Universal type space

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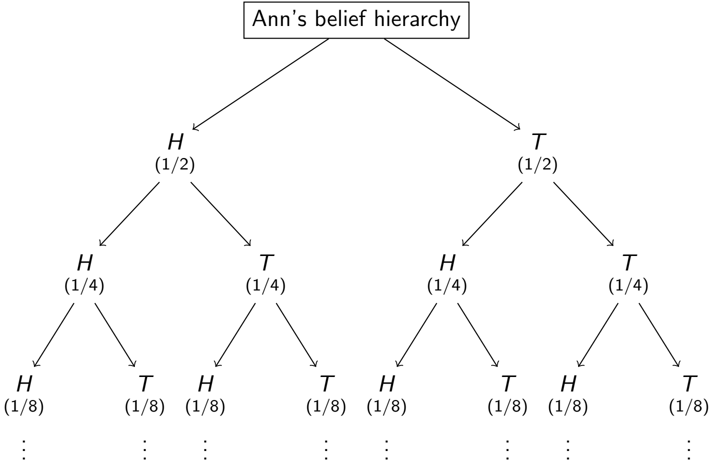
What is a belief hierarchy?

- Our main object of study is **belief hierarchies**.
- A belief hierarchy describes the player's beliefs about
  - the opponents' choices (1st order beliefs),
  - the opponents' choices and 1st order beliefs (2nd order beliefs),
  - the opponents' choices and 1st order beliefs and 2nd order beliefs (3rd order beliefs),
  - ...
- Formally it is an infinite sequence of probability measures.
- Belief hierarchies are actual objects (contrary to the types).
- So far, we have started with the types.
- Now, we start directly from the belief hierarchies.
- **First question:** Can I construct a type space that induces all the belief hierarchies? YES.
- **Second question:** If I take a finite type space, is it a special case of the "large" type space that I constructed above? YES.

Matching pennies example

- Ann's 1st order beliefs: she believes (with equal prob) that
  - Bob will play H
  - Bob will play T
- Ann's 2nd order beliefs: she believes (with equal prob) that
  - Bob will play H and he believes that Ann will play H
  - Bob will play H and he believes that Ann will play T
  - Bob will play T and he believes that Ann will play H
  - Bob will play T and he believes that Ann will play T
- Ann's 3nd order beliefs: she believes (with equal prob) that
  - Bob will play H and he believes that Ann will play H and he believes that she believes that Bob will play H
  - Bob will play H and he believes that Ann will play H and he believes that she believes that Bob will play T
  - Bob will play H and he believes that Ann will play T and he believes that she believes that Bob will play H
  - Bob will play H and he believes that Ann will play T and he believes that she believes that Bob will play T
  - and so on.

Matching pennies example



Formal definition

$$\Theta_a^0 \quad := \quad C_b$$

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1st: $\pi_a^1 \in \Delta(\Theta_a^0) \quad := \quad \Delta(C_b)$

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$$\text{1st: } \pi_a^1 \in \Delta(\Theta_a^0) := \Delta(C_b)$$
$$\Theta_a^1 := \Theta_a^0 \times \Delta(\Theta_b^0)$$

Formal definition

$$
\begin{aligned}
\text{1st: } \pi_a^1 \in \Delta(\Theta_a^0) &:= \Delta(C_b) \\
\text{2nd: } \pi_a^2 \in \Delta(\Theta_a^1) &:= \Delta(\Theta_a^0 \times \Delta(\Theta_b^0))
\end{aligned}
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&:= \Delta(\underbrace{\Theta_a^0 \times \Delta(\Theta_b^0)}_{\Theta_a^1} \times \Delta(\Theta_b^1)) \\
&\vdots \\
\Theta_a^k &:= \Theta_a^{k-1} \times \Delta(\Theta_b^{k-1}) \\
&:= \underbrace{\Theta_a^0 \times \Delta(\Theta_b^0) \times \cdots \times \Delta(\Theta_b^{k-2})}_{\Theta_a^{k-1}} \times \Delta(\Theta_b^{k-1})
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&\;\vdots \\
(k+1)\text{-th: } \pi_a^{k+1} \in \Delta(\Theta_a^k) &:= \Delta(\Theta_a^{k-1} \times \Delta(\Theta_b^{k-1})) \\
&:= \Delta(\underbrace{\Theta_a^0 \times \Delta(\Theta_b^0) \times \cdots \times \Delta(\Theta_b^{k-2})}_{\Theta_a^{k-1}} \times \Delta(\Theta_b^{k-1})) \\
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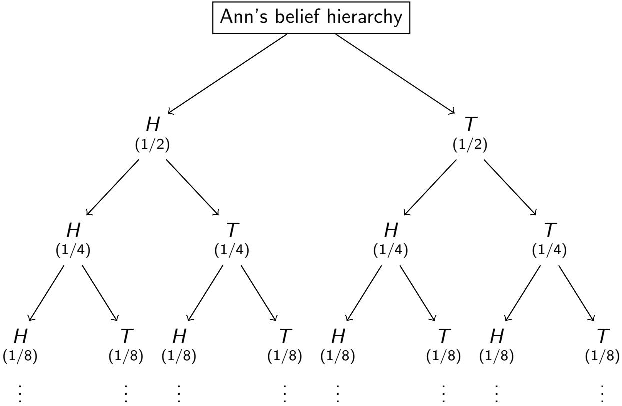
&\vdots
\end{aligned}$$

