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# Lexicographic Beliefs Part III: Assumption of Rationality

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## Introduction

- Two ways of cautious reasoning have been presented so far:
  - Common Primary Belief in (Caution & Rationality)
  - Common Full Belief in (Caution & Respect of Preferences)
- Respect of preferences imposes restrictions not only on the primary but also on deeper lexicographic levels!
- However, there are other reasonable conditions that could be put on the various lexicographic levels.



Assumption of Rationality

Common Assumption of Rationality

Algorithm



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#### Assumption of Rationality

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# Example: Spy Game

#### Story

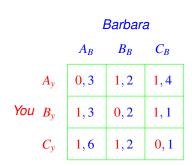
- *You* would like to go to a pub to read your book.
- Barbara is going to a pub as well, but you forgot to ask her to which one.
- The only objective for you is to avoid Barbara, since you would like to read your book in silence.
- Barbara prefers Pub A to Pub B, and Pub B to Pub C.
- Besides, Barbara suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- Barbara derives additional utility of 3 from spying.
- Question: Which pub should you go to?

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#### Example: Spy Game



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# Example: Spy Game

			Barbara		
		$A_B$	BB	$C_B$	
	$A_y$	0, 3	1,2	1,4	
You	$B_y$	1,3	0, 2	1,1	
	$C_y$	1,6	1, 2	0, 1	

Under common full belief in (caution & respect of preferences), you go to Pub C:

- As Barbara prefers A<sub>B</sub> to B<sub>B</sub> and you respect her preferences, you must deem her choice A<sub>B</sub> infinitely more likely than B<sub>B</sub>.
- Then, you prefer B<sub>y</sub> to A<sub>y</sub>.
- Hence, you believe that Barbara deems your choice  $B_{y}$  infinitely more likely than  $A_{y}$ .
- Consequently, you believe that Barbara prefers *B<sub>B</sub>* to *C<sub>B</sub>*, and you must deem *B<sub>B</sub>* infinitely more likely than *C<sub>B</sub>*.
- But then the unique optimal choice for you is C<sub>y</sub>.
- However, this is not the only plausible way to reason about Barbara!

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## Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
	$A_y$	0,3	1, 2	1,4
You	$B_y$	1,3	0, 2	1,1
	$C_y$	1,6	1, 2	0,1

An alternative way of reasoning:

- For Barbara, both A<sub>B</sub> and C<sub>B</sub> can be optimal for some cautious lexicographic belief, but B<sub>B</sub> can never be optimal.
- Therefore, you deem Barbara's choice  $A_B$  and  $C_B$  infinitely more likely than  $B_B$ .
- But then, your unique optimal choice is By!

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# **The Underlying Intuition**

- If player j's choice c<sub>j</sub> is optimal for some cautious lexicographic belief, while his choice c'<sub>j</sub> is not optimal for any cautious lexicographic belief, then player i must deem c<sub>j</sub> infinitely more likely than c'<sub>j</sub>.
- Player i is then said to assume rationality.
- In other words, player *i* deems his opponent *j*'s good choices infinitely more likely than *j*'s bad choices.

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### How Can this Intuition Be Formalized?

- How can the idea of assuming rationality be formalized in an epistemic model?
- Attempt: Type *t<sub>i</sub>* must deem all choice-type pairs (*c<sub>j</sub>*, *t<sub>j</sub>*), where *c<sub>j</sub>* is optimal for *t<sub>j</sub>* and *t<sub>j</sub>* is cautious, infinitely more likely than all choice-type pairs (*c'<sub>i</sub>*, *t'<sub>j</sub>*) that do not have this property.

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# The Attempt Does Not Work!

			Barbara		
		$A_B$	BB	$C_B$	
	$A_y$	0, 3	1, 2	1,4	
You	$B_y$	1,3	0, 2	1,1	
	$C_y$	1,6	1, 2	0,1	

- **Attempt:** Type  $t_i$  must deem all choice-type pairs  $(c_j, t_j)$ , where  $c_j$  is optimal for  $t_j$  and  $t_j$  is cautious, infinitely more likely than all choice-type pairs  $(c'_i, t'_j)$  that do not have this property.
- Consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B\}$ Beliefs:  $b_y(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$  and  $b_B(t_B) = ((C_y, t_y); (B_y, t_y); (A_y, t_y))$ 

- Your type ty satisfies the condition, but does not assume rationality in the intended way.
- Problem: Choice C<sub>B</sub> can be optimal for Barbara for some cautious type, but your type t<sub>y</sub> does not deem possible any type for Barbara for which C<sub>B</sub> is indeed optimal.
- Remedy: it is additionally required that you must deem possible a cautious type for Barbara for which C<sub>B</sub> is optimal!

