Lexicographic Beliefs

Part III: Assumption of Rationality

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Introduction

- Two ways of cautious reasoning have been presented so far:
  - Common Primary Belief in (Caution & Rationality)
  - Common Full Belief in (Caution & Respect of Preferences)
- Respect of preferences imposes restrictions not only on the primary but also on deeper lexicographic levels!
- However, there are other reasonable conditions that could be put on the various lexicographic levels.

Agenda

- Assumption of Rationality
- Common Assumption of Rationality
- Algorithm
- Existence

Agenda

- **Assumption of Rationality**
- Common Assumption of Rationality
- Algorithm
- Existence

Example: Spy Game

Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but you forgot to ask her to which one.
- The only objective for *you* is to avoid Barbara, since you would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- Question: Which pub should *you* go to?

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

Example: Spy Game

		Barbara A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

Under common full belief in (caution & respect of preferences), you go to Pub C:

- As Barbara prefers A_B to B_B and you respect her preferences, you must deem her choice A_B infinitely more likely than B_B.
- Then, you prefer B_y to A_y.
- Hence, you believe that Barbara deems your choice B_y infinitely more likely than A_y.
- Consequently, you believe that Barbara prefers B_B to C_B, and you must deem B_B infinitely more likely than C_B.
- But then the unique optimal choice for you is C_y.
- However, this is not the only plausible way to reason about Barbara!

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

An alternative way of reasoning:

- For Barbara, both A_B and C_B can be optimal for some cautious lexicographic belief, but B_B can never be optimal.
- Therefore, you deem Barbara's choice A_B and C_B infinitely more likely than B_B.
- But then, your unique optimal choice is B_y!

The Underlying Intuition

- If player j's choice c_j is optimal for some cautious lexicographic belief, while his choice c_j' is not optimal for any cautious lexicographic belief, then player i must deem c_j infinitely more likely than c_j'.
- Player i is then said to **assume rationality**.
- In other words, player i deems his opponent j' s good choices infinitely more likely than j's bad choices.

How Can this Intuition Be Formalized?

- How can the idea of assuming rationality be formalized in an epistemic model?

- **Attempt:** Type t_i must deem all choice-type pairs (c_j, t_j), where c_j is optimal for t_j and t_j is cautious, infinitely more likely than all choice-type pairs (c_j', t_j') that do not have this property.

The Attempt Does Not Work!

		Barbara A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- **Attempt:** Type t_i must deem all choice-type pairs (c_j, t_j), where c_j is optimal for t_j and t_j is cautious, infinitely more likely than all choice-type pairs (c_j', t_j') that do not have this property.

- Consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B\}$

  Beliefs: $b_y(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$ and $b_B(t_B) = ((C_y, t_y); (B_y, t_y); (A_y, t_y))$

- Your type t_y satisfies the condition, but does not assume rationality in the intended way.

- **Problem:** Choice C_B can be optimal for Barbara for some cautious type, but your type t_y does not deem possible any type for Barbara for which C_B is indeed optimal.

- **Remedy:** it is additionally required that you must deem possible a cautious type for Barbara for which C_B is optimal!

More Types Are Needed

		Barbara		
		A_B	B_B	C_B
You	A_y	$0, 3$	$1, 2$	$1, 4$
	B_y	$1, 3$	$0, 2$	$1, 1$
	C_y	$1, 6$	$1, 2$	$0, 1$

- Consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B, t'_B\}$

  Beliefs: $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (B_B, t_B); (A_B, t'_B); (B_B, t'_B))$,

  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$, and $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$

- For Barbara choices A_B and C_B can be optimal for some cautious type.

- Your type t_y deems possible the cautious type t_B for which A_B is optimal as well as the cautious type t'_B for which C_B is optimal.

- Your type t_y deems all choice-type pairs where the type is cautious and the choice is optimal for the type infinitely more likely than all choice-type pairs that do not have this property.

- Indeed, type t_y assumes rationality in the intended way!

Assumption of Rationality

Definition

A cautious type t_i ***assumes rationality***, whenever

- for every choice c_j that is optimal for some cautious type, t_i deems possible a cautious type t_j for which c_j is indeed optimal,
- t_i deems all choice-type pairs (c_j, t_j), where t_j is cautious and c_j optimal for t_j, infinitely more likely than all choice-type pairs not satisfying this property.

