Common Belief in Rationality in Games with Unawareness

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Andrés Perea*
Maastricht University

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Abstract

This paper investigates static games with unawareness, where players may be unaware of some of the choices that can be made by other players. That is, different players may have different *views* on the game. We propose an epistemic model that encodes players' belief hierarchies on choices and views, and use it to formulate the basic reasoning concept of *common belief in rationality*. We do so for two scenarios: one in which we do not fix the players' belief hierarchies on views, and one in which we do. For both scenarios we design a recursive elimination procedure that yields for every possible view the choices that can rationally be made under common belief in rationality.

Keywords: Unawareness, common belief in rationality, epistemic game theory, elimination procedure

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1 Introduction

A standard assumption in game theory is that all ingredients of the game – the players, their choices and their utility functions – are perfectly transparent to everybody involved. However,

**Address:* Epicenter and Dept. of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. *E-mail:* a.perea@maastrichtuniversity.nl *Web:* http://www.epicenter.name/Perea/

there are many situations of interest in which players may not be fully informed about some of these ingredients. For instance, a player may be uncertain about the precise utility functions of his opponents. Such situations may be modelled as *games with incomplete information,* and Harsanyi (1967–1968) opened the door towards a formal analysis of this class of games. In some cases the lack of information may even be more basic, as a player may be unaware of certain choices that his opponents can make, or may even be unaware of the presence of certain players in the game. This type of situations has recently given rise to the study of *games with unawareness.* For an overview of the relatively young literature in this field, see Schipper (2014).

In terms of reasoning there is a crucial difference between these two classes of games. In a game with incomplete information, a player may not be informed about the true utility function of an opponent, yet at the same time may reason about all the possible utility functions that this opponent may have. And he may reason about an opponent reasoning about all the possible utility functions that some third player may have, and so on. That is, if we list all the possible utility functions that the players may have, then there is no limit to the players' reasoning about these utility functions.

The same is not true for games with unawareness, however. If a player is unaware of an opponent's choice c, then he cannot reason about other players who are aware of c. In a sense, the choice c is not part of his language, or state space, and hence this choice c cannot enter at any level of his reasoning. These endogenous constraints on the players' reasoning constitute the key factor that distinguishes games with unawareness from other classes of games.

At the same time, this reasoning about the level of unawareness of other players is at the central stage of games with unawareness. Indeed, if a player in a game with unawareness must decide what to do, then he must base his choice not only on his own (possibly partial) view of the game, but also on what he believes about the opponents' views of the game, what he believes that his opponents believe about the views of other players, and so on. In other words, a player holds a belief hierarchy on the players' views of the game, and bases his choice upon this belief hierarchy.

In that light, the reasoning of players in games with unawareness is considerably more complex than in standard games, as a player must form beliefs about the opponents' choices *and* the opponents' views, where his beliefs about the opponents' choices will depend on his belief about their views. In the literature, the reasoning about views has typically been disentangled from the reasoning about choices, as most models for games with unawareness *exogenously* specify a belief hierarchy on views for every player. The strategic reasoning is then modelled by using an equilibrium or rationalizability concept that assumes these fixed belief hierarchies on views.

In this paper we take a different approach by *combining* the players' reasoning about views and choices into *one* belief hierarchy that models both. More precisely, we propose a model of static games with unawareness that no longer fixes the players' belief hierarchies on views, and subsequently encode the players' belief hierarchies on *choices and views* by an appropriately designed epistemic model with types. Types in this epistemic model thus simultaneously describe the players' reasoning about views and their strategic reasoning – something that proves to be very convenient for an epistemic analysis. Another difference with most of the existing literature is that we allow for *probabilistic* beliefs about the opponents' views, and not only deterministic beliefs. We find this important, as a player who is truly uncertain about the level of unawareness of his opponent may well ascribe positive probability to *various* possible views for this opponent. Such probabilistic beliefs on views can naturally be captured by our choice of an epistemic model.

We then use this epistemic model to epistemically investigate the strategic reasoning of players in games with unawareness, which is the main purpose of this paper. To do so, we focus on the central yet basic reasoning concept of *common belief in rationality* (Tan and Werlang (1988), Brandenburger and Dekel (1987)) which in standard static games characterizes *rationalizability* (Bernheim (1984), Pearce (1984)) and the iterated strict dominance procedure. In the context of games with unawareness, this concept states that a player believes that his opponents choose optimally *given their views of the game,* that a player believes that his opponents believe that the other players choose optimally *given their views of the game,* and so on. It turns out that this concept can naturally be formulated within the language of our epistemic model which, as we saw, encodes belief hierarchies on choices and views.

A natural question is whether we can find a recursive elimination procedure à la *iterated strict dominance* that characterizes precisely those choices that can rationally be made under common belief in rationality. We indeed propose such a procedure, and call it *iterated strict dominance for unawareness.* The main difference with the standard strict dominance procedure is that, in every round and for every player, we eliminate choices for *every possible view* that this player can hold in the game. More precisely, at a given view v_i for player i we first eliminate those choices for opponent j that have not survived the previous round for *any possible view of player j that player i can reason about when holding the view v_i.* Subsequently, at view v_i we eliminate for player i those choices that are strictly dominated, given the current set of opponents' choices.

We show in Theorem 4.2 that this procedure selects, for every player and every view, precisely those choices that this player can make with this particular view under common belief in rationality. Since the procedure always yields a non-empty output, it immediately follows that for every static game with unawareness there is for every player and every view at least one belief hierarchy on choices and views that expresses common belief in rationality. The procedure is very similar to the *generalized iterated strict dominance procedure* (Bach and Perea (2017)) which has been designed for static games with *incomplete information.* The main difference is that in the latter procedure, choices are being eliminated at every possible *utility function* that a player can have in the game, instead of at every possible *view* that a player can hold.

As a second step, we reconcile the concept of common belief in rationality with the common assumption that the players' belief hierarchies on views are fixed. The new concept then selects, for every view and every fixed belief hierarchy on views, those choices that a player can rationally make under common belief in rationality if he holds this particular view and belief hierarchy on views. Also for this concept we design a recursive elimination procedure, called *iterated strict dominance with fixed beliefs on views,* that yields precisely these choices. See our Theorem 5.2.

