

Exercises: Primary belief in the opponent's rationality

Niels Mourmans

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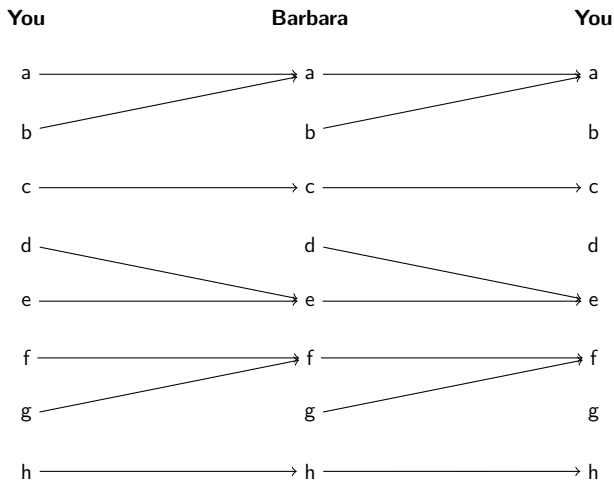
The Game: A walk through the forest

- You and Barbara are walking through the park. Suddenly you lose each other.
- There are open spots a, b, \dots, h where you can wait for Barbara and where Barbara can wait for you.
- You can see as far as the next spot, but no further.
- If both you and Barbara are waiting at spots where you can see each other, you both get a utility of 5.
- At spot c there is moreover a bench; waiting there gives an additional utility of 1.
- Where to wait for Barbara?

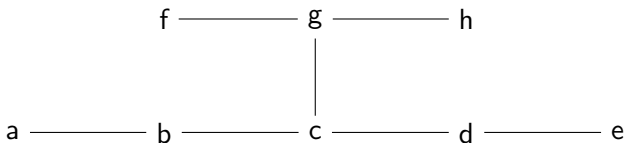
a) Where to wait for Barbara under CBR

You	Barbara	You
a	a	a
b	b	b
c	c	c
d	d	d
e	e	e
f	f	f
g	g	g
h	h	h

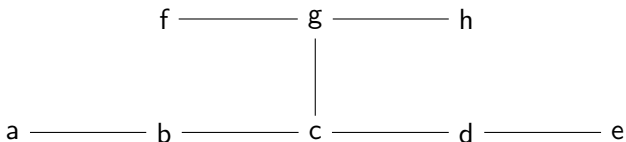
a) Where to wait for Barbara under CBR



b) Rational choices under cautious lexicographic belief

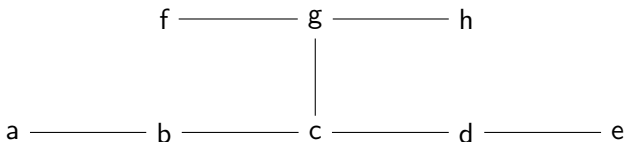


b) Rational choices under cautious lexicographic belief



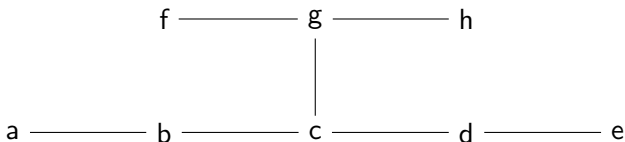
- If I can see Barbara from a, I can also see her from b. However, if Barbara is at c, I can still see her from b but not from a.

b) Rational choices under cautious lexicographic belief



- If I can see Barbara from a, I can also see her from b. However, if Barbara is at c, I can still see her from b but not from a.
- a is weakly dominated.

b) Rational choices under cautious lexicographic belief



- If I can see Barbara from a, I can also see her from b. However, if Barbara is at c, I can still see her from b but not from a.
- a is weakly dominated.
- Similarly, e, f and h are weakly dominated.

b) Rational choices under cautious lexicographic belief

Rational choices under cautious lexicographic beliefs:

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Rational choices under cautious lexicographic beliefs:

- b: optimal for $b_y = (a; c; b; d; e; f; g; h)$. For the first level, a and b give same utility, but for the second level b gives higher utility.

b) Rational choices under cautious lexicographic belief

Rational choices under cautious lexicographic beliefs:

- b: optimal for $b_y = (a; c; b; d; e; f; g; h)$. For the first level, a and b give same utility, but for the second level b gives higher utility.
- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.

b) Rational choices under cautious lexicographic belief

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- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.