Intuition:
A player deems good choices infinitely more likely than bad choices.

Remark:
Assumption of rationality can only be defined for cautious types.

Assumption and Primary Belief in Rationality

Observation. If $Alice$ is cautious and assumes Bob's rationality, then she also primarily believes in Bob's rationality.

- Suppose that t_{Alice} is cautious and assumes Bob's rationality.
- Then, t_{Alice} considers all choice-type pairs where the choice is optimal for the type infinitely more likely than other choice-type pairs.
- In particular, the support of t_{Alice}'s primary belief can then only contain choice-type pairs such that the choice is optimal for the type.

Assumption and Respect of Preferences

Observation. There is no general relationship between assuming rationality and respecting preferences.

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- Consider the following lexicographic epistemic model:
  Types: $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B\}$
  Beliefs: $b_y(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$ and $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$
- Your type t_y respects Barbara's preferences, but does not assume her rationality.

- Consider the following lexicographic epistemic model:
  Types: $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B, t_B'\}$
  Beliefs: $b_y(t_y) = ((A_B, t_B); (C_B, t_B'); (C_B, t_B); (B_B, t_B); (A_B, t_B'); (B_B, t_B'))$,
  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$, and $b_B(t_B') = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$
- Your type t_y assumes Barbara's rationality, but does not respect her preferences.
- Indeed, for t_B choice B_B is better than C_B, yet t_y deems (C_B, t_B) infinitely more likely than (B_B, t_B).

Remark

- It is always possible to satisfy respect of preferences and assumption of rationality.
- **Intuition:** A type's lexicographic belief deems optimal choices infinitely more likely than the non-optimal choices, yet orders the non-optimal choices as required by respect of preference.

Agenda

- Assumption of Rationality
- **Common Assumption of Rationality**
- Algorithm
- Existence

Assuming (Rationality & Assumption of Rationality)

Definition

A cautious type t_i ***assumes (rationality & assumption of rationality)***, whenever

- for every choice c_j that is optimal for some cautious type that assumes i's rationality, type t_i deems possible a cautious type t_j that assumes i's rationality and for which c_j is indeed optimal;
- type t_i deems all choice-type pairs (c_j, t_j), where t_j is cautious, assumes i's rationality, and c_j is optimal for t_j, infinitely more likely than all choice-type pairs not satisfying this property.

Common Assumption of Rationality

Definition

- A cautious type t_i expresses ***1-fold assumption of rationality***, whenever t_i assumes rationality.
- For all $k \geq 2$, a cautious type t_i expresses ***k-fold assumption of rationality***, whenever
  - for every choice c_j that is optimal for some cautious type that expresses up to $(k-1)$-fold assumption of rationality, type t_i deems possible a cautious type t_j that expresses up to $(k-1)$-fold assumption of rationality and for which c_j is indeed optimal;
  - type t_i deems all choice-type pairs (c_j, t_j) where t_j is cautious, expresses up to $(k-1)$-fold assumption of rationality, and c_j is optimal for t_j, infinitely more likely than all choice-type pairs not satisfying this property.
- A cautious type t_i expresses ***common assumption of rationality***, whenever t_i expresses k-fold assumption of rationality for all $k \geq 1$.

Example: Spy Game

Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but you forgot to ask her to which one.
- The only objective for *you* is to avoid Barbara, since you would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- Question: Which pub should *you* go to?

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
	A_y	$0, 3$	$1, 2$	$1, 4$
You	B_y	$1, 3$	$0, 2$	$1, 1$
	C_y	$1, 6$	$1, 2$	$0, 1$

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- Consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B, t'_B\}$

  Beliefs: $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (B_B, t_B); (A_B, t'_B); (B_B, t'_B))$,

  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$, and $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$

- Your type t_y assumes Barbara's rationality.

- Barbara's type t_B does not assume your rationality: although your choices A_y and C_y are optimal for some cautious belief, t_B does not deem possible types for you for which A_y and C_y are optimal. (analogous for type t'_B)

- Thus, type t_y only deems possible types for Barbara that do not assume rationality.

- However, Barbara's choice A_B is optimal for some type that is cautious and assumes your rationality.

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y^A, t_y^B, t_y^C\}$ and $T_{Barbara} = \{t_B^A, t_B^C\}$

  Beliefs for you: $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots)$, $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots)$, and $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \ldots)$

  Beliefs for Barbara: $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots)$ and $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots)$

- Type t_B^A does assume your rationality, and Barbara's choice A_B is optimal for t_B^A.