To define this procedure we first encode the given belief hierarchy on views by means of an epistemic model with types, similar to the one mentioned above. The difference is that there is no reference to choices in this epistemic model, only to views. Types in this epistemic model are called *view-types,* as they encode belief hierarchies on views only. More precisely, every view-type in the model can be identified with a probability distribution on the opponents' views and view-types. The new procedure is more refined as above, as it now eliminates, in every round and for every player, choices at every possible view *and every possible view-type* for that player. Moreover, at a given view v_i and view-type r_i for player i, the opponents' choices that can be eliminated at (v_i, r_i) are based on the probability distribution that r_i induces on the opponents' views and view-types. In that sense, the procedure is closely related to the *interim correlated rationalizability procedure* (Dekel, Fudenberg and Morris (2007)) for static games with *incomplete information.* The key difference is that in the latter procedure, choices are being eliminated at pairs of *utility functions* and belief hierarchies on *utility functions*, whereas in this paper choices are eliminated at pairs of *views* and belief hierarchies on *views.*

With these two procedures we thus characterize the behavioral consequences of common belief in rationality in games with unawareness, both in a scenario where the belief hierarchies on views are unrestricted, and a scenario where these belief hierarchies are fixed. Moreover, if the belief hierarchies on views are fixed and *deterministic*, then our procedure becomes equivalent to the *extensive-form rationalizability* procedure in Heifetz, Meier and Schipper (2013) when applied to *static* games with unawareness. Our analysis is also closely related to Feinberg (2012) who investigates the concept of *rationalizability* for static games with unawareness. Most other papers on games with unawareness investigate *equilibrium* concepts instead of rationalizability concepts.

The rest of the paper is organized as follows. In Section 2 we provide our definition of static games with unawareness. In Section 3 we encode belief hierarchies on choices and views by means of an epistemic model with types, and use it to formally define common belief in rationality for static games with unawareness. In Section 4 we present the *iterated strict dominance procedure for unawareness* and show that it characterizes the behavioral consequences of common belief in rationality. In Section 5 we impose a fixed belief hierarchy on views for every player, present the *iterated strict dominance procedure with fixed beliefs on views,* and show that it characterizes the behavioral consequences of common belief in rationality with fixed belief hierarchies on views. In Section 6 we relate our work to other papers on unawareness in the literature. We conclude in Section 7. The appendix (Section 8) contains all proofs, and shows how to formally derive belief hierarchies on views from types in an epistemic model.

2 Static Games with Unawareness

In this paper we restrict to *static games,* and focus on unawareness about the possible *choices* that the players can make. That is, a player may be unaware of certain choices that he, or his opponents, can make in the game. Feinberg (2012) allows players, in addition, to be unaware of some of the other *players* in the game. Such unawareness, however, will not be part of our framework.

Before we can analyze games with unawareness, we must first establish how we *describe* the possible unawareness of players about some of the choices in the game. We will do so by defining, for every player, a collection of *partial descriptions* of the full game, which contain some – but not necessarily all – possible choices that can be made. These partial descriptions will be called the possible *views* that the player can hold. Every view can thus be interpreted as a personal, and possibly incomplete, perception of the full game.

Formally, a *static game* is a tuple $G = (C_i, u_i)_{i \in I}$ where I is a finite set of players, C_i is a finite set of choices, and $u_i : \times_{j \in I} C_j \to \mathbb{R}$ is a utility function for every player i. A *view* for player i of the game G is a tuple $v_i = (D_i, D_{-i})$ where $D_i \subseteq C_i$ is a possibly reduced set of choices and $D_{-i} \subseteq C_{-i}$ is a possibly reduced set of opponents' choice combinations. Here, by D_{-i} we mean the Cartesian product $\times_{j \neq i} D_j$, and similarly for C_{-i}. We implicitly assume that a player i with view v_i believes that the utilities induced by the choice combinations in v_i coincide with those of the game G. For that reason, it is not necessary to specify a new utility function for a view. For any two views $v_i = (D_i, D_{-i})$ and $v'_j = (D'_j, D'_{-j})$, belonging to possibly different players i and j, we say that v_i is *contained* in v'_j if $D_i \times D_{-i} \subseteq D'_j \times D'_{-j}$. That is, all choices considered possible in v_i are also considered possible in v'_j. An important principle in this – and any other – paper on unawareness is that a player with view v_i can only reason about opponents' views that are contained in v_i.

We can now define a static game with unawareness as a tuple consisting of a full static game, containing all choices that the players can possibly make, and for every player a finite collection of possible views of the full game.

Definition 2.1 (Static game with unawareness) *A static game with unawareness is a tuple $G^u = (G^{base}, (V_i)_{i \in I})$ where G^{base} is a static game, and V_i is a finite collection of views for player i of the game G^{base}. Moreover, for every player i, every view v_i in V_i, and every opponent $j \neq i$ there must be a view in V_j that is contained in v_i.*

Here, we refer to G^{base} as the *base game.* The condition above thus guarantees that for every possible view $v_i \in V_i$ that player i can have, there is for every opponent j at least one view $v_j \in V_j$ that player i can reason about.

The main difference between this model and other definitions for games with unawareness, such as Feinberg (2012), Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013)[1], is that the latter fix for every player a view and a *belief hierarchy on views,* whereas we do not. That is, these papers exogenously describe, for every player, the view he holds on the game, what he player believes about the opponents' views, what he believes about the opponents' beliefs about the views by the other players, and so on. In contrast, we allow players to hold

[1] Other papers that model games with unawareness can be found in Section 6.

any view and any belief hierarchy on views they wish, as long as these only use views from the collections $(V_i)_{i \in I}$.

Moreover, we allow such belief hierarchies on views to be *probabilistic,* whereas Feinberg (2012) and Heifetz, Meier and Schipper (2013) restrict to deterministic belief hierarchies on views. Rêgo and Halpern (2012), in turn, do allow for probabilistic belief hierarchies on views through the introduction of chance moves.