$$\boxed{H_a^0 := \Delta(\Theta_a^0) \times \Delta(\Theta_a^1) \times \cdots \times \Delta(\Theta_a^k) \times \cdots}$$

Matching pennies example



- Which is Ann's 1st order belief?
- Which is Ann's 2nd order belief?
- Which is Ann's 3rd order belief?

Coherency

- Take Ann's belief hierarchy such that:
  - $\pi_a^1 = \left(\frac{1}{2} \otimes H; \frac{1}{2} \otimes T\right)$
  - $\pi_a^2 = \left(\frac{1}{2} \otimes (H, \pi_b^1); \frac{1}{4} \otimes (H, \tilde{\pi}_b^1); \frac{1}{4} \otimes (T, \tilde{\pi}_b^1)\right)$

What is wrong with this?

Coherency

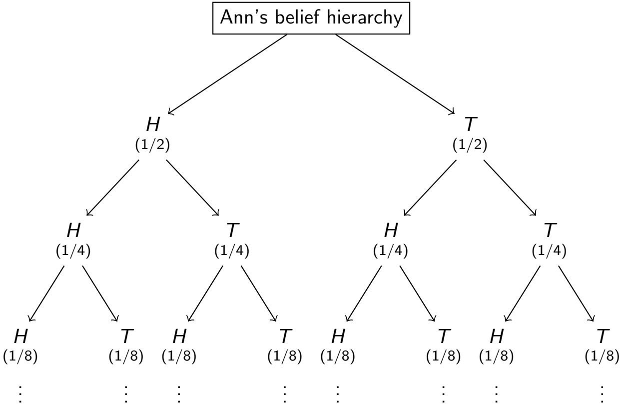
- Take Ann's belief hierarchy such that:
  - $\pi_a^1 = \left(\frac{1}{2} \otimes H; \frac{1}{2} \otimes T\right)$
  - $\pi_a^2 = \left(\frac{1}{2} \otimes (H, \pi_b^1); \frac{1}{4} \otimes (H, \tilde{\pi}_b^1); \frac{1}{4} \otimes (T, \tilde{\pi}_b^1)\right)$

  What is wrong with this?
- Higher order beliefs should not contradict lower order beliefs.
- A belief hierarchy $(\pi_a^1, \pi_a^2, \dots)$ is **coherent** if for every $k \geq 0$

$$\boxed{\text{marg}_{\Theta_a^k} \pi_a^{k+2} = \pi_a^{k+1}}$$

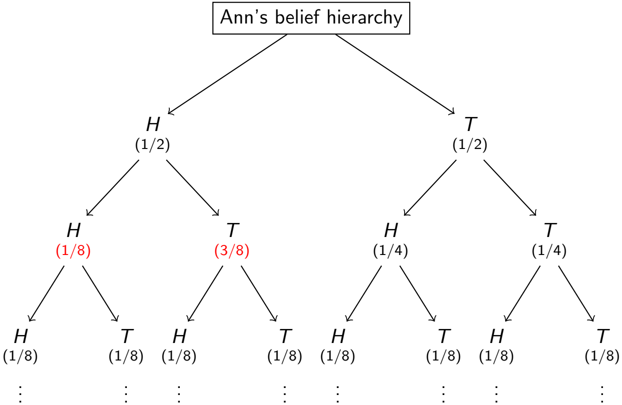
- The set of coherent belief hierarchies is denoted by H_a^c.
- Obviously $H_a^c \subseteq H_a^0$.

