

# Lexicographic Beliefs

## Part III: Assumption of Rationality

Christian W. Bach

EPICENTER & University of Liverpool



# Introduction

- Two ways of **cautious reasoning** have been presented so far:
  - Common Primary Belief in (Caution & Rationality)
  - Common Full Belief in (Caution & Respect of Preferences)
- Respect of preferences imposes restrictions not only on the primary but also on deeper lexicographic levels!
- However, there are other reasonable conditions that could be put on the various lexicographic levels.

# Agenda

- Assumption of Rationality
- Common Assumption of Rationality
- Algorithm
- Existence

# Agenda

- **Assumption of Rationality**
- Common Assumption of Rationality
- Algorithm
- Existence

# Example: Spy Game

## Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but you forgot to ask her to which one.
- The only objective for *you* is to avoid Barbara, since you would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- **Question:** Which pub should *you* go to?

# Example: Spy Game

*Barbara*

		$A_B$	$B_B$	$C_B$
<i>You</i>	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

Under **common full belief in (caution & respect of preferences)**, you go to Pub C:

- As Barbara prefers  $A_B$  to  $B_B$  and you respect her preferences, you must deem her choice  $A_B$  infinitely more likely than  $B_B$ .
- Then, you prefer  $B_y$  to  $A_y$ .
- Hence, you believe that Barbara deems your choice  $B_y$  infinitely more likely than  $A_y$ .
- Consequently, you believe that Barbara prefers  $B_B$  to  $C_B$ , and you must deem  $B_B$  infinitely more likely than  $C_B$ .
- But then the unique optimal choice for you is  $C_y$ .
- However, this is not the only plausible way to reason about Barbara!

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

An alternative way of reasoning:

- For Barbara, both  $A_B$  and  $C_B$  can be optimal for some cautious lexicographic belief, but  $B_B$  can never be optimal.
- Therefore, you deem Barbara's choice  $A_B$  and  $C_B$  infinitely more likely than  $B_B$ .
- But then, your unique optimal choice is  $B_y$ !



# The Underlying Intuition

- If player  $j$ 's choice  $c_j$  is **optimal for some cautious lexicographic belief**, while his choice  $c'_j$  is **not optimal for any cautious lexicographic belief**, then player  $i$  must deem  $c_j$  **infinitely more likely** than  $c'_j$ .
- Player  $i$  is then said to **assume rationality**.
- In other words, player  $i$  deems his opponent  $j$ 's **good choices infinitely more likely** than  $j$ 's **bad choices**.

# How Can this Intuition Be Formalized?

- How can the idea of assuming rationality be formalized in an epistemic model?
- **Attempt:** Type  $t_i$  must deem all choice-type pairs  $(c_j, t_j)$ , where  $c_j$  is **optimal** for  $t_j$  and  $t_j$  is **cautious**, **infinitely more likely** than all choice-type pairs  $(c'_j, t'_j)$  that do **not** have this property.

# The Attempt Does Not Work!

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Attempt:** Type  $t_i$  must deem all choice-type pairs  $(c_j, t_j)$ , where  $c_j$  is **optimal** for  $t_j$  and  $t_j$  is **cautious**, **infinitely more likely** than all choice-type pairs  $(c'_j, t'_j)$  that do **not** have this property.
- Consider the following lexicographic epistemic model:  
 Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B\}$   
 Beliefs:  $b_y(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$  and  $b_B(t_B) = ((C_y, t_y); (B_y, t_y); (A_y, t_y))$
- Your type  $t_y$  satisfies the condition, but does not assume rationality in the intended way.
- Problem:** Choice  $C_B$  can be optimal for Barbara for some cautious type, but your type  $t_y$  does not deem possible any type for Barbara for which  $C_B$  is indeed optimal.
- Remedy:** it is additionally required that you must deem possible a cautious type for Barbara for which  $C_B$  is optimal!

# More Types Are Needed

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B, t'_B\}$

Beliefs:  $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (B_B, t_B); (A_B, t'_B); (B_B, t'_B))$ ,

$b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$ , and  $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$

- For Barbara choices  $A_B$  and  $C_B$  can be optimal for some cautious type.
- Your type  $t_y$  deems possible the cautious type  $t_B$  for which  $A_B$  is optimal as well as the cautious type  $t'_B$  for which  $C_B$  is optimal.
- Your type  $t_y$  deems all choice-type pairs where the type is cautious and the choice is optimal for the type infinitely more likely than all choice-type pairs that do not have this property.
- Indeed, type  $t_y$  assumes rationality in the intended way!