A last difference we wish to outline is that the models by Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013) were specifically designed for *dynamic* games with unawareness. But their definitions capture static games as a special case.

We now illustrate the definition of a static game with unawareness by means of an example.

Example 1. Going to a party, version 1

This example is based on the example "Going to a party" in Perea (2012). You and Barbara have been invited for a party tonight, and must decide which color to wear: *blue, green, red* or *yellow.* You prefer *blue* to *green*, *green* to *red*, and *red* to *yellow*, whereas Barbara prefers *red* to *yellow*, *yellow* to *blue*, and *blue* to *green.* However, you both dislike wearing the same color as the other. Suppose that until now Barbara has only seen you in *red* or *yellow* clothes, and therefore you believe that Barbara is unaware of you having any *blue* or *green* outfits. This situation can be represented by the game with unawareness in Table 1, where G^{base} is the base game, $V_1 = \{v_1, v_1'\}$ contains the possible views for you, and $V_2 = \{v_2\}$ contains the only possible view for Barbara you consider.

In the base game, your choices are in the rows and Barbara's choices are in the columns. In both of your possible views v_1 and v_1', your choices are in the rows and Barbara's choices in the columns. In the corresponding cells we have put your utilities. In Barbara's view v_2 we have put her choices in the rows and your choices in the columns, and have written the induced utilities for her in the cells. This is a general convention we adopt for depicting views of a player i: we always put i's choices in the rows, the opponents' choice combinations in the columns, and the induced utilities for player i in the corresponding cells.

The interpretation of the views is as follows: Suppose you hold view v_1, and hence are aware of all possible color choices by Barbara and you. According to the story, you believe that Barbara is only aware of your choices *red* and *yellow,* and therefore you believe that Barbara holds view v_2. On the other hand, if Barbara holds view v_2 she cannot believe you hold view v_1, as she is unaware of your choices *blue* and *green.* One possibility is that Barbara believes you hold the same view as she does, that is, she believes your view is v_1'. This is precisely the possibility we have adopted in Table 1.

It may be verified that Table 1 yields a well-defined static game with unawareness, meeting the condition on views as specified in Definition 2.1. Indeed, for both of your views v_1 and v_1' there is the view v_2 for Barbara that is contained in v_1 and v_1', and for Barbara's view v_2 there is the view v_1' for you that is contained in v_2.

Base game

G^{base}	blue	green	red	yellow
blue	$0, 0$	$4, 1$	$4, 4$	$4, 3$
green	$3, 2$	$0, 0$	$3, 4$	$3, 3$
red	$2, 2$	$2, 1$	$0, 0$	$2, 3$
yellow	$1, 2$	$1, 1$	$1, 4$	$0, 0$

Your views

v_1	blue	green	red	yellow
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2
yellow	1	1	1	0

v_1'	blue	green	red	yellow
red	2	2	0	2
yellow	1	1	1	0

Barbara's views

v_2	red	yellow
blue	2	2
green	1	1
red	0	4
yellow	3	0

Table 1: Going to a party, version 1

Note that in Table 1 there is only one belief hierarchy on views possible for you: irrespective of whether you hold view v_1 or v_1', you believe that Barbara holds view v_2, believe that Barbara believes you hold view v_1', believe that Barbara believes that you believe that Barbara holds view v_2, and so on.

3 Common Belief in Rationality

The idea of *common belief in rationality* (Tan and Werlang (1988), Brandenburger and Dekel (1987)) is that a player believes that every opponent chooses optimally given his view, that he believes that every opponent believes that every other player chooses optimally given his view, and so on. In order to formally define this idea for static games with unawareness, we must specify (i) what a player believes about the possible choices and views of his opponents, (ii) what he believes about the opponents' beliefs about their opponents' choices and views, and so on. Such belief hierarchies can be encoded by means of epistemic models with types, where every type holds a probabilistic belief about the opponents' choices, views and types.

Definition 3.1 (Epistemic model) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$. An epistemic model for G^u is a tuple $M = (T_i, b_i)_{i\in I}$ where T_i is the finite set of types for player i, and b_i is a belief mapping that assigns to every type $t_i \in T_i$ some probabilistic belief $b_i(t_i) \in \Delta(C_{-i} \times V_{-i} \times T_{-i})$.*
Moreover, the belief mappings b_i should be such that $b_i(t_i)((c_j, v_j, t_j)_{j\neq i}) > 0$ only if for every player $j \neq i$
(a) choice c_j is part of the view v_j, and
(b) $b_j(t_j)$ only assigns positive probability to opponents' views that are contained in v_j.

Here, C_{-i} is a short-hand for $\times_{j\neq i} C_j$, and denotes the set of opponents' choice combinations. Similarly for V_{-i} and T_{-i}. For every finite set X, we denote by $\Delta(X)$ the set of probability distributions on X. Hence, $\Delta(C_{-i} \times V_{-i} \times T_{-i})$ denotes the set of probability dsitributions on $C_{-i} \times V_{-i} \times T_{-i}$.

Condition (a) states that a player can only assign positive probability to opponent's choices that are in fact feasible for the opponent's view considered by the player. Condition (b), on the other hand, reflects the fact that an opponent with view v_j can only reason about views that are contained in v_j. In other words, if you are unaware of certain choices, you cannot reason about the event that another player is aware of these choices.

Condition (b) is indispensable for modelling unawareness, and similar conditions can be found in other papers on games with unawareness. Indeed, condition (b) corresponds to Condition 2 in Feinberg (2012), condition C2 in Rêgo and Halpern (2012) and condition I4 in Heifetz, Meier and Schipper (2013).

Note that a player i with view v_i may not have mental access to *all* types in the epistemic model, since he is only able to reason about views that are contained in v_i. Consequently, such

Types	$T_1 = \{t_1, t'_1\}$ $T_2 = \{t_2, t'_2\}$
Beliefs for you	$b_1(t_1) = (red, v_2, t_2)$ $b_1(t'_1) = (blue, v_2, t'_2)$
Beliefs for Barbara	$b_2(t_2) = (yellow, v'_1, t_1)$ $b_2(t'_2) = (0.6) \cdot (red, v'_1, t'_1) + (0.4) \cdot (yellow, v'_1, t_1)$

Table 2: An epistemic model for "Going to a party, version 1"

a player will only have access to types in the model that only reason about views that are contained in v_i. In this spirit, we say that a type $t_i \in T_i$ is *feasible* for a view $v_i \in V_i$ if $b_i(t_i)$ only assigns positive probability to opponents' views that are contained in v_i. Note that in this case, the belief hierarchy of type t_i, at each of its different layers, only reasons about views that are contained in v_i, as it should be. This follows from condition (b) above.

