Cautious reasoning in psychological games

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Abstract

Caution is an integral part of many solution concepts in traditional game theory and is commonly modelled using lexicographic beliefs. We show here that lexicographic beliefs lack the expressive power to model caution once we extend traditional games to psychological games. Quantification of the relation of 'deeming an event infinitely more likely than another event' is necessary, which can be accomplished by using non-standard beliefs.

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1 Introduction

Cautious reasoning is an essential component of various concepts of traditional game theory. It encapsulates the idea that a decision-maker has beliefs that do not disregard any of his opponents' options and explains the epistemics behind solution concepts such as elimination of weakly dominated strategies (Luce & Raiffa, 1957; Blume, Brandenburger & Dekel, 1991a), perfect equilibrium (Selten, 1975; Blume, Brandenburger & Dekel, 1991b), proper equilibrium (Myerson, 1978; Blume, Brandenburger & Dekel, 1991b) and permissibility (Brandenburger, 1992; Börgers, 1994). To give meaning to rational decision-making in scenarios with caution, cautious beliefs are typically modelled using lexicographic probability systems.

A *lexicographic belief* is a finite sequence of beliefs (or "theories") in which the first belief in the sequence is deemed infinitely more important than the second, the second belief infinitely more important than the third, and so on. A player can derive preferences over choices for every belief in the sequence, where preferences based on earlier beliefs in the sequence take precedence over preferences based on beliefs later in the sequence. Under such an interpretation, for a choice to be considered optimal under a lexicographic belief in a traditional game, it must be optimal given the first belief in the sequence. In case of a tie, the choice must then, among those choices with which there was a tie at the previous level, also be considered optimal given the second belief in the sequence. And so, until the

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Table 1: *Game of King and Queen: Gift*

	Beliefs Queen *Games*	*Land*
Games	0	1
Land	1	0

tie is resolved or the sequence ends. Belief in the opponent's rationality can also still be defined in a traditional game with lexicographic beliefs. For instance, we may say a player with lexicographic beliefs believes in his opponent's rationality if in his first belief in the sequence he only considers rational alternatives for his opponents. Then the decision-maker is left free to consider irrational alternatives in beliefs that are 'further' in the sequence.

For similar reasons as in traditional games, one may want to model cautious beliefs for decision-makers in *psychological games*. Psychological games differ from traditional games in the sense that they are able to model belief-dependent motivations as well [Section 2]. That is, utilities in psychological games may depend explicitly on the full belief hierarchy instead of just what a player believes his opponents will choose.

To illustrate this idea, consider the game of a King and a Queen in Table 1. Being fellow monarchs of neighboring lands, the King finds it a good idea to improve his personal relationship with the Queen by surprising her with a gift during her next visit to the Kingdom. He is thinking of either organizing gladiator games in her honor, lasting for half a year, or transferring a large portion of the coastal lands of his Kingdom that are littered with beaches. The Queen is aware the King has the intention of surprising her with either of these two options. If the King organizes the Games while the Queen expected the transfer of the coastal lands, the King receives a utility of 1. If the King transfers the Land while the Queen expected the Games, the King also receives a utility of 1 by surprising her. All other extreme scenarios result in a utility of 0. If with some probability p he expects the Queen to be incorrect in her prediction of what he will do and with probability $1 - p$ to be correct in her belief, then the King receives a utility of p.

Let us assume that the King knows the Queen is a cautious reasoner and will not disregard either of the two options the King has. Also let us assume that the King with probability a half thinks that the Queen considers it infinitely more likely that the King will transfer the coastal lands compared to organizing Games and with probability one half that the Queen considers it infinitely more likely that the King will organize Games instead of transfer the lands. These orderings can be represented by the King's second-order lexicographic beliefs.

In traditional games, in case of a tie in terms of preferences over choices given a previous belief in a sequence, a player would move further down in his lexicographic belief until the end of the sequence or until the tie is resolved. The case here is that the King's preferences depend on his second-order beliefs, i.e. in particular his beliefs about the Queen's lexicographic beliefs about his choice. In fact, the King deems possible two different first-order lexicographic beliefs for the Queen. Each of these lexicographic beliefs specify the relative importance of her theories *within* each lexicographic belief. However, we cannot make comparisons between theories from different lexicographic beliefs without imposing extra assumptions [Section 3]. For instance, it is well possible that deeming something 'infinitely more likely' means something different probabilistically in the two possible lexicographic beliefs for the Queen that the King considers. In this regard, we can ask ourselves the following question: does the King expect the Queen to be more likely to believe the King will choose Games or Land? At face value, the answer to this is not obvious, yet is relevant for determining the King's preferences.