Matching pennies example

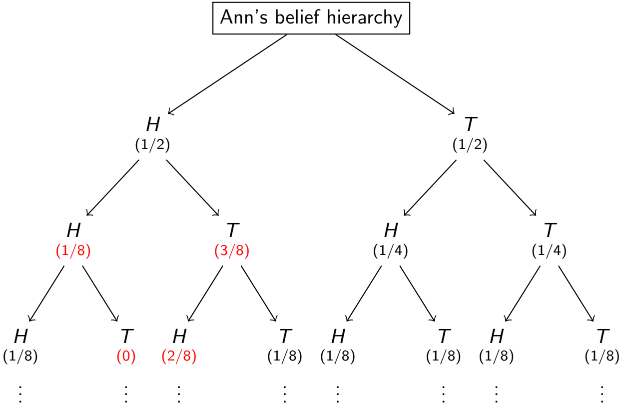


- Is this belief hierarchy coherent?

Matching pennies example



- Is this belief hierarchy coherent?
- How about this one?



- Is this belief hierarchy coherent?
- How about this one?
- And this one?

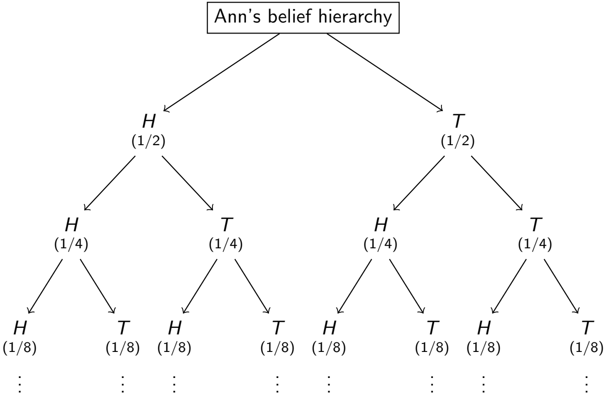
First result (Brandenburger & Dekel, 1993)

Proposition (informal statement)

(i) *If a belief hierarchy of Ann is coherent, it can be uniquely associated with a belief of hers over combinations of Bob's choices and belief hierarchies.*

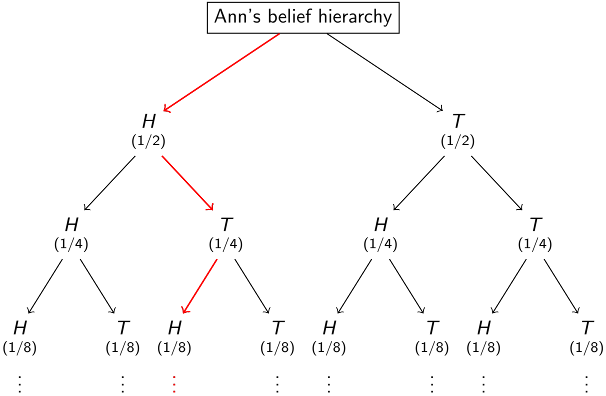
(ii) *For every belief of Ann over combinations of Bob's choices and belief hierarchies, there is a unique belief hierarchy of hers associated with it.*

Matching pennies example



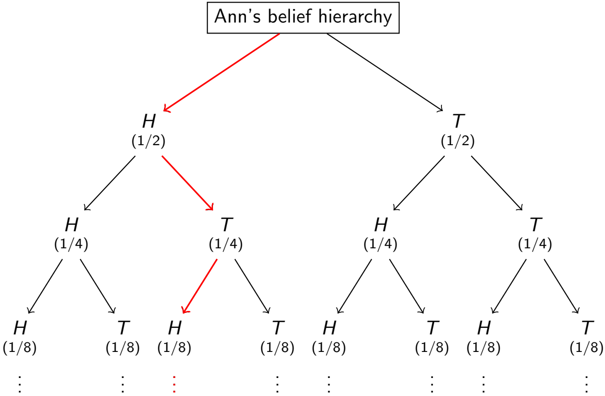
- What does the previous result say?

Matching pennies example



- What does the previous result say?
- Each leaf describes a choice and a belief hierarchy of Bob. Which belief hierarchy?