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#### More Types Are Needed

		Barbara		
		$A_B$	BB	$C_B$
	$A_y$	0,3	1,2	1,4
You	$B_y$	1,3	0, 2	1,1
	$C_y$	1,6	1, 2	0,1

- Consider the following lexicographic epistemic model: Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B, t'_B\}$ Beliefs:  $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (A_B, t'_B); (B_B, t'_B)),$  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y)),$  and  $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$
- For Barbara choices A<sub>B</sub> and C<sub>B</sub> can be optimal for some cautious type.
- Your type t<sub>y</sub> deems possible the cautious type t<sub>B</sub> for which A<sub>B</sub> is optimal as well as the cautious type t'<sub>B</sub> for which C<sub>B</sub> is optimal.
- Your type ty deems all choice-type pairs where the type is cautious and the choice is optimal for the type infinitely more likely than all choice-type pairs that do not have this property.
- Indeed, type t<sub>y</sub> assumes rationality in the intended way!

# **Assumption of Rationality**

#### Definition

A cautious type t<sub>i</sub> assumes rationality, whenever

- for every choice c<sub>j</sub> that is optimal for some cautious type, t<sub>i</sub> deems possible a cautious type t<sub>j</sub> for which c<sub>j</sub> is indeed optimal,
- t<sub>i</sub> deems all choice-type pairs (c<sub>j</sub>, t<sub>j</sub>), where t<sub>j</sub> is cautious and c<sub>j</sub> optimal for t<sub>j</sub>, infinitely more likely than all choice-type pairs not satisfying this property.

#### Intuition:

A player deems good choices infinitely more likely than bad choices.

#### **Remark:**

Assumption of rationality can only be defined for cautious types.

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## Assumption and Primary Belief in Rationality

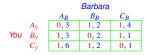
**Observation.** If *Alice* is cautious and assumes *Bob*'s rationality, then she also primarily believes in *Bob*'s rationality.

- Suppose that *t*<sub>Alice</sub> is cautious and assumes *Bob*'s rationality.
- Then, t<sub>Alice</sub> considers all choice-type pairs where the choice is optimal for the type infinitely more likely than other choice-type pairs.
- In particular, the support of t<sub>Alice</sub>'s primary belief can then only contain choice-type pairs such that the choice is optimal for the type.

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#### Assumption and Respect of Preferences

Observation. There is no general relationship between assuming rationality and respecting preferences.



- Consider the following lexicographic epistemic model: Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B\}$ Beliefs:  $b_y(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$  and  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$
- Your type ty respects Barbara's preferences, but does not assume her rationality.

- Consider the following lexicographic epistemic model: Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B, t'_B\}$ Beliefs:  $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (A_B, t'_B); (B_B, t'_B)),$  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y)),$  and  $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$
- Your type ty assumes Barbara's rationality, but does not respect her preferences.
- Indeed, for  $t_B$  choice  $B_B$  is better than  $C_B$ , yet  $t_y$  deems  $(C_B, t_B)$  infinitely more likely than  $(B_B, t_B)$ .

#### Remark

It is always possible to satisfy respect of preferences and assumption of rationality.

Intuition: A type's lexicographic belief deems optimal choices infinitely more likely than the non-optimal choices, yet orders the non-optimal choices as required by respect of preference.



Assumption of Rationality

#### Common Assumption of Rationality

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# Assuming (Rationality & Assumption of Rationality)

#### Definition

A cautious type *t<sub>i</sub>* **assumes** (*rationality & assumption of rationality*), whenever

- for every choice c<sub>j</sub> that is optimal for some cautious type that assumes i's rationality, type t<sub>i</sub> deems possible a cautious type t<sub>j</sub> that assumes i's rationality and for which c<sub>j</sub> is indeed optimal;
- type t<sub>i</sub> deems all choice-type pairs (c<sub>j</sub>, t<sub>j</sub>), where t<sub>j</sub> is cautious, assumes i's rationality, and c<sub>j</sub> is optimal for t<sub>j</sub>, infinitely more likely than all choice-type pairs not satisfying this property.