As an illustration, consider the epistemic model in Table 2 for the game "Going to a party, version 1". The beliefs for the types should be read as follows: Type t_1 for you assigns probability 1 to the event that Barbara chooses *red,* holds view v_2 and has type t_2. Type t'_2 for Barbara assigns probability 0.6 to the event that you choose *red,* hold view v'_1 and have type t'_1, and assigns probability 0.4 to the event that you choose *yellow,* hold view v'_1 and have type t_1. Similarly for the other types.

For every type, we can now derive the full belief hierarchy about choices and views it encodes. Consider, for instance, type t_1 for you which believes that Barbara chooses *red* while having view v_2, believes that Barbara believes that you choose *yellow* while having view v'_1, believes that Barbara believes that you believe that Barbara chooses *red* while having view v_2, and so on. Similarly for the other types. It may be verified that the conditions (a) and (b) above are satisfied, and hence Table 2 offers a well-defined epistemic model.

Now that we know how to encode belief hierarchies on choices and views, the next step towards a formal definition of common belief in rationality is to define optimal choice for a particular view, and belief in the opponents' rationality. For a given type t_i in an epistemic model, and a choice c_i, we denote by

$$u_i(c_i, t_i) := \sum_{(c_{-i}, v_{-i}, t_{-i}) \in C_{-i} \times V_{-i} \times T_{-i}} b_i(t_i)(c_{-i}, v_{-i}, t_{-i}) \cdot u_i(c_i, c_{-i})$$

the *expected utility* induced by choice c_i under t_i's first-order belief about the opponents' choice combinations.

Now, consider a view $v_i \in V_i$ such that t_i is feasible for v_i. By $C_i(v_i)$ we denote the set of choices that player i has available at view v_i. We say that choice c_i is *optimal* for type t_i and view v_i if

$$u_i(c_i, t_i) \geq u_i(c_i', t_i) \text{ for all } c_i' \in C_i(v_i).$$

We next define what it means to believe in the opponents' rationality. In words, it means that you only deem possible combinations of choices, views and types for the opponent where the choice is optimal for the type and the view.

Definition 3.2 (Belief in the opponents' rationality) *Consider a static game with unawareness G^u, an epistemic model $M = (T_i, b_i)_{i \in I}$ for G^u, and a type $t_i \in T_i$. We say that type t_i believes in the opponents' rationality if $b_i(t_i)((c_j, v_j, t_j)_{j \neq i}) > 0$ only if for every player $j \neq i$, the choice c_j is optimal for type t_j and view v_j.*

In the epistemic model of Table 2, it may be verified that all types believe in the opponent's rationality. With this definition at hand, we can now define common belief in rationality in an iterative fashion.

Definition 3.3 (Common belief in rationality) *Consider a static game with unawareness G^u and an epistemic model $M = (T_i, b_i)_{i \in I}$ for G^u.*

(1-fold) A type in M expresses 1-fold belief in rationality if it believes in the opponents' rationality.

(k-fold) For $k \geq 2$, a type t_i in M expresses k-fold belief in rationality if $b_i(t_i)$ only assigns positive probability to opponents' types that express $(k-1)$-fold belief in rationality.

A type in M expresses common belief in rationality if it expresses k-fold belief in rationality for every $k \geq 1$.

Now, consider a choice c_i for player i and a view v_i for player i that contains c_i. We say that c_i can *rationally be chosen under common belief in rationality with the view v_i* if there is an epistemic model $M = (T_j, b_j)_{j \in I}$ and a type $t_i \in T_i$ that is feasible for v_i and expresses common belief in rationality, such that c_i is optimal for the type t_i and the view v_i.

To illustrate these notions, consider again the epistemic model from Table 2. As all types believe in the opponent's rationality, it follows that all types in the epistemic model express *common* belief in rationality as well. Note that *blue* is optimal for your type t_1 and the view v_1, and *green* is optimal for your type t_1' and the view v_1. As such, with the view v_1 you can rationally choose *blue* and *green* under common belief in rationality. In the next section we will see that these are also the *only* choices you can rationally make under common belief in rationality while holding the view v_1.

4 Recursive Procedure

In this section we wish to characterize the choices a player can rationally make under common belief in rationality while holding a particular view. To that purpose we introduce a recursive elimination procedure, called *iterated strict dominance for unawareness,* which iteratively eliminates choices from every possible view in the game. We show that the procedure delivers, for every view, exactly those choices that can rationally be made under common belief in rationality with that particular view.

4.1 Definition

To formally define the procedure, we need some additional terminology. Consider a view $v_i = (D_i, D_{-i})$ for player i and a choice $c_i \in D_i$. We say that c_i is *strictly dominated* for the view v_i if there is some randomized choice $\rho_i \in \Delta(D_i)$ such that

$$u_i(c_i, c_{-i}) < \sum_{c_i' \in D_i} \rho_i(c_i') \cdot u_i(c_i', c_{-i}) \text{ for all } c_{-i} \in D_{-i}.$$

Recall that for every view $v_i = (D_i, D_{-i})$ we denote by $C_i(v_i) := D_i$ the set of choices available for player i in v_i. Similarly, we denote by $C_{-i}(v_i) := D_{-i}$ the set of opponents' choice combinations possible in v_i. Any pair (D_i', D_{-i}') with $D_i' \subseteq D_i$ and $D_{-i}' \subseteq D_{-i}$ is called a *reduction of view* v_i, or simply a *reduced view.*

Definition 4.1 (Iterated strict dominance for unawareness) *Consider a static game with unawareness* $G^u = (G^{base}, (V_i)_{i\in I})$.

