Lexicographic Beliefs Part I: Primary Belief in Rationality

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Introduction

- Thus far, a player's belief about his opponents' choices has been modelled by a probability distribution.
- Ways of reasoning have been described in which some choices are completely discarded by receiving probability 0.
- Now, cautious reasoning is considered: some choices can be deemed much more likely than others, while at the same time no choice is completely discarded.

Tool used to model cautious reasoning in Epistemic GT: lexicographic beliefs

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Introduction

- In Classical GT the idea of cautious reasoning is modelled by converging sequences of (full support) mixed choices.
- **Example**: suppose a game where some player *i* chooses between three choices *a*, *b*, and *c*.
 - Caution modelled classically:

$$\left(\left(1-\frac{1}{n}-\frac{1}{n^2}\right)\cdot a+\frac{1}{n}\cdot b+\frac{1}{n^2}\cdot c\right)_{n\in\mathbb{N}}$$

Caution modelled epistemically:

(a,b,c)

- Intuitively, the epistemic model of caution could be seen as a one shot representation of the classical model of caution.
- For details of how to go from "epistemic caution" to "classical caution" and vice versa: Blume et al. (1991a) and (1991b).

Introduction

Three ways of cautious reasoning based on lexicographic beliefs are presented in this part of the course:

1 Common Primary Belief in (Caution & Rationality) (Brandenburger, 1992; Börgers, 1994)

Classical Analogue: Dekel-Fudenberg-Procedure (Dekel & Fudenberg, 1990)

- Related Equilibrium Concept: Perfect Equilibrium (Selten, 1975)
- Common Full Belief in (Caution & Respect of Preferences) (Schuhmacher, 1999; Asheim, 2001)

Classical Analogue: Iterated Addition of Preference Restrictions (Perea, 2011)

- **Related Equilibrium Concept:** Proper Equilibrium (Myerson, 1978)
- 3 Common Assumption of Rationality (Brandenburger et al., 2008)

Classical Analogue: Iterated Weak Dominance (Luce & Raiffa, 1957)

Related Equilibrium Concept: none in the literature

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Agenda

Lexicographic Beliefs

Lexicographic Epistemic Models

Common Primary Belief in (Caution & Rationality)

Existence



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Algorithm

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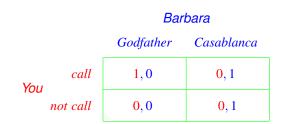
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Story

- Tonight *Barbara* will go to the cinema.
- You can join if you wish, but Barbara decides on the movie.
- There is the choice between *The Godfather* and *Casablanca*.
- You prefer The Godfather (utility 1) to Casablanca (utility 0).
- Barbara's movie preferences are inverse to yours.
- Staying at home yields you utility 0.
- Barbara goes to the cinema in any case.
- Question: Should you call Barbara or not?

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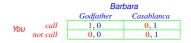
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Intuitively, the unique best choice for you is call!

standard beliefs

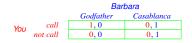
However, if you believe in Barbara's rationality with standard beliefs, then you must assign probability 0 to her choice Godfather.

Consequently, both of your choices would be optimal for *you*.

lexicographic beliefs

- A state of mind can be modelled in which you deem Barbara choosing Casablanca infinitely more likely than her picking Godfather.
- Yet, the possibility of *Barbara* choosing *Godfather* is not completely discarded.

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- Suppose you hold the following lexicographic belief on Barbara's choice:
 - primary belief: you believe Barbara to choose Casablanca.
 secondary belief: you believe Barbara to choose Godfather.
- You then deem the event that Barbara chooses Casablanca infinitely more likely than the event that she picks Godfather.
 - Yet, given this lexicographic belief, the unique optimal choice for you is then call!

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Lexicographic Beliefs

Definition

A lexicographic belief on some set S is a finite sequence

$$b^{lex} = (b^1, b^2, \dots, b^k)$$

of distinct probability measures on S, where

 \blacksquare b^1 is called *level-1 belief*,

 \blacksquare b^2 is called *level-2 belief*,

...

 \bullet *b^k* is called *level-k belief*.

Remark.

Some authors require the probability measures in b^{lex} to have disjoint supports.

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Intuition

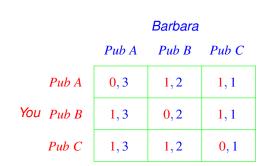
- An event can be deemed infinitely more likely than another event, without completely discarding the latter!
- **Example:** lexicographic beliefs about the solar system
 - primary belief: the earth rotates around the sun
 - secondary belief: the sun rotates around the earth
 - tertiary belief: the sun and the earth both rotate around a hidden star
- A player *i* is said to deem an opponent *j*'s choice c_j infinitely more likely than some choice c'_j for *j*, if c_j receives positive probability at an earlier lexicographic level than c'_j under his lexicographic belief b_i^{lex} .

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Story

- You would like to go to a pub to read your book.
- Barbara is going to a pub as well, but you forgot to ask her to which one.
- Your only objective is to avoid *Barbara*, since *you* would like to read your book in silence.
- Barbara prefers Pub A to Pub B, and Pub B to Pub C.
- Question: Which pub should you go to?

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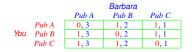
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			Barbara	
		Pub A	Pub B	Pub C
	Pub A	0,3	1,2	1, 1
You	Pub B	1,3	0,2	1, 1
	Pub C	1,3	1,2	0,1

- Intuitively, the unique best choice for you is Pub C, since it is the least preferred pub for Barbara!
- However, if you believe in Barbara's rationality with standard beliefs, then you must assign probability 0 to her choosing Pub B and Pub C.
- Consequently, both Pub B and Pub C are optimal for you.
- Indeed, with standard beliefs you cannot believe in Barbara's rationality, while at the same time deeming her choice Pub C less likely than Pub B.

