Correlation in games

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EpiCenter Spring Course on Epistemic Game Theory
June 2018

Roadmap

1. Preliminaries from probability theory
2. Correlation in beliefs

Independence of two events

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- Two events $A, B \subseteq \Omega$ are **independent** whenever

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- Pairwise independence and independence are not the same.

Pairwise independence vs. independence

- We throw two dies simultaneously, and consider the events:
  - $A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$: the sum of the dies is 7.
  - $B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$: the outcome of the first die is 3.
  - $C = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$: the outcome of the second die is 4.

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- The three events are pairwise independent:
  - $\pi(A) = \pi(B) = \pi(C) = 1/6$
  - $\pi(A \cap B) = \pi(A \cap C) = \pi(B \cap C) = 1/36$

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- The three events are pairwise independent:
  - $\pi(A) = \pi(B) = \pi(C) = 1/6$
  - $\pi(A \cap B) = \pi(A \cap C) = \pi(B \cap C) = 1/36$
- The three events are not independent:
  - $\pi(A) \cdot \pi(B) \cdot \pi(C) = 1/216$
  - $\pi(A \cap B \cap C) = 1/36$

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$$\begin{aligned}[A_k] &:= \Omega_1 \times \cdots \times \Omega_{k-1} \times A_k \times \Omega_{k+1} \times \cdots \times \Omega_n \\ &= \{(\omega_1, \dots, \omega_n) \in \Omega : \omega_k \in A_k\}\end{aligned}$$

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- A probability measure π over Ω is called a **product measure** whenever for every $A_1 \subseteq \Omega_1, \dots, A_n \subseteq \Omega_n$ it is the case that $[A_1], \dots, [A_n]$ are independent, i.e.,

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- If π is a product measure, we say that the marginal probability measures $(\mathrm{marg}_{\Omega_1} \pi, \dots, \mathrm{marg}_{\Omega_n} \pi)$ are independent. Otherwise, we say that they are correlated.

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- The probability of each event in Ω depends on which coin we choose to flip at each round.

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  - We flip the fair coin second, *irrespective of the outcome of the first coin.*

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H	3/16	1/16
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    - We flip the fair coin second, *irrespective of the outcome of the first coin.*
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  - We flip the fair coin second, *irrespective of the outcome of the first coin.*
  - We flip the biased coin second, *irrespective of the outcome of the first coin.*
- Not a product measure (correlated flips):
  - We flip the biased coin *after observing heads*, and we flip the fair coin *after observing tail*.

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	H	T
H	3/16	1/16
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	H	T
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- Then, the corresponding probabilities are shown below.

	H	T
H	$1/8$	$1/8$
T	$1/8$	$1/8$

H

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- Suppose that we flip three times. We always flip the fair coin, unless we observe tails in both the first and the second round, in which case we flip the biased coin at the third round.
- Then, the corresponding probabilities are shown below.
- Observe that the events "heads at round 2" and "heads at round 3" are not independent events, but they are conditionally independent given the event "heads at round 1".

	H	T
H	1/8	1/8
T	1/8	1/8

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T

Probability measures in game theory

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  - **Mixed strategies** (objective uncertainty): $\sigma_i \in \Delta(C_i)$

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- There are two types of uncertainty modeled with probability measures in game theory.
  - **Beliefs** (subjective uncertainty): $\mu_i \in \Delta(C_{-i})$
  - **Mixed strategies** (objective uncertainty): $\sigma_i \in \Delta(C_i)$
- Today, we are going to focus on the consequences of correlation in beliefs (correlation in mixed strategies leads to new concepts, viz., most well-known, correlated equilibrium).

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Correlation in first order beliefs

- A (first order) belief is a probability measure μ_i over the product space

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- A belief $\mu_i \in \Delta(C_{-i})$ is **independent** whenever it is a product measure. It is **correlated** otherwise.
- Obviously, in two-player games there is no distinction. Thus, we focus on games with three (or more) players.

Rationality

- In the following example, the numbers correspond to utilities of the matrix player.

$$
\begin{array}{c|c|c|}
 & C & D \\ \hline
A & 2 & 2 \\ \hline
B & 2 & 0 \\ \hline
\multicolumn{3}{c}{L}
\end{array}
\qquad
\begin{array}{c|c|c|}
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A & 0 & 2 \\ \hline
B & 2 & 2 \\ \hline
\multicolumn{3}{c}{M}
\end{array}
\qquad
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 & C & D \\ \hline
A & 1 & 1 \\ \hline
B & 1 & 1 \\ \hline
\multicolumn{3}{c}{R}
\end{array}
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Rationality

- In the following example, the numbers correspond to utilities of the matrix player.
- R is rational given $\mu_a = \left(\frac{1}{2} \otimes (A, C), \frac{1}{2} \otimes (B, D)\right)$, which is a **correlated belief**.