# Assumption of Rationality

## Definition

A cautious type  $t_i$  **assumes rationality**, whenever

- for every choice  $c_j$  that is optimal for some cautious type,  $t_i$  deems possible a cautious type  $t_j$  for which  $c_j$  is indeed optimal,
- $t_i$  deems all choice-type pairs  $(c_j, t_j)$ , where  $t_j$  is cautious and  $c_j$  optimal for  $t_j$ , infinitely more likely than all choice-type pairs not satisfying this property.

## Intuition:

A player deems **good** choices infinitely more likely than **bad** choices.

## Remark:

**Assumption of rationality** can only be defined for **cautious types**.

# Assumption and Primary Belief in Rationality

**Observation.** If *Alice* is cautious and assumes *Bob's* rationality, then she also primarily believes in *Bob's* rationality.

- Suppose that  $t_{Alice}$  is cautious and assumes *Bob's* rationality.
- Then,  $t_{Alice}$  considers all choice-type pairs where the choice is optimal for the type infinitely more likely than other choice-type pairs.
- In particular, the support of  $t_{Alice}$ 's primary belief can then only contain choice-type pairs such that the choice is optimal for the type.

# Assumption and Respect of Preferences

**Observation.** There is no general relationship between assuming rationality and respecting preferences.

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B\}$

Beliefs:  $b_y(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$  and  $b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$

- Your type  $t_y$  respects Barbara's preferences, but does not assume her rationality.

- Consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B, t'_B\}$

Beliefs:  $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (B_B, t_B); (A_B, t'_B); (B_B, t'_B))$ ,

$b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$ , and  $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$

- Your type  $t_y$  assumes Barbara's rationality, but does not respect her preferences.
- Indeed, for  $t_B$  choice  $B_B$  is better than  $C_B$ , yet  $t_y$  deems  $(C_B, t_B)$  infinitely more likely than  $(B_B, t_B)$ .

# Remark

- It is always possible to satisfy **respect of preferences** and **assumption of rationality**.
- **Intuition:** A type's lexicographic belief deems optimal choices infinitely more likely than the non-optimal choices, yet orders the non-optimal choices as required by respect of preference.



# Agenda

- Assumption of Rationality
- **Common Assumption of Rationality**
- Algorithm
- Existence

# Assuming (Rationality & Assumption of Rationality)

## Definition

A cautious type  $t_i$  **assumes (rationality & assumption of rationality)**, whenever

- for every choice  $c_j$  that is optimal for some cautious type that assumes  $i$ 's rationality, type  $t_i$  deems possible a cautious type  $t_j$  that assumes  $i$ 's rationality and for which  $c_j$  is indeed optimal;
- type  $t_i$  deems all choice-type pairs  $(c_j, t_j)$ , where  $t_j$  is cautious, assumes  $i$ 's rationality, and  $c_j$  is optimal for  $t_j$ , infinitely more likely than all choice-type pairs not satisfying this property.

# Common Assumption of Rationality

## Definition

- A cautious type  $t_i$  expresses **1-fold assumption of rationality**, whenever  $t_i$  assumes rationality.
- For all  $k \geq 2$ , a cautious type  $t_i$  expresses  **$k$ -fold assumption of rationality**, whenever
  - for every choice  $c_j$  that is optimal for some cautious type that expresses up to  $(k - 1)$ -fold assumption of rationality, type  $t_i$  deems possible a cautious type  $t_j$  that expresses up to  $(k - 1)$ -fold assumption of rationality and for which  $c_j$  is indeed optimal;
  - type  $t_i$  deems all choice-type pairs  $(c_j, t_j)$  where  $t_j$  is cautious, expresses up to  $(k - 1)$ -fold assumption of rationality, and  $c_j$  is optimal for  $t_j$ , infinitely more likely than all choice-type pairs not satisfying this property.
- A cautious type  $t_i$  expresses **common assumption of rationality**, whenever  $t_i$  expresses  $k$ -fold assumption of rationality for all  $k \geq 1$ .

# Example: Spy Game

## Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but you forgot to ask her to which one.
- The only objective for *you* is to avoid Barbara, since you would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- **Question:** Which pub should *you* go to?