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- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.
- g: optimal for $b_y = (f; g; h; a; b; c; d; f)$.

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- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.
- g: optimal for $b_y = (f; g; h; a; b; c; d; f)$.

Irrational choices under cautious lexicographic beliefs:

b) Rational choices under cautious lexicographic belief

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- b: optimal for $b_y = (a; c; b; d; e; f; g; h)$. For the first level, a and b give same utility, but for the second level b gives higher utility.
- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.
- g: optimal for $b_y = (f; g; h; a; b; c; d; f)$.

Irrational choices under cautious lexicographic beliefs:

- a: weakly dominated by b

b) Rational choices under cautious lexicographic belief

Rational choices under cautious lexicographic beliefs:

- b: optimal for $b_y = (a; c; b; d; e; f; g; h)$. For the first level, a and b give same utility, but for the second level b gives higher utility.
- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.
- g: optimal for $b_y = (f; g; h; a; b; c; d; f)$.

Irrational choices under cautious lexicographic beliefs:

- a: weakly dominated by b
- e: weakly dominated by d

b) Rational choices under cautious lexicographic belief

Rational choices under cautious lexicographic beliefs:

- b: optimal for $b_y = (a; c; b; d; e; f; g; h)$. For the first level, a and b give same utility, but for the second level b gives higher utility.
- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.
- g: optimal for $b_y = (f; g; h; a; b; c; d; f)$.

Irrational choices under cautious lexicographic beliefs:

- a: weakly dominated by b
- e: weakly dominated by d
- f: weakly dominated by g

b) Rational choices under cautious lexicographic belief

Rational choices under cautious lexicographic beliefs:

- b: optimal for $b_y = (a; c; b; d; e; f; g; h)$. For the first level, a and b give same utility, but for the second level b gives higher utility.
- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.
- g: optimal for $b_y = (f; g; h; a; b; c; d; f)$.

Irrational choices under cautious lexicographic beliefs:

- a: weakly dominated by b
- e: weakly dominated by d
- f: weakly dominated by g
- h: weakly dominated by g

b) Rational choices under cautious lexicographic belief

Rational choices under cautious lexicographic beliefs:

- b: optimal for $b_y = (a; c; b; d; e; f; g; h)$. For the first level, a and b give same utility, but for the second level b gives higher utility.
- c: optimal for $b_y = (c; a; b; d; e; f; g; h)$.
- d: optimal for $b_y = (e; c; a; b; d; f; g; h)$.
- g: optimal for $b_y = (f; g; h; a; b; c; d; f)$.

Irrational choices under cautious lexicographic beliefs:

- a: weakly dominated by b
- e: weakly dominated by d
- f: weakly dominated by g
- h: weakly dominated by g

Because of symmetry, we have exactly the same rational and irrational choices for Barbara.

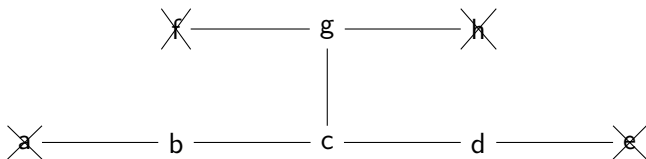
c) Rational choices under common full belief in caution and primary belief in rationality

We can use the Dekel-Fudenberg procedure.

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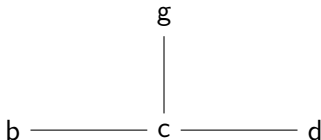
Step 1 Delete all weakly dominated choices for both players in the original game.



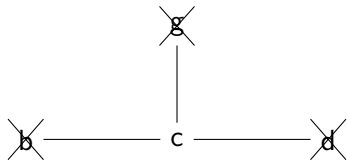
- a, e, f and h are never optimal under a cautious belief.

c) Rational choices under common full belief in caution and primary belief in rationality

Step 2 Delete all strictly dominated choices for both players in the reduced game.