Preferably, to compare theories from two different sequences of beliefs, one wants to be able to quantify what it means for one theory to be 'infinitely more important' than another. This would involve assigning infinitely small numbers to events that are deemed infinitely less likely to happen than others. This is a feature that non-standard analysis accomplishes by constructing *infinitesimals* (Robinson, 1973) [Section 4].

In the remainder of this note we will elaborate further on the shortcomings of lexicographic beliefs in modeling cautious reasoning in psychological games. Moreover, the expressive power of non-standard analysis will be contrasted to this. In Section 2, we define the framework of psychological games. In Section 3, the problem of lexicographic beliefs in psychological games will be discussed. Section 4 concludes by illustrating the necessary expressive power of non-standard, cautious beliefs in psychological games and shows how these beliefs can be used to define the cautious reasoning concept of permissibility for such games.

2 Psychological games

A psychological game is a generalisation of a traditional game (Geanakoplos et al., 1989; Battigalli & Dufwenberg, 2009). In a traditional game a player's preferences are shaped exclusively by decisions made by himself and his first-order belief about what others will choose. In a psychological game the player's utility depends on his own decision and can depend linearly or non-linearly on any *higher*-order belief. That is, the player's preferences may depend on what he believes an opponent believes about what the player is going to do, what he believes the opponent believes about what he believes the opponent is going to do, et cetera. We can formally define a psychological game as follows (Jagau & Perea, 2017).

Definition 1. *A* ***static psychological game*** *is a tuple* $G = (C_i, B_i, u_i)_{i \in I}$ *with* I *denoting the finite set of players,* C_i *representing the finite set of choices for player* i*,*[1] B_i *the set of belief hierarchies for player* i *that express coherency and common belief in coherency, and* $u_i : C_i \times B_i \to \mathbb{R}$ *representing player* i*'s utility function.*

A belief hierarchy $b_i \in B_i$ for a player i represents an infinite chain of beliefs, with an increasing order. The first element in this chain represents the first-order belief about the opponents' choices, the second represents the second-order belief about the opponents' choices combined with the opponents' beliefs about their opponents' choices and the third represents the third-order belief about the combination of opponents' choices, opponents' first-order beliefs and the opponents' second-order beliefs. And so on. In the game of the King and Queen in the introduction, only part of the second-order beliefs specifically matter for the King. Namely, he only cares for what he *expects* the Queen to believe about what he will choose to do. Similarly, in the game yet to be considered in the next section, only the Queen's second-order beliefs, that is, what she believes the King believes she will do, matter.

[1] C_i may well be a singleton set, indicating a situation where player i does not have any choices to make but where his beliefs matter for the utilities of other players.

3 The problem with lexicographic beliefs

Following Blume et al. (1991a), we can define a lexicographic belief as follows.

Definition 2. *A* ***lexicographic belief*** b_i *for player* i *on a finite set* X *is a finite sequence of beliefs* $(b_i^1, ..., b_i^n)$*, where each element specifies a probability distribution on* X*. We call* b_i^1 *the primary theory,* b_i^2 *the secondary theory, and so on.*

Thus, a lexicographic belief captures an ordering of beliefs, where the $(l-1)^{th}$ belief is deemed infinitely more important than the l^{th} belief. We could impose natural conditions on these sequences of beliefs, such as that $b_i^l \neq b_i^m$ for every $m \neq l$. This ensures no inconsistencies in the ordering occur. That is, we cannot have a scenario where e.g. $b_i^l = b_i^{l+2}$ is deemed infinitely more important than b_i^{l+1} and vice versa. However, as this section will show, irregardless of whether lexicographic beliefs meet such conditions, problems may occur in analysing psychological games.

Since we are explicitly dealing with higher-order beliefs or even full belief hierarchies, it is helpful to simplify notation by usage of types (Harsanyi, 1967-1968). To every belief hierarchy of player i, we can assign a type $t_i \in T_i$, where T_i is a finite set of types. Doing this for every player in the game, one can construct an epistemic model.