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#### **Common Assumption of Rationality**

#### Definition

- A cautious type t<sub>i</sub> expresses 1-fold assumption of rationality, whenever t<sub>i</sub> assumes rationality.
- For all  $k \ge 2$ , a cautious type  $t_i$  expresses k-fold assumption of rationality, whenever
  - for every choice c<sub>i</sub> that is optimal for some cautious type that expresses up to (k 1)-fold assumption of rationality, type t<sub>i</sub> deems possible a cautious type t<sub>j</sub> that expresses up to (k 1)-fold assumption of rationality and for which c<sub>i</sub> is indeed optimal;
  - type  $t_i$  deems all choice-type pairs  $(c_i, t_j)$  where  $t_j$  is cautious, expresses up to (k 1)-fold assumption of rationality, and  $c_j$  is optimal for  $t_j$ , infinitely more likely than all choice-type pairs not satisfying this property.
- A cautious type t<sub>i</sub> expresses common assumption of rationality, whenever t<sub>i</sub> expresses k-fold assumption of rationality for all k ≥ 1.

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# Example: Spy Game

#### Story

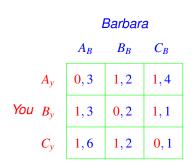
- *You* would like to go to a pub to read your book.
- Barbara is going to a pub as well, but you forgot to ask her to which one.
- The only objective for you is to avoid Barbara, since you would like to read your book in silence.
- Barbara prefers Pub A to Pub B, and Pub B to Pub C.
- Besides, Barbara suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- Barbara derives additional utility of 3 from spying.
- Question: Which pub should you go to?

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#### Example: Spy Game



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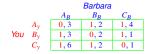
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### Example: Spy Game



- Consider the following lexicographic epistemic model: Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B, t'_B\}$ Beliefs:  $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (A_B, t'_B); (B_B, t'_B)),$  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y)),$  and  $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$
- Your type t<sub>v</sub> assumes Barbara's rationality.
- Barbara's type  $t_B$  does not assume your rationality: although your choices  $A_y$  and  $C_y$  are optimal for some cautious belief,  $t_B$  does not deem possible types for you for which  $A_y$  and  $C_y$  are optimal. (analogous for type  $t'_B$ )
- Thus, type ty only deems possible types for Barbara that do not assume rationality.
- However, Barbara's choice A<sub>B</sub> is optimal for some type that is cautious and assumes your rationality.

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# Example: Spy Game

		Barbara		
		$A_B$	BB	$C_B$
	$A_y$	0,3	1,2	1,4
You	$B_{y}$	1,3	0, 2	1,1
	$\dot{C_y}$	1,6	1, 2	0,1

Indeed, consider the following lexicographic epistemic model:

$$\begin{split} & \text{Types:} \ T_{yout} = \{f_y^A, t_g^B, t_y^C\} \text{ and } T_{Barbara} = \{t_B^A, t_B^C\} \\ & \text{Beliefs for you:} \ b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, A_B^A); \ldots), b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots), \\ & \text{and } b_y(t_y^C) = ((A_B, A_B^A); (B_B, A_B^A); (C_B, t_B^C); \ldots) \\ & \text{Beliefs for Barbara:} \ b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots) \text{ and } \\ & b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots) \end{split}$$

- Type  $t_B^A$  does assume your rationality, and Barbara's choice  $A_B$  is optimal for  $t_B^A$ .
- Thus, Barbara's choice A<sub>B</sub> is optimal for some cautious type that assumes your rationality.
- Note that type t<sup>C</sup><sub>B</sub> also assumes your rationality.
- Observe that your type  $t_y^B$  assumes Barbara's rationality, but your types  $t_y^A$  and  $t_y^C$  do not assume her rationality.

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#### Example: Spy Game

		Barbara		
		$A_B$	BB	$C_B$
	$A_y$	0,3	1, 2	1,4
You	$B_y$	1,3	0, 2	1, 1
	$C_y$	1,6	1, 2	0, 1

Indeed, consider the following lexicographic epistemic model:

$$\begin{split} & \text{Types:} \ T_{you} = \{f_y^A, t_y^B, t_y^C\} \text{ and } T_{Barbara} = \{t_B^A, t_B^C\} \\ & \text{Beliefs for you:} \ b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots), b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots), \\ & \text{and } b_y(t_y^C) = ((A_B, A_B^A); (B_B, A_B^A); (C_B, t_B^C); \ldots) \\ & \text{Beliefs for Barbara:} \ b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots) \text{ and } \\ & b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots) \end{split}$$