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- A strategy is not strictly dominated if and only if is rational given some belief, *independent or correlated*.

Correlated rationalizability

- Take the following sequence of strategy-type pairs.

$$
\begin{aligned}
CR_i^0 &:= \{(c_i, t_i) : c_i \text{ is rational given } b_i^1(t_i)\} \\
CR_i^1 &:= \{(c_i, t_i) : b_i(t_i)(CR_1^0 \times \cdots \times CR_{i-1}^0 \times CR_{i+1}^0 \times \cdots \times CR_n^0) = 1\} \\
&\vdots \\
CR_i^k &:= \{(c_i, t_i) : b_i(t_i)(CR_1^{k-1} \times \cdots \times CR_{i-1}^{k-1} \times CR_{i+1}^{k-1} \times \cdots \times CR_n^{k-1}) \\
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- Then, $RCBCR_i := \bigcap_{k \geq 0} CR_i^k$ does not impose any restriction on whether the beliefs are correlated or independent.

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- Then, $RCBCR_i := \bigcap_{k \geq 0} CR_i^k$ does not impose any restriction on whether the beliefs are correlated or independent.
- $CR_i := \text{proj}_{C_i}\, CBCR_i$ is the set of **correlated rationalizable strategies** (Brandenburg & Dekel, 1987; Tan & Werlang, 1988).

Independent rationalizability

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- Then, $RCBIR_i := \bigcap_{k\geq 0} IR_i^k$ contains the action-type pairs that satisfy rationality (given independent beliefs) and common belief in rationality (given independent beliefs).

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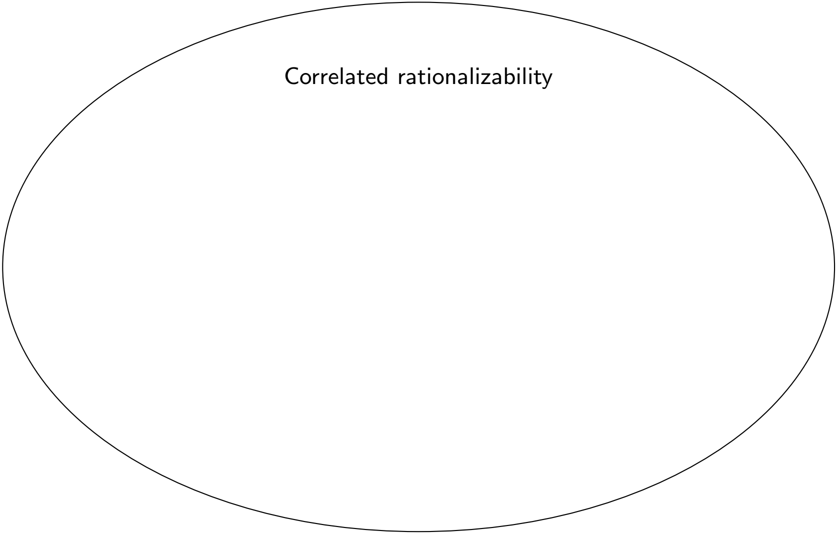
- Then, $RCBIR_i := \bigcap_{k \geq 0} IR_i^k$ contains the action-type pairs that satisfy rationality (given independent beliefs) and common belief in rationality (given independent beliefs).
- $IR_i := \text{proj}_{C_i} RCBIR_i$ is the set of **(independent) rationalizable strategies** (Bernheim, 1984; Pearce, 1984).

Relation between solution concepts



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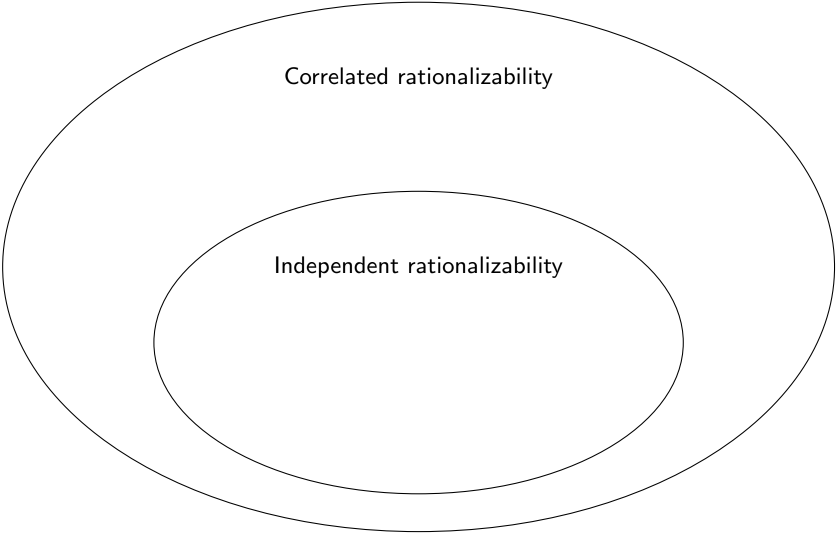