(Initial step) For every player i and every view $v_i \in V_i$, define $C_{-i}^0(v_i) := C_{-i}(v_i)$ and $C_i^0(v_i) := C_i(v_i)$.

(Inductive step) For $k \geq 1$, every player i, and every view $v_i \in V_i$, define

$$C_{-i}^k(v_i) := \{(c_j)_{j\neq i} \in C_{-i}^{k-1}(v_i) \mid \text{ for all } j \neq i \text{ choice } c_j \text{ is in } C_j^{k-1}(v_j) \text{ for some view } v_j \in V_j \text{ that is contained in } v_i\},$$

and

$$C_i^k(v_i) := \{c_i \in C_i^{k-1}(v_i) \mid c_i \text{ not strictly dominated within the reduced view } (C_i^{k-1}(v_i), C_{-i}^k(v_i))\}.$$

A choice-view pair (c_i, v_i) is said to survive the procedure if $c_i \in C_i^k(v_i)$ for every $k \geq 0$.

Hence, in this procedure we recursively restrict, for every view v_i, the possible beliefs that player i can hold about his opponents' choices, through the sets $C_{-i}^k(v_i)$, and the possible choices that player i can make himself, through the sets $C_i^k(v_i)$. In that sense, it is very similar to the *generalized iterated strict dominance procedure* (Bach and Perea (2017)) for static games with *incomplete information.* The latter procedure recursively restricts such beliefs and choices for every possible *utility function* that player i can have in the game with incomplete information, instead of for every possible *view* in the game, as we do here. This similarity supports Feinberg's (2012) view that games with unawareness and games with incomplete information have a lot in common.

In the following subsection we will show that this procedure always delivers a non-empty set of choices for every possible view, and indeed characterizes precisely those choice-view pairs where the choice is possible for the view under common belief in rationality.

4.2 Non-Empty Output and Characterization Result

We first show that the iterated strict dominance procedure for unawareness always yields a non-empty output. More precisely, we show that for every possible view in the game, there is always at least one choice for the respective player that survives the procedure.

Theorem 4.1 (Non-empty output) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i and every view $v_i \in V_i$ there is some choice $c_i \in C_i$ such that (c_i, v_i) survives the iterated strict dominance procedure for unawareness.*

We next present the main result in this section, showing that the iterated strict dominance procedure for unawareness selects for every view precisely those choices that can rationally be made under common belief in rationality.

Theorem 4.2 (Characterization of common belief in rationality) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i, every view $v_i \in V_i$ and every choice $c_i \in C_i(v_i)$, choice c_i can rationally be made under common belief in rationality with the view v_i, if and only if, (c_i, v_i) survives the procedure of iterated strict dominance for unawareness.*

One direction of this theorem thus states that if (c_i, v_i) survives the procedure, then we can always find an epistemic model, and a type t_i for player i within that epistemic model, such that the type t_i is feasible for the view v_i, expresses common belief in rationality, and the choice c_i is optimal for the type t_i with the view v_i. For the construction of this epistemic model we rely on Theorem 4.1, which guarantees that for every player j, and every view v_j, there is at least one choice c_j that survives the procedure together with v_j.

In particular, this direction implies that for every finite static game with unawareness, we can always construct for every player i, and every view v_i, a type that is feasible for this view v_i, and that expresses common belief in rationality.

Corollary 4.1 (Common belief in rationality is always possible) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i and every view $v_i \in V_i$, there is* *an epistemic model $M = (T_j, b_j)_{j \in I}$ and a type $t_i \in T_i$, such that the type t_i is feasible for the view v_i and t_i expresses common belief in rationality.*

In other words, for every view it is always possible to reason in accordance with common belief in rationality, while respecting the bounds set by the view.

4.3 Example

In this subsection we will illustrate the iterated strict dominance procedure for unawareness by means of the example "Going to a party, version 1", introduced in the previous section and depicted in Table 1. To save space, we use the abbreviations b, g, r and y for the four colors. At the beginning of the procedure we have the initial views given by

$$\begin{array}{ll} C_1^0(v_1) = \{b, g, r, y\}, & C_{-1}^0(v_1) = \{b, g, r, y\}, \\ C_1^0(v_1') = \{r, y\}, & C_{-1}^0(v_1') = \{b, g, r, y\} \end{array}$$

$$C_2^0(v_2) = \{b, g, r, y\}, \quad C_{-2}^0(v_2) = \{r, y\}.$$

Round 1. By definition we have that $C_{-1}^1(v_1) = C_{-1}^0(v_1)$, $C_{-1}^1(v_1') = C_{-1}^0(v_1')$ and $C_{-2}^1(v_2) = C_{-2}^0(v_2)$. Note that y is strictly dominated for you by the randomized choice $(0.5) \cdot b + (0.5) \cdot g$ within the view $(C_1^0(v_1), C_{-1}^{-1}(v_1))$, and that g is strictly dominated for Barbara by b within her view $(C_2^0(v_2), C_{-2}^1(v_2))$. No other choices are strictly dominated in this round. We can therefore eliminate your choice y from $C_1^0(v_1)$ and Barbara's choice g from $C_2^0(v_2)$, yielding the reduced views

$$\begin{array}{ll} C_1^1(v_1) = \{b, g, r\}, & C_{-1}^1(v_1) = \{b, g, r, y\}, \\ C_1^1(v_1') = \{r, y\}, & C_{-1}^1(v_1') = \{b, g, r, y\} \end{array}$$

$$C_2^1(v_2) = \{b, r, y\}, \quad C_{-2}^1(v_2) = \{r, y\}.$$

Round 2. As Barbara's choice g is not in her reduced view at v_2 anymore, and v_2 is her unique view, we can eliminate Barbara's choice g from your reduced views at v_1 and v_1'. That is, $C_{-1}^2(v_1) = \{b, r, y\}$ and $C_{-1}^2(v_1') = \{b, r, y\}$. Note that we cannot eliminate your choice y from Barbara's reduced view at v_2, since at v_2 Barbara can only deem possible your view v_1' at which your choice y is still present. We thus have that $C_{-2}^2(v_2) = \{r, y\}$.