Matching pennies example



- What does the previous result say?
- Each leaf describes a choice and a belief hierarchy of Bob. Which belief hierarchy?
- Set of leaves: $C_b \times H_b^0 = \Theta_a^0 \times \Delta(\Theta_b^0) \times \Delta(\Theta_b^1) \times \cdots$

First result (Brandenburger & Dekel, 1993)

- Set of leaves: $\boxed{C_b \times H_b^0 = \Theta_a^0 \times \Delta(\Theta_b^0) \times \Delta(\Theta_b^1) \times \cdots}$
- Take a belief $\pi_a \in \Delta(C_a \times H_b^0)$ (it resembles a type)
- **BD (part ii):** We can retrieve one hierarchy:
  - $\pi_a^1 = \text{marg}_{\Theta_a^0} \pi_a$
  - $\pi_a^2 = \text{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0)} \pi_a$
  - $\pi_a^3 = \text{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0) \times \Delta(\Theta_b^1)} \pi_a$
- Take a coherent belief hierarchy $(\pi_a^1, \pi_a^2, \dots)$.
- **BD (part i):** We can find one belief $\pi_a \in \Delta(C_a \times H_b^0)$:
  - $\pi_a^1 = \text{marg}_{\Theta_a^0} \pi_a$
  - $\pi_a^2 = \text{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0)} \pi_a$
  - $\pi_a^3 = \text{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0) \times \Delta(\Theta_b^1)} \pi_a$

Proposition (formal statement)

There exists a homeomorphism $f_a : H_a^c \to \Delta(C_b \times H_b^0)$.

Certainty in the opponent's coherency

- Since we have associated each coherent belief hierarchy in H_a^c with a belief in $\Delta(C_b \times H_b^0)$ we can work with the latter. Why is this useful?
- **Advantage:** A belief $\pi_a \in \Delta(C_b \times H_b^0)$ expresses Ann's beliefs about Bob's entire belief hierarchies, whereas $(\pi_a^1, \pi_a^2, \dots) \in H_a^c$ only expresses Ann's beliefs about Bob's (finite) orders of beliefs.
- **Observation:** Since $\pi_a \in \Delta(C_b \times H_b^0)$, Ann may believe that Bob's belief hierarchy is incoherent. Why?
- We want to rule this out.

Common certainty in coherency

- **Additional restrictions:** Ann believes that
  - Bob is coherent
  - Bob believes that Ann is coherent
  - Bob believes that Ann believes that he is coherent
  - and so on
- We can express these conditions using BD's result.

$$
\begin{aligned}
H_a^1 &:= \{h_a \in H_a^c : f_a(h_a)(C_b \times H_b^c) = 1\} \\
H_a^2 &:= \{h_a \in H_a^c : f_a(h_a)(C_b \times H_b^1) = 1\} \\
&\vdots \\
H_a^k &:= \{h_a \in H_a^c : f_a(h_a)(C_b \times H_b^{k-1}) = 1\} \\
&\vdots
\end{aligned}
$$

- **Common certainty in coherency:** $\boxed{H_a := \bigcap_{k=1}^{\infty} H_a^k}$

Main result (Brandenburger & Dekel, 1993)

Theorem (informal statement)

(i) *If a belief hierarchy of Ann satisfies common certainty in coherency, it can be uniquely associated with a belief of hers over combinations of Bob's choices and belief hierarchies that satisfy common certainty in coherency.*

(ii) *For every belief of Ann over combinations of Bob's choices and belief hierarchies that satisfy common certainty in coherency, there is a unique belief hierarchy of hers that satisfies common certainty in coherency and is associated with it.*

Main result (Brandenburger & Dekel, 1993)

- Take a belief $\pi_a \in \Delta(C_a \times H_b)$ (it resembles a type)
- **BD (part ii):** We can retrieve one hierarchy in H_a:
  - $\pi_a^1 = \mathrm{marg}_{\Theta_a^0} \pi_a$
  - $\pi_a^2 = \mathrm{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0)} \pi_a$
  - $\pi_a^3 = \mathrm{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0) \times \Delta(\Theta_b^1)} \pi_a$
- Take a coherent belief hierarchy $(\pi_a^1, \pi_a^2, \dots) \in H_a$.
- **BD (part i):** We can find one belief $\pi_a \in \Delta(C_a \times H_b)$:
  - $\pi_a^1 = \mathrm{marg}_{\Theta_a^0} \pi_a$
  - $\pi_a^2 = \mathrm{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0)} \pi_a$
  - $\pi_a^3 = \mathrm{marg}_{\Theta_a^0 \times \Delta(\Theta_b^0) \times \Delta(\Theta_b^1)} \pi_a$

Theorem (formal statement)

There exists a homeomorphism $g_a : H_a \to \Delta(C_b \times H_b)$.