- It is now shown that type t<sup>B</sup><sub>v</sub> expresses common assumption of rationality.
- Type t<sup>B</sup><sub>v</sub> expresses 1-fold assumption of rationality:
  - Only Barbara's choices  $A_B$  and  $C_B$  can be optimal for a cautious belief: type  $t_y^B$  deems possible cautious types  $t_B^A$  and  $t_C^C$  for which  $A_B$  and  $C_B$ , respectively, are optimal.
  - Type  $t_{y}^{B}$  deems  $(A_{B}, t_{R}^{A})$  and  $(C_{B}, t_{R}^{C})$  infinitely more likely than the rest.
- Note that only choice B<sub>y</sub> can be optimal for you, if you express 1-fold assumption of rationality.

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#### Example: Spy Game

			Barbara		
		$A_B$	BB	$C_B$	
	$A_y$	0, 3	1, 2	1,4	
You	$B_{y}$	1,3	0, 2	1,1	
	$\dot{C_y}$	1,6	1, 2	0,1	

Indeed, consider the following lexicographic epistemic model:

$$\begin{split} & \text{Types: } T_{you} = \{f_y^A, t_y^B, t_y^C\} \text{ and } T_{Barbara} = \{t_B^A, t_B^C\} \\ & \text{Beliefs for you: } b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, A_B^A); \ldots), b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots), \\ & \text{and } b_y(t_y^C) = ((A_B, A_B^A); (B_B, A_B^A); (C_B, t_B^C); \ldots) \\ & \text{Beliefs for Barbara: } b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots) \text{ and } \\ & b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots) \end{split}$$

- Type  $t_v^B$  expresses 2-fold assumption of rationality:
  - Barbara's types  $t_B^A$  and  $t_B^C$  express 1-fold assumption of rationality
  - Thus, Barbara's choices A<sub>B</sub> and C<sub>B</sub> are optimal for cautious types that express 1-fold assumption of rationality.

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- Type  $t_v^B$  deems possible these types  $t_B^A$  and  $t_B^C$ .
- Type  $t_y^B$  deems  $(A_B, t_B^A)$  and  $(C_B, t_B^C)$  infinitely more likely than the rest.

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#### Example: Spy Game

		Barbara		
		$A_B$	BB	$C_B$
	$A_y$	0, 3	1,2	1,4
You	$B_y$	1,3	0, 2	1,1
	$C_y$	1,6	1, 2	0,1

Indeed, consider the following lexicographic epistemic model:

$$\begin{split} & \text{Types:} \ T_{yout} = \{ f_y^A, t_y^B, t_y^C \} \text{ and } T_{Barbara} = \{ r_B^A, t_B^C \} \\ & \text{Beliefs for you:} \ b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots), b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots), \\ & \text{and } b_y(t_y^C) = ((A_B, A_B^A); (B_B, A_B^A); (C_B, t_B^C); \ldots) \\ & \text{Beliefs for Barbara:} \ b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots) \text{ and } \\ & b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_C^C); \ldots) \end{split}$$

- Note that in order to express 2-fold assumption of rationality Barbara must deem your choice B<sub>y</sub> infinitely more likely than your other choices.
- Barbara's type t<sup>A</sup><sub>B</sub> expresses 2-fold assumption of rationality:

Only your choice *B<sub>y</sub>* is optimal for a cautious type that expresses 1-fold assumption of rationality.

- Type  $t_B^A$  deems possible your type  $t_y^B$  that is cautious, expresses 1-fold assumption of rationality, and for which your choice  $B_y$  is optimal.
- Type  $t_B^A$  deems  $(B_y, t_y^B)$  infinitely more likely than the rest.

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# Example: Spy Game

		Barbara		
		$A_B$	BB	$C_B$
	$A_y$	0, 3	1, 2	1,4
You	$B_y$	1,3	0, 2	1,1
	$C_y$	1,6	1,2	0,1

- Indeed, consider the following lexicographic epistemic model: Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$ Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots), b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots),$ and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \ldots)$ Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots)$  and  $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots)$
- Type  $t_v^B$  expresses 3-fold assumption of rationality:
  - Barbara can only rationally make choice A<sub>B</sub> under up to 2-fold assumption of rationality.
  - Type  $t_y^B$  deems possible Barbara's type  $t_B^A$  that is cautious, expresses up to 2-fold assumption of rationality, and for which  $A_B$  is optimal.
  - **Type**  $t_{v}^{B}$  deems  $(A_{B}, t_{B}^{A})$  infinitely more likely than the rest.
- By continuing in this fashion, it can be concluded that your type  $t_y^B$  expresses *k*-fold assumption of rationality for every  $k \ge 1$ : hence,  $t_y^B$  entertains common assumption of rationality.
- Consequently, you can rationally and cautiously only go to Pub B.