In your reduced view $(C_1^1(v_1), C_{-1}^2(v_1)) = (\{b, g, r\}, \{b, r, y\})$ at v_1, your choice r is strictly dominated by g, and can thus be eliminated from $C_1^1(v_1)$. No other choices can be eliminated in this round. We thus obtain the reduced views

$$\begin{array}{ll} C_1^2(v_1) = \{b, g\}, & C_{-1}^2(v_1) = \{b, r, y\}, \\ C_1^2(v_1') = \{r, y\}, & C_{-1}^2(v_1') = \{b, r, y\} \end{array}$$

$$C_2^2(v_2) = \{b, r, y\}, \quad C_{-2}^2(v_2) = \{r, y\}.$$

After this round no further choices can be eliminated at any of the possible views, and hence the procedure terminates at the end of round 2. The choice-view pairs that survive for you are $(b, v_1), (g, v_1), (r, v_1')$ and (y, v_1'), whereas the choice-view pairs surviving for Barbara are $(b, v_2), (r, v_2)$ and (y, v_2).

Hence, in view of Theorem 4.2**,** these are exactly the choice-view pairs that are possible under common belief in rationality. That is, under common belief in rationality, you can rationally choose *blue* and *green* with the view v_1, you can rationally choose *red* and *yellow* with the view v_1', and Barbara can rationally choose *blue, red* and *yellow* with the view v_2.

5 Fixed Beliefs on Views

In the literature on games with unawareness, it is typically assumed that every player holds some *exogenously given belief hierarchy on views*. See, for instance, Feinberg (2012), Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013). Following this approach, we reconcile in this section the concept of common belief in rationality with the assumption that the *belief hierarchy on views is fixed.* One important difference with Feinberg (2012) and Heifetz, Meier and Schipper (2013) is that we allow for *truly probabilistic* belief hierarchies on views, and not only belief hierarchies consisting of probability 1 beliefs on views. The reason is that we wish to allow for situations in which a player is *uncertain* about the precise view adopted by his opponent, and therefore assigns positive probability to various possible views for this opponent.

5.1 Common Belief in Rationality with Fixed Beliefs on Views

Different from Heifetz (2012), Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013), we decide to encode belief hierarchies on views by means of epistemic models with *types.* The reason is that such encodings are easy to work with, and turn out to be convenient for designing proofs and an associated elimination procedure as well. Such an epistemic model may be seen as a reduced version of the one used in Section 3, since now a type only holds a belief about the opponents' *views* and types, instead of the opponents' *choices, views* and types.

Definition 5.1 (Epistemic model for views) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i\in I})$. An epistemic model for views is a tuple $M^{view} = (R_i, p_i)_{i\in I}$ where R_i is the finite set of types for player i, and p_i is a belief mapping that assigns to every type $r_i \in R_i$ some probabilistic belief $p_i(r_i) \in \Delta(V_{-i} \times R_{-i})$. Moreover, the belief mapping p_i should be such that $p_i(r_i)((v_j, r_j)_{j\neq i}) > 0$ only if for every player $j \neq i$, the belief $p_j(r_j)$ only assigns positive probability to opponents' views that are contained in v_j.*

We call the types in this model *view-types,* since they generate belief hierarchies on views. Note that the condition at the end mimicks condition (b) in Definition 3.1. Similarly as before, we say that a view-type $r_i \in R_i$ is *feasible* for a view $v_i \in V_i$ if $p_i(r_i)$ only assigns positive probability to opponents' views that are contained in v_i. For every view-type $r_i \in R_i$, let $h_i^{view}(r_i)$ be the belief hierarchy on views induced by r_i. The precise construction of this belief hierarchy can be found in Section 8.2.1 of the appendix. Note that if the view-type r_i is feasible for the view v_i, then the belief hierarchy $h_i^{view}(r_i)$ on views only considers, at each of its layers, views that are contained in v_i, as it should be.

Compare this to the epistemic models we considered in Definition 3.1, used to encode belief hierarchies on *choices and views.* In such an epistemic model $M = (T_i, b_i)_{i \in I}$, every type t_i induces a belief hierarchy on choices and views, and hence also on views alone. Let $h_i^{view}(t_i)$ be the induced belief hierarchy on views. The precise construction of $h_i^{view}(t_i)$ can be found in Section 8.2.2 of the appendix.

With these definitions at hand, we can now formally define what we mean by common belief in rationality with fixed beliefs on views.

Definition 5.2 (Common belief in rationality with fixed beliefs on views) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$, an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views, a view $v_i \in V_i$ for player i and a view-type $r_i \in R_i$ that is feasible for r_i. A choice $c_i \in C_i(v_i)$ can rationally be made under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i, if there is an epistemic model $M = (T_j, b_j)_{j \in I}$ for choices and views, and a type $t_i \in T_i$ such that $h_i^{view}(t_i) = h_i^{view}(r_i)$, type t_i expresses common belief in rationality, and c_i is optimal for t_i and v_i.*

Note that in this definition, every type t_i with $h_i^{view}(t_i) = h_i^{view}(r_i)$ is automatically feasible for view v_i, since r_i is feasible for v_i and t_i holds the same belief hierarchy on views as r_i.

5.2 Recursive Procedure

We will now present a recursive elimination procedure, called *iterated strict dominance with fixed beliefs on views,* that characterizes precisely those choices that can rationally be made, with every possible view, under common belief in rationality with a fixed belief hierarchy on views. Not surprisingly, the procedure is quite similar to *iterated strict dominance for unawareness* (without fixed belief hierarchies on views). There are two important differences. The first is that choice sets will now be defined for every view v_i *and every view-type* r_i feasible for v_i, where $r_i \in R_i$ is taken from the epistemic model for views. Moreover, the sets $C_{-i}^k(v_i)$ of opponents' choice combinations as defined in iterated strict dominance with unawareness, restricting the possible beliefs that player i can hold at round k, will now be replaced by sets of possible probabilistic beliefs $B_i^k(v_i, r_i)$, representing the possible probabilistic beliefs that player i can hold at round k if he holds view v_i and has the belief hierarchy on views induced by r_i.