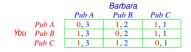
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Scenario 1: Consider the lexicographic belief (Pub A; Pub B; Pub C) for you about Barbara's choice

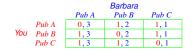
- primary belief: you believe Barbara to choose Pub A.
- secondary belief: you believe Barbara to choose Pub B.
- tertiary belief: you believe Barbara to choose Pub C.
- Interpretation: you deem Barbara's choice Pub A infinitely more likely than Pub B and Pub B infinitely more likely than Pub C, yet you consider all her choices possible.
- Given this lexicographic belief, the unique optimal choice for you is Pub C!

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- Scenario 2: Consider the lexicographic belief (Pub A; ¹/₃Pub B + ²/₃Pub C) for for you about Barbara's choice
 - primary belief: you believe Barbara to choose Pub A.
 - **secondary belief**: *you* believe with probability $\frac{1}{3}$ *Barbara* to choose *Pub B* and with probability $\frac{2}{3}$ her to choose *Pub C*.
- Given this lexicographic belief, the unique optimal choice for you is Pub B!

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- Scenario 3: Consider the lexicographic belief (Pub C; Pub B; Pub A) for you about Barbara's choice
 - primary belief: you believe Barbara to choose Pub C.
 - secondary belief: you believe Barbara to choose Pub B.
 - tertiary belief: you believe Barbara to choose Pub A.
- Given this lexicographic belief, the unique optimal choice for you is Pub A!

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Expected Utility under Lexicographic Beliefs

Let $\Gamma = (\{i, j\}, (C_i, C_j), (U_i, U_j))$ be a two player game.

- Suppose that player *i* entertains a lexicographic belief $b_i^{lex} = (b_i^1, b_i^2, \dots, b_i^K)$ about *j*'s choice.
- For every level $k \in \{1, 2, ..., K\}$ and for every choice $c_i \in C_i$ the *k*-level expected utility for player *i* of picking c_i is given by

$$u_i^k(c_i, b_i^{lex}) = \sum_{c_j \in C_j} \left(b_i^k(c_j) \cdot U_i(c_i, c_j) \right)$$

■ Hence, every choice c_i ∈ C_i for player i induces a sequence of expected utilities: lexicographic expected utility

$$u_i^{lex}(c_i, b_i^{lex}) = \left(u_i^1(c_i, b_i^{lex}), u_i^2(c_i, b_i^{lex}), \dots, u_i^K(c_i, b_i^{lex})\right)$$

Preferences Induced by Lexicographic Beliefs

Definition

A player *i* with lexicographic belief b_i^{lex} **prefers** some choice c_i to c'_i , if there exists some lexicographic level *k* such that

1 $u_i^k(c_i, b_i^{lex}) > u_i^k(c'_i, b_i^{lex})$ and

2 $u_i^l(c_i, b_i^{lex}) = u_i^l(c'_i, b_i^{lex})$ for all lexicographic levels l < k.

Useful Fact: Note that the binary relation *prefer* is transitive on the respective agent's choice set!

Definition

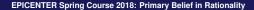
Given a lexicographic belief b_i^{lex} a choice c_i is called **optimal**, if there exists no choice $c_i^* \in C_i$ such that *i* prefers c_i^* to c_i .

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Rationality under Lexicographic Beliefs

Definition

A choice c_i is called **rational**, if there exists some lexicographic belief b_i^{lex} such that c_i is optimal.



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		Barbara		
		Pub A	Pub B	Pub C
	Pub A	0 , 3	1,2	1, 1
You	Pub B	1,3	0, 2	1, 1
	Pub C	1,3	1,2	0, 1

Consider lexicographic belief $b_{you}^{lex} = (Pub A; Pub B; Pub C)$

- under the primary belief: $u_{you}^{1}(Pub A, b_{you}^{lex}) = 0, u_{you}^{1}(Pub B, b_{you}^{lex}) = 1, u_{you}^{1}(Pub C, b_{you}^{lex}) = 1$
- under the secondary belief: $u_{you}^2(Pub \ B, b_{you}^{lex}) = 0, \ u_{you}^2(Pub \ C, b_{you}^{lex}) = 1$
- Hence, you prefer Pub C to Pub B, and Pub B to Pub A.

Given b_{vou}^{lex} the unique optimal choice is *Pub C* for *you*!

			Barbara	
		Pub A	Pub B	Pub C
	Pub A	0, 3	1, 2	1, 1
You	Pub B	1,3	0, 2	1, 1
	Pub C	1, 3	1, 2	0, 1

Consider lexicographic belief $b_{you}^{lex'} = (Pub \ A; \frac{1}{3}Pub \ B + \frac{2}{3}Pub \ C)$

■ under the primary belief: $u_{you}^{1}(Pub A, b_{you}^{lex'}) = 0, \ u_{you}^{1}(Pub B, b_{you}^{lex'}) = 1, \ u_{you}^{1}(Pub C, b_{you}^{lex'}) = 1$

• under the secondary belief: $u_{you}^2(Pub \ B, b_{you}^{lex'}) = \frac{2}{3}, \ u_{you}^2(Pub \ C, b_{you'}^{lex'}) = \frac{1}{3}$

Hence, you prefer Pub B to Pub C, and Pub C to Pub A.

Given $b_{you}^{lex'}$, the unique optimal choice is *Pub B* for *you*!