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#### **Assumption of Rationality**

#### Definition

A cautious type t<sub>i</sub> assumes rationality, whenever

- for every choice c<sub>j</sub> that is optimal for some cautious type, t<sub>i</sub> deems possible a cautious type t<sub>j</sub> for which c<sub>j</sub> is indeed optimal,
- t<sub>i</sub> deems all choice-type pairs (c<sub>j</sub>, t<sub>j</sub>), where t<sub>j</sub> is cautious and c<sub>j</sub> optimal for t<sub>j</sub>, infinitely more likely than all choice-type pairs not satisfying this property.

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#### **Common Assumption of Rationality**

#### Definition

- A cautious type t<sub>i</sub> expresses 1-fold assumption of rationality, whenever t<sub>i</sub> assumes rationality.
- For all  $k \ge 2$ , a cautious type  $t_i$  expresses k-fold assumption of rationality, whenever
  - for every choice c<sub>i</sub> that is optimal for some cautious type that expresses up to (k 1)-fold assumption of rationality, type t<sub>i</sub> deems possible a cautious type t<sub>j</sub> that expresses up to (k 1)-fold assumption of rationality and for which c<sub>i</sub> is indeed optimal;
  - type  $t_i$  deems all choice-type pairs  $(c_i, t_i)$  where  $t_j$  is cautious, expresses up to (k 1)-fold assumption of rationality, and  $c_j$  is optimal for  $t_j$ , infinitely more likely than all choice-type pairs not satisfying this property.
- A cautious type t<sub>i</sub> expresses common assumption of rationality, whenever t<sub>i</sub> expresses k-fold assumption of rationality for all k ≥ 1.

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# **Towards an Algorithm**

Step 1. 1-fold assumption of rationality: What choices can *i* rationally and cautiously make when assuming rationality?

- First, note that i does not choose by Lexicographic Pearce's Lemma a weakly dominated choice due to caution.
- If i assumes j's rationality, then i deems all choices that are optimal for some cautious belief infinitely more likely than all choices that are not optimal for any cautious belief.
- Again by Lexicographic Pearce's Lemma optimal choices under caution are equivalent with non-weakly-dominated choices.
- Hence, if i assumes j's rationality, then i deems all non-weakly-dominated choices of j infinitely more likely than all weakly dominated choices of j.
- Let C<sup>1</sup><sub>j</sub> be the set of non-weakly-dominated choices for j: Then, i deems all choices inside C<sup>1</sup><sub>j</sub> infinitely more likely than all choices outside C<sup>1</sup><sub>i</sub>
- Let  $b_i^{lex} = (b_i^1; b_i^2 \dots; b_i^K)$  be *i*'s lexicographic belief.
- Then, there exists some level *L* < *K* such that
  - 1 the beliefs  $b_i^1, \ldots, b_i^L$  only assign positive probability to choices in  $C_i^1$
  - 2 all

all choices in  $C_i^1$  receive positive probability in some belief from  $b_i^1, \ldots, b_i^L$ 

- Consequently, (*b*<sup>1</sup><sub>*i*</sub>; . . . ; *b*<sup>L</sup><sub>*i*</sub>) forms a cautious lexicographic belief on *C*<sup>1</sup><sub>*i*</sub>.
- Moreover, every choice  $c_i$  which is optimal under  $b_i^{lex}$  must also be optimal under the truncated cautious belief  $(b_i^1; \ldots; b_i^L)$  on  $C_j^1$ , i.e. must not be weakly dominated on  $C_j^1$ !

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# **Towards an Algorithm**

**Conclusion:** If *i* is cautious and assumes *j*'s rationality, then every optimal choice  $c_i$ 

- must not be weakly dominated in the original game
- must not be weakly dominated in the reduced game, obtained after 1 round of weak dominance
- i.e. every optimal choice  $c_i$  survives 2 rounds of weak dominance!