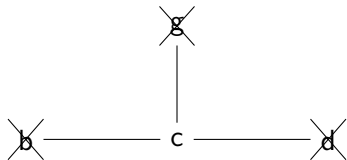


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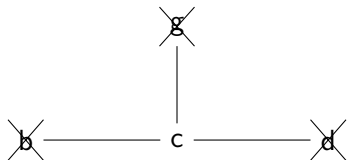
- If I can see Barbara from b, I can also see her from c. However, I can also see Barbara if she is at g or d if I am at c, but not if I am at b. Moreover, being at c gives me an extra utility of 1 because of the resting spot.

c) Rational choices under common full belief in caution and primary belief in rationality



- If I can see Barbara from b , I can also see her from c . However, I can also see Barbara if she is at g or d if I am at c , but not if I am at b . Moreover, being at c gives me an extra utility of 1 because of the resting spot.
- Spot b is strictly dominated.
- Similarly, g and d are strictly dominated.

c) Rational choices under common full belief in caution and primary belief in rationality



- So c for you and c for Barbara are only left. Thus in Step 3 of the Dekel-Fudenberg Procedure there is nothing left to be eliminated.
- Only c is a rational choices under **common full belief in caution and primary belief in rationality**.

d) Epistemic model for solution in c) satisfying CFBCPBR

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Types	$T_y = \{t_y^c\}$ $T_B = \{t_b^c\}$
Beliefs for You	$b_y(t_y^c) = [(c, t_b^c); \frac{1}{7}((a, t_b^c) + \dots + (h, t_b^c))]$
Beliefs for Barbara	$b_b(t_b^c) = [(c, t_y^c); \frac{1}{7}((a, t_y^c) + \dots + (h, t_y^c))]$

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- The belief for both you and Barbara assign positive probability to each choice-type combination given the types we deem possible. → Cautious.
- c is indeed optimal if the other chooses c. Moreover, it is the ONLY optimal choice given the first level of the lexicographic belief, because of the resting spot.

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- c appears in the primary level for both $b_y(t_y^c)$ and $b_b(t_b^c)$ with the respective types t_b^c and t_y^c

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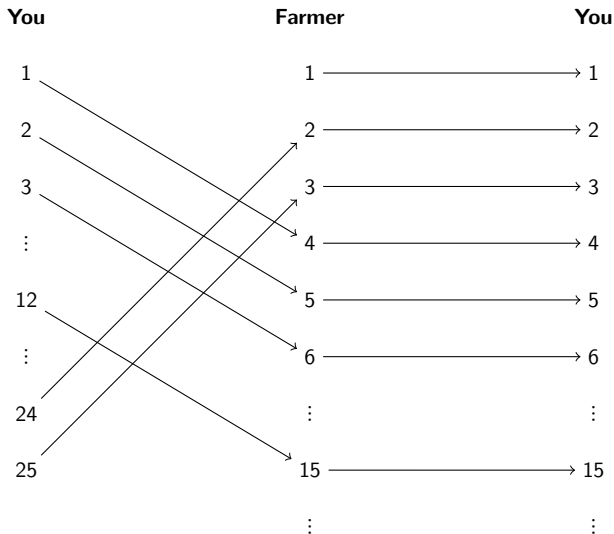
- The belief for both you and Barbara assign positive probability to each choice-type combination given the types we deem possible. → Cautious.
- c is indeed optimal if the other chooses c. Moreover, it is the ONLY optimal choice given the first level of the lexicographic belief, because of the resting spot.
- c appears in the primary level for both $b_y(t_y^c)$ and $b_b(t_b^c)$ with the respective types t_b^c and t_y^c → primary belief in rationality.
- All types express caution and primary belief in rationality → common full belief in caution and primary belief in rationality.

The Game: Stealing an apple

- Five minutes ago you stole an apple from a farmer. You fled into a nearby castle.
- There are 25 rooms in the castle for you to hide in.
- If you end up in the same room as the farmer, or in an adjacent room, you will be caught by the farmer (corners are also adjacent).
- If the farmer catches you, you get a utility of 0. If you manage to save yourself, you get a utility of 1.
- If the farmer catches you, he gets a utility of 1. If he cannot find you, the farmer gets a utility of 0.
- Where to hide from the farmer?

a) Where can we rationally hide from farmer?

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a) Where can we rationally hide from farmer?

- All potential rooms have outgoing, **solid lines**, for both the farmer and yourself.
- Each choice can be rationally made under some first-order belief, for both the farmer and you.
- Each room is rational for you to hide in under some belief expressing common belief in rationality.