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			Barbara	
		Pub A	Pub B	Pub C
	Pub A	0, 3	1, 2	1, 1
You	Pub B	1,3	0, 2	1, 1
	Pub C	1,3	1,2	0, 1

Consider lexicographic belief $b_{you}^{lex''} = (\frac{1}{2}Pub A + \frac{1}{2}Pub B; \frac{1}{3}Pub B + \frac{2}{3}Pub C)$

> under the primary belief: $u_{you}^1(Pub \ A, b_{you}^{lex "}) = \frac{1}{2}, \ u_{you}^1(Pub \ B, b_{you}^{lex "}) = \frac{1}{2}, \ u_{you}^1(Pub \ C, b_{you}^{lex "}) = 1$

• under the secondary belief: $u_{you}^2(Pub \ A, b_{you}^{lex}'') = 1, \ u_{you}^2(Pub \ B, b_{you}^{lex}'') = \frac{2}{3}$

Hence, *you* prefer *Pub C* to *Pub A*, and *Pub A* to *Pub B*.

Given b_{you}^{lex} , the unique optimal choice is *Pub C* for *you*!

Agenda

Lexicographic Beliefs

Lexicographic Epistemic Model

Common Primary Belief in (Caution & Rationality)

Algorithm

Algorithm

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Reasoning with Lexicographic Beliefs

- When reasoning about his opponents a player does not only entertain a belief about his opponents' choices but also about their beliefs, their beliefs about their opponents' beliefs, etc., i.e. a full belief hierarchy.
- A full belief hierarchy with standard beliefs is modelled by types in an epistemic model: a type induces a standard belief about his opponents' choice-type combinations.
- Analogously, a full belief hierarchy with lexicographic beliefs is now modelled by types in a lexicographic epistemic model: a type induces a lexicographic belief about his opponents' choice-type combinations.

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Epistemic Model with Lexicographic Beliefs

Definition

A *lexicographic epistemic model* is a tuple $M_l = \langle (T_i)_{i \in I}, (b_i^{lex})_{i \in I} \rangle$ such that

 \blacksquare T_i is a set of types for player *i*,

every type $t_i \in T_i$ induces a lexicographic belief $b_i^{lex}(t_i)$ on the opponents' choice-type combinations $\times_{j \in I \setminus \{i\}} (C_j \times T_j)$.

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Formalizing Caution

- Intuition: No opponent's choice is excluded from consideration, yet some opponent's choice can be deemed infinitely more likely than some other choice of his.
- A type t_i is said to deem possible an opponent's type t_j, whenever there exists some lexicographic level k such that t_j receives positive probability under b^k_i.

Definition

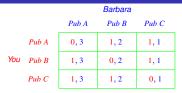
A type t_i is *cautious*, whenever, if t_i deems possible some opponent's type t_j , then t_i also deems possible the choice-type pair (c_j, t_j) for all $c_j \in C_j$.

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Interpretation

Agent *i* is cautious, if for every mental set-up ("type") that *i* deems possible for *j* to entertain, *i* does not exclude any feasible act.

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Consider the following lexicographic epistemic model:

Type Spaces:

$$T_{you} = \{t_y, t'_y\}$$
$$T_{Barbara} = \{t_B, t'_B\}$$

Beliefs for You:

$$\begin{split} b_{you}^{lex}(t_y) &= ((Pub \, A, t_B); \frac{1}{3}(Pub \, B, t'_B) + \frac{2}{3}(Pub \, C, t'_B)) \\ b_{you}^{lex}(t'_y) &= (\frac{1}{2}(Pub \, A, t_B) + \frac{1}{2}(Pub \, B, t'_B); (Pub \, C, t'_B)) \end{split}$$

Beliefs for Barbara:

$$\begin{split} b_{Barbara}^{lex}(t_B) &= ((Pub \, A, t_y); \frac{3}{4}(Pub \, A, t_y') + \frac{1}{4}(Pub \, C, t_y)) \\ b_{Barbara}^{lex}(t_B') &= ((Pub \, A, t_y'); (Pub \, B, t_y); (Pub \, C, t_y')) \end{split}$$

No type in this lexicographic epistemic model is cautious!

A lexicographic epistemic model with a cautious type for you:

Type Spaces:

 $T_{you} = \{t_y, t'_y, t''_y\}$ $T_{Barbara} = \{t_B, t'_B\}$

Beliefs for You:

$$\begin{split} b_{you}^{lex}(t_y) &= ((Pub\ A, t_B); \frac{1}{3}(Pub\ B, t'_B) + \frac{2}{3}(Pub\ C, t'_B)) \\ b_{you}^{lex}(t'_y) &= (\frac{1}{2}(Pub\ A, t_B) + \frac{1}{2}(Pub\ B, t'_B); (Pub\ C, t'_B)) \\ b_{you}^{lex}(t''_y) &= ((Pub\ A, t_B); (Pub\ A, t'_B); \frac{1}{3}(Pub\ B, t_B) + \frac{2}{3}(Pub\ C, t'_B); \frac{1}{3}(Pub\ B, t'_B) + \frac{2}{3}(Pub\ C, t_B)) \end{split}$$

Beliefs for Barbara:

$$\begin{split} b_{Barbara}^{lex}(t_B) &= ((Pub \ A, t_y); \frac{3}{4}(Pub \ A, t'_y) + \frac{1}{4}(Pub \ C, t_y)) \\ b_{Barbara}^{lex}(t'_B) &= ((Pub \ A, t'_y); (Pub \ B, t_y); (Pub \ C, t'_y)) \end{split}$$

• Your type t''_{y} is cautious!

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Rationality in Lexicographic Epistemic Models

Definition

A choice c_i is called **rational**, if there exists some lexicographic epistemic model \mathcal{M}_l with a type t_i such that c_i is optimal for the induced lexicographic first-order belief of t_i .

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Being Cautious and Believing in Rationality

Caution and belief in the opponents' rationality at all lexicographic levels is generally impossible!

Indeed, caution requires every choice – including non-rational ones (i.e. choices that are not optimal for any belief) – to receive positive probability at some lexicographic level.