Definition 3. *Consider a psychological game* G*. An* ***epistemic model with lexicographic beliefs*** $M = (T_i, b_i)_{i \in I}$ *for* G *specifies for every player* i *a finite set* T_i *of possible types. Moreover, for every player* i *and every type* $t_i \in T_i$ *the epistemic model specifies a lexicographic belief* $b_i(t_i) = (b_i^1(t_i); b_i^2(t_i); ...; b_i^n(t_i))$ *over the set* $C_{-i} \times T_{-i}$ *of opponents' choice-type combinations.*

Thus every type specifies a first-order lexicographic belief about the opponents' choice-type pairs, whose types specify a first-order lexicographic belief about their opponents' choice-type pairs. Continuing this process, we can retrieve for each type a hierarchy of lexicographic beliefs. Finally, we assume that a player i makes his decisions in line with subjective expected utility maximization. In a *traditional* game (where utilities only depend on first-order beliefs) with lexicographic beliefs this implies that a choice c_i is weakly preferred over another alternative $c_i' \in C_i$ if:

$u_i(c_i, b_i^1) > u_i(c_i', b_i^1)$, or
$u_i(c_i, b_i^1) = u_i(c_i', b_i^1)$ and $u_i(c_i, b_i^2) > u_i(c_i', b_i^2)$, or
...
$u_i(c_i, b_i^1) = u_i(c_i', b_i^1)$ and $u_i(c_i, b_i^2) = u_i(c_i', b_i^2)$ and ... and $u_i(c_i, b_i^n) \geq u_i(c_i', b_i^n)$.
We say the choice c_i is optimal for b_i if the decision maker does not prefer any choice to c_i.

Extending the above intuition of optimality of a choice to psychological games is problematic however. To understand why, let us return to the Queen and the King in Table 1. The beliefs of the King and Queen are presented in Table 2. One interpretation of these beliefs could be that organising Games is deemed infinitely more likely *to a higher degree* than lands for the Queen's belief induced by t_2', than Lands is deemed infinitely more likely than Games for the Queen's belief induced by t_2. Then it is clear that choosing Land is optimal for the King. However, the opposite is possible as well, under which Games is the only optimal choice for the King.

Now, suppose the King had decided to organize the gladiator games for the Queen. However, the spending during the half-year of the games were so exorbitantly high that the people started a rebellion against the King. The King does not have the required number

Table 2: Epistemic model with lexicographic beliefs, Gift game

Type Queen	$T_1 = \{t_1\}$		
Types King	$T_2 = \{t_2, t_2'\}$		
Queen's beliefs	$b_1(t_1)$	$=$	$\frac{1}{2}t_2 + \frac{1}{2}t_2'$
Queen's beliefs	$b_2(t_2)$	$=$	$((Lands, t_1); (Games, t_1))$
	$b_2(t_2')$	$=$	$((Games, t_1); (Lands, t_1))$

Table 3: *Game of King and Queen: Aid.*

	Beliefs King		
	Largeforce	*No aid*	*Smallforce*
Largeforce	−2	3	−2
No aid	−2	0	−2
Smallforce	−1	4	−6

of faithful soldiers to protect his position. The Queen knows that sending a large force of soldiers of her own to protect her colleague will easily quell the rebellion, whereas a small force will also subdue the rebels, though at the larger cost of losing more King's soldiers. This game is presented in Table 3. The Queen has two separate motivations here: to lose as few soldiers as possible and to be respected by her fellow ruler the King. Sending a large force will cost her 2 units of utility, whereas a small force will cost her 1 unit of utility. The respect the Queen cares for can come in three forms: if she sends no aid at all whereas the King believes her to send some soldiers to beat the rebellion, she loses 2 units of utility due to an *expected loss* in respect with the King. More important to her is however to see the King be elated with her helping presence. As such, she believes she will be perceived as the 'unexpected savior' by the King. Thus, sending a force (large or small) to subdue the rebels, while she expected the King to believe she would not send any help, will give the Queen a utility of 5. However, sending a small force specifically may also have an exactly opposite effect for the Queen. That is, she certainly does not wish to *reinforce* any existing reservations the King may have about her sending troops *solely* to be perceived as the unexpected savior instead of also caring for his safety and mental well-being. This would occur if the King believes the Queen to only send a small force instead of a large force to quickly quell the rebellion, while the Queen in fact indeed sends such a small force. Such a circumstance would cause her to lose 5 units of utility.