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

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- The key here is that we want to minimize our visibility to the farmer.
- Some rooms make you in expectation less visible than others.
- For example: Suppose you hide room 6. Then the farmer will be able to spot you if he goes to room

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- Some rooms make you in expectation less visible than others.
- For example: Suppose you hide room 6. Then the farmer will be able to spot you if he goes to room 1, 2, 6, 7, 11 or 12.

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

- The key here is that we want to minimize our visibility to the farmer.
- Some rooms make you in expectation less visible than others.
- For example: Suppose you hide room 6. Then the farmer will be able to spot you if he goes to room 1, 2, 6, 7, 11 or 12.
- If you instead hide in room 1, the farmer will only be able to find you if he goes to room 1, 2, 6 or 7, but not when he goes to room 11 or 12.

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

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- Some rooms make you in expectation less visible than others.
- For example: Suppose you hide room 6. Then the farmer will be able to spot you if he goes to room 1, 2, 6, 7, 11 or 12.
- If you instead hide in room 1, the farmer will only be able to find you if he goes to room 1, 2, 6 or 7, but not when he goes to room 11 or 12.
- Room 6 is Weakly dominated.

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

1	2	3	4	5
	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

1	2	3	4	5
	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

- By symmetry of the game, it is the case that Rooms 10, 16 and 20 on the one hand and 2, 4, 22 and 24 are weakly dominated as well.

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

1		3		5
	7	8	9	
11	12	13	14	15
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1		3		5
	7	8	9	
11	12	13	14	15
	17	18	19	
21		23		25

- Room 8 is also weakly dominated for me: If I am in Room 8, I am visible for the farmer if he is in room 2, 3, 5, 7, 8, 9, 12, 13 or 14.
- I expect to be less visible in room 3, where the farmer can only spot me from room 2, 3, 4, 7, 8 or 9 and not room 12, 13 or 14.

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- By symmetry, also rooms 12, 14 and 18 are weakly dominated.

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

1		3		5
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b) Where can we rationally hide from farmer under a cautious lexicographic belief?

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11		13		15
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21		23		25

- Room 7 is also weakly dominated for me: If I am in Room 7, I am visible for the farmer if he is in room 1, 2, 3, 6, 7, 8, 11, 12 or 13.
- I expect to be less visible in room 1, where the farmer can only spot me from room 1, 2, 6 or 7 and not room 3, 8, 11, 12 or 13.

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- I expect to be less visible in room 1, where the farmer can only spot me from room 1, 2, 6 or 7 and not room 3, 8, 11, 12 or 13.
- By symmetry, also rooms 9, 17 and 19 are weakly dominated.

b) Where can we rationally hide from farmer under a cautious lexicographic belief?

1		3		5
11		13		15
21		23		25

b) Accompanying beliefs for optimal choices

Room Choice	Belief
1	(8;10;11;19;21;1;...)
3	(6;10;17;19;13;...)
5	(7;19;21;14;8;...)
⋮	

b) Accompanying beliefs for optimal choices

Room Choice	Belief
1	(8;10;11;19;21;1;...)
3	(6;10;17;19;13;...)
5	(7;19;21;14;8;...)
⋮	

- For the belief for which Room 1 is optimal: the rooms considered in the first 5 levels of the lexicographic belief *combined*, allow the farmer to see you in any room *but* room 1.

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1	(8;10;11;19;21;1;...)
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- For the belief for which Room 1 is optimal: the rooms considered in the first 5 levels of the lexicographic belief *combined*, allow the farmer to see you in any room *but* room 1.
- Then definitely Room 1 is the best choice for You, as there is the least chance to get seen under such a lexicographic belief.

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⋮	

- For the belief for which Room 1 is optimal: the rooms considered in the first 5 levels of the lexicographic belief *combined*, allow the farmer to see you in any room *but* room 1.
- Then definitely Room 1 is the best choice for You, as there is the least chance to get seen under such a lexicographic belief.
- Only in the sixth level of the lexicographic belief, a room is considered in which the farmer can see you if you are in Room 1.

c) Rational choices for the farmer

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
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c) Rational choices for the farmer

1	2	3	4	5
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- The farmer's goal is to find you.

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- The farmer's goal is to find you.
- Room 7 weakly dominates being in Room 1, 2 or 6 for him.