To define the procedure formally, we need some additional notation. Consider some Euclidean space $\mathbb{R}^n$, some subsets $A_1, ..., A_K$ of $\mathbb{R}^n$, and some numbers $x_1, ..., x_K \in \mathbb{R}$. Then, by

$$\sum_{k \in \{1,...,K\}} x_k \cdot A_k := \{ \sum_{k \in \{1,...,K\}} x_k \cdot a_k \mid a_k \in A_k \text{ for all } k \in \{1, ..., K\}\}$$

we define the corresponding "linear combination" of these sets $A_1, ..., A_K$.

Definition 5.3 (Iterated strict dominance with fixed beliefs on views) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views.*

(Initial step) For every player i, every view $v_i \in V_i$ and every view-type $r_i \in R_i$ feasible for v_i, define

$$B_i^0(v_i, r_i) := \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \Delta(\times_{j \neq i} C_j(v_j)),$$

and $C_i^0(v_i, r_i) := C_i(v_i)$.

(Inductive step) For $k \geq 1$, every player i, every view $v_i \in V_i$ and every view-type $r_i \in R_i$ feasible for v_i, define

$$B_i^k(v_i, r_i) := \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \Delta(\times_{j \neq i} C_j^{k-1}(v_j, r_j)),$$

and

$$C_i^k(v_i, r_i) := \{c_i \in C_i^{k-1}(v_i, r_i) \mid \begin{array}{l} c_i \text{ is optimal for some belief } \beta_i \in B_i^k(v_i, r_i) \\ \text{among choices in } C_i^{k-1}(v_i, r_i)\}. \end{array}$$

A triple (c_i, v_i, r_i), consisting of a choice, view and view-type, is said to survive the procedure if $c_i \in C_i^k(v_i, r_i)$ for every $k \geq 0$.

More precisely, this procedure is the iterated strict dominance procedure with fixed beliefs on views *as given by* M^{view}. As a short-hand, we will refer to this procedure as the *iterated strict dominance procedure for* M^{view}.

Consider now the special case where every view-type in M^{view} assigns probability 1 to one specific view for every opponent. Then, it may be verified that the procedure above is equivalent to the *extensive-form rationalizability procedure* in Heifetz, Meier and Schipper (2013), when applied to the special case of static games. The procedure in Heifetz, Meier and Schipper (2013) is designed for *dynamic* games with unawareness, and hence can also be applied to *static* games.

Our procedure above is quite similar to the *interim correlated rationalizability* procedure (Dekel, Fudenberg and Morris (2007)) for games with *incomplete information*, which in turn is analogous to the concept of *interim (independent) rationalizability* in Ely and Pęski (2006). Also interim correlated rationalizability assumes a fixed belief hierarchy, not on views but on *utility functions*. The interim correlated rationalizability procedure then recursively restricts, for every possible *utility function* and every belief hierarchy on *utilities*, the set of choices for the respective player. In turn, we recursively restrict the player's set of choices for every possible *view* and belief hierarchy on *views* (as encoded by a view-type r_i). This similarity again supports Feinberg's (2012) view that games with unawareness are close, in spirit, to games with incomplete information.

5.3 Non-Empty Output and Characterization Result

Like in Section 4, we first show that the procedure always delivers a non-empty output, and subsequently prove that the procedure yields, for every view and view-type, exactly those choices that can rationally be made under common belief in rationality with this particular view and view-type.

Theorem 5.1 (Non-empty output) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views. Then, for every player i, every view $v_i \in V_i$ and every view-type $r_i \in R_i$ that is feasible for r_i, there is some choice $c_i \in C_i$ such that (c_i, v_i, r_i) survives the iterated strict dominance procedure for M^{view}.*

The reader will note that the proof for this result is very similar to one we gave for Theorem 4.1. We thus conclude that, no matter which belief hierarchy on views we impose, it is always possible for a player to reason in accordance with this particular belief hierarchy on views, while respecting common belief in rationality.

We next show that the procedure selects, for every view and every belief hierarchy on views encoded by M^{view}, exactly those choices that can rationally be made under common belief in rationality for this specific view and belief hierarchy on views.

Theorem 5.2 (Characterization of common belief in rationality) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views. Then, for every player i, every choice $c_i \in C_i$, every view $v_i \in V_i$ and every view-type $r_i \in R_i$ that is feasible for v_i, player i can rationally choose c_i under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i, if and only if, (c_i, v_i, r_i) survives the iterated strict dominance procedure for M^{view}.*

Also here, the proof follows a similar structure as the one for Theorem 4.2. From Theorem 5.1 we know that the procedure always delivers a non-empty set of choices for every possible view and view-type in the game. The "if" direction of Theorem 5.2 therefore implies that for every view v_i and view-type r_i we can always construct an epistemic model and a type t_i within it that expresses common belief in rationality, is feasible for the view v_i, and which holds the belief hierarchy on views induced by r_i. The following result thus obtains.

Corollary 5.1 (Common belief in rationality is always possible) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views. Then, for every player i, every view $v_i \in V_i$ and view-type $r_i \in R_i$, there is an epistemic model $M = (T_j, b_j)_{j \in I}$ and a type $t_i \in T_i$, such that the type t_i is feasible for the view v_i, has the belief hierarchy on views induced by r_i, and expresses common belief in rationality.*

In other words, it is always possible to reason in accordance with common belief in rationality, while respecting the bounds set by a fixed view and a fixed belief hierarchy on views.

5.4 Example

To see how the procedure of *iterated strict dominance with fixed beliefs on views* works, consider the following variant of "Going to a party".

Example 2. Going to party, version 2

The story is almost the same as in "Going to a party, version 1". Similarly as in version 1, Barbara has only seen you in *red* or *yellow* oufits so far. However – and this is the novel part – you have recently bought some new *blue* and *green* clothes, but you do not remember whether you told Barbara about it or not. More precisely, you deem the event that you told Barbara about it equally like than the event that you did not. In case you told Barbara about it, you believe that Barbara believes that you are indeed uncertain about whether you told her or not (with equal belief probabilities). In case you did not tell Barbara about your new clothes, you believe she is unaware of you having *blue* or *green* clothes in your wardrobe, and you believe that Barbara believes that there is common belief in the fact that you can only choose between *red* and *yellow*. This naturally gives rise to the game with unawareness in Table 3, with the fixed belief hierarchy on views induced by your view-type r_1 at the bottom of Table 3. This belief hierarchy on views is also graphically represented by the arrows between the various views. Indeed, if you have view v_1 and view-type r_1, then the induced belief hierarchy on views matches exactly the story above.