Let us make the assumption that the Queen believes the King is a cautious reasoner. We can model the King's cautious beliefs by lexicographic probabilities over the set of the Queen's choices.[2] Caution implies here that for each type of the Queen considered by the King, positive probability is assigned to each possible choice for the Queen somewhere in the lexicographic belief. The Queen by definition satisfies caution, as there are no choices by the King to be cautious about.

To illustrate the problems regarding lexicographic beliefs in psychological games, a simple epistemic model will suffice. A leading example is presented in Table 4. The Queen considers two *types* of the King, each representing one of the King's possible lexicographic beliefs. The King considers a single type for the Queen which induces a non-lexicographic belief. The

[2] Caution can also be defined over the strategy-type space, instead of just over the strategy-space (as adopted in this paper). However, this distinction is irrelevant for the problem discussed here.

Table 4: Epistemic model with lexicographic beliefs, V1

Type Queen	$T_1 = \{t_1\}$
Types King	$T_2 = \{t_2, t_2'\}$
Queen's beliefs	$b_1(t_1) = \frac{4}{5}t_2 + \frac{1}{5}t_2'$
King's beliefs	$b_2(t_2) = ((LF, t_1); \frac{4}{10}(SF, t_1) + \frac{6}{10}(NA, t_1))$
	$b_2(t_2') = ((SF, t_1); \frac{4}{10}(LF, t_1) + \frac{6}{10}(NA, t_1))$

Queen believes here with probability $\frac{4}{5}$ that the King deems it infinitely more likely that the Queen will send out a large force of soldiers (LF) than a small force (SF) or no aid (NA) at all. Yet, the latter two choices are still considered by the King in the secondary theory of his lexicographic belief with probability $\frac{4}{10}$ and $\frac{6}{10}$ respectively. In a similar manner, the Queen believes with probability $\frac{1}{5}$ that the King deems it infinitely more likely than either a large force or no force at all that a small force will be sent to aid him. However, the Queen also believes with probability $\frac{1}{5}$ that the King still deems a large force or no aid possible in his secondary theory with probability $\frac{4}{10}$ and $\frac{6}{10}$ respectively.

Were the Queen only to consider the primary theories of each of the King's two possible lexicographic beliefs, then it is clear that the Queen is indifferent between all her options. Namely, we would have that $u_Q(LF) = u_Q(NA) = u_Q(SF) = -2$. Similarly, were the Queen only to look at the secondary theories in the King's lexicographic beliefs, then it is clear that only LF is an optimal choice for her. That is, we would have

$$u_Q(LF) = \frac{4}{5}(\frac{6}{10} \cdot 3 + \frac{4}{10} \cdot (-2)) + \frac{1}{5}(\frac{6}{10} \cdot 3 + \frac{4}{10} \cdot (-2)) = \frac{4}{5} \cdot 1 + \frac{1}{5} \cdot 1 = 1$$

for her choice LF,

$$u_Q(NA) = \frac{4}{5}(\frac{6}{10} \cdot 0 + \frac{4}{10} \cdot (-2)) + \frac{1}{5}(\frac{6}{10} \cdot 0 + \frac{4}{10} \cdot (-2)) = -\frac{4}{5}$$

for her choice NA, and for her choice SF

$$u_Q(SF) = \frac{4}{5}(\frac{6}{10} \cdot 4 + \frac{4}{10} \cdot (-6)) + \frac{1}{5}(\frac{6}{10} \cdot 4 + \frac{4}{10} \cdot (-1)) = \frac{4}{5} \cdot 0 + \frac{1}{5} \cdot 2 = \frac{2}{5}.$$

We cannot make the statement however that this makes the choice to send a large force LF optimal for the Queen by the observations above. By Definition 2, we know that $b_2(t_2)$ and $b_2(t_2')$ are both sequences of beliefs on $C_1 \times T_1$ where the beliefs are decreasing in importance. However, we need something even more expressive than this. That is to say, at face value we cannot recover from a lexicographic belief whether 'deeming a choice infinitely less likely' means probabilistically the same thing in $b_2(t_2)$ as in $b_2(t_2')$. It could well be that the secondary theory in $b_2(t_2)$ receives infinitely less weight compared to the secondary theory in $b_2(t_2')$ (or vice versa) with the information we have now. Such information is however relevant for the Queen, since her utility depends on her second-order expectations.

