Characterizing Assumption of Rationality by Incomplete Information*

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Shuige Liu†

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Abstract

We characterize common assumption of rationality of 2-person games within an incomplete information framework. We use the lexicographic model with incomplete information and show that a belief hierarchy expresses common assumption of rationality within a complete information framework if and only if there is a belief hierarchy within the corresponding incomplete information framework that expresses common full belief in caution, rationality, every good choice is supported, and prior belief in the original utility functions.

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Keywords: *epistemic game theory*, *lexicographic belief*, *assumption of rationality*, *incomplete information*

1. Introduction

Assumption of rationality is a concept in epistemic game theory introduced by Brandenburger et al. [9] and studied in Perea [12] by using lexicographic belief. A lexicographic belief is said to *assume the opponents' rationality* means that a "good" choice always occurs in front of a "bad" one. Here by good we mean a choice of the opponent can be supported by a *cautious* belief of him, that is, a belief that does not exclude any choice of the opponents; by bad we mean it cannot be supported by any such belief.

Like other concepts in epistemic game theory such as permissibility (Brandenburger [8]) and proper rationalizability (Schuhmacher [15], Ascheim [1]), iterative admissibility is defined partly to alleviate the tension between caution and rationality (Blume et al. [5], Brandenburger [8],

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†Faculty of Political Science and Economics, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-Ku, 169-8050, Tokyo, Japan (shuige_liu@aoni.waseda.jp).

Börgers [6], Samuelson [14], Börgers and Samuelson [7]) by sacrificing rationality. Caution requires that every choice, be it rational or not, should appear in a belief; assumption of rationality only requires that those rational choices should occur in front of those irrational ones but cannot exclude the irrational ones. On the other hand, since rationality is a basic assumption on human behavior in game theory, it seems desirable to find an approach to have a "complete" rationality while to keep the definition of iterative admissibility.

One approach is to use an incomplete information framework introduced by Perea and Roy [13] which defined standard probabilistic epistemic model with incomplete information and used it to characterized ε-proper rationalizability. Following their approach, Liu [11] defined lexicographic epistemic models with incomplete information, constructed a mapping between them and models with complete information, and characterized permissibility and proper rationalizability. In this paper, we still use the construction in Liu [11] and characterize assumption of rationality. We show that a choice is optimal for a belief hierarchy which expresses common assumption of rationality within a complete information framework if and only if it is optimal for a belief hierarchy within the corresponding incomplete information framework that expresses common full belief in caution, rationality, every good choice is supported, and prior belief in the original utility functions.

This paper is organized as follows. Section 2 gives a survey of assumption of rationality in epistemic models with complete information and the lexicographic epistemic models with incomplete information. Section 3 gives the characterization result and their proofs. Section 4 gives some concluding remarks on the relationship between the result of this paper and characterization of permissbility in Section 4.6 of Liu [11].

2. Models

2.1. Complete information model

In this subsection, we give a survey of lexicographic epistemic model with complete information and define iterative admissibility within it. We adopt the approach of Perea [12], Chapters 5 and 7. See Brandenburger et al. [9] for an alternative approach.

Consider a finite 2-person static game $\Gamma = (C_i, u_i)_{i\in I}$ where $I = \{1, 2\}$ is the set of players, C_i is the finite set of choices and $u_i : C_1 \times C_2 \to \mathbb{R}$ is the utility function for player $i \in I$. In the following we sometimes denote $C_1 \times C_2$ by C. We assume that each player has a lexicographic belief on the opponent's choices, a lexicographic belief on the opponent's lexicographic belief on her, and so on. This belief hierarchy is described by a lexicographic epistemic model with types.

Definition 2.1 (Epistemic model with complete information). Consider a finite 2-person static game $\Gamma = (C_i, u_i)_{i\in I}$. A finite *lexicographic epistemic model* for Γ is a tuple $M^{co} = (T_i, b_i)_{i\in I}$ where

(a) T_i is a finite set of types, and

(b) b_i is a mapping that assigns to each $t_i \in T_i$ a lexicographic belief over $\Delta(C_j \times T_j)$, i.e., $b_i(t_i) = (b_{i1}, b_{i2}, ..., b_{iK})$ where $b_{ik} \in \Delta(C_j \times T_j)$ for $k = 1, ..., K$.

Consider $t_i \in T_i$ with $b_i(t_i) = (b_{i1}, b_{i2}, ..., b_{iK})$. Each b_{ik} $(k = 1, ..., K)$ is called t_i's *level-k belief.* For each $(c_j, t_j) \in C_j \times T_j$, we say t_i *deems* (c_j, t_j) *possible* iff $b_{ik}(c_j, t_j) > 0$ for some $k \in \{1, ..., K\}$. We say t_i *deems* $t_j \in T_j$ *possible* iff t_i deems (c_j, t_j) possible for some $c_j \in C_j$. For each $t_i \in T_i$, we denote by $T_j(t_i)$ the set of types in T_j deemed possible by t_i. A type $t_i \in T_i$ is *cautious* iff for each $c_j \in C_j$ and each $t_j \in T_j(t_i)$, t_i deems (c_j, t_j) possible. That is, t_i takes into account each choice of player j for every belief hierarchy of j deemed possible by t_i.

For each $c_i \in C_i$, let $u_i(c_i, t_i) = (u_i(c_i, b_{i1}).,.., u_i(c_i, b_{iK}))$ where for each $k = 1, ..., K$, $u_i(c_i, b_{ik}) := \Sigma_{(c_j,t_j)\in C_j\times T_j} b_{ik}(c_j, t_j)u_i(c_i, c_j)$, that is, each $u_i(c_i, b_{ik})$ is the expected utility for c_i

over b_{ik} and $u_i(c_i, t_i)$ is a vector of expected utilities. For each $c_i, c_i' \in C_i$, we say that t_i *prefers* c_i *to* c_i', denoted by $u_i(c_i, t_i) > u_i(c_i', t_i)$, iff there is $k \in \{0, ..., K-1\}$ such that the following two conditions are satisfied:

(a) $u_i(c_i, b_{i\ell}) = u_i(c_i', b_{i\ell})$ for $\ell = 0, ..., k$, and

(b) $u_i(c_i, b_{i,k+1}) > u_i(c_i', b_{i,k+1})$.