- **Implication:** (H_a, H_b, g_a, g_b) can be seen as a type space inducing all belief hierarchies that satisfy common certainty in coherency. It is called **universal type space**.

From type spaces to belief hierarchies

- Previously we constructed a type space that encompasses all belief hierarchies that satisfy common certainty in coherency.
- **First question:** What is the relationship between the universal type space and any (finite) type space?
- **Second question:** If we take an arbitrary type space, do we obtain belief hierarchies that satisfy common certainty in coherency?
- If yes, then any type space can be embedded in the universal type space (via what we call a "type morphism").

From types to belief hierarchies

- Take a finite type space (T_a, T_b, b_a, b_b).
- A type $t_a \in T_a$ is associated with $\big(\pi_a^1(t_a), \pi_a^2(t_a), \dots\big)$:

$$\begin{aligned}
\pi_a^1(t_a)(c_b) &:= b_a(t_a)\big(\{(c_b', t_b') \in C_b \times T_b : c_b' = c_b\}\big) \\
\pi_a^2(t_a)(c_b, \pi_b^1) &:= b_a(t_a)\big(\{(c_b', t_b') \in C_b \times T_b : (c_b', \pi_b^1(t_b')) = (c_b, \pi_b^1)\}\big) \\
\pi_a^3(t_a)(c_b, \pi_b^1, \pi_b^2) &:= b_a(t_a)\big(\{(c_b', t_b') \in C_b \times T_b : (c_b', \pi_b^1(t_b'), \pi_b^2(t_b')) = (c_b, \pi_b^1, \pi_b^2)\}\big) \\
&\ \ \vdots
\end{aligned}$$

Coherency and common certainty in coherency

Proposition

$\left(\pi_a^1(t_a), \pi_a^2(t_a), \dots\right) \in H_a$ *for every* $t_a \in T_a$.

Coherency and common certainty in coherency

Proposition

$\big(\pi_a^1(t_a), \pi_a^2(t_a), \ldots\big) \in H_a$ *for every* $t_a \in T_a$.

Proof.

It suffices to prove $\big(\pi_a^1(t_a), \pi_a^2(t_a), \ldots\big) \in H_a^c$ for all $t_a \in T_a$. Why?

$$\begin{aligned} & \operatorname{marg}_{\Theta_b^k} \pi_a^{k+2}(t_a)(c_b, \pi_b^1, \ldots, \pi_b^k) \\ = \; & \pi_a^{k+2}(t_a)(\{c_b, \pi_b^1, \ldots, \pi_b^k\} \times \Delta(\Theta_b^k)) \\ = \; & b_a(t_a)(\{(c_b', t_b') : (c_b', \pi_b^1(t_b'), \ldots, \pi_b^k(t_b')) = (c_b, \pi_b^1, \ldots, \pi_b^k)\}) \\ = \; & \pi_a^{k+1}(t_a)(c_b, \pi_b^1, \ldots, \pi_b^k). \end{aligned}$$

□

Embedding via type morphisms

- Since every $t_a \in T_a$ satisfies common certainty in coherency, there exists a unique $(\pi_a^1, \pi_a^2, \dots) \in H_a$ such that

$$(\pi_a^1(t_a), \pi_a^2(t_a), \dots) = (\pi_a^1, \pi_a^2, \dots)$$

- Why is this the case?

Embedding via type morphisms

- Since every $t_a \in T_a$ satisfies common certainty in coherency, there exists a unique $(\pi_a^1, \pi_a^2, \dots) \in H_a$ such that

$$(\pi_a^1(t_a), \pi_a^2(t_a), \dots) = (\pi_a^1, \pi_a^2, \dots)$$

- Why is this the case?
- Therefore, we define the function (type morphism) $\phi_a : T_a \to H_a$ that embeds the finite type space in the universal type space.
- This means that whenever we work with finite type spaces, we essentially work in a subset of the universal type space.

Terminology

- A type space is **terminal** if for every $(\pi_a^1, \pi_a^2, \dots) \in H_a$ there is some $t_a \in T_a$ such that $\big(\pi_a^1(t_a), \pi_a^2(t_a), \dots\big) = (\pi_a^1, \pi_a^2, \dots)$.
- A type space is **complete** if for every $\pi_a \in \Delta(C_b \times T_b)$ there is some $t_a \in T_a$ such that $b_a(t_a) = \pi_a$.

Questions???