## Towards an Algorithm

Step 2. up to 2-fold assumption of rationality: What choices can *i* rationally and cautiously make under up to 2-fold assumption of rationality?

- If c<sub>j</sub> is optimal for some cautious belief b<sup>lex</sup><sub>j</sub> that assumes is rationality, while c'<sub>j</sub> is not, then i deems c<sub>j</sub> infinitely more likely than c'<sub>i</sub>.
- Let C<sub>j</sub><sup>2</sup> be the set of j's choices that are optimal for some cautious belief that assumes i's rationality: Then, i deems all choices inside C<sub>j</sub><sup>2</sup> infinitely more likely than all choices outside C<sub>j</sub><sup>2</sup>
- Then, by Lexicographic Pearce's Lemma, every optimal choice for *i* must not be weakly dominated on  $C_i^2$ .
- By Step 1  $C_i^2$  contains precisely those choices that survive 2 rounds of weak dominance.
- Therefore, every optimal choice for *i* must not be weakly dominated within the reduced game obtained after 2 rounds of weak dominance, i.e. must survive 3 rounds of weak dominance.

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#### **Towards an Algorithm**

**In general:** If *i* is cautious and expresses up to k-fold assumption of rationality, then every optimal choice for *i* must survive (k+1) rounds of weak dominance.

#### **Iterated Weak Dominance**

- Step 1. Within the original game, eliminate all choices that are weakly dominated.
- **Step 2.** Within the reduced game obtained after step 1, eliminate all choices that are weakly dominated.
- etc, until no further choices can be eliminated.

# Algorithmic Characterization

#### Theorem

For all  $k \ge 1$ , the choices that can rationally be made by a cautious type that expresses up to *k*-fold assumption of rationality are exactly those choices that survive the first k + 1 rounds of Iterated Weak Dominance.

#### Corollary

The choices that can rationally be made by a cautious type that expresses common assumption of rationality are exactly those choices that survive Iterated Weak Dominance.

### **Properties of the Algorithm**

- Iterated Weak Dominance stops after finitely many rounds.
- Iterated Weak Dominance always yields a non-empty set of choices for both players.
- The order and speed of elimination crucially matter for the eventual output of the algorithm!

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## Example: Spy Game

#### Story

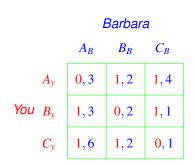
- *You* would like to go to a pub to read your book.
- Barbara is going to a pub as well, but you forgot to ask her to which one.
- The only objective for you is to avoid Barbara, since you would like to read your book in silence.
- Barbara prefers Pub A to Pub B, and Pub B to Pub C.
- Besides, Barbara suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- Barbara derives additional utility of 3 from spying.
- Question: Which pub should you go to?

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### Example: Spy Game



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### **Example: Spy Game**

		Barbara			
		$A_B$	$B_B$	CB	
	$A_y$	0, 3	1, 2	1,4	
You	$B_y$	1,3	0, 2	1, 1	
	Cy	1,6	1, 2	0, 1	

#### First Order of Elimination

Step 1. Eliminate B<sub>B</sub>

Existence

### **Example: Spy Game**

		Daibaia		
		A <sub>B</sub>	$C_B$	
	$A_y$	0, 3	1,4	
You	$B_y$	1,3	1, 1	
	Cy	1,6	0, 1	

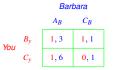
Rarbara

#### First Order of Elimination

Step 2. Only eliminate A<sub>v</sub>

Existence

# **Example: Spy Game**



First Order of Elimination

Step 3. Eliminate CB

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# Example: Spy Game



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First Order of Elimination

 $B_{y}$  and  $C_{y}$  survive for you!

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### **Example: Spy Game**

		Barbara			
		$A_B$	BB	$C_B$	
	$A_y$	0, 3	1, 2	1,4	
You	$B_y$	1,3	0, 2	1, 1	
	$C_y$	1,6	1, 2	<b>0</b> , 1	

#### Second Order of Elimination

Step 1. Eliminate B<sub>B</sub>

Existence

### **Example: Spy Game**

		Barbara			
		$A_B$	$C_B$		
	$A_y$	0, 3	1,4		
You	$B_y$	1,3	1, 1		
	Cy	1,6	0, 1		

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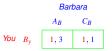
#### Second Order of Elimination

Step 2. Eliminate  $A_y$  and  $C_y$ 

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# Example: Spy Game



#### Second Order of Elimination

Step 3. Eliminate CB

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# Example: Spy Game



#### Second Order of Elimination

Only By survives for you!