We say that t_i *is indifferent between* c_i *and* c_i', denoted by $u_i(c_i, t_i) = u_i(c_i', t_i)$, iff $u_i(c_i, b_{ik}) = u_i(c_i', b_{ik})$ for each $k = 1, ..., K$. It can be seen that the preference relation on C_i under each type t_i is a linear order. c_i is *rational* (or *optimal*) for t_i iff t_i does not prefer any choice to c_i.

For $(c_j, t_j), (c_j', t_j') \in C_j \times T_j$, we say that t_i *deems* (c_j, t_j) *infinitely more likely than* (c_j', t_j') iff there is $k \in \{0, ..., K-1\}$ such that the following two conditions are satisfied:

(a) $b_{i\ell}(c_j, t_j) = b_{i\ell}(c_j', t_j') = 0$ for $\ell = 1, ..., k$, and

(b) $b_{i,k+1}(c_j, t_j) > 0$ and $b_{i,k+1}(c_j', t_j') = 0$.

Definition 2.2 (Assumption of rationality) A cautious type $t_i \in T_i$ *assumes the* j*'s rationality* iff the following two conditions are satisfied:

(A1) for all of player j's choices c_j that are optimal for some cautious belief, t_i deems possible some type t_j for which c_j is optimal;

(A2) t_i deems all choice-type pairs (c_j, t_j) where t_j is cautious and c_j is optimal for t_j infinitely more likely than any choice-type pairs (c_j', t_j') that does not have this property.

Informally speaking, assumption of the opponent's rationality is that t_i puts all "good" choices in front of those "bad" choices.The following definition extends assumption of rationality inductively into n-fold for any $n \in \mathbb{N}$.

Definition 2.3 (n-fold assumption of rationality) Consider a finite lexicographic epistemic model $M^{co} = (T_i, b_i)_{i\in I}$ for a game $\Gamma = (C_i, u_i)_{i\in I}$. A cautious type $t_i \in T_i$ *expresses* 1*-fold assumption of rationality* iff it assumes j's rationality. For any $n \in \mathbb{N}$, we say that a cautious type $t_i \in T_i$ expresses $(n+1)$*-fold assumption of rationality* iff the following two conditions are satisfied:

(nA1) whenever a choice c_j of player j is optimal for some cautious type (not necessarily in M^{co}) that expresses up to n-fold assumption of rationality, t_i deems possible some cautious type t_j for player j which expresses up to n-fold assumption of rationality for which c_j is optimal;

(nA2) t_i deems all choice-type pair (c_j, t_j), where t_j is cautious and expresses up to n-fold assumption of rationality and c_j is optimal for t_j, infinitely more likely than any choice-type pairs (c_j', t_j') that does not satisfy this property.

We say that t_i *expresses common assumption of rationality* iff it expresses n-fold assumption of rationality for every $n \in \mathbb{N}$.

2.2. Incomplete information model

In this subsection, we give a survey of lexicographic epistemic model with incomplete information defined in Liu [11] which is the counterpart of the probabilistic epistemic model with incomplete information introduced by Battigalli [2] and further developed in Battigalli and Siniscalchi [3], [4], and Dekel and Siniscalchi [10]. We also define some conditions on types in such a model.

Definition 2.4 (Lexicographic epistemic model with incomplete information). Consider a finite 2-person static game form $G = (C_i)_{i\in I}$. For each $i \in I$, let V_i be the set of utility functions $v_i : C_1 \times C_2 \to \mathbb{R}$. A *finite lexicographic epistemic model for* G *with incomplete information* is a tuple $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ where

(a) Θ_i is a finite set of types,

(b) w_i is a mapping that assigns to each $\theta_i \in \Theta_i$ a utility function $w_i(\theta_i) \in V_i$, and

(c) β_i is a mapping that assigns to each $\theta_i \in \Theta_i$ a lexicographic belief over $\Delta(C_j \times \Theta_j)$, i.e., $\beta_i(\theta_i) = (\beta_{i1}, \beta_{i2}, ..., \beta_{iK})$ where $\beta_{ik} \in \Delta(C_j \times \Theta_j)$ for $k = 1, ..., K$.

Concepts such as "θ_i deems (c_j, θ_j) possible" and "θ_i deems (c_j, θ_j) infinitely more likely than (c'_j, θ'_j)" can be defined in a similar way as in Section 2.1. For each $\theta_i \in \Theta_i$, we use $\Theta_j(\theta_i)$ to denote the set of types in Θ_j deemed possible by θ_i. For each $\theta_i \in \Theta_i$ and $v_i \in V_i$, $\theta_i^{v_i}$ is the auxiliary type satisfying that $\beta_i(\theta_i^{v_i}) = \beta_i(\theta_i)$ and $w_i(\theta_i^{v_i}) = v_i$.

For each $c_i \in C_i, v_i \in V_i$, and $\theta_i \in \Theta_i$ with $\beta_i(\theta_i) = (\beta_{i1}, \beta_{i2}, ..., \beta_{iK})$, let $v_i(c_i, \theta_i) = (v_i(c_i, \beta_{i1}), ..., v_i(c_i, \beta_{iK}))$ where for each $k = 1, ..., K$, $v_i(c_i, \beta_{ik}) := \Sigma_{(c_j,\theta_j) \in C_j \times \Theta_j} \beta_{ik}(c_j, \theta_j) v_i(c_i, c_j)$. For each $c_i, c'_i \in C_i$ and $\theta_i \in \Theta_i$, we say that θ_i *prefers* c_i *to* c'_i iff $w_i(\theta_i)(c_i, \theta_i) > w_i(\theta_i)(c'_i, \theta_i)$. As in Section 2.1, this is also the lexicographic comparison between two vectors. c_i is *rational* (or *optimal*) for θ_i iff θ_i does not prefer any choice to c_i.

Definition 2.5 (Caution). $\theta_i \in \Theta_i$ is *cautious* iff for each $c_j \in C_j$ and each $\theta_j \in \Theta_j(\theta_i)$, there is some utility function $v_j \in V_j$ such that θ_i deems $(c_j, \theta_j^{v_j})$ possible.