To see this, first consider the secondary *theory* in $b_2(t_2)$ to be deemed infinitely less likely to occur by the Queen in her second-*order* belief than the secondary theory in $b_2(t_2')$. Say the Queen would fully focus on what she believes would be the King's secondary theories to determine her preferences. This would have as a consequence that the Queen expects the King to believe her choosing SF is infinitely less likely to occur than LF. As a result, the Queen has little concern that the King believes she will only send a small force, leading

Table 5: Epistemic model with lexicographic beliefs, V2

Type Queen	$T_1 = \{t_1\}$		
Types King	$T_2 = \{t_2, t_2'\}$		
Queen's beliefs	$b_1(t_1)$	$=$	$(\frac{4}{5}t_2 + \frac{1}{5}t_2';\ t_2)$
King's beliefs	$b_2(t_2)$	$=$	$((LF, t_1); \frac{4}{10}(SF, t_1) + \frac{6}{10}(NA, t_1))$
	$b_2(t_2')$	$=$	$((SF, t_1); \frac{4}{10}(LF, t_1) + \frac{6}{10}(NA, t_1))$

SF to be the only optimal choice. However, if we reverse the relation between $b_2(t_2)$ and $b_2(t_2')$, it would mean that, as far as secondary theories go, the Queen expects the King to deem SF to be infinitely *more* likely to occur than LF. In such a scenario, sending a large force would be the Queen's only optimal choice. In more general words, lexicographic beliefs carry insufficient information to consistently determine a player's preference over his own choices in a psychological game.

An assumption that one could impose is that 'deeming a belief infinitely more important' means the same thing across lexicographic beliefs of a particular player. However, such a resolution may also not suffice. Consider the scenario in which the Queen also has a lexicographic belief by simply extending her belief $b_1(t_1)$ in Table 4 to one as portrayed in Table 5. Now, in the Queen's secondary theory she fully believes that the King is of type t_2 and thus primarily believes the Queen will play LF. Under such beliefs, SF would be the only optimal choice. It is clear that the Queen's preferences over her choices shaped by her primary theory about the King's primary theories have precedence over the preferences shaped by second-order beliefs relating to all other combinations of theories. It is also obvious that the preferences shaped by the Queen's secondary theory about the King's secondary theories ought to be assigned the least importance. However, with the information provided as of yet it is unclear whether the preferences shaped by the Queen's primary theory about the King's secondary theories should take precedence over those shaped by the Queen's secondary theory about the King's primary theory. This does however matter for determining the Queen's optimal choice: in the former case LF is the optimal choice, but in the latter SF.

This would leave us with two options to go forward from here. First, we could take an axiomatic approach and define an additional *choice rule*. For instance, we could define the ordering of the preferences such that the preferences shaped by the Queen's secondary (and if she had any: her tertiary, quaternary etc.) theory about the King's primary theory (or theories) are deemed infinitely more important than the ones shaped by the Queen's beliefs about non-primary theories of the King. This ordering is however rather random in the sense that we might as well propose a different ordering rule that may be equally intuitive. Instead, we can also look at the primitives of the model of expected utility maximization. We may represent cautious beliefs such that it still allows for deriving optimal choices in psychological games in a non-ambiguous manner and without the need to specify extra choice rules. This implies we need to be able to give a clear description of one event being 'deemed infinitely more likely' than another, a matter elaborated on in the following section.

Table 6: Epistemic model with non-standard beliefs, V1

Type Queen	$T_1 = \{t_1\}$		
Types King	$T_2 = \{t_2, t_2'\}$		
Queen's beliefs	$b_1(t_1)$	$=$	$\frac{4}{5}t_2 + \frac{1}{5}t_2'$
King's beliefs	$b_2(t_2)$	$=$	$(1-\epsilon)(LF, t_1)$
		$+$	$\epsilon(\frac{4}{10}(SF, t_1) + \frac{6}{10}(NA, t_1))$
	$b_2(t_2')$	$=$	$(1-\epsilon)(SF, t_1)$
		$+$	$\epsilon(\frac{4}{10}(LF, t_1) + \frac{6}{10}(NA, t_1))$