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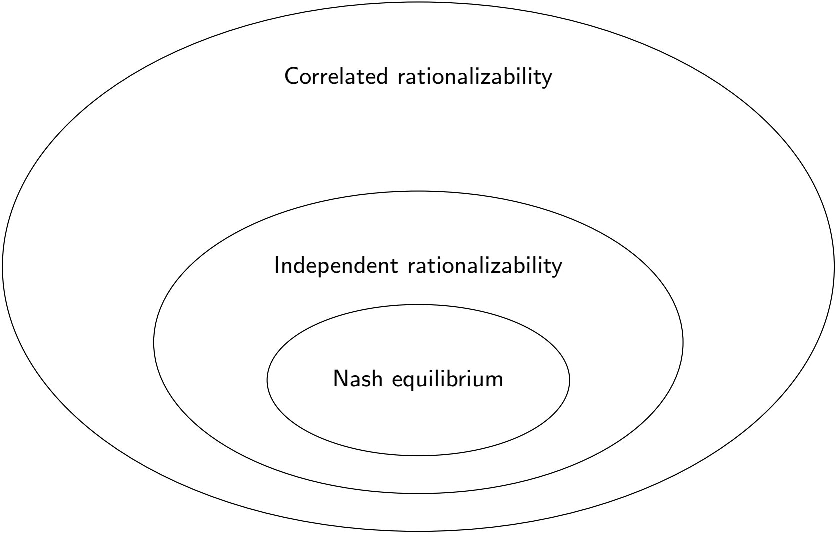

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B	2,2,4	0,2,2

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B	2,2,4	2,2,2

R

	C	D
A	1,3,3	1,3,3
B	1,3,3	1,3,3

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- Correlated rationalizability yields the entire strategy space.

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- Independent rationalizability

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R	C	D
A	~~1,3,3~~	~~1,3,3~~
B	~~1,3,3~~	~~1,3,3~~

- Correlated rationalizability yields the entire strategy space.
- Independent rationalizability

Is $IR_i \subseteq CR_i$ a strict inclusion?

Proposition

The inclusion $IR_i \subseteq CR_i$ can be strict.

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L

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- Does the distinction matter for our predictions?

Modelling intrinsic correlation

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- Intuitively, Ann thinks that Bob and Carol think alike, e.g., they took the same game theory course.

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where $[h_{-a}] := \{(c_{-a}, t_{-a}) : h_j(t_j) = h_j,\ \forall j \neq a\}$.

Conditional Independence: an example

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- For every $j \neq a$, let t_j and t'_j yield a different hierarchy.
- Then, t_a's hierarchy satisfies sufficiency.

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Proposition (Brandenburger & Friedenberg, 2008)

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- In other words, under CI and SUFF, Ann's beliefs about Bob's and Carol's strategies are correlated only if the correlation is intrinsic.

Correlated rationalizability with intrinsic correlation

Definition

We say that a correlated rationalizable strategy c_i is **consistent with intrinsic correlation**, and we write $c_i \in ICR_i$, if there is some $t_i \in T_i^*$ such that

(i) $(c_i, t_i) \in RCBCR_i$, and

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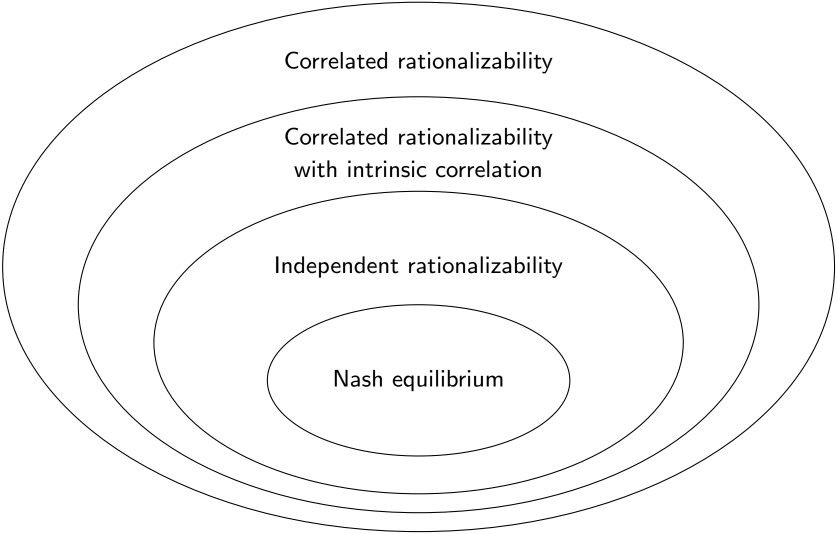
Proposition (Brandenburger & Friedenberg, 2008)

$IR_i \subseteq ICR_i \subseteq CR_i$.