This is a faithful translation of Perea and Roy [13]'s definition of caution in probabilistic model (p.312) into lexicographic model. It is the counterpart of caution defined within the complete information framework in Section 2.1; the only difference is that in incomplete information models we allow different utility functions since c_j will be required to be rational for the paired type.

Definition 2.6 (Belief in rationality). $\theta_i \in \Theta_i$ *believes in j's rationality* iff θ_i deems (c_j, θ_j) possible only if c_j is rational for θ_j.

The following lemma shows that caution and a belief of full rationality can be satisfied simultaneously in an incomplete information model because each type is assigned with a belief on the opponent's choice-type pairs as well as a payoff function. The consistency of caution and full rationality is the essential difference between models with incomplete information and those with complete information.

Lemma 2.1 (Belief in rationality can be satisfied). Consider a static game form $G = (C_i)_{i \in I}$, $C'_i \in C_i$, and $\beta_i = (\beta_{i1}, \beta_{i2}, ..., \beta_{iK})$ such that $\beta_{ik} \in \Delta(C_j)$ for each $k = 1, ..., K$. Then there is $v_i \in V_i$ such that each $c_i \in C'_i$ is optimal in v_i for β_i.

Proof. There are various way to construct such a v_i. Here we provide a simple one. For each $c \in C$, let

$$v_i(c) = \begin{cases} 1 \text{ if } c_i \in C'_i \text{ and } c_j \in \text{supp}\beta_{i1}; \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

It can be seen that each $c_i \in C'_i$ is optimal in v_i for β_i. //

Caution and belief in rationality can be extended into k-fold for any $k \in \mathbb{N}$ as follows. Let P be an arbitrary property of lexicographic beliefs. We define that

(CP1) $\theta_i \in \Theta$ *expresses* 0*-fold full belief in* P iff θ_i satisfies P;

(CP2) For each $n \in \mathbb{N}$ with $n \geq 2$, $\theta_i \in T_i$ *expresses* n*-fold full belief in* P iff θ_i only deems possible j's types that express n-fold full belief in P.

θ_i *expresses common full belief in* P iff it expresses n-fold full belief in P for each $n \in \mathbb{N}$. By replacing P with "caution" or "rationality" we can obtain common full belief in caution or in rationality.

The following two conditions are important in characterizing assumption of rationality.

Definition 2.7 (Every good choice is supported). Consider a static game form $G = (C_i)_{i \in I}$, a lexicographic epistemic model $M^{in} = (\Theta_i, w_i, \beta_i)_{i \in I}$ for G with incomplete information, and a pair $u = (u_i)_{i \in I}$ of utility functions. A cautious type $\theta_i \in \Theta_i$ *believes in that every good choice of j is supported* iff for each c_j that is optimal for some cautious type of j (may not be in M^{in}) with u_j as its assigned utility function, θ_i deems possible a cautious type $\theta_j \in \Theta_j$ such that

$w_j(\theta_j) = u_j$ and c_j is optimal for θ_j.

Definition 2.8 (Prior belief in u). Consider a static game form $G = (C_i)_{i \in I}$, a lexicographic epistemic model $M^{in} = (\Theta_i, w_i, \beta_i)_{i \in I}$ for G with incomplete information, and a pair $u = (u_i)_{i \in I}$ of utility functions. $\theta_i \in \Theta_i$ *priorly believes in* u iff for any (c_j, θ_j) with θ_j cautious deemed possible by θ_i satisfying that $w_j(\theta_j) = u_j$, then θ_i deems (c_j, θ_j) infinitely more likely than any pair does not satisfy that property.

Common full belief in that every good choice is supported and prior belief in u is different from that in caution or rationality. We have the following definition.

Definition 2.9 (n-fold belief in that every good choice is supported and prior belief in u) Consider a static game form $G = (C_i)_{i \in I}$, a lexicographic epistemic model $M^{in} = (\Theta_i, w_i, \beta_i)_{i \in I}$ for G with incomplete information, and a pair $u = (u_i)_{i \in I}$ of utility functions. $\theta_i \in \Theta_i$ *express* 1*-fold belief in that every good choice is supported and prior belief in* u iff it believes that every good choice of j is supported and has prior belief in u. For any $n \in \mathbb{N}$, we say that a cautious type $\theta_i \in \Theta_i$ expresses $(n+1)$*-fold belief in prior belief in that every good choice is supported and prior belief in* u iff the following two conditions are satisfied:

(nP1) whenever a choice c_j of player j is optimal for some cautious type (not necessarily in M^{in}) with u_j as its assigned utility function that expresses up to n-fold belief in that every good choice is supported, θ_i deems possible some cautious type θ_j with $w_j(\theta_j) = u_j$ for player j which expresses up to n-fold belief in that every good choice is supported and prior belief in u for which c_j is optimal.

(nP2) θ_i deems all choice-type pair (c_j, θ_j), where θ_j is cautious and expresses up to n-fold belief in that prior belief in u and every good choice is supported and satisfies $w_j(\theta_j) = u_j$, infinitely more likely than any choice-type pairs (c'_j, θ'_j) that does not satisfy this property.

We say that t_i *expresses common full belief in that every good choice is supported and prior belief in* u iff it expresses n-fold belief in that every good choice is supported and prior belief in u for every $n \in \mathbb{N}$.

3. Characterization

So far we have introduced two different groups of concepts for static games: one includes assumption of rationality within a complete information framework, the other contains some conditions on types within an incomplete information framework. In this section we will show that there is correspondence between them.

3.1. The statement and an example

In this subsection we give the characterization result and an illustrative example.

Theorem 3.1 (Characterization of iterative admissibility). Consider a finite 2-person static game $\Gamma = (C_i, u_i)_{i \in I}$ and the corresponding game form $G = (C_i)$. $c_i^* \in C_i$ is optimal to some type expressing common full belief in caution and common assumption of rationality within some finite epistemic model with complete information if and only if there is some finite epistemic model $M^{in} = (\Theta_i, w_i, \beta_i)_{i \in I}$ with incomplete information for G and some $\theta_i^* \in \Theta_i$ with $w_i(\theta_i^*) = u_i$ such that

(a) c_i^* is rational for θ_i^*, and

(b) θ_i^* expresses common full belief in caution, rationality, that every good choice is supported, and prior belief in u.