Table 7: Epistemic model with non-standard beliefs, V2

Type Queen	$T_1 = \{t_1\}$		
Types King	$T_2 = \{t_2, t_2'\}$		
Queen's beliefs	$b_1(t_1)$	$=$	$\frac{4}{5}t_2 + \frac{1}{5}t_2'$
King's beliefs	$b_2(t_2)$	$=$	$(1-\epsilon^2)(LF, t_1)$
		$+$	$\epsilon^2(\frac{4}{10}(SF, t_1) + \frac{6}{10}(NA, t_1))$
	$b_2(t_2')$	$=$	$(1-\epsilon)(SF, t_1)$
		$+$	$\epsilon(\frac{4}{10}(LF, t_1) + \frac{6}{10}(NA, t_1))$

4 Non-standard beliefs as a solution

As the previous discussion has shown, psychological games with cautious beliefs including infinitely small weights on choices will require one to *quantify* what it means for a player to deem one event 'infinitely more or less likely' than another. Such a quantification can be provided by using *non-standard analysis* (going back to at least Robinson (1973)). The main idea of this form of analysis is that one extends the line of reals $\mathbb{R}$ to a non-Archimedean field of hyperreals $\mathbb{R}^*$, which includes *infinitesimals* yet retains the first-order structure of the line of reals. A strictly positive number $\epsilon \in \mathbb{R}^*$ is called an infinitesimal if $\epsilon \cdot r < 1$ for every $r \in \mathbb{R}$.[3] One important feature of infinitesimals will be highlighted here. If we have $r, s \in \mathbb{R}^*$ with r and s strictly positive such that $\frac{s}{r}$ is an infinitesimal, then we can say that s is infinitely smaller than r. This allows us to capture the intuition of lexicographic beliefs in the sense that we can still define what it means for one event to be deemed infinitely less likely than another, but also allows us to quantify this relation.

Using *non-standard probabilities*, we can transform the beliefs in the epistemic model of Table 4. Let a non-standard probability distribution p on X assign probabilities $p(x) \in \mathbb{R}^*$, where $p(x) \geq 0$, such that $\sum_{x \in X} p(x) = 1$. An epistemic model with non-standard beliefs is then as follows.

Definition 4. *Consider a psychological game G. An* ***epistemic model*** *$M = (T_i, b_i)_{i \in I}$* ***with non-standard beliefs*** *for G specifies for every player i a finite set T_i of possible types. Moreover, for every player i and every type $t_i \in T_i$ the epistemic model specifies a non-standard probability distribution $b_i(t_i)$ on the set $C_{-i} \times T_{-i}$ of opponents' choice-type combinations.*

The result of adapting Table 4 to account for this is found in Table 6, where $\epsilon > 0$ is an infinitesimal. For both his types the King is cautious, as in both cases he deems possible every choice, be it with a positive real number or an infinitesimal number. Note that in this particular example we modelled the King's cautious beliefs such that deeming one event infinitely more likely than another means the same thing for both his types t_2 and t_2', as in both this relation is defined using the same infinitesimal ϵ.

Crucial here is that non-standard probability measures are defined on a non-Archimedean, *ordered* field and higher-order, non-standard beliefs are defined, in a sense, over the set of such probability measures. The resulting probability measure is then still defined on an

[3] An infinitesimal may be constructed from a fixed sequence that converges to 0 using ultrafilters or using a polynomial construction as in Robinson (1973). For an in-depth discussion about non-standard analysis in (traditional) game theory, see Hammond (1994) and Halpern (2010).

ordered field. This is in contrast to e.g. a second-order lexicographic belief, which is defined over the set of *sequences* of standard probability measures. The way in which higher-order, non-standard beliefs are constructed, thus allows us to define the expected utility for a player in a similar manner as how one would define expected utility with standard beliefs. The only distinction is that the utility function now is a mapping $u_i : C_i \times B_i^* \to \mathbb{R}^*$, where B_i^* is the set of non-standard belief hierarchies. Taking into account an epistemic model, utilities can be defined as a function of choices and types: $u_i(c_i, t_i)$. This utility itself may also involve infinitesimal numbers. An optimal choice for a type with non-standard beliefs can subsequently be defined as follows.