Relation between solution concepts



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$$\{(M, t_a), (M, t'_a)\} \subseteq CR_a^0 \quad \Rightarrow \quad b_a^1(t_a) = b_a^1(t'_a) = \Big(\frac{1}{2} \otimes (A, D), \frac{1}{2} \otimes (B, E)\Big)$$

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$$\begin{aligned}
\{(M,t_a),(M,t_a')\} \subseteq CR_a^0 \quad &\Rightarrow \quad b_a^1(t_a) = b_a^1(t_a') = \left(\frac{1}{2}\otimes(A,D), \frac{1}{2}\otimes(B,E)\right) \\
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$$\{(M,t_a),(M,t'_a)\} \subseteq CR^0_a \quad \Rightarrow \quad b^1_a(t_a) = b^1_a(t'_a) = (\frac{1}{2} \otimes (A,D), \frac{1}{2} \otimes (B,E))$$

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The only belief hierarchy h_a for which M is rational is $h_a(t_a)$ in the type space model $T_a = \{t_a\}$, $T_b = \{t_b\}$, $T_c = \{T_c\}$ with

$$\begin{aligned} b_a(t_a) &= \Big(\frac{1}{2} \otimes \big((A,t_b),(D,t_c)\big), \frac{1}{2} \otimes \big((B,t_b),(E,t_c)\big)\Big) \\ b_b(t_b) &= \Big(1 \otimes \big((M,t_a),(D,t_c)\big)\Big) \\ b_c(t_c) &= \Big(1 \otimes \big((M,t_a),(A,t_b)\big)\Big) \end{aligned}$$

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	D	E	F
A	0,0,0	0,0,0	0,0,1
B	2,0,0	2,0,0	2,0,1
C	2,1,0	2,1,0	2,1,1

R

The only belief hierarchy h_a for which M is rational is $h_a(t_a)$ in the type space model $T_a = \{t_a\}$, $T_b = \{t_b\}$, $T_c = \{T_c\}$ with

$$\begin{aligned} b_a(t_a) &= \left(\frac{1}{2} \otimes ((A, t_b), (D, t_c)), \frac{1}{2} \otimes ((B, t_b), (E, t_c))\right) \\ b_b(t_b) &= \left(1 \otimes ((M, t_a), (D, t_c))\right) \\ b_c(t_c) &= \left(1 \otimes ((M, t_a), (A, t_b))\right) \end{aligned}$$

$ICR_i \subseteq CR_i$ can be a strict inclusion

	D	E	F
A	2,0,0	2,0,0	2,0,1
B	0,0,0	0,0,0	0,0,1
C	2,1,0	2,1,0	2,1,1

L

	D	E	F
A	1,1,1	0,0,1	0,0,1
B	0,1,0	1,0,0	0,0,1
C	0,1,0	0,1,0	0,1,1

M

	D	E	F
A	0,0,0	0,0,0	0,0,1
B	2,0,0	2,0,0	2,0,1
C	2,1,0	2,1,0	2,1,1

R

The only belief hierarchy h_a for which M is rational is $h_a(t_a)$ in the type space model $T_a = \{t_a\}$, $T_b = \{t_b\}$, $T_c = \{T_c\}$ with

$$\begin{aligned}
b_a(t_a) &= \Big(\frac{1}{2} \otimes ((A, t_b), (D, t_c)), \frac{1}{2} \otimes ((B, t_b), (E, t_c))\Big) \\
b_b(t_b) &= \Big(1 \otimes ((M, t_a), (D, t_c))\Big) \\
b_c(t_c) &= \Big(1 \otimes ((M, t_a), (A, t_b))\Big)
\end{aligned}$$

However, $h_a(t_a)$ does <u>not</u> satisfy CI:

$$b_a(t_a)\Big([A]\cap[D] \,\Big|\, [t_b]\cap[t_c]\Big) \neq b_a(t_a)\Big([A] \,\Big|\, [t_b]\cap[t_c]\Big) \cdot b_a(t_a)\Big([D] \,\Big|\, [t_b]\cap[t_c]\Big)$$

Questions???