To show this statement, we need to construct a correspondence between states of complete and incomplete information models. Before we go to the formal proof, we use the following three examples to show the intuition.

Example 3.1. Consider the following game Γ (Perea [12], p.262):

$u_1\backslash u_2$	D	E	F
A	$0,3$	$1,2$	$1,4$
B	$1,3$	$0,2$	$1,1$
C	$1,6$	$1,2$	$0,1$

and the lexicographic epistemic model $M^{co}=(T_i,b_i)_{i\in I}$ with complete information for Γ where $T_1=\{t_1,t_1',t_1''\}$, $T_2=\{t_2,t_2'\}$, and

$$\begin{aligned}
b_1(t_1) &= ((F,t_2'),(E,t_2'),(D,t_2),...),\ b_1(t_1')=((D,t_2),(F,t_2'),(E,t_2),...),\\
b_1(t_1'') &= ((D,t_2),(E,t_2),(F,t_2'),...),\\
b_2(t_2) &= ((B,t_1'),(C,t_1''),(A,t_1),...),\ b_2(t_2')=((A,t_1),(B,t_1'),(C,t_1''),...).
\end{aligned}$$

It can be seen that B is iteratively admissible since B is optimal to t_1' and t_1' expresses common full assumption of rationality (cf. Perea [12], pp. 312-314). We can construct a corresponding lexicographic epistemic model $M^{in}=(\Theta_i,w_i,\beta_i)_{i\in I}$ with incomplete information where $\Theta_1=\{\theta_{11},\theta_{11}',\theta_{11}'',\theta_{12},\theta_{12}',\theta_{12}'',\theta_{13},\theta_{13}',\theta_{13}''\}$, $\Theta_2=\{\theta_{21},\theta_{21}',\theta_{21}'',\theta_{22},\theta_{22}',\theta_{22}''\}$, and

$$\begin{aligned}
w_1(\theta_{11}) &= v_{11}=u_1,\ w_1(\theta_{11}')=v_{11}',w_1(\theta_{11}'')=v_{11}'',\\
\beta_1(\theta_{11}) &= \beta_1(\theta_{11}')=\beta_1(\theta_{11}'')=((F,\theta_{22}),(E,\theta_{22}''),(D,\theta_{21}),...),\\
w_1(\theta_{12}) &= v_{12}=u_1,\ w_1(\theta_{12}')=v_{12}',w_1(\theta_{12}'')=v_{12}'',\\
\beta_1(\theta_{12}) &= \beta_1(\theta_{12}')=\beta_1(\theta_{12}'')=((D,\theta_{21}),(F,\theta_{22}),(E,\theta_{22}''),...),\\
w_1(\theta_{13}) &= v_{13}=u_1,\ w_1(\theta_{13}'')=v_{13}',w_1(\theta_{12}')=v_{13}'',\\
\beta_1(\theta_{12}) &= \beta_1(\theta_{12}')=\beta_1(\theta_{12}'')=((D,\theta_{21}),(E,\theta_{22}''),(F,\theta_{22}),...),\\
w_2(\theta_{21}) &= v_{21}=u_2,\ w_2(\theta_{21}')=v_{21}',w_1(\theta_{21}'')=v_{21}'',\\
\beta_2(\theta_{21}) &= \beta_2(\theta_{21}')=\beta_2(\theta_{21}'')=((B,\theta_{12}),(C,\theta_{13}),(A,\theta_{11}),...),\\
w_2(\theta_{22}) &= v_{22}=u_2,\ w_2(\theta_{22}')=v_{22}',w_1(\theta_{22}'')=v_{21}'',\\
\beta_2(\theta_{22}) &= \beta_2(\theta_{22}')=\beta_2(\theta_{22}'')=((A,\theta_{11}),(B,\theta_{12}),(C,\theta_{13}),...)
\end{aligned}$$

where

v_{11}'	D	E	F
A	0	1	1
B	1	0	2
C	1	1	0

,

v_{11}''	D	E	F
A	0	1	1
B	1	0	1
C	1	1	2

,

v_{12}'	D	E	F
A	0	1	1
B	1	0	1
C	2	1	0

,

v_{12}''	D	E	F
A	2	1	1
B	1	0	1
C	1	1	0

,

v_{13}'	D	E	F
A	0	1	1
B	2	0	1
C	1	1	0

,

v_{13}''	D	E	F
A	0	1	1
B	1	0	2
C	1	1	0

,

v_{21}'	D	E	F
A	3	2	4
B	3	4	1
C	6	2	1

,

v_{21}''	D	E	F
A	3	2	4
B	3	2	4
C	6	2	1

,

v_{22}'	D	E	F
A	5	2	4
B	3	2	1
C	6	2	1

,

v_{22}''	D	E	F
A	3	5	4
B	3	3	1
C	6	2	1

.

It can be seen that θ_{12} expresses common full belief in caution, rationality, that every good choice is supported, and prior belief in u. Also, since θ_{12} generates the same belief hierarchy as t_1' does and $w_1(\theta_{12})=u_1$, B is optimal for θ_{12}.

3.2. Proof of Theorem 3.1

To show Theorem 3.1, we construct the mappings between finite lexicographic epistemic models with complete information and those with incomplete information. First, consider $\Gamma=(C_i,u_i)_{i\in I}$

and a finite lexicographic epistemic model $M^{co} = (T_i, b_i)_{i\in I}$ with complete information for Γ. We first define types in a model with incomplete information in the following two steps:

Step 1. For each $i \in I$ and $t_i \in T_i$, let $\Pi_i(t_i) = (C_{i1}, ..., C_{iL})$ be the partition of C_i defined in Lemma 2.1, that is, $\Pi_i(t_i)$ is the sequence of equivalence classes of choices in C_i arranged from the most preferred to the least preferred under t_i. We define $v_{i\ell}(t_i) \in V_i$ for each $\ell = 1, ..., L$. We let $v_{i1}(t_i) = u_i$. By Lemma 2.1, for each $C_{i\ell}$ with $\ell > 1$ there is some $v_{i\ell}(t_i) \in V_i$ such that each choice in $C_{i\ell}$ is rational at $v_{i\ell}(t_i)$ under t_i.