Definition 5. *Consider an epistemic model $M = (T_i, b_i)_{i \in I}$ with* ***non-standard beliefs*** *for G and a type t_i for player i in such a model. A* ***choice*** *c_i is* ***optimal*** *for type t_i of player i if $\forall c_i' \in C_i : u_i(c_i, t_i) \geq u_i(c_i', t_i)$.*

With these tools in hand, we can show that using non-standard beliefs the issue of determining an optimal choice specifically can be resolved. Returning to the game in Table 3, we already established the expected utilities for the Queen given the primary theory and given the secondary theory in the previous section. If we take the weighted sum of these, where $(1 - \epsilon)$ is assigned to the primary theory and ϵ in the secondary for both $b_2(t_2)$ and $b_2(t_2')$, we have $u_Q(LF) = (1-\epsilon) \cdot (-2) + \epsilon$ for her choice LF, $u_Q(NA) = (1-\epsilon) \cdot (-2) + \epsilon \cdot (-\frac{4}{5})$ for her choice NA, and $u_Q(SF) = (1 - \epsilon) \cdot (-2) + \epsilon \cdot (\frac{4}{10})$ for her choice SF. Thus, we have unambiguously derived that under this specific second-order belief sending a large force is optimal for the Queen.

In a similar manner, we can transform the epistemic model in Table 4 into another epistemic model with non-standard beliefs that induces the same lexicographic beliefs, but in which sending a small force would be the only optimal choice. This is depicted in Table 7. Note however that now the relation of one event being infinitely more likely than another in $b_2(t_2)$ is denoted by ϵ^2 (that of $b_2(t_2')$ is still given by ϵ), where ϵ^2 is infinitely smaller than ϵ. Since the two non-standard beliefs of the King are characterized by two different infinitesimals, we need to take the weighted sum of the utilities given each combination of theories and types to derive expected utilities for the Queen. We then acquire $u_Q(LF) = \frac{4}{5}(-2(1 - \epsilon^2) + \epsilon^2) + \frac{1}{5}(-2(1 - \epsilon) + \epsilon) = -\frac{8}{5} + \frac{8}{5}\epsilon^2 + \frac{4}{5}\epsilon^2 - \frac{2}{5} + \frac{2}{5}\epsilon + \frac{1}{5}\epsilon = -2 + \frac{3}{5}\epsilon + \frac{12}{5}\epsilon^2$. In a similar way, we can get for her choice NA that $u_Q(NA) = -2 - \frac{6}{25}\epsilon + \frac{24}{25}\epsilon^2$ and $u_Q(SF) = -2 + \frac{11}{5}\epsilon + \frac{4}{5}\epsilon^2$ for her choice SF. Clearly then, sending a small force would be optimal. This in part because the 'primary theory' in $b_2(t_2)$ is deemed infinitely more important than its secondary theory to a slightly higher degree than that the primary theory in $b_2(t_2')$ is deemed infinitely more important than its secondary theory. Hence, the Queen expects the King to believe in his primary theories that the Queen is slightly more likely to choose LF than SF. Also important is however that the secondary theory of $b_2(t_2)$ is assigned an infinitely smaller probability than the secondary theory of $b_2(t_2')$. It follows that the Queen expects the King to believe in his secondary theories that LF and NA are infinitely more likely to be chosen than SF. This explains why SF is the only optimal choice.

Once utility (and thus optimality) is defined, we can extend solution concepts from traditional games to psychological games. For notions where caution is an integral part, such as permissibility (Brandenburger, 1992; Borgers, 1994) and common full belief in caution and primary belief in rationality (Perea, 2012), we can use non-standard beliefs to do so.

Definition 6. *Consider an epistemic model $M = (T_i, b_i)_{i \in I}$ with non-standard beliefs and a type t_i for player i. Player i has a* ***cautious type*** *if, whenever he deems possible an* *opponent's type t_j for some player j, then for every $c_j \in C_j$ it assigns positive probability in $\mathbb{R}^*$ to (c_j, t_j). This probability may be real or an infinitesimal.*

A type is deemed possible if it is assigned positive probability in $\mathbb{R}^*$ in the belief. Assuming a player to have a full-support belief and believing in the opponent's rationality may be incompatible. However, we can impose the following condition for a choice to be considered rational.

Definition 7. *Consider an epistemic model $M = (T_i, b_i)_{i \in I}$ with non-standard beliefs and a type t_i for player i.* ***Type t_i primarily believes in the opponent's rationality*** *if for every opponent $j \neq i$ we have $b_i(t_i)(c_j, t_j) \in \mathbb{R}_+$ only if c_j is optimal for t_j, where $\mathbb{R}_+$ is the set of all positive real numbers.*

We can iterate the arguments of believing an opponent is cautious and primarily believing in an opponent's rationality.