# Example: Spy Game

		<i>Barbara</i>		
		$A_B$	$B_B$	$C_B$
<i>You</i>	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y\}$  and  $T_{Barbara} = \{t_B, t'_B\}$

Beliefs:  $b_y(t_y) = ((A_B, t_B); (C_B, t'_B); (C_B, t_B); (B_B, t_B); (A_B, t'_B); (B_B, t'_B))$ ,

$b_B(t_B) = ((B_y, t_y); (C_y, t_y); (A_y, t_y))$ , and  $b_B(t'_B) = ((A_y, t_y); (B_y, t_y); (C_y, t_y))$

- Your type  $t_y$  assumes Barbara's rationality.
- Barbara's type  $t_B$  does not assume your rationality: although your choices  $A_y$  and  $C_y$  are optimal for some cautious belief,  $t_B$  does not deem possible types for you for which  $A_y$  and  $C_y$  are optimal. (analogous for type  $t'_B$ )
- Thus, type  $t_y$  only deems possible types for Barbara that do not assume rationality.
- However, Barbara's choice  $A_B$  is optimal for some type that is cautious and assumes your rationality.

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$

Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \dots)$ ,  $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \dots)$ ,  
and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \dots)$

Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \dots)$  and  
 $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \dots)$

- Type  $t_B^A$  does assume your rationality, and Barbara's choice  $A_B$  is optimal for  $t_B^A$ .
- Thus, Barbara's choice  $A_B$  is optimal for some cautious type that assumes your rationality.
- Note that type  $t_B^C$  also assumes your rationality.
- Observe that your type  $t_y^B$  assumes Barbara's rationality, but your types  $t_y^A$  and  $t_y^C$  do not assume her rationality.

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$

Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \dots)$ ,  $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \dots)$ ,  
and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \dots)$

Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \dots)$  and  
 $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \dots)$

- It is now shown that type  $t_y^B$  expresses common assumption of rationality.
- Type  $t_y^B$  expresses 1-fold assumption of rationality:
  - Only Barbara's choices  $A_B$  and  $C_B$  can be optimal for a cautious belief: type  $t_y^B$  deems possible cautious types  $t_B^A$  and  $t_B^C$  for which  $A_B$  and  $C_B$ , respectively, are optimal.
  - Type  $t_y^B$  deems  $(A_B, t_B^A)$  and  $(C_B, t_B^C)$  infinitely more likely than the rest.
- Note that only choice  $B_y$  can be optimal for you, if you express 1-fold assumption of rationality.



# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$

Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \dots)$ ,  $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \dots)$ ,  
and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \dots)$

Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \dots)$  and  
 $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \dots)$

- Type  $t_y^B$  expresses 2-fold assumption of rationality:
  - Barbara's types  $t_B^A$  and  $t_B^C$  express 1-fold assumption of rationality
  - Thus, Barbara's choices  $A_B$  and  $C_B$  are optimal for cautious types that express 1-fold assumption of rationality.
  - Type  $t_y^B$  deems possible these types  $t_B^A$  and  $t_B^C$ .
  - Type  $t_y^B$  deems  $(A_B, t_B^A)$  and  $(C_B, t_B^C)$  infinitely more likely than the rest.

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$

Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \dots)$ ,  $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \dots)$ ,  
and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \dots)$

Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \dots)$  and  
 $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \dots)$

- Note that in order to express 2-fold assumption of rationality Barbara must deem your choice  $B_y$  infinitely more likely than your other choices.
- Barbara's type  $t_B^A$  expresses 2-fold assumption of rationality:
  - Only your choice  $B_y$  is optimal for a cautious type that expresses 1-fold assumption of rationality.
  - Type  $t_B^A$  deems possible your type  $t_y^B$  that is cautious, expresses 1-fold assumption of rationality, and for which your choice  $B_y$  is optimal.
  - Type  $t_B^A$  deems  $(B_y, t_y^B)$  infinitely more likely than the rest.