Step 2. We define $\Theta_i(t_i) = \{\theta_{i1}(t_i), ..., \theta_{iL}(t_i)\}$ where for each $\ell = 1, ..., L$, the type $\theta_{i\ell}(t_i)$ satisfies that (1) $w_i(\theta_{i\ell}(t_i)) = v_{i\ell}(t_i)$, and (2) $\beta_i(\theta_{i\ell}(t_i))$ is obtained from $b_i(t_i)$ by replacing every (c_j, t_j) with $c_j \in C_{jr} \in \Pi_j(t_j)$ for some r with (c_j, θ_j) where $\theta_j = \theta_{jr}(t_j)$, that is, $w_j(\theta_j)$ is the utility function among those corresponding to $\Pi_j(t_j)$ in which c_j is the rational for t_i.

For each $i \in I$, let $\Theta_i = \cup_{t_i\in T_i}\Theta_i(t_i)$. Here we have constructed a finite lexicographic epistemic model $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ for the corresponding game form $G = (C_i)_{i\in I}$ with incomplete information. In the following example we show how this construction goes.

Example 3.2. Consider the following game Γ (Perea [12], p.188):

$u_1\backslash u_2$	C	D
A	$1, 0$	$0, 1$
B	$0, 0$	$0, 1$

and the lexicographic epistemic model $M^{co} = (T_i, b_i)_{i\in I}$ Γ where $T_1 = \{t_1\}$, $T_2 = \{t_2\}$, and

$$b_1(t_1) = ((D, t_2), (C, t_2)),\ b_2(t_2) = ((A, t_1), (B, t_1)).$$

We show how to construct a corresponding model $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$. First, by Step 1 it can be seen that $\Pi_1(t_1) = (\{A\}, \{B\})$ and $\Pi_2(t_2) = (\{D\}, \{C\})$. We let $v_{11}(t_1) = u_1$ where A is rational for t_1 and $v_{12}(t_1)$ where B is rational for t_1 as follows. Similarly, we let $v_{21}(t_2) = u_2$ where D is rational under t_2 and $v_{22}(t_2)$ where C is rational under t_2 as follows:

$v_{12}(t_1)$	C	D
A	1	0
B	0	1

,

$v_{22}(t_2)$	C	D
A	2	1
B	0	1

.

Then we go to Step 2. It can be seen that $\Theta_1(t_1) = \{\theta_{11}(t_1), \theta_{12}(t_1)\}$, where

$$\begin{aligned} w_1(\theta_{11}(t_1)) &= v_{11}(t_1),\ \beta_1(\theta_{11}(t_1)) = ((D, \theta_{21}(t_2)), (C, \theta_{22}(t_2))), \\ w_1(\theta_{12}(t_1)) &= v_{12}(t_1),\ \beta_1(\theta_{12}(t_1)) = ((D, \theta_{21}(t_2)), (C, \theta_{22}(t_2))). \end{aligned}$$

Also, $\Theta_2(t_2) = \{\theta_{21}(t_2), \theta_{22}(t_2)\}$, where

$$\begin{aligned} w_2(\theta_{21}(t_2)) &= v_{21}(t_2),\ \beta_2(\theta_{21}(t_2)) = ((A, \theta_{11}(t_1)), (B, \theta_{12}(t_1))), \\ w_2(\theta_{22}(t_2)) &= v_{22}(t_2),\ \beta_2(\theta_{22}(t_2)) = ((A, \theta_{11}(t_1)), (B, \theta_{12}(t_1))). \end{aligned}$$

Let $M^{co} = (T_i, b_i)_{i\in I}$ and $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ be constructed from M^{co} by the two steps above. We have the following observations.

Observation 3.1 (Redundancy). For each $t_i \in T_i$ and each $\theta_i, \theta_i' \in \Theta_i(t_i)$, $\beta_i(\theta_i) = \beta_i(\theta_i')$.

Observation 3.2 (Rationality). Eeach $\theta_i \in \Theta_i(t_i)$ believes in j's rationality.

We omit their proofs since they hold by construction. Observation 3.1 means that the difference between any two types in a $\Theta_i(t_i)$ is in the utility functions assigned to them. Observation 3.2 means that in an incomplete information model constructed from one with complete information, each type has (full) belief in the opponent's rationality. This is because in the construction, we requires that for each pair (c_j, t_j) occurring in a belief, its counterpart in the incomplete information replaces t_j by the type in $\Theta_j(t_j)$ with the utility function in which c_j is optimal for $b_i(t_j)$. It follows from Observation 3.2 that each $\theta_i \in \Theta_i(t_i)$ expresses common full belief in rationality.

The following lemma shows that caution is preserved in this construction.

Lemma 3.1 (Caution^{co} $\rightarrow$ Caution^{in}). Let $M^{co} = (T_i, b_i)_{i\in I}$ and $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ be constructed from M^{co} by the two steps above. If $t_i \in T_i$ expresses common full belief in caution, so does each $\theta_i \in \Theta_i(t_i)$.

Proof. We show this statement by induction. First we show that if t_i is cautious, then each $\theta_i \in \Theta_i(t_i)$ is also cautious. Let $c_j \in C_j$ and $\theta_j \in \Theta_j(\theta_i)$. By construction, it can be seen that the type $t_j \in T_j$ satisfying the condition that $\theta_j \in \Theta_j(t_j)$ is in $T_j(t_i)$. Since t_i is cautious, t_i deems (c_j, t_j) possible. Consider the pair (c_j, θ'_j) in $\beta_i(\theta_i)$ corresponding to (c_j, t_j). Since both θ_j and θ'_j are in $\Theta_j(t_j)$, it follows from Observation 3.1 that $\beta_j(\theta_j) = \beta_j(\theta'_j)$. Hence $(c_j, \theta_j^{w_j(\theta'_j)})$ is deemed possible by θ_i. Here we have shown that θ_i is cautious.