- Thus, Barbara's choice A_B is optimal for some cautious type that assumes your rationality.

- Note that type t_B^C also assumes your rationality.

- Observe that your type t_y^B assumes Barbara's rationality, but your types t_y^A and t_y^C do not assume her rationality.

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	$0, 3$	$1, 2$	$1, 4$
	B_y	$1, 3$	$0, 2$	$1, 1$
	C_y	$1, 6$	$1, 2$	$0, 1$

- Indeed, consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y^A, t_y^B, t_y^C\}$ and $T_{Barbara} = \{t_B^A, t_B^C\}$

  Beliefs for you: $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots)$, $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots)$, and $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \ldots)$

  Beliefs for Barbara: $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots)$ and $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots)$

- It is now shown that type t_y^B expresses common assumption of rationality.

- Type t_y^B expresses 1-fold assumption of rationality:
  - Only Barbara's choices A_B and C_B can be optimal for a cautious belief: type t_y^B deems possible cautious types t_B^A and t_B^C for which A_B and C_B, respectively, are optimal.
  - Type t_y^B deems (A_B, t_B^A) and (C_B, t_B^C) infinitely more likely than the rest.

- Note that only choice B_y can be optimal for you, if you express 1-fold assumption of rationality.

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y^A, t_y^B, t_y^C\}$ and $T_{Barbara} = \{t_B^A, t_B^C\}$

  Beliefs for you: $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots)$, $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots)$, and $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \ldots)$

  Beliefs for Barbara: $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots)$ and $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots)$

- Type t_y^B expresses 2-fold assumption of rationality:
  - Barbara's types t_B^A and t_B^C express 1-fold assumption of rationality
  - Thus, Barbara's choices A_B and C_B are optimal for cautious types that express 1-fold assumption of rationality.
  - Type t_y^B deems possible these types t_B^A and t_B^C.
  - Type t_y^B deems (A_B, t_B^A) and (C_B, t_B^C) infinitely more likely than the rest.

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	0, 3	1, 2	1, 4
	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y^A, t_y^B, t_y^C\}$ and $T_{Barbara} = \{t_B^A, t_B^C\}$

  Beliefs for you: $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots)$, $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots)$, and $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \ldots)$

  Beliefs for Barbara: $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots)$ and $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots)$

- Note that in order to express 2-fold assumption of rationality Barbara must deem your choice B_y infinitely more likely than your other choices.

- Barbara's type t_B^A expresses 2-fold assumption of rationality:
  - Only your choice B_y is optimal for a cautious type that expresses 1-fold assumption of rationality.
  - Type t_B^A deems possible your type t_y^B that is cautious, expresses 1-fold assumption of rationality, and for which your choice B_y is optimal.
  - Type t_B^A deems (B_y, t_y^B) infinitely more likely than the rest.

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
You	A_y	$0,3$	$1,2$	$1,4$
	B_y	$1,3$	$0,2$	$1,1$
	C_y	$1,6$	$1,2$	$0,1$

- Indeed, consider the following lexicographic epistemic model:

  Types: $T_{you} = \{t_y^A, t_y^B, t_y^C\}$ and $T_{Barbara} = \{t_B^A, t_B^C\}$

  Beliefs for you: $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots)$, $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots)$, and $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \ldots)$

  Beliefs for Barbara: $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots)$ and $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots)$

- Type t_y^B expresses 3-fold assumption of rationality:
  - Barbara can only rationally make choice A_B under up to 2-fold assumption of rationality.
  - Type t_y^B deems possible Barbara's type t_B^A that is cautious, expresses up to 2-fold assumption of rationality, and for which A_B is optimal.
  - Type t_y^B deems (A_B, t_B^A) infinitely more likely than the rest.

- By continuing in this fashion, it can be concluded that your type t_y^B expresses k-fold assumption of rationality for every $k \geq 1$: hence, t_y^B entertains common assumption of rationality.

- Consequently, you can rationally and cautiously only go to Pub B.

