *interim (independent) rationalizability* (Ely and Peşki (2006)) and *interim correlated rationalizability* (Dekel, Fudenberg and Morris (2007)). The main difference is that the latter two papers fix an *arbitrary* belief hierarchy on utilities for every player, whereas we consider a collection of *simple* belief hierarchies on utilities for the players that are derived from the *same* collection of beliefs $p = (p_i)_{i \in I}$ on utilities. Indeed, by imposing p, your belief about the opponents' utilities is given by p_{-i}, you believe that every opponent j holds the belief p_{-j} about the other players' utilities, and so on. That is, your belief hierarchy on utilities is the *simple* belief hierarchy generated by p. The two papers mentioned above do not insist on simple belief hierarchies about utilities, but fix arbitrary, possibly non-simple, belief hierarchies for the players. They then formulate iterated elimination procedures for these fixed belief hierarchies on utilities, similar to our *generalized iterated strict dominance procedure with fixed beliefs on utilities.* This similarity is confirmed by Battigalli, Di Tillio, Grillo and Penta (2011) who show that the concept of interim correlated rationalizability yields, for every player and every utility function, precisely those choices that can rationally be made under common belief in rationality with the fixed belief hierarchy on utilities. This closely resembles the epistemic conditions in Theorem 5.5.1 that characterize our *generalized iterated strict dominance procedure with fixed beliefs on utilities.*

Common belief in rationality in auctions. The example "Chris' drawings" from Section 5.4.4 is an example of a *first-price auction.* Such auctions play a very important role in economic theory and economic design. However, most theoretical papers on auctions use the concept of *Bayesian equilibrium* (see Chapter 6) rather than *common belief in rationality* to study the behavior of the bidders in such auctions, and to see what the expected revenue for the auctioneer will be. Some exceptions are the papers by Battigalli and Sinicalchi (2003b), Dekel and Wolinsky (2003), Robles and Shimoji (2012) and Shimoji (2017). Battigalli and Siniscalchi (2003b) use the concept of *interim correlated rationalizability* discussed above, whereas Dekel and Wolinsky (2003) and Robles and Shimoji (2012) use common belief in rationality with some exogenous restrictions on the beliefs that players have about the opponents' valuations. Shimoji (2017) is different, as it uses the iterated elimination of *weakly* dominated choices.