Suppose we have shown that, for each $i \in I$, if t_i expresses n-fold full belief in caution then so does each $\theta_i \in \Theta_i(t_i)$. Now suppose that t_i expresses $(n+1)$-fold full belief in caution, i.e., each $t_j \in T_j(t_i)$ expresses n-fold full belief in caution. By construction, for each $\theta_i \in \Theta_i(t_i)$ and each $\theta_j \in \Theta_j(\theta_i)$ there is some $t_j \in T_j(t_i)$ such that $\theta_j \in \Theta_j(t_i)$, and, by inductive assumption, each $\theta_j \in \Theta_j(\theta_i)$ expresses n-fold full belief in caution. Therefore, each $\theta_i \in \Theta_i(t_i)$ expresses $(n+1)$-fold full belief in caution. //

We also need a mapping from epistemic models with incomplete information to those with complete information. Consider a finite 2-person static game $\Gamma = (C_i, u_i)_{i\in I}$, the corresponding game form $G = (C_i)_{i\in I}$, and a finite epistemic model $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ for G with incomplete information. We construct a model $M^{co} = (T_i, b_i)_{i\in I}$ for Γ with complete information as follows. For each $\theta_i \in \Theta_i$, we define $E_i(\theta_i) = \{\theta'_i \in \Theta_i : \beta_i(\theta'_i) = \beta(\theta_i)\}$. In this way Θ_i is partitioned into some equivalence classes $\mathbb{E}_i = \{E_{i1}, ..., E_{iL}\}$ where for each $\ell = 1, .., L$, $E_{i\ell} = E_i(\theta_i)$ for some $\theta_i \in \Theta_i$. To each $E_i \in \mathbb{E}_i$ we use $t_i(E_i)$ to represent a type. We define $b_i(t_i(E_i))$ to be a lexicographic belief which is obtained from $\beta_i(\theta_i)$ by replacing each occurrence of (c_j, θ_j) by $(c_j, t_j(E_j(\theta_j)))$; in other words, $b_i(t_i(E_i))$ has the same distribution on choices at each level as $\beta_i(\theta_i)$ for each $\theta_i \in E_i$, while each $\theta_j \in \Theta_j(\theta_i)$ is replaced by $t_j(E_j(\theta_j))$. For each $i \in I$, let $T_i = \{t_i(E_i)\}_{E_i\in\mathbb{E}_i}$. We have constructed from M^{in} a finite epistemic model $M^{co} = (T_i, b_i)_{i\in I}$ with complete information for Γ.

It can be seen that this is the reversion of the previous construction. That is, let $M^{co} = (T_i, b_i)_{i\in I}$ satisfying that $b_i(t_i) \neq b_i(t'_i)$ for each $t_i, t'_i \in T_i$ with $t_i \neq t'_i$, and $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ be constructed from M^{co} by the previous two steps. Then $\mathbb{E}_i = \{\Theta_i(t_i)\}_{t_i\in T_i}$ and $t_i(\Theta_i(t_i)) = t_i$ for each $i \in I$.

In the following example we show how this construction goes.

Example 3.3. Consider the game Γ in Example 3.2 and the model $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ for the corresponding game form where $\Theta_1 = \{\theta_{11}, \theta_{12}\}$, $\Theta_2 = \{\theta_{21}, \theta_{22}\}$, and

$$\begin{aligned} w_1(\theta_{11}) &= u_1,\ \beta_1(\theta_{11}) = ((D, \theta_{21}), (C, \theta_{22})), \\ w_1(\theta_{12}) &= v_1,\ \beta_1(\theta_{12}) = ((D, \theta_{21}), (C, \theta_{22})), \\ w_2(\theta_{21}) &= u_2,\ \beta_2(\theta_{21}) = ((A, \theta_{11}), (B, \theta_{12})), \\ w_2(\theta_{22}) &= v_2,\ \beta_2(\theta_{22}) = ((A, \theta_{11}), (B, \theta_{12})). \end{aligned}$$

where $v_1 = v_{12}(t_1)$ and $v_2 = v_{22}(t_2)$ in Example 3.2. It can be seen that $\mathbb{E}_1 = \{\{\theta_{11}, \theta_{12}\}\}$ since $\beta_1(\theta_{11}) = \beta_1(\theta_{12})$ and $\mathbb{E}_2 = \{\{\theta_{21}, \theta_{22}\}\}$ since $\beta_2(\theta_{21}) = \beta_2(\theta_{22})$. Corresponding to those equivalence classes we have $t_1(\{\theta_{11}, \theta_{12}\})$ and $t_2(\{\theta_{21}, \theta_{22}\})$, and

$$\begin{aligned} b_1(t_1(\{\theta_{11}, \theta_{12}\})) &= ((D, t_2(\{\theta_{21}, \theta_{22}\})), (C, t_2(\{\theta_{21}, \theta_{22}\}))), \\ b_2(t_2(\{\theta_{21}, \theta_{22}\})) &= ((A, t_1(\{\theta_{11}, \theta_{12}\})), (B, t_1(\{\theta_{11}, \theta_{12}\}))). \end{aligned}$$

We have the following lemmas.

Lemma 3.2 (Caution^{in} $\to$ Caution^{co}). Let $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ and $M^{co} = (T_i, b_i)_{i\in I}$ be constructed from M^{in} by the above approach. If $\theta_i \in \Theta_i$ expresses common full belief in caution, so does $t_i(E_i(\theta_i))$.

Proof. We show this statement by induction. First we show that if θ_i is cautious, then $t_i(E_i(\theta_i))$ is also cautious. Let $c_j \in C_j$ and $t_j \in T_j(t_i(E_i(\theta_i)))$. By construction, $t_j = t_j(E_j)$ for some $E_j \in \mathbb{E}_j$, and there is some $\theta_j \in E_j$ which is deemed possible by θ_i. Since θ_i is cautious, there is some θ'_j with $\beta_j(\theta'_j) = \beta_j(\theta_j)$, i.e., $\theta'_j \in E_j$, such that (c_j, θ'_j) is deemed possible by θ_i. By construction, (c_j, t_j) is deemed possible by $t_i(E_i(\theta_i))$.