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Indeed, consider the following lexicographic epistemic model:

Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$

Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \dots)$ ,  $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \dots)$ ,  
and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \dots)$

Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \dots)$  and  
 $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \dots)$

- Type  $t_y^B$  expresses 3-fold assumption of rationality:
  - Barbara can only rationally make choice  $A_B$  under up to 2-fold assumption of rationality.
  - Type  $t_y^B$  deems possible Barbara's type  $t_B^A$  that is cautious, expresses up to 2-fold assumption of rationality, and for which  $A_B$  is optimal.
  - Type  $t_y^B$  deems  $(A_B, t_B^A)$  infinitely more likely than the rest.
- By continuing in this fashion, it can be concluded that your type  $t_y^B$  expresses  $k$ -fold assumption of rationality for every  $k \geq 1$ : hence,  $t_y^B$  entertains common assumption of rationality.
- Consequently, you can rationally and cautiously only go to Pub B.

# Agenda

- Assumption of Rationality
- Common Assumption of Rationality
- **Algorithm**
- Existence

# Assumption of Rationality

## Definition

A cautious type  $t_i$  **assumes rationality**, whenever

- for every choice  $c_j$  that is optimal for some cautious type,  $t_i$  deems possible a cautious type  $t_j$  for which  $c_j$  is indeed optimal,
- $t_i$  deems all choice-type pairs  $(c_j, t_j)$ , where  $t_j$  is cautious and  $c_j$  optimal for  $t_j$ , infinitely more likely than all choice-type pairs not satisfying this property.

# Common Assumption of Rationality

## Definition

- A cautious type  $t_i$  expresses **1-fold assumption of rationality**, whenever  $t_i$  assumes rationality.
- For all  $k \geq 2$ , a cautious type  $t_i$  expresses  **$k$ -fold assumption of rationality**, whenever
  - for every choice  $c_j$  that is optimal for some cautious type that expresses up to  $(k - 1)$ -fold assumption of rationality, type  $t_i$  deems possible a cautious type  $t_j$  that expresses up to  $(k - 1)$ -fold assumption of rationality and for which  $c_j$  is indeed optimal;
  - type  $t_i$  deems all choice-type pairs  $(c_j, t_j)$  where  $t_j$  is cautious, expresses up to  $(k - 1)$ -fold assumption of rationality, and  $c_j$  is optimal for  $t_j$ , infinitely more likely than all choice-type pairs not satisfying this property.
- A cautious type  $t_i$  expresses **common assumption of rationality**, whenever  $t_i$  expresses  $k$ -fold assumption of rationality for all  $k \geq 1$ .

# Towards an Algorithm

**Step 1.** 1-fold assumption of rationality: What choices can  $i$  rationally and cautiously make when assuming rationality?

- First, note that  $i$  does not choose – by Lexicographic Pearce’s Lemma – a weakly dominated choice due to caution.
- If  $i$  assumes  $j$ ’s rationality, then  $i$  deems all choices that are optimal for some cautious belief infinitely more likely than all choices that are not optimal for any cautious belief.
- Again – by Lexicographic Pearce’s Lemma – optimal choices under caution are equivalent with non-weakly-dominated choices.
- Hence, if  $i$  assumes  $j$ ’s rationality, then  $i$  deems all non-weakly-dominated choices of  $j$  infinitely more likely than all weakly dominated choices of  $j$ .
- Let  $C_j^1$  be the set of non-weakly-dominated choices for  $j$ : Then,  $i$  deems all choices inside  $C_j^1$  infinitely more likely than all choices outside  $C_j^1$
- Let  $b_i^{lex} = (b_i^1; b_i^2 \dots; b_i^K)$  be  $i$ ’s lexicographic belief.
- Then, there exists some level  $L < K$  such that
  - 1** the beliefs  $b_i^1, \dots, b_i^L$  only assign positive probability to choices in  $C_j^1$
  - 2** all choices in  $C_j^1$  receive positive probability in some belief from  $b_i^1, \dots, b_i^L$
- Consequently,  $(b_i^1; \dots; b_i^L)$  forms a cautious lexicographic belief on  $C_j^1$ .
- Moreover, every choice  $c_i$  which is optimal under  $b_i^{lex}$  must also be optimal under the truncated cautious belief  $(b_i^1; \dots; b_i^L)$  on  $C_j^1$ , i.e. must not be weakly dominated on  $C_j^1$ !

# Towards an Algorithm

**Conclusion:** If  $i$  is cautious and assumes  $j$ 's rationality, then every optimal choice  $c_i$

- must not be weakly dominated in the original game
- must not be weakly dominated in the reduced game, obtained after 1 round of weak dominance

i.e. every optimal choice  $c_i$  survives 2 rounds of weak dominance!