Primary Belief in Rationality

A type t_i is said to primarily believe in some property, if t_i's primary belief only assigns positive probability to j's choice-type pairs that satisfy this property.

Definition

A type t_i **primarily believes in rationality**, whenever t_i 's level-1 belief only assigns positive probability to opponent choice-type pairs (c_j, t_j) such that c_j is optimal for t_j .

Remark.

Note that no conditions are put on any lexicographic level deeper than the primary one!

		Barbara		
		Pub A	Pub B	Pub C
	Pub A	0, 3	1,2	1, 1
You	Pub B	1,3	0, 2	1, 1
	Pub C	1,3	1, 2	0, 1

Type Spaces:

 $T_{you} = \{t_y, t'_y\}$ $T_{Barbara} = \{t_B, t'_B\}$

Beliefs for You:

$$b_{you}(t_y) = ((Pub A, t_B); \frac{1}{3}(Pub B, t'_B) + \frac{2}{3}(Pub C, t'_B))$$

$$b_{you}(t'_y) = (\frac{1}{2}(Pub A, t_B) + \frac{1}{2}(Pub B, t'_B); (Pub C, t'_B))$$

Beliefs for Barbara:

 $b_{Barbara}(t_B) = ((Pub \ B, t_y); \frac{3}{4}(Pub \ A, t'_y) + \frac{1}{4}(Pub \ C, t_y))$ $b_{Barbara}(t'_B) = ((Pub \ A, t'_y); (Pub \ B, t_y); (Pub \ C, t'_y))$

- If you primarily believe in Barbara's rationality, then your primary belief must only assign positive probability to Barbara's choice Pub A.
- Type t_y primarily believes in *Barbara's* rationality and t'_y does not.
- Type t_B primarily believes in *your* rationality and t'_B does not.

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Common Primary Belief in (Caution & Rationality)

Definition

A type *t_i* expresses *common primary belief in (caution & rationality)*, whenever

- *t_i* expresses 1-fold primary belief in (caution & rationality), i.e. *t_i* primarily believes in *j*'s caution and rationality, i.e. primarily only deems possible choice type pairs (*c_j*, *t_j*) such that *t_j* is cautious and *c_j* is optimal for *t_i*,
- t_i expresses 2-fold primary belief in (caution & rationality), i.e. t_i primarily only deems possible types t_j that express 1-fold primary belief in (caution & rationality),
- t_i expresses 3-fold primary belief in (caution & rationality), i.e. t_i primarily only deems possible types t_j that express 2-fold primary belief in (caution & rationality),

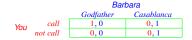
etc.

Note that all restrictions on the belief hierarchies are put on the *first lexicographic level*.

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Example: Should I call or not?



Type Spaces:

 $T_{you} = \{t_y\} \\ T_{Barbara} = \{t_B\}$

Beliefs for You:

 $b_{you}(t_y) = ((Casablanca, t_B); (Godfather, t_B))$

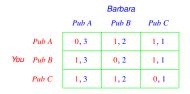
Beliefs for Barbara:

 $b_{Barbara}(t_B) = ((call, t_y); (not call, t_y))$

- If you are cautious then your only optimal choice is call.
- Your type t_y is cautious thus call is optimal for him and expresses common primary belief in (caution & rationality).
- Hence, you can rationally and cautiously choose call under common primary belief in (caution & rationality).

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Example: Where to read my book?



- If you primarily believe in Barbara's rationality, then your primary belief must assign probability 1 to Barbara's choice Pub A.
- Hence, *Pub A* cannot be optimal for you.
- Which of your remaining choices Pub B and Pub C can you rationally choose under caution and common primary belief in (caution & rationality)?

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Example: Where to read my book?

		Barbara				
		Pub A	Pub B	Pub C		
	Pub A	<mark>0</mark> , 3	1, 2	1, 1		
You	Pub B	1,3	0 , 2	1, 1		
	Pub C	1,3	1, 2	0, 1		

Type Spaces:

 $T_{you} = \{t_y\} \\ T_{Barbara} = \{t_B\}$

Beliefs for You:

 $b_{you}(t_y) = ((Pub A, t_B); \frac{1}{3}(Pub B, t_B) + \frac{2}{3}(Pub C, t_B))$

Beliefs for Barbara:

 $b_{Barbara}(t_B) = ((Pub B, t_y); \frac{1}{2}(Pub A, t_y) + \frac{1}{2}(Pub C, t_Y))$

- Your type t_y is cautious and expresses common primary belief in (caution & rationality).
- Your choice Pub B is optimal for type t_y.
- Hence, you can rationally and cautiously choose Pub B under common primary belief in (caution & rationality).

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Example: Where to read my book?

		Barbara				
		Pub A	Pub B	Pub C		
	Pub A	<mark>0</mark> , 3	1, 2	1, 1		
You	Pub B	1,3	0 , 2	1, 1		
	Pub C	1,3	1, 2	0, 1		

Type Spaces:

 $T_{you} = \{t_y\} \\ T_{Barbara} = \{t_B\}$

Beliefs for You:

 $b_{you}(t_y) = ((Pub A, t_B); \frac{2}{3}(Pub B, t_B) + \frac{1}{3}(Pub C, t_B))$

Beliefs for Barbara:

 $b_{Barbara}(t_B) = ((Pub \ C, t_y); \frac{1}{2}(Pub \ A, t_y) + \frac{1}{2}(Pub \ C, t_Y))$

- Your type t_y is cautious and expresses common primary belief in (caution & rationality).
- Your choice Pub C is optimal for type t_y.
- Hence, you can rationally and cautiously choose Pub C under common primary belief in (caution & rationality).