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- By symmetry of the game, we have that **9** weakly dominates 4,5 and 10; **17** weakly dominates 16, 21 and 22; **19** weakly dominates 20, 24 and 25.

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- Room 7 weakly dominates being in Room 1, 2 or 6 for him.
- By symmetry of the game, we have that **9** weakly dominates 4,5 and 10; **17** weakly dominates 16, 21 and 22; **19** weakly dominates 20, 24 and 25.
- **12** weakly dominates 11. By symmetry **8** weakly dominates 3; **14** weakly dominates 15; **18** weakly dominates 23.

c) Rational choices for the farmer

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6	7	8	9	10
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16	17	18	19	20
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- The farmer's goal is to find you.
- Room 7 weakly dominates being in Room 1, 2 or 6 for him.
- By symmetry of the game, we have that **9** weakly dominates 4,5 and 10; **17** weakly dominates 16, 21 and 22; **19** weakly dominates 20, 24 and 25.
- **12** weakly dominates 11. By symmetry **8** weakly dominates 3; **14** weakly dominates 15; **18** weakly dominates 23.
- We are left with 7, 9, 17, 19, 12, 8, 14, 18 and 13

c) Rational choices for the farmer

	7	8	9	
	12	13	14	
	17	18	19	

c) Accompanying beliefs for optimal choices of farmer

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Room Choice	Belief
7	(1;13;...)
8	(2;4;13...)
13	(8;12;14;18...)
⋮	

c) Accompanying beliefs for optimal choices of farmer

Room Choice	Belief
7	(1;13;...)
8	(2;4;13...)
13	(8;12;14;18...)
⋮	

- Take the belief for Room 7: Room 13 here is a tie-breaker between rooms 1, 2, 6 and 7.

d) Rational hiding places under common full belief in caution and primary belief in rationality

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	7	8	9	
	12	13	14	
	17	18	19	

- Then staying at 13 will always lead to a zero utility for me. Room 13 is strictly dominated e.g. by a randomization over all corner rooms: 1, 5, 21 and 25: $\frac{1}{4}(1 + 5 + 21 + 25) \geq \frac{3}{4}$.

d) Rational hiding places under common full belief in caution and primary belief in rationality

- If I express common full belief in caution and primary belief in rationality, I must believe the farmer will NEVER choose any of his weakly dominated choices.

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- Then staying at 13 will always lead to a zero utility for me. Room 13 is strictly dominated e.g. by a randomization over all corner rooms: 1, 5, 21 and 25: $\frac{1}{4}(1 + 5 + 21 + 25) \geq \frac{3}{4}$.
- This falls under Step 2 of the Dekel-Fudenberg Procedure.
- We cannot eliminate anything for the farmer at this point.

d) Rational hiding places under common full belief in caution and primary belief in rationality

Your remaining choices after Step 2:

1		3		5
11				15
21		23		25

d) Rational hiding places under common full belief in caution and primary belief in rationality

Your remaining choices after Step 2:

1		3		5
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21		23		25

- After eliminating 13 for You, we can eliminate room 13 for the farmer. It is strictly dominated by $\frac{1}{4}(7 + 9 + 17 + 19)$.

d) Rational hiding places under common full belief in caution and primary belief in rationality

Your remaining choices after Step 2:

1		3		5
11				15
21		23		25

- After eliminating 13 for You, we can eliminate room 13 for the farmer. It is strictly dominated by $\frac{1}{4}(7 + 9 + 17 + 19)$.

d) Rational hiding places under common full belief in caution and primary belief in rationality

Farmer's remaining choices after Step 3:

	7	8	9	
	12		14	
	17	18	19	

d) Rational hiding places under common full belief in caution and primary belief in rationality

Farmer's remaining choices after Step 3:

	7	8	9	
	12		14	
	17	18	19	

- After eliminating 13 for the farmer, there is nothing left to eliminate for You or the farmer.

d) Rational hiding places under common full belief in caution and primary belief in rationality

Farmer's remaining choices after Step 3:

	7	8	9	
	12		14	
	17	18	19	

- After eliminating 13 for the farmer, there is nothing left to eliminate for You or the farmer.
- Elimination procedure ends: You can rationally hide in rooms 1, 3, 5, 11, 15, 21, 23 and 25.