# Towards an Algorithm

**Step 2.** up to 2-fold assumption of rationality: What choices can  $i$  rationally and cautiously make under up to 2-fold assumption of rationality?

- If  $c_j$  is optimal for some cautious belief  $b_j^{lex}$  that assumes  $i$ 's rationality, while  $c'_j$  is not, then  $i$  deems  $c_j$  infinitely more likely than  $c'_j$ .
- Let  $C_j^2$  be the set of  $j$ 's choices that are optimal for some cautious belief that assumes  $i$ 's rationality: Then,  $i$  deems all choices inside  $C_j^2$  infinitely more likely than all choices outside  $C_j^2$ .
- Then, by Lexicographic Pearce's Lemma, every optimal choice for  $i$  must not be weakly dominated on  $C_j^2$ .
- By Step 1  $C_j^2$  contains precisely those choices that survive 2 rounds of weak dominance.
- Therefore, every optimal choice for  $i$  must not be weakly dominated within the reduced game obtained after 2 rounds of weak dominance, i.e. must survive 3 rounds of weak dominance.

# Towards an Algorithm

**In general:** If  $i$  is cautious and expresses up to  $k$ -fold assumption of rationality, then every optimal choice for  $i$  must survive  $(k+1)$  rounds of weak dominance.

# Algorithm

## Iterated Weak Dominance

- **Step 1.** Within the original game, eliminate all choices that are weakly dominated.
- **Step 2.** Within the reduced game obtained after step 1, eliminate all choices that are weakly dominated.
- etc, until no further choices can be eliminated.

# Algorithmic Characterization

## Theorem

*For all  $k \geq 1$ , the choices that can rationally be made by a cautious type that expresses up to  $k$ -fold assumption of rationality are exactly those choices that survive the first  $k + 1$  rounds of Iterated Weak Dominance.*

## Corollary

*The choices that can rationally be made by a cautious type that expresses common assumption of rationality are exactly those choices that survive Iterated Weak Dominance.*

# Properties of the Algorithm

- Iterated Weak Dominance stops after **finitely** many rounds.
- Iterated Weak Dominance always yields a **non-empty** set of choices for both players.
- The **order** and **speed** of elimination **crucially matter** for the eventual output of the algorithm!

# Example: Spy Game

## Story

- *You* would like to go to a pub to read your book.
- *Barbara* is going to a pub as well, but you forgot to ask her to which one.
- The only objective for *you* is to avoid Barbara, since you would like to read your book in silence.
- *Barbara* prefers *Pub A* to *Pub B*, and *Pub B* to *Pub C*.
- Besides, *Barbara* suspects you to have an affair and would thus like to spy on you.
- Spying is only possible from *Pub A* to *Pub C*, or vice versa.
- *Barbara* derives additional utility of 3 from spying.
- **Question:** Which pub should *you* go to?

# Example: Spy Game

*Barbara*

		$A_B$	$B_B$	$C_B$
<i>You</i>	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

# Example: Spy Game

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

## First Order of Elimination

Step 1. Eliminate  $B_B$



# Example: Spy Game

		<i>Barbara</i>	
		$A_B$	$C_B$
<i>You</i>	$A_y$	0, 3	1, 4
	$B_y$	1, 3	1, 1
	$C_y$	1, 6	0, 1

## First Order of Elimination

Step 2. Only eliminate  $A_y$

# Example: Spy Game

		<i>Barbara</i>	
		$A_B$	$C_B$
<i>You</i>	$B_y$	1, 3	1, 1
	$C_y$	1, 6	0, 1

## First Order of Elimination

Step 3. Eliminate  $C_B$

# Example: Spy Game

		<i>Barbara</i>	
		$A_B$	
<i>You</i>	$B_y$	1, 3	
	$C_y$	1, 6	

## First Order of Elimination

$B_y$  and  $C_y$  survive for you!

# Example: Spy Game

		<i>Barbara</i>		
		$A_B$	$B_B$	$C_B$
<i>You</i>	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

## Second Order of Elimination

Step 1. Eliminate  $B_B$

# Example: Spy Game

		<i>Barbara</i>	
		$A_B$	$C_B$
<i>You</i>	$A_y$	0, 3	1, 4
	$B_y$	1, 3	1, 1
	$C_y$	1, 6	0, 1

## Second Order of Elimination

Step 2. Eliminate  $A_y$  and  $C_y$

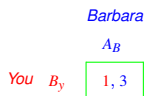
# Example: Spy Game

		<i>Barbara</i>	
		$A_B$	$C_B$
<i>You</i> $B_y$		1, 3	1, 1

## Second Order of Elimination

Step 3. Eliminate  $C_B$

# Example: Spy Game



## Second Order of Elimination

Only  $B_y$  survives for you!