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A Way of Cautious Reasoning

A lexicographic cautious way of reasoning – Common Primary Belief in (Caution & Rationality) – has been introduced.

Accordingly, a type

primarily only deems possible choice type pairs such that the type is cautious and the choice is optimal for the type,

[= 1-fold primary belief in (caution & rationality]

primarily only deems possible opponent types that primarily only deem possible choice type pairs such that the type is cautious and the choice is optimal for the type,

[= 2-fold primary belief in (caution & rationality)]

only primarily deems possible opponent types that primarily only deem possible opponent types that primarily only deem possiblechoice type pairs such that the type is cautious and the choice is optimal for the type,

[= 3-fold primary belief in (caution & rationality)]

etc.

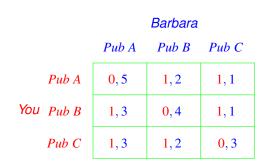
Two remaining key questions:

existence and algorithmic characterization

Story

- You would like to go to a pub to read your book.
- Barbara is going to a pub as well, but you forgot to ask her to which one.
- You would like to avoid Barbara, in order to enjoy reading your book in silence.
- Barbara prefers Pub A to Pub B, and Pub B to Pub C, and would also like to talk to you.
- Question: Which pub should you go to?

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Is common primary belief in (caution & rationality) possible in this game?

- Consider some arbitrary cautious lexicographic belief for you about Barbara's choice, e.g. (A_B; B_B; C_B).
- Given this belief, the choice C_y is optimal for you.
- Consider the belief $(C_y; A_y; B_y)$ for *Barbara* about your choice.
- Given this belief, the choice A_B is optimal for Barbara.
- Consider the belief $(A_B; B_B; C_B)$ for you about Barbara's choice.
- A chain of lexicographic beliefs has thus been formed which has entered in a cylce: $(A_B; B_B; C_B) \rightarrow (C_y; A_y; B_y) \rightarrow (A_B; B_B; C_B)$

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		Barbara				
		A_B	BB	C_B		
	A_y	0, 5	1, 2	1, 1		
You	B_y	1,3	0,4	1, 1		
	C _y	1,3	1, 2	0, 3		

- The cycle (A_B; B_B; C_B) → (C_y; A_y; B_y) → (A_B; B_B; C_B) is now transformed into a lexicographic epistemic model.
- **Type Spaces:** $T_{you} = \{t_y\}$ and $T_{Barbara} = \{t_B\}$
- Beliefs for You: $b_{you}^{lex}(t_y) = ((A_B, t_B); (B_B, t_B); (C_B, t_B))$
- Beliefs for Barbara: $b_{Barbara}^{lex}(t_B) = ((C_y, t_y); (A_y, t_y); (B_y, t_y))$
- Both types in the epistemic model t_y and t_B are cautious and primarily believe in rationality.
- Hence, both types t_v and t_B express common primary belief in (caution & rationality).
- Concluding, common primary belief in (caution & rationality) is indeed possible in the Hide and Seek game.

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Generalizing the Construction for Existence

- Fix some finite game and consider an arbitrary cautious lexicographic belief b_i^{lex1} for player i about j's choice.
- Let c_i^1 be optimal given this belief.
- Consider some cautious lexicographic belief b^{tex2}_j for player j about i's choice such that the primary belief assigns probability 1 to c¹_j and also probability 1 to some choice at all deeper levels.
- Let c_i^2 be optimal given this belief.
- Consider some cautious lexicographic belief b_i^{lex3} for player *i* about *j*'s choice such that the primary belief assigns probability 1 to c_i² and also probability 1 to some choice at all deeper levels.
- Let c_i³ be optimal given this belief.
- etc.
- The sequence of lexicographic beliefs thus constructed bears the following property: The unique choice in the support of the primary belief of any element of the sequence is optimal given the immediate predecessor lexicographic belief in the sequence.
- Since there are only finitely many choices and the same choices can always be specified for the support of all belief levels beyond level 1, respectively, the sequence of lexicographic beliefs must eventually enter into a cycle of lexicographic beliefs.

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From Lexicographic Beliefs to Types

- Suppose some cycle of lexicographic beliefs: $b_i^{lex^1} \rightarrow b_j^{lex^2} \rightarrow b_i^{lex^3} \rightarrow \ldots \rightarrow b_j^{lexK} \rightarrow b_i^{lex^1}$
- This cycle can be transformed into an lexicographic epistemic model:

- In such an epistemic model, every type is cautious and primarily believes in rationality.
- Hence, all types express common primary belief in (caution & rationality)!

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Existence

Theorem

Let Γ be some finite two player game. Then, there exists a lexicographic epistemic model such that

- every type in the model is cautious and expresses common primary belief in (caution & rationality),
- every type in the model deems possible only one opponent's type, and assigns at each lexicographic level probability 1 to one of the opponent's choices.

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Agenda

Lexicographic Beliefs

Lexicographic Epistemic Models

Common Full Belief in (Caution & Primary Belief in Rationality)

Existence

Algorithm

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Towards Characterizing Cautious Reasoning

Definition

A choice c_i of player *i* is **weakly dominated** by some randomized choice $r_i \in \Delta(C_i)$, whenever

- $\blacksquare U_i(c_i,c_j) \le V_i(r_i,c_j) \text{ for all } c_j \in C_j,$
- there exists $c_i^* \in C_j$ such that $U_i(c_i, c_j^*) < V_i(r_i, c_j^*)$.

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Characterizing Cautious Reasoning

An analogy to Pearce's Lemma for lexicographic beliefs:

Theorem

A choice c_i of player *i* can **optimally** be chosen under a **cautious lexicographic belief** if and only if c_i is **not weakly dominated** by some randomized choice r_i .