Agenda

- Assumption of Rationality
- Common Assumption of Rationality
- **Algorithm**
- Existence

Assumption of Rationality

Definition

A cautious type t_i ***assumes rationality***, whenever

- for every choice c_j that is optimal for some cautious type, t_i deems possible a cautious type t_j for which c_j is indeed optimal,
- t_i deems all choice-type pairs (c_j, t_j), where t_j is cautious and c_j optimal for t_j, infinitely more likely than all choice-type pairs not satisfying this property.

Common Assumption of Rationality

Definition

- A cautious type t_i expresses ***1-fold assumption of rationality***, whenever t_i assumes rationality.
- For all $k \geq 2$, a cautious type t_i expresses ***k-fold assumption of rationality***, whenever
  - for every choice c_j that is optimal for some cautious type that expresses up to $(k-1)$-fold assumption of rationality, type t_i deems possible a cautious type t_j that expresses up to $(k-1)$-fold assumption of rationality and for which c_j is indeed optimal;
  - type t_i deems all choice-type pairs (c_j, t_j) where t_j is cautious, expresses up to $(k-1)$-fold assumption of rationality, and c_j is optimal for t_j, infinitely more likely than all choice-type pairs not satisfying this property.
- A cautious type t_i expresses ***common assumption of rationality***, whenever t_i expresses k-fold assumption of rationality for all $k \geq 1$.

Towards an Algorithm

Step 1. 1-fold assumption of rationality: What choices can i rationally and cautiously make when assuming rationality?

- First, note that i does not choose – by Lexicographic Pearce's Lemma – a weakly dominated choice due to caution.
- If i assumes j's rationality, then i deems all choices that are optimal for some cautious belief infinitely more likely than all choices that are not optimal for any cautious belief.
- Again – by Lexicographic Pearce's Lemma – optimal choices under caution are equivalent with non-weakly-dominated choices.
- Hence, if i assumes j's rationality, then i deems all non-weakly-dominated choices of j infinitely more likely than all weakly dominated choices of j.
- Let C_j^1 be the set of non-weakly-dominated choices for j: Then, i deems all choices inside C_j^1 infinitely more likely than all choices outside C_j^1
- Let $b_i^{lex} = (b_i^1; b_i^2 \ldots; b_i^K)$ be i's lexicographic belief.
- Then, there exists some level $L < K$ such that
  1. the beliefs $b_i^1, \ldots, b_i^L$ only assign positive probability to choices in C_j^1
  2. all choices in C_j^1 receive positive probability in some belief from $b_i^1, \ldots, b_i^L$
- Consequently, $(b_i^1; \ldots; b_i^L)$ forms a cautious lexicographic belief on C_j^1.
- Moreover, every choice c_i which is optimal under b_i^{lex} must also be optimal under the truncated cautious belief $(b_i^1; \ldots; b_i^L)$ on C_j^1, i.e. must not be weakly dominated on C_j^1!

Towards an Algorithm

Conclusion: If i is cautious and assumes j's rationality, then every optimal choice c_i

- must not be weakly dominated in the original game
- must not be weakly dominated in the reduced game, obtained after 1 round of weak dominance

i.e. every optimal choice c_i survives 2 rounds of weak dominance!

Towards an Algorithm

Step 2. up to 2-fold assumption of rationality: What choices can i rationally and cautiously make under up to 2-fold assumption of rationality?

- If c_j is optimal for some cautious belief b_j^{lex} that assumes i's rationality, while c_j' is not, then i deems c_j infinitely more likely than c_j'.
- Let C_j^2 be the set of j's choices that are optimal for some cautious belief that assumes i's rationality: Then, i deems all choices inside C_j^2 infinitely more likely than all choices outside C_j^2
- Then, by Lexicographic Pearce's Lemma, every optimal choice for i must not be weakly dominated on C_j^2.
- By Step 1 C_j^2 contains precisely those choices that survive 2 rounds of weak dominance.
- Therefore, every optimal choice for i must not be weakly dominated within the reduced game obtained after 2 rounds of weak dominance, i.e. must survive 3 rounds of weak dominance.

Towards an Algorithm

In general: If i is cautious and expresses up to k-fold assumption of rationality, then every optimal choice for i must survive (k+1) rounds of weak dominance.

Algorithm

Iterated Weak Dominance

- **Step 1.** Within the original game, eliminate all choices that are weakly dominated.
- **Step 2.** Within the reduced game obtained after step 1, eliminate all choices that are weakly dominated.
- etc, until no further choices can be eliminated.