Definition 8. *Consider an epistemic model $M = (T_i, b_i)_{i \in I}$ with non-standard beliefs and a type t_i for player i. Type t_i expresses 1-fold full belief in caution if it only deems possible opponents' types that are cautious. For every $k > 1$, every player i, and every type $t_i \in T_i$, we say that type t_i expresses k-fold full belief in caution if t_i only deems possible opponents' types that express $(k-1)$-fold full belief in caution.*
Type t_i expresses ***common full belief in caution*** *if t_i expresses k-fold full belief in caution for every k.*

Definition 9. *Consider an epistemic model $M = (T_i, b_i)_{i \in I}$ with non-standard beliefs and a type t_i for player i. Type t_i expresses 1-fold full belief in primary belief in rationality if t_i primarily believes in the opponent's rationality. For every $k > 1$, every player i, and every type $t_i \in T_i$, we say that type t_i expresses k-fold full belief in primary belief in rationality if t_i only deems possible opponents' types that express $(k-1)$-fold full belief in primary belief in rationality.*
Type t_i expresses ***common full belief in primary belief in rationality*** *if t_i expresses k-fold full belief in primary belief in rationality for every k.*

Then, a *rational* choice under common full belief in caution and primary belief in rationality entails that the choice is optimal for a type t_i that expresses common full belief in caution and primary belief in rationality. Note that common full belief in caution and primary belief in rationality is similar to the solution concept of permissibility. The only difference between the two is that under permissibility a player assigns infinitesimal probability to opponent's types that do not primarily believe in the opponent's rationality.

It is clear that there is no type in either the epistemic model of Table 6 or the epistemic model of Table 7 that expresses common full belief in caution and primary belief in rationality. Namely, the Queen only has a single type, that considers type t_2 and t_2' for the King. Type t_2 primarily believes the Queen will choose LF and type t_2' primarily believes the Queen will choose SF. We already established that in the situation of Table 6 LF is the only optimal course of action and in the situation of Table 7 SF is the only optimal choice. The Queen expects the King however to believe with some positive probability in $\mathbb{R}_+$ that she will choose LF and with some positive probability in $\mathbb{R}_+$ that the she will choose SF. However, by the previous argument LF and SF cannot be optimal choices at the same time in these two situations. Hence, the Queen does not believe the King always primarily believes in her rationality. This does not mean there is no epistemic model in which common full

Table 8: Epistemic model with non-standard beliefs, V3

Type Queen	$T_1 = \{t_1, t_1'\}$		
Types King	$T_2 = \{t_2, t_2'\}$		
Queen's beliefs	$b_1(t_1)$	$=$	t_2
	$b_1(t_1')$	$=$	t_2'
King's beliefs	$b_2(t_2)$	$=$	$(1-\epsilon-\epsilon^2)(LF, t_1')$
		$+$	$\epsilon(SF, t_1') + \epsilon^2(NA, t_1')$
	$b_2(t_2')$	$=$	$(1-\epsilon-\epsilon^2)(SF, t_1)$
		$+$	$\epsilon(NA, t_1) + \epsilon^2(LF, t_1))$

belief in caution and primary belief in rationality is satisfied. One possibility is a scenario in which the Queen believes the King to (partially) believe she is of a different type than she actually is. Namely, one of the motivations of the Queen is to gain respect from the King by surprisingly coming to his aid. Surprise and correct beliefs are incompatible. Table 8 provides an epistemic model in which SF is optimal for type t_1 and LF is optimal for type t_1', where both types express common full belief in caution and primary belief in rationality. The analysis is left to the reader. Note finally that if caution is assumed, NA cannot be optimal, as it is weakly dominated.

The distinctive feature of a belief modelled by non-standard probabilities compared to a lexicographic belief, is that in case of the latter we know that it is derived from *some* sequence of beliefs, whereas in case of the former we have more information about which *specific* sequence of beliefs it would correspond to. That is, for each lexicographic belief we can find an equivalent non-standard belief that quantifies the probabilistic structure behind one state being 'infinitely more likely' than another. In some cases this extra information evidently is crucial in unambiguously deriving preferences over choices. However, it also stresses that in psychological games, depending on the types of beliefs held by all players, more sorts of information have to be accounted for. If cautious reasoning is involved, the decision-maker needs to be aware of what 'infinitely more likely' means in one considered belief hierarchy of the opponent compared to another.

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