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Existence



Assumption of Rationality

Cautious Reasoning

Algorithm

#### Existence

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Existence

### **Existence**

- There is no easy iterative procedure delivering a type that expresses common assumption of rationality.
- Since the non-emptyness of the algorithm ensures the existence of a choice surviving it which in turn can be made under common assumption of rationality by the preceding theorem, it is always possible to construct an epistemic model containing a type that expresses common assumption of rationality!

#### Theorem

Let  $\Gamma$  be some finite two player game. Then, there exists a lexicographic epistemic model which contains a type  $t_i$  that expresses common assumption of rationality.

Existence

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### Example: Take a Seat

#### Story

- Barbara and you are the only ones to take an exam.
- Both must choose a seat.
- If both choose the same seat, then with probability 0.5 you get the seat you want, and with probability 0.5 you get the one horizontally next to it.
- In order to pass the exam you must be able copy from Barbara, and the same applies to her.
- A person can only copy from the other person if seated horizontally next or diagonally behind the latter.

Existence

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### Example: Take a Seat

#### Story (continued)

The probabilities of successful copying for the respective seats are given in percentages:

a = 0, b = 10, c = d = 20, e = f = 45, g = h = 95

- The objective is to maximize the expected percentage of successful copying.
- Question: What seats can you rationally and cautiously choose under common assumption of rationality?

Existence

### **Example: Take a Seat**

		Barbara							
		$a_B$	$b_B$	$c_B$	$d_B$	$e_B$	f <sub>B</sub>	8B	h <sub>B</sub>
	a <sub>Y</sub>	5, 5	<mark>0</mark> , 10	0,0	<mark>0</mark> , 20	0, 0	0, 0	<b>0</b> , <b>0</b>	0,0
	$b_Y$	10, 0	5, 5	<mark>0</mark> , 20	0, 0	<b>0</b> , 0	0, 0	<b>0</b> , 0	0,0
	cy	<mark>0</mark> , 0	20, 0	20, 20	20, 20	0, 0	<b>0</b> , 45	<b>0</b> , <b>0</b>	0,0
You	$d_Y$	20, 0	<mark>0</mark> , 0	20, 20	20, 20	<mark>0</mark> , 45	0, 0	<b>0</b> , <b>0</b>	0,0
100	ey	<b>0</b> , 0	0, 0	0,0	45,0	45, 45	45, 45	0, 0	<b>0</b> , 95
	$f_Y$	<mark>0, 0</mark>	<mark>0</mark> , 0	45, <mark>0</mark>	0, 0	45, 45	45, 45	<mark>0</mark> , 95	0,0
	g <sub>Y</sub>	<mark>0</mark> , 0	<mark>0</mark> , 0	0,0	0, 0	0, 0	95, <mark>0</mark>	95, 95	95, 95
	$h_Y$	0, 0	0, 0	0,0	0, 0	95,0	0, 0	95, 95	95, 95

Barbara

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		Barbara							
		$a_B$	$b_B$	$c_B$	$d_B$	$e_B$	f <sub>B</sub>	8B	hB
	ay	5,5	<mark>0</mark> , 10	0,0	<mark>0</mark> , 20	0, 0	0, 0	<mark>0, 0</mark>	0,0
	$b_Y$	10, <mark>0</mark>	5, 5	<mark>0</mark> , 20	0, 0	0, 0	0, 0	<mark>0, 0</mark>	0,0
	cy	0,0	20, 0	20, 20	20, 20	0, 0	<mark>0</mark> , 45	<mark>0, 0</mark>	0,0
You	$d_Y$	20, 0	<mark>0</mark> , 0	20, 20	20, 20	<mark>0</mark> , 45	0, 0	<mark>0, 0</mark>	0,0
100	ey	0,0	<mark>0</mark> , 0	0,0	45, <mark>0</mark>	45, 45	45, 45	<mark>0</mark> , 0	<b>0</b> , 95
	$f_Y$	0,0	<b>0</b> , 0	45, <mark>0</mark>	<b>0</b> , 0	45, 45	45, 45	<mark>0</mark> , 95	0,0
	gy	0,0	<b>0</b> , 0	0,0	0, 0	<b>0</b> , 0	95, <mark>0</mark>	95, 95	95, 95
	$h_Y$	0, 0	<mark>0</mark> , 0	0,0	0, 0	95, 0	0, 0	95, 95	95, 95

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#### Round 1.