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Randomized Choices and Lexicographic Expected Utility

The *k*-level expected utility $v_i^k(r_i, b_i^{lex})$ of a randomized choice $r_i \in \Delta(C_i)$ is defined as

$$v_i^k(r_i, b_i^{lex}) := \sum_{c_j \in C_j} b_i^k(c_j) \Big(\sum_{c_i \in C_i} \big(r_i(c_i) \cdot U_i(c_i, c_j) \big) \Big)$$

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A basic lemma

Basic-Lemma I

Let *I* be some index set, $0 \le \alpha_i \le 1$ for all $i \in I$ such that $\sum_{i \in I} \alpha_i = 1$, $x \in \mathbb{R}$, and $y_i \in \mathbb{R}$ for all $i \in I$. If $x < \sum_{i \in I} \alpha_i y_i$, then there exists $i^* \in I$ such that $x < y_{i^*}$.

Proof:

- Towards a contradiction suppose that x ≥ y_i for all i ∈ I.
- Then, $\alpha_i x \ge \alpha_i y_i$ holds for all $i \in I$.
- It directly follows that $1 \cdot x = \sum_{i \in I} \alpha_i x \ge \sum_{i \in I} \alpha_i y_i$, a contradiction.

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A second basic lemma

Basic-Lemma II

Let *I* be some index set, $0 < \alpha_i < 1$ for all $i \in I$ such that $\sum_{i \in I} \alpha_i = 1$, $x \in \mathbb{R}$, and $y_i \in \mathbb{R}$ for all $i \in I$. If $x \leq \sum_{i \in I} \alpha_i y_i$, then (there exists $i^* \in I$ such that $x < y_{i^*}$) or ($x = y_i$ for all $i \in I$).

Proof:

- By contraposition, suppose that $x \ge y_i$ for all $i \in I$ and that there exists $i' \in I$ such that $x \ne y_{i'}$.
- Then, $x > y_{i'}$.
- As $0 < \alpha_i < 1$ holds for all $i \in I$, it is the case that $\alpha_{i'} x > \alpha_{i'} y_{i'}$ and $\alpha_i x \ge \alpha_i y_i$ for all $i \in I \setminus \{i'\}$.

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Proof of the *only if* (\Rightarrow) Direction of the Theorem

- The proof proceeds by contraposition.
- Let $c_i \in C_i$ be weakly dominated by some randomized choice $r_i \in \Delta(C_i)$.
- Thus, $U_i(c_i, c_j) \leq \sum_{c_i \in C_i} (r_i(c_i) \cdot U_i(c_i, c_j))$ for all $c_j \in C_j$ and there exists some choice $c_j^* \in C_j$ such that $U_i(c_i, c_j^*) < \sum_{c_i \in C_i} (r_i(c_i) \cdot U_i(c_i, c_j^*))$.
- Suppose that player *i* holds some cautious lexicographic belief $b_i^{lex} = (b_i^1, b_i^2, \dots, b_i^K)$.
- Then, for all levels k

$$\sum_{c_j \in C_j} \left(b_i^k(c_j) \cdot U_i(c_i, c_j) \right) \le \sum_{c_j \in C_j} \left(b_i^k(c_j) \sum_{c_i \in C_i} \left(r_i(c_i) \cdot U_i(c_i, c_j) \right) \right)$$

i.e.

$$u_i^k(c_i, b_i^{lex}) \le \sum_{c_i' \in C_i} r_i(c_i') u_i^k(c_i', b_i^{lex}) = v_i^k(r_i, b_i^{lex}),$$

and, by caution there exists a level k^* such that $c_j^* \in \mathsf{supp}(b_i^{k^*})$ and thus

$$\sum_{c_j \in C_j} \left(b_i^{k^*}(c_j) \cdot U_i(c_i, c_j) \right) < \sum_{c_j \in C_j} \left(b_i^{k^*}(c_j) \sum_{c_i \in C_i} \left(r_i(c_i) \cdot U_i(c_i, c_j) \right) \right)$$

i.e.

$$u_{i}^{k^{*}}(c_{i}, b_{i}^{lex}) < \sum_{c_{i}^{\prime} \land C_{i}} r_{i}(c_{i}^{\prime}) u_{i}^{k^{*}}(c_{i}^{\prime}, b_{i}^{lex}) = v_{i}^{k^{*}}(r_{i}, b_{i}^{lex}).$$

Proof of the *only if* (\Rightarrow) Direction of the Theorem (continued)

- Consider the set supp $(r_i) \subseteq C_i$ of *i*'s choices to which r_i assigns positive probability and level-1 belief b_i^1 .
- Then, by Basic-Lemma II, either (a) there exists some $c'_i \in \text{supp}(r_i)$ such that $u_i^1(c_i, b_i^{lex}) < u_i^1(c'_i, b_i^{lex})$, or (b) $u_i^1(c_i, b_i^{lex}) = u_i^1(c'_i, b_i^{lex})$ for all $c'_i \in \text{supp}(r_i)$.
- If case (a) holds, then player *i* prefers c'_i to c_i , and c_i is thus not optimal.
- If case (b) holds, i.e., $u_i^1(c_i, b_i^{lex}) = u_i^1(c'_i, b_i^{lex})$ for all $c'_i \in \text{supp}(r_i)$, then consider b_i^2 .
- Then, again by Basic-Lemma II, either (a) there exists some $c'_i \in \text{supp}(r_i)$ such that $u_i^2(c_i, b_i^{\text{tex}}) < u_i^2(c'_i, b_i^{\text{tex}})$, or (b) $u_i^2(c_i, b_i^{\text{tex}}) = u_i^2(c'_i, b_i^{\text{tex}})$ for all $c'_i \in \text{supp}(r_i)$.
- If case (a) holds, then $u_i^1(c_i, b_i^{lex}) = u_i^1(c'_i, b_i^{lex})$ and $u_i^2(c_i, b_i^{lex}) < u_i^2(c'_i, b_i^{lex})$, and consequently player *i* prefers c'_i to c_i , implying that c_i is not optimal.
- If case (b) holds, i.e., $u_i^1(c_i, b_i^{lex}) = u_i^1(c_i', b_i^{lex})$ and $u_i^2(c_i, b_i^{lex}) = u_i^2(c_i', b_i^{lex})$ for all $c_i' \in \text{supp}(r_i)$, then consider b_i^3 .
- etc.
- As $u_i^{k^*}(c_i, b_i^{lex}) < v_i^{k^*}(r_i, b_i^{lex})$ there must eventually be some level l' such that by Basic-Lemma I it is the case that $u_i^l(c_i, b_i^{lex}) < u_i^{l'}(c_i', b_i^{lex})$ for some $c_i' \in \text{supp}(r_i)$.
- Hence, there exists some choice $c'_i \in \text{supp}(r_i)$ that player *i* prefers to c_i , and therefore c_i is not optimal.