Algorithmic Characterization

Theorem

For all $k \geq 1$, the choices that can rationally be made by a cautious type that expresses up to k-fold assumption of rationality are exactly those choices that survive the first $k+1$ rounds of Iterated Weak Dominance.

Corollary

The choices that can rationally be made by a cautious type that expresses common assumption of rationality are exactly those choices that survive Iterated Weak Dominance.

Properties of the Algorithm

- Iterated Weak Dominance stops after finitely many rounds.
- Iterated Weak Dominance always yields a non-empty set of choices for both players.
- The order and speed of elimination crucially matter for the eventual output of the algorithm!

Example: Spy Game

Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but you forgot to ask her to which one.
- The only objective for *you* is to avoid Barbara, since you would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- Question: Which pub should *you* go to?

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
	A_y	0, 3	1, 2	1, 4
You	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
	A_y	0, 3	1, 2	1, 4
You	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

First Order of Elimination

Step 1. Eliminate B_B

Example: Spy Game

		Barbara	
		A_B	C_B
	A_y	$0, 3$	$1, 4$
You	B_y	$1, 3$	$1, 1$
	C_y	$1, 6$	$0, 1$

First Order of Elimination

Step 2. Only eliminate A_y

Example: Spy Game

		Barbara A_B	Barbara C_B
You	B_y	1, 3	1, 1
You	C_y	1, 6	0, 1

First Order of Elimination

Step 3. Eliminate C_B

Example: Spy Game

You		Barbara A_B
	B_y	1, 3
	C_y	1, 6

First Order of Elimination

B_y and C_y survive for you!

Example: Spy Game

		Barbara		
		A_B	B_B	C_B
	A_y	0, 3	1, 2	1, 4
You	B_y	1, 3	0, 2	1, 1
	C_y	1, 6	1, 2	0, 1

Second Order of Elimination

Step 1. Eliminate B_B

Example: Spy Game

		Barbara A_B	Barbara C_B
You	A_y	0, 3	1, 4
You	B_y	1, 3	1, 1
You	C_y	1, 6	0, 1

Second Order of Elimination

Step 2. Eliminate A_y and C_y

Example: Spy Game

		Barbara	
		A_B	C_B
You	B_y	1, 3	1, 1

Second Order of Elimination

Step 3. Eliminate C_B

Example: Spy Game

		Barbara
		A_B
You	B_y	$1, 3$

Second Order of Elimination

Only B_y survives for you!

Agenda

- Assumption of Rationality
- Cautious Reasoning
- Algorithm
- **Existence**

Existence

- There is no easy iterative procedure delivering a type that expresses common assumption of rationality.
- Since the non-emptyness of the algorithm ensures the existence of a choice surviving it which in turn can be made under common assumption of rationality by the preceding theorem, it is always possible to construct an epistemic model containing a type that expresses common assumption of rationality!

Theorem

Let Γ be some finite two player game. Then, there exists a lexicographic epistemic model which contains a type t_i that expresses common assumption of rationality.

Example: Take a Seat

Story

- *Barbara* and *you* are the only ones to take an exam.
- Both must choose a seat.
- If both choose the same seat, then with probability 0.5 you get the seat you want, and with probability 0.5 you get the one horizontally next to it.
- In order to pass the exam *you* must be able copy from *Barbara*, and the same applies to her.
- A person can only copy from the other person if seated horizontally next or diagonally behind the latter.

Example: Take a Seat

Story (continued)

- The probabilities of successful copying for the respective seats are given in percentages:
  $a = 0$, $b = 10$, $c = d = 20$, $e = f = 45$, $g = h = 95$
- The objective is to maximize the expected percentage of successful copying.
- Question: What seats can you rationally and cautiously choose under common assumption of rationality?