# Agenda

- Assumption of Rationality
- Cautious Reasoning
- Algorithm
- **Existence**



# Existence

- There is no easy iterative procedure delivering a type that expresses common assumption of rationality.
- Since the non-emptiness of the algorithm ensures the existence of a choice surviving it which in turn can be made under common assumption of rationality by the preceding theorem, it is always possible to construct an epistemic model containing a type that expresses common assumption of rationality!

## Theorem

*Let  $\Gamma$  be some finite two player game. Then, there exists a lexicographic epistemic model which contains a type  $t_i$  that expresses common assumption of rationality.*

# Example: Take a Seat

## Story

- *Barbara* and *you* are the only ones to take an exam.
- Both must choose a seat.
- If both choose the same seat, then with probability 0.5 you get the seat you want, and with probability 0.5 you get the one horizontally next to it.
- In order to pass the exam *you* must be able copy from *Barbara*, and the same applies to her.
- A person can only copy from the other person if seated horizontally next or diagonally behind the latter.

# Example: Take a Seat

## Story (continued)

- The probabilities of successful copying for the respective seats are given in percentages:  
 $a = 0, b = 10, c = d = 20, e = f = 45, g = h = 95$
- The objective is to maximize the expected percentage of successful copying.
- **Question:** What seats can you **rationally** and **cautiously** choose under **common assumption of rationality**?

# Example: Take a Seat

		Barbara							
		$a_B$	$b_B$	$c_B$	$d_B$	$e_B$	$f_B$	$g_B$	$h_B$
You	$a_Y$	5, 5	0, 10	0, 0	0, 20	0, 0	0, 0	0, 0	0, 0
	$b_Y$	10, 0	5, 5	0, 20	0, 0	0, 0	0, 0	0, 0	0, 0
	$c_Y$	0, 0	20, 0	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
	$d_Y$	20, 0	0, 0	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
	$e_Y$	0, 0	0, 0	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
	$f_Y$	0, 0	0, 0	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
	$g_Y$	0, 0	0, 0	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
	$h_Y$	0, 0	0, 0	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

# Example: Take a Seat

		Barbara							
		$a_B$	$b_B$	$c_B$	$d_B$	$e_B$	$f_B$	$g_B$	$h_B$
You	$a_Y$	5, 5	0, 10	0, 0	0, 20	0, 0	0, 0	0, 0	0, 0
	$b_Y$	10, 0	5, 5	0, 20	0, 0	0, 0	0, 0	0, 0	0, 0
	$c_Y$	0, 0	20, 0	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
	$d_Y$	20, 0	0, 0	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
	$e_Y$	0, 0	0, 0	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
	$f_Y$	0, 0	0, 0	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
	$g_Y$	0, 0	0, 0	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
	$h_Y$	0, 0	0, 0	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

## Round 1.

- In the full game  $a_Y$  and  $b_Y$  are weakly dominated by  $\frac{1}{2}c_Y + \frac{1}{2}d_Y$ .
- Eliminate  $a_Y$  and  $b_Y$ , as well as  $a_B$  and  $b_B$  by symmetry.

# Example: Take a Seat

		Barbara					
		$c_B$	$d_B$	$e_B$	$f_B$	$g_B$	$h_B$
You	$c_Y$	20, 20	20, 20	0, 0	0, 45	0, 0	0, 0
	$d_Y$	20, 20	20, 20	0, 45	0, 0	0, 0	0, 0
	$e_Y$	0, 0	45, 0	45, 45	45, 45	0, 0	0, 95
	$f_Y$	45, 0	0, 0	45, 45	45, 45	0, 95	0, 0
	$g_Y$	0, 0	0, 0	0, 0	95, 0	95, 95	95, 95
	$h_Y$	0, 0	0, 0	95, 0	0, 0	95, 95	95, 95

## Round 2.

- In the reduced game  $c_Y$  and  $d_Y$  are weakly dominated by  $\frac{1}{2}e_Y + \frac{1}{2}f_Y$ .
- Eliminate  $c_Y$  and  $d_Y$ , as well as  $c_B$  and  $d_B$  by symmetry.