Suppose we have shown that, for each $i \in I$, if θ_i expresses n-fold full belief in caution then so does $t_i(E_i(\theta_i))$. Now suppose that θ_i expresses $(n+1)$-fold full belief in caution, i.e., each $\theta_j \in \Theta_j(\theta_i)$ expresses n-fold full belief in caution. Since, by construction, for each $t_j \in T_j(t_i(E_i(\theta_i)))$, there is some $\theta_j \in \Theta_j(\theta_i)$ such that $t_j = t_j(E_j(\theta_j))$, by inductive assumption t_j expresses n-fold full belief in caution. Therefore, $t_i(E_i(\theta_i))$ expresses $(n+1)$-fold full belief in caution. //

Lemma 3.3 (Assumption of rationality $\longleftrightarrow$ every good choice is supportedprior + belief in u). Let $M^{co} = (T_i, b_i)_{i\in I}$ and $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ be constructed from M^{co}. If $t_i \in T_i$ expresses common assumption of rationality, then each $\theta_i \in \Theta_i(t_i)$ expresses common full belief in that every good choice is supported and prior belief in u.

On the other hand, let $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ and $M^{co} = (T_i, b_i)_{i\in I}$ be constructed from M^{in}. If $\theta_i \in \Theta_i$ expresses common full belief in that every good choice is supported and prior belief in u, then $t_i(E_i(\theta_i))$ expresses common full assumption of rationality.

Proof. We show this statement by induction. Let $\theta_i \in \Theta_i(t_i)$. First we show that if t_i assumes in j's rationality, θ_i believes that every good choice is supported and prior belief in u. Let $c_j \in C_j$ be optimal for some cautious type of j whose assigned utility function is u_j within an epistemic model with incomplete information. It is easy to see that c_j is optimal for its corresponding type, which is also cautious by Lemma 3.2, in any complete information model constructed from the one with incomplete information by our approach above. Since t_i assumes j's rationality, t_i deems possible a cautious type t_j for which c_j is optimal. By construction, some $\theta_j \in \Theta_j(t_j)$ is deemed possible by θ_i. Since t_i is cautious, (c_j, t_j) is deemed possible by t_i, and, by construction $(c_j, \theta_{j1}(t_j))$ is deemed possible by θ_i. Since $w_j(\theta_{j1}(t_j)) = u_j$ and c_j is optimal for $\theta_{j1}(t_j)$, it follows that θ_i believes in that every good choice is supported.

Let (c_j, θ_j) with θ_j cautious deemed possible by θ_i satisfying $w_j(\theta_j) = u_j$ and (c'_j, θ'_j) a pair which does not satisfy that condition. Let (c_j, t_j) and (c'_j, t'_j) be the pairs occuring in the belief of t_i corresponding to (c_j, θ_j) and (c'_j, θ'_j). Since c_j is rational to θ_j and $w_j(\theta_j) = u_j$, it follows that c_j is optimal for t_j. On the other hand, c'_j is not optimal for t'_j. Since t_i assumes j's rationality, t_i deems (c_j, t_j) infinitely more likely than (c'_j, t'_j). By construction, θ_i deems (c_j, θ_j) infinitely more likely than (c'_j, θ'_j). Here we have shown that θ_i priorly believes in u.

Now we show the other direction: suppose that if $\theta_i \in \Theta_i$ believes in that every good choice is supported and priorly believes in u, we prove that $t_i(E_i(\theta_i))$ assumes j's rationality. Suppose that c_j is optimal for some cautious type within some epistemic model with complete information. It can be seen by construction that c_j is optimal for some cautious type with u_i as its assigned utility function within some epistemic model with incomplete information which corresponds to that complete information model. Since θ_i believes in that every good choice is supported, θ_i deems possible a cautious type θ_j such that $w_j(\theta_j) = u_j$ and c_j is optimal for θ_j. By construction it follows that $t_i(E_i(\theta_i))$ deems $t_j(E_j(\theta_j))$ possible for which c_j is optimal.

Let (c_j, t_j) with t_j cautious be a pair which is deemed possible by $t_i(E_i(\theta_i))$ satisfying that c_j is optimal for t_j, and (c'_j, t'_j) be a pair deemed possible by $t_i(E_i(\theta_i))$ which does not satisfy that condition. Let (c_j, θ_j) and (c'_j, θ'_j) be the corresponding pairs occuring in the belief of θ_i. Since θ_i believes in rationality, by construction it follows that $u_j(\theta_j) = u_j$ while $u_j(\theta'_j) \neq u_j$. Since θ_i priorly believes in u, θ_i deems (c_j, θ_j) infinitely more likely than (c'_j, θ'_j). It follows that $t_i(E_i(\theta_i))$ deems (c_j, t_j) infinitely more likely than (c'_j, t'_j). Here we have shown that $t_i(E_i(\theta_i))$ assumes j's rationality.

Suppose that, for some $n \in \mathbb{N}$, we have shown that for each $k \leq n$,

(n1) if $t_i \in T_i$ expresses k-fold assumption of rationality, then each $\theta_i \in \Theta_i(t_i)$ expresses k-fold full belief in that every good choice is supported and prior belief in u;

(n2) If $\theta_i \in \Theta_i$ expresses k-fold full belief in that every good choice is supported and prior belief in u, then $t_i(E_i(\theta_i))$ expresses k-fold assumption of rationality.

Now we show that these two statements hold for $n+1$. First, suppose that $t_i \in T_i$ expresses $(n+1)$-fold assumption of rationality. Let $c_j \in C_j$ be a choice of j optimal for some cautious type whose assigned utility function is u_j that expresses up to n-fold belief in that every good choice is supported. Then it is easy to see that (1) by inductive assumption, in the constructed complete information model the corresponding type expresses n-fold assumption of rationality, and (2) c_j is optimal for that type. Since t_i expresses $(n+1)$-fold assumption of rationality, t_i deems possible a cautious type t_j that expresses up to n-fold assumption of rationality and for which c_j is optimal. By construction, it follows that θ_i deems possible some $\theta_j \in \Theta_j(t_j)$. By inductive assumption it follows that each $\theta_j \in \Theta_j(t_j)$ expresses n-fold belief in that every good choice is supported. Since θ_i expresses common belief in caution and rationality it follows that θ_i deems (c_j, θ_{j1}) for $\theta_{j1} \in \Theta_j(t_j)$ (that is, $w_j(\theta_{j1}) = u_j$).