Towards an Algorithm

It is desirable to algorithmically characterize the choices under

rationality (=optimality given the agent's beliefs),

caution,

common primary belief in (caution & rationality).

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Lexicographic Optimality and Standard Optimality

Lemma

If a choice c_i is lexicographically-optimal given a lexicographic belief b_i^{lex} , then c_i is standard-optimal given b_i^1 .

Proof:

- Towards a contradiction suppose that c_i is lexicographically-optimal given b^{lex}_i, but not standard-optimal given b¹_i.
- Then, there exists a choice $c_i^* \in C_i$ such that $u_i^1(c_i, b_i^{lex}) = u_i(c_i, b_i^1) < u_i(c_i^*, b_i^1) = u_i^1(c_i^*, b_i^{lex}).$
- However, this contradicts lexicographic optimality of c_i according to which there exists no choice $c'_i \in C_i$ such that $u^k_i(c_i, b^{lex}_i) < u^k_i(c'_i, b^{lex}_i)$ for some level k and $u^k_i(c_i, b^{lex}_i) = u^k_i(c'_i, b^{lex}_i)$ for all levels l < k.

Step 1

1-fold primary belief in (caution & rationality)

- Which choices can optimally and cautiously be made under 1-fold primary belief in (caution & rationality)?
- Suppose that type t_i is cautious and expresses 1-fold primary belief in (caution & rationality).
- Then, by the Theorem, t_i's primary belief assigns probability 0 to all weakly dominated choices for j.
- Note that due to t_i being cautious, t_i cannot optimally choose any weakly dominated choice himself.
- Let Γ¹ be the reduced game that remains after eliminating all weakly dominated choices from the game: t_i's primary belief is concentrated on Γ¹.
- Hence, every optimal choice for t_i must be optimal for some lexicographic belief with primary belief restricted to Γ¹, i.e. standard-optimal given the primary belief.
- Thus, by Pearce's Lemma applied to Γ^1 , every optimal choice for t_i must not be strictly dominated on Γ^1 .
- Let Γ^2 be the reduced game that remains after eliminating all strictly dominated choices from Γ^1 .
- Then, every optimal choice for t_i must be in Γ^2 .
- Conclusion: If type t_i is cautious and expresses 1-fold primary belief in (caution & rationality), then every optimal choice for t_i must be in Γ².
- Note that Γ² is obtained by first eliminating all weakly dominated choices, and then eliminating all strictly dominated choices.

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Step 2

Up to 2-fold primary belief in (caution & rationality)

- Which choices can optimally and cautiously be made under up to 2-fold primary belief in (caution & rationality)?
- Suppose that type t_i is cautious and expresses up to 2-fold primary belief in (caution & rationality).
- Then, t_i's primary belief only assigns positive probability to choice-type pairs (c_j, t_j) such that c_j is optimal for t_i, and t_i expresses 1-fold primary belief in (caution & rationality).
- From Step 1 it follows that all such choices c_i receiving positive probability by t_i 's primary belief are in Γ^2 .
- As t_i satisfies 1-fold primary belief in (caution & rationality), every optimal choice for t_i is in Γ^2 .
- Hence, every optimal choice for t_i must be optimal for some lexicographic belief with primary belief restricted to r², i.e. standard-optimal given the primary belief.
- Thus, by Pearce's Lemma applied to Γ^2 , every optimal choice for t_i must not be strictly dominated in Γ^2 .
- Let Γ^3 be the reduced game that remains after eliminating all strictly dominated choices from Γ^2 .
- Then, every optimal choice for t_i must be in Γ^3 .
- **Conclusion:** If type t_i is cautious and expresses up to 2-fold primary belief in (caution & rationality), then every optimal choice for t_i must be in Γ^3 .
- Note that Γ³ is obtained by first eliminating all weakly dominated choices, and then applying two-fold strict dominance.

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Algorithm

Definition (Dekel-Fudenberg-Procedure)

Step 1. Eliminate all choices that are weakly dominated in the game.

Step 2. Within the reduced game after Step 1, apply iterated strict dominance.

- The algorithm stops after finitely many steps.
- The algorithm returns a non-empty set.
- The order and speed in which choices are eliminated after Step 1 is not relevant for the set it returns.

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Algorithmic Characterization

Theorem

For all $k \ge 1$, the choices that can rationally be made by a cautious type that expresses up to *k*-fold primary belief in (caution & rationality) are exactly those choices that survive the first k + 1 steps of the Dekel-Fudenberg-Procedure.

Corollary

The choices that can rationally be made by a cautious type that expresses common primary belief in (caution & rationality) are exactly those choices that survive the Dekel-Fudenberg-Procedure.