# Example: Take a Seat

		Barbara			
		$e_B$	$f_B$	$g_B$	$h_B$
You	$e_Y$	45, 45	45, 45	0, 0	0, 95
	$f_Y$	45, 45	45, 45	0, 95	0, 0
	$g_Y$	0, 0	95, 0	95, 95	95, 95
	$h_Y$	95, 0	0, 0	95, 95	95, 95

## Round 3.

- In the reduced game  $e_Y$  and  $f_Y$  are weakly dominated by  $\frac{1}{2}g_Y + \frac{1}{2}h_Y$ .
- Eliminate  $e_Y$  and  $f_Y$ , as well as  $e_B$  and  $f_B$  by symmetry.

# Example: Take a Seat

		<i>Barbara</i>	
		<i>g<sub>B</sub></i>	<i>h<sub>B</sub></i>
<i>You</i>	<i>g<sub>Y</sub></i>	95, 95	95, 95
	<i>h<sub>Y</sub></i>	95, 95	95, 95

## Round 4.

- No more choices can be eliminated.
- You can rationally and cautiously choose seats  $g$  and  $h$  under common assumption of rationality.



# Example: Take a Seat

**Intuition:** Why does **common assumption of rationality** lead to a different conclusion as **common full belief in (caution & respect of preferences)**?

## ■ First step of reasoning

- Not that both choices  $a$  and  $b$  are irrational, yet  $b$  is better than  $a$ .
- Under **common assumption of rationality** it is thus not distinguished between  $a$  and  $b$ , however under **common full belief in (caution & respect of preferences)** it is.

## ■ Second step of reasoning

- If you believe Barbara to reason in line with the first step, then  $c$  and  $d$  can no longer be optimal, yet  $c$  is better than  $d$ .
- Under **common assumption of rationality** it is not distinguished between  $c$  and  $d$ , however under **common full belief in (caution & respect of preferences)** it is.

## ■ Third step of reasoning

- If you believe Barbara to reason in line with the first and the second step, then  $e$  and  $f$  can no longer be optimal, yet  $f$  is better than  $e$ .
- Under **common assumption of rationality** it is not distinguished between  $e$  and  $f$ , however under **common full belief in (caution & respect of preferences)** it is.

## ■ Fourth step of reasoning

- If you believe Barbara to reason in line with the first, the second and the fourth step, then  $g$  and  $h$  can no longer be optimal, yet  $g$  is better than  $h$ .
- Under **common assumption of rationality**  $g$  and  $h$  are both optimal, while under **common full belief in (caution & respect of preferences)** only  $g$  remains optimal.

# There Exists No Related Equilibrium Notion

- The **correct beliefs** assumption implicit in any **equilibrium** notion seems to be **at odds** with **common assumption of rationality**.
- As illustration consider the lexicographic epistemic model of the *Spy Game* again.

		Barbara		
		$A_B$	$B_B$	$C_B$
You	$A_y$	0, 3	1, 2	1, 4
	$B_y$	1, 3	0, 2	1, 1
	$C_y$	1, 6	1, 2	0, 1

- Types:  $T_{you} = \{t_y^A, t_y^B, t_y^C\}$  and  $T_{Barbara} = \{t_B^A, t_B^C\}$   
 Beliefs for you:  $b_y(t_y^A) = ((C_B, t_B^C); (B_B, t_B^C); (A_B, t_B^A); \dots)$ ,  $b_y(t_y^B) = ((A_B, t_B^A); (C_B, t_B^C); (B_B, t_B^A); \dots)$ ,  
 and  $b_y(t_y^C) = ((A_B, t_B^A); (B_B, t_B^A); (C_B, t_B^C); \dots)$   
 Beliefs for Barbara:  $b_B(t_B^A) = ((B_y, t_y^B); (C_y, t_y^C); (A_y, t_y^A); \dots)$  and  
 $b_B(t_B^C) = ((A_y, t_y^A); (B_y, t_y^B); (C_y, t_y^C); \dots)$
- Recall that  $t_y^B$  express **common assumption of rationality**.
- However,  $t_y^B$  deems it possible that Barbara is **not correct** about the his type!