Let (c_j, θ_j) with θ_j cautious deemed possible by θ_i satisfying that θ_j expresses up tp n-fold belief in prior belief in u and that every good choice is supported and $w_j(\theta_j) = u_j$ and (c'_j, θ'_j) a pair which does not satisfy those conditions. Let (c_j, t_j) and (c'_j, t'_j) be the pairs occuring in the belief of t_i corresponding to (c_j, θ_j) and (c'_j, θ'_j). Since c_j is rational for θ_j and $w_j(\theta_j) = u_j$, it follows that c_j is optimal for t_j. Also, by inductive assumption, it follows that t_j expresses up to n-fold assumption of rationality. On the other hand, it can be seen that (c'_j, t'_j) does not satisfy these conditions. Since t_i expresses $(n+1)$-fold of assumptions of rationality, t_i deems (c_j, t_j) infinitely more likely than (c'_j, t'_j). By construction, θ_i deems (c_j, θ_j) infinitely more likely than (c'_j, θ'_j). Here we have shown that θ_i expresses $(n+1)$-fold full belief in that every good choice is supported and prior belief in u.

Now suppose that $\theta_i \in \Theta_i$ expresses $(n+1)$-fold full belief in that every good choice is supported and prior belief in u. Let $c_j \in C_j$ be a choice of j optimal for some cautious type that expresses to n-fold assumption of rationality. By inductive assumption it follows that the corresponding type within some incomplete information model also expresses n-fold full belief in that every good choice is supported and prior belief in u. It can be seen that c_j is optimal to the constructed type having u_j as its utility functionand the type expresses up to n-fold full belief in that every good choice is supported and prior belief in u. Then θ_i deems possible a type θ_j with $w_j(\theta_j) = u_j$ for player j which expresses up to n-fold belief in that every good choice is supported for which c_j is optimal. By inductive assumption it follows that $t_i(E_i(\theta_i))$ deems possible $t_j(E_j(\theta_j))$ which expresses n-fold assumption of rationality and for which c_j is optimal.

Let (c_j, t_j) be a pair with t_j cautious deemed possible by $t_i(E_i(\theta_i))$ where t_j expresses up to n-fold assumption of rationality and c_j is optimal for t_j, and let (c'_j, t'_j) be a pair that does not satisfy this property. Let (c_j, θ_j) and (c'_j, θ'_j) be the corresponding pairs occurring in the belief of θ_i. By inductive assumption and by construction, θ_j is cautious and expresses up to n-fold belief in that prior belief in u and every good choice is supported and $w_j(\theta_j) = u_j$, while (c'_j, θ'_j) does not satisfy this property. Therefore θ_i deems (c_j, θ_j) infinitely more likely than

(c'_j, θ'_j), which implies that $t_i(E_i(\theta_i))$ deems (c_j, t_j) infinitely more likely than (c'_j, t'_j). Here we have shown that $t_i(E_i(\theta_i))$ expresses $(n+1)$-fold assumption of rationality. //

Proof of Theorem 3.1. **(Only-if)** Let $M^{co} = (T_i, b_i)_{i\in I}$, $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$ be constructed from M^{co} by the two steps above, $c_i^* \in C_i$ be a permissible choice, and $t_i^* \in T_i$ be a type expressing common full belief in caution and common assumption of rationality such that c_i^* is rational for t_i^*. Let $\theta_i^* = \theta_{i1}(t_i^*)$. By definition, $w_i(\theta_i^*) = u_i$ and $\beta_i(\theta_i^*)$ has the same distribution on j's choices at each level as $b_i(t_i^*)$. Hence c_i^* is rational for θ_i^*. Also, it follows from Observation 3.2, Lemmas 3.1, and 3.3 that θ_i^* expresses common full belief in caution, rationality,that a good choice is supported, and prior belief in u.

(If). Let $M^{in} = (\Theta_i, w_i, \beta_i)_{i\in I}$, $M^{co} = (T_i, b_i)_{i\in I}$ be constructed from M^{in} by the above approach, and $c_i^* \in C_i$ be rational for some θ_i^* with $w_i(\theta_i^*) = u_i$ which expresses common full belief in caution, rationality, that a good choice is supported, and prior belief in u. Consider $t_i(E_i(\theta_i^*))$. Since $w_i(\theta_i^*) = u_i$ and $b_i(t_i(E_i(\theta_i^*)))$ has the same distribution on j's choices at each level as $\beta_i(\theta_i^*)$, c_i^* is rational for $t_i(E_i(\theta_i^*))$. Also, by Lemmas 3.2 and 3.3, $t_i(E_i(\theta_i^*))$ expresses common full belief in caution and common assumption of rationality. //

4. Concluding Remarks

Assumption of rationality is a refinement of permissibility (See Perea [12]). This can also be seen within the framework of incomplete information. Comparing our characterization of the former of the characterization of the latter in Section 4.6 in Liu [11] it can be seen that there is correspondence between the conditions. Section 4.6 in Liu [11] characterizes permissibility by weak caution, rationality, and primary belief in u within the incomplete information framework. The characteization of assumption of rationality shares rationality with it, while caution and prior belief are stronger than weak caution and primary belief in u, respectively.

An interesting phenomenon is the role of rationality. Liu [11] provides two ways to characterize permissibility, one with rationality and one without it. The characterization of proper rationality there is a stronger version of the latter, while the characterization in this paper a stronger version of the former. So far, it seems that using or not using rationality in the characterization differentiate the two refinements of permissibility, that is, assumption of rationality and proper rationalizability, within the incomplete information framework. It would be interesting that any future research would confirm this statement or provide any counterexample, that is, show that proper rationalizability can be characterized with rationality while assumption of rationality can be done without it.

On the other hand, as shown in Liu [11] (and the construction here), it is always possible to construct epistemic models with incomplete information which satisfies rationality as well as all conditions for characterization of proper rationalizability. Further, prior belief in u is voguely a condition between primary belief in u and u-centered belief which is used in Theorem 3.2 of Liu [11] to characterize proper rationalizability. Those seem to correspond to the fact within the complete information framework that there is always possible to construct belief hierarchy which both assumes the opponent's rationality and respects the opponent's preferences.

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