Story

- It is Friday and your teacher announces a surprise exam for next week.
- You must decide on what day you will start preparing for the exam.
- In order to pass the exam you must study for at least two days.
- For a perfect exam and a subsequent compliment by your father you need to study for at least six days.
- Passing the exam increases your utility by 5.
- Failing the exam increases the teacher's utility by 5.
- Every day you study decreases your utility by 1, but increases the teacher's utility by 1.
- A compliment by your father increases your utility by 4.

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	Teacher					
	Mon	Tue	Wed	Thu	Fri	
Sat	3,2	2,3	1,4	0,5	3,6	
Sun	-1,6	3,2	2,3	1,4	0,5	
You Mon	0,5	-1,6	3,2	2,3	1,4	
Tue	0,5	0,5	-1,6	3,2	2,3	
Wed	0,5	0,5	0,5	-1,6	3,2	

Teacher

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			reacher		
	Mon	Tue	Wed	Thu	Fri
Sat	3, 2	2, 3	1,4	0, 5	3,6
Sun	-1,6	3, 2	2, 3	1, 4	0, 5
You Mon	0 , 5	-1,6	3, 2	2, 3	1,4
Tue	0 , 5	0, 5	-1,6	3, 2	2, 3
Wed	0, 5	0, 5	0, 5	-1,6	3, 2

Teacher

- With standard beliefs under common belief in rationality you can rationally choose any day.
- With standard beliefs under common belief in rationality and a simple belief hierarchy you can only rationally pick Saturday or Wednesday.
- What days can you rationally and cautiously choose under common primary belief in (caution & rationality)?

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		Teacher						
		Mon	Tue	Wed	Thu	Fri		
	Sat	3, 2	2, 3	1,4	0, 5	3,6		
	Sun	-1,6	3, 2	2, 3	1, 4	0, 5		
You	Mon	0, 5	-1,6	3, 2	2, 3	1,4		
	Tue	0, 5	0, 5	-1,6	3, 2	2, 3		
	Wed	0, 5	0, 5	0, 5	-1,6	3, 2		

Step 1.

- Your choice Wednesday is weakly dominated by your choice Saturday.
- Eliminate your choice Wednesday from the original game.

		Teacher						
		Mon	Tue	Wed	Thu	Fri		
	Sat	3, 2	2, 3	1,4	0, 5	3,6		
You	Sun	-1,6	3, 2	2, 3	1, 4	0, 5		
	Mon	0, 5	-1,6	3, 2	2, 3	1,4		
	Tue	0, 5	0, 5	-1,6	3, 2	2, 3		

Step 2.

- The teacher's choice Thursday is strictly dominated by Friday.
- Eliminate the teacher's choice Friday from the reduced game after Step 1.

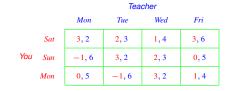
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		Teacher					
		Mon	Tue	Wed	Fri		
	Sat	3, 2	2, 3	1,4	3,6		
You	Sun	-1,6	3,2	2, 3	0, 5		
	Mon	0, 5	-1,6	3, 2	1,4		
	Tue	0, 5	0, 5	-1,6	2, 3		

Step 3.

- Your choice Tuesday is strictly dominated by Saturday.
- Eliminate the your choice Tuesday from the reduced game after Step 2.

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Step 4.

- The teacher's choice Wednesday is strictly dominated by Friday.
- Eliminate the *teacher*'s choice Wednesday from the reduced game after Step 3.

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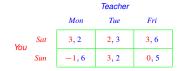
Teacher Mon Tue Fri Sat 3,2 2,3 3,6 You Sun 3.2 0.5 -1.6Mon 0,5 -1, 61,4

Step 5.

- Sour choice Monday is strictly dominated by Saturday.
- Eliminate your choice Monday from the reduced game after Step 4.

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Step 6.

- The teacher's choice Tuesday is strictly dominated by Friday.
- Eliminate the *teacher*'s choice *Tuesday* from the reduced game after Step 5.

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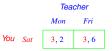
Step 7.

- Sour choice Sunday is strictly dominated by Saturday.
- Eliminate your choice Sunday from the reduced game after Step 6.

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Step 8.

- The teacher's choice Monday is strictly dominated by Friday.
- Eliminate the *teacher*'s choice *Monday* from the reduced game after Step 7.

The algorithm stops.



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		Teacher						
		Mon	Tue	Wed	Thu	Fri		
	Sat	3,2	2,3	1,4	0, 5	3,6		
	Sun	-1, 6	3,2	2,3	1,4	0, 5		
You	Mon	0, 5	-1, 6	3, 2	2, 3	1,4		
	Tue	0, 5	-1, 6	3, 2	2, 3	1,4		
	Wed	0,5	0,5	0, 5	-1, 6	3, 2		

Type Spaces:

 $T_{you} = \{t_y\} \\ T_{Teacher} = \{t_B\}$

Beliefs for You:

 $b_{you}(t_y) = ((Fri, t_T); \frac{1}{4}(Mon, t_T) + \frac{1}{4}(Tue, t_T) + \frac{1}{4}(Wed, t_T) + \frac{1}{4}(Thu, t_T))$

Beliefs for Teacher:

 $b_{Teacher}(t_T) = ((Sat, t_y); \frac{1}{4}(Sun, t_y) + \frac{1}{4}(Mon, t_y) + \frac{1}{4}(Tue, t_y) + \frac{1}{4}(Wed, t_y))$

- Your type t_y is cautious and expresses common full belief in caution and primary belief in rationality.
- Your choice Saturday is optimal for type t_y.
- Hence, you can indeed cautiously and rationally choose Saturday under common primary belief in (caution & rationality).

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