Example: Take a Seat

You \ Barbara	a_B	b_B	c_B	d_B	e_B	f_B	g_B	h_B
a_Y	5, 5	0, 10	0, 0	0, 20	0, 0	0, 0	0, 0	0, 0
b_Y	10, 0	5, 5	0, 20	0, 0	0, 0	0, 0	0, 0	0, 0
c_Y	0, 0	20, 0	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
d_Y	20, 0	0, 0	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
e_Y	0, 0	0, 0	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
f_Y	0, 0	0, 0	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
g_Y	0, 0	0, 0	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
h_Y	0, 0	0, 0	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

Example: Take a Seat

		Barbara							
		a_B	b_B	c_B	d_B	e_B	f_B	g_B	h_B
You	a_Y	5, 5	0, 10	0, 0	0, 20	0, 0	0, 0	0, 0	0, 0
	b_Y	10, 0	5, 5	0, 20	0, 0	0, 0	0, 0	0, 0	0, 0
	c_Y	0, 0	20, 0	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
	d_Y	20, 0	0, 0	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
	e_Y	0, 0	0, 0	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
	f_Y	0, 0	0, 0	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
	g_Y	0, 0	0, 0	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
	h_Y	0, 0	0, 0	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

Round 1.

- In the full game a_Y and b_Y are weakly dominated by $\frac{1}{2}c_Y + \frac{1}{2}d_Y$.
- Eliminate a_Y and b_Y, as well as a_B and b_B by symmetry.

Example: Take a Seat

You \ Barbara	c_B	d_B	e_B	f_B	g_B	h_B
c_Y	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
d_Y	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
e_Y	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
f_Y	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
g_Y	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
h_Y	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

Round 2.

- In the reduced game c_Y and d_Y are weakly dominated by $\frac{1}{2}e_Y + \frac{1}{2}f_Y$.
- Eliminate c_Y and d_Y, as well as c_B and d_B by symmetry.

Example: Take a Seat

Barbara

		e_B	f_B	g_B	h_B
You	e_Y	45, 45	45, 45	0, 0	0, 95
	f_Y	45, 45	45, 45	0, 95	0, 0
	g_Y	0, 0	95, 0	95, 95	95, 95
	h_Y	95, 0	0, 0	95, 95	95, 95

Round 3.

- In the reduced game e_Y and f_Y are weakly dominated by $\frac{1}{2}g_Y + \frac{1}{2}h_Y$.
- Eliminate e_Y and f_Y, as well as e_B and f_B by symmetry.

Example: Take a Seat

		Barbara g_B	Barbara h_B
You	g_Y	95, 95	95, 95
You	h_Y	95, 95	95, 95

Round 4.

- No more choices can be eliminated.
- You can rationally and cautiously choose seats g and h under common assumption of rationality.

Example: Take a Seat

Intuition: Why does common assumption of rationality lead to a different conclusion as common full belief in (caution & respect of preferences)?

- First step of reasoning
  - Not that both choices a and b are irrational, yet b is better than a.
  - Under common assumption of rationality it is thus not distinguished between a and b, however under common full belief in (caution & respect of preferences) it is.
- Second step of reasoning
  - If you believe Barbara to reason in line with the first step, then c and d can no longer be optimal, yet c is better than d.
  - Under common assumption of rationality it is not distinguished between c and d, however under common full belief in (caution & respect of preferences) it is.
- Third step of reasoning
  - If you believe Barbara to reason in line with the first and the second step, then e and f can no longer be optimal, yet f is better than e.
  - Under common assumption of rationality it is not distinguished between e and f, however under common full belief in (caution & respect of preferences) it is.
- Fourth step of reasoning
  - If you believe Barbara to reason in line with the first, the second and the fourth step, then g and h can no longer be optimal, yet g is better than h.
  - Under common assumption of rationality g and h are both optimal, while under common full belief in (caution & respect of preferences) only g remains optimal.

There Exists No Related Equilibrium Notion

- The correct beliefs assumption implicit in any **equilibrium** notion seems to be at odds with **common assumption of rationality**.

- As illustration consider the lexicographic epistemic model of the *Spy Game* again.

You \ Barbara	A_B	B_B	C_B
A_y	0, 3	1, 2	1, 4
B_y	1, 3	0, 2	1, 1
C_y	1, 6	1, 2	0, 1

- Types: $T_{you} = \{t_y^A, t_y^B, t_y^C\}$ and $T_{Barbara} = \{t_B^A, t_B^C\}$

  Beliefs for you: $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \ldots)$, $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \ldots)$, and $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \ldots)$

  Beliefs for Barbara: $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \ldots)$ and $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \ldots)$

- Recall that t_y^B express common assumption of rationality.

- However, t_y^B deems it possible that Barbara is **not correct** about the his type!

- Bach & Jagau (2022) generalize such insights to an **incompatibility theorem** about equilibrium and IWD: "compatibility implies one round of weak dominance only".