- In the full game  $a_Y$  and  $b_Y$  are weakly dominated by  $\frac{1}{2}c_Y + \frac{1}{2}d_Y$ .
- Eliminate  $a_Y$  and  $b_Y$ , as well as  $a_B$  and  $b_B$  by symmetry.

		$c_B$	$d_B$	$e_B$	$f_B$	$g_B$	h <sub>B</sub>
	cy	20, 20	20, 20	0, 0	<b>0</b> , 45	0, 0	0, 0
	$d_Y$	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
You	ey	0,0	45,0	45, 45	45, 45	0, 0	0,95
100	$f_Y$	45,0	0,0	45, 45	45, 45	<mark>0</mark> , 95	0, 0
	$g_Y$	0,0	0,0	0, 0	95,0	95, 95	95, 95
	$h_Y$	0,0	0,0	95,0	0, 0	95, 95	95, 95

#### Barbara

#### Round 2.

- In the reduced game  $c_Y$  and  $d_Y$  are weakly dominated by  $\frac{1}{2}e_Y + \frac{1}{2}f_Y$ .
- Eliminate  $c_Y$  and  $d_Y$ , as well as  $c_B$  and  $d_B$  by symmetry.

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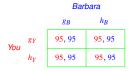
		Barbara				
		eB	f <sub>B</sub>	8B	h <sub>B</sub>	
	$e_Y$	45, 45	45, 45	<b>0</b> , <b>0</b>	<mark>0</mark> , 95	
You	$f_Y$	45, 45	45, 45	<mark>0</mark> , 95	0, 0	
	g <sub>Y</sub>	0, 0	95, <mark>0</mark>	95, 95	95, 95	
	$h_Y$	95,0	<b>0</b> , 0	95, 95	95, 95	

#### Round 3.

- In the reduced game  $e_Y$  and  $f_Y$  are weakly dominated by  $\frac{1}{2}g_Y + \frac{1}{2}h_Y$ .
- Eliminate e<sub>Y</sub> and f<sub>Y</sub>, as well as e<sub>B</sub> and f<sub>B</sub> by symmetry.

Existence

#### Example: Take a Seat



Round 4.

- No more choices can be eliminated.
- Vou can rationally and cautiously choose seats g and h under common assumption of rationality.

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Intuition: Why does common assumption of rationality lead to a different conclusion as common full belief in (caution & respect of preferences)?

First step of reasoning



Not that both choices a and b are irrational, yet b is better than a.

Under common assumption of rationality it is thus not distinguished between a and b, however under common full belief in (caution & respect of preferences) it is.

#### Second step of reasoning

If you believe Barbara to reason in line with the first step, then c and d can no longer be optimal, yet c is better than d.

Under common assumption of rationality it is not distinguished between c and d, however under common full belief in (caution & respect of preferences) it is.

#### Third step of reasoning

- If you believe Barbara to reason in line with the first and the second step, then e and f can no longer be optimal, yet f is better than e.
- Under common assumption of rationality it is not distinguished between e and f, however under common full belief in (caution & respect of preferences) it is.

#### Fourth step of reasoning

- If you believe Barbara to reason in line with the first, the second and the fourth step, then g and h can no longer be optimal, yet g is better than h.
- Under common assumption of rationality g and h are both optimal, while under common full belief in (caution & respect of preferences) only g remains optimal.

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### There Exists No Related Equilibrium Notion

- The correct beliefs assumption implicit in any equilibrium notion seems to be at odds with common assumption of rationality.
- As illustration consider the lexicographic epistemic model of the Spy Game again.

			Barbara	
		$A_B$	BB	$C_B$
	$A_{y}$	0, 3	1, 2	1,4
You	$\dot{B_y}$	1,3	0, 2	1,1
	$C_y$	1,6	1, 2	0,1

- Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$ Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); ...), b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); ...),$ and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); ...)$ Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^P); (C_y, t_y^C); (A_y, t_y^A); ...)$  and  $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^P); (C_y, t_y^C); ...)$
- Recall that  $t_y^B$  express common assumption of rationality.
- However,  $t_v^B$  deems it possible that Barbara is **not correct** about the his type!
- Bach & Jagau (2022) generalize such insights to an incompatibility theorem about equilibrium and IWD: "compatibility implies one round of weak dominance only".

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