The *iterated strict dominance procedure for* M^{view} proceeds as follows.

Initial step. Note that, given the epistemic model for views M^{view}, the only relevant pairs of views and view-types are $(v_1, r_1), (v_1', r_1'), (v_2, r_2)$ and (v_2', r_2'). The initial sets of beliefs are given by

$$\begin{aligned} B_1^0(v_1, r_1) &= (0.5) \cdot \Delta(C_2(v_2)) + (0.5) \cdot \Delta(C_2(v_2')) = \Delta(\{b, g, r, y\}), \\ B_1^0(v_1', r_1') &= \Delta(C_2(v_2')) = \Delta(\{b, g, r, y\}), \\ B_2^0(v_2, r_2) &= \Delta(C_1(v_1)) = \Delta(\{b, g, r, y\}), \\ B_2^0(v_2', r_2') &= \Delta(C_1(v_1')) = \Delta(\{r, y\}), \end{aligned}$$

whereas the initial sets of choices are

$$\begin{aligned} C_1^0(v_1, r_1) &= \{b, g, r, y\},\ C_1^0(v_1', r_1') = \{r, y\}, \\ C_2^0(v_2, r_2) &= C_2^0(v_2', r_2') = \{b, g, r, y\}. \end{aligned}$$

Round 1. By definition, the sets of beliefs remain the same as in the initial step. Note that choice y is not optimal for you at view v_1 for any belief in $B_1^1(v_1, r_1)$, and that Barbara's choice

Base game

G^{base}	blue	green	red	yellow
blue	$0, 0$	$4, 1$	$4, 4$	$4, 3$
green	$3, 2$	$0, 0$	$3, 4$	$3, 3$
red	$2, 2$	$2, 1$	$0, 0$	$2, 3$
yellow	$1, 2$	$1, 1$	$1, 4$	$0, 0$

Your views

v_1	blue	green	red	yellow
blue	0	4	4	4
green	3	0	3	3
red	2	2	0	2
yellow	1	1	1	0

v_1'	blue	green	red	yellow
red	2	2	0	2
yellow	1	1	1	0

$\downarrow$ (0.5) (0.5) $\searrow$ $\downarrow$

$\uparrow$ $\uparrow$

Barbara's views

v_2	blue	green	red	yellow
blue	0	2	2	2
green	1	0	1	1
red	4	4	0	4
yellow	3	3	3	0

v_2'	red	yellow
blue	2	2
green	1	1
red	0	4
yellow	3	0

Epistemic model for views M^{view}

$R_1 = \{r_1, r_1'\}, \quad R_2 = \{r_2, r_2'\}$

$$\begin{aligned} p_1(r_1) &= (0.5) \cdot (v_2, r_2) + (0.5) \cdot (v_2', r_2') \\ p_1(r_1') &= (v_2', r_2') \end{aligned}$$

$$\begin{aligned} p_2(r_2) &= (v_1, r_1) \\ p_2(r_2') &= (v_1', r_1') \end{aligned}$$

Table 3: Going to a party, version 2

g is not optimal for her at view v_2 for any belief in $B_2^1(v_2, r_2)$, nor optimal for her at view v_2' for any belief in $B_2^1(v_2', r_2')$. Hence, we obtain

$$\begin{aligned} C_1^1(v_1, r_1) &= \{b, g, r\},\ C_1^1(v_1', r_1') = \{r, y\}, \\ C_2^1(v_2, r_2) &= \{b, r, y\},\ C_2^1(v_2', r_2') = \{b, r, y\}. \end{aligned}$$

Round 2. The new sets of beliefs are

$$\begin{aligned} B_1^2(v_1, r_1) &= (0.5) \cdot \Delta(C_2^1(v_2, r_2)) + (0.5) \cdot \Delta(C_2^1(v_2', r_2')) = \Delta(\{b, r, y\}), \\ B_1^2(v_1', r_1') &= \Delta(C_2^1(v_2', r_2')) = \Delta(\{b, r, y\}), \\ B_2^2(v_2, r_2) &= \Delta(C_1^1(v_1, r_1)) = \Delta(\{b, g, r\}), \\ B_2^2(v_2', r_2') &= \Delta(C_1^1(v_1', r_1')) = \Delta(\{r, y\}). \end{aligned}$$

Then, your choice r is not optimal at your view v_1 for any belief in $B_1^2(v_1, r_1)$. Moreover, Barbara's choice b is not optimal at her view v_2 for any belief in $B_2^2(v_2, r_2)$. The new sets of choices are thus given by

$$\begin{aligned} C_1^2(v_1, r_1) &= \{b, g\},\ C_1^2(v_1', r_1') = \{r, y\}, \\ C_2^2(v_2, r_2) &= \{r, y\},\ C_2^2(v_2', r_2') = \{b, r, y\}. \end{aligned}$$

Round 3. The new sets of beliefs are

$$\begin{aligned} B_1^3(v_1, r_1) &= (0.5) \cdot \Delta(C_2^2(v_2, r_2)) + (0.5) \cdot \Delta(C_2^2(v_2', r_2')) \\ &= (0.5) \cdot \Delta(\{r, y\}) + (0.5) \cdot \Delta(\{b, r, y\}) \\ &= \{\beta_1 \in \Delta(\{b, r, y\}) \mid \beta_1(b) \leq 0.5\}, \\ B_1^3(v_1', r_1') &= \Delta(C_2^2(v_2', r_2')) = \Delta(\{b, r, y\}), \\ B_2^3(v_2, r_2) &= \Delta(C_1^2(v_1, r_1)) = \Delta(\{b, g\}), \\ B_2^3(v_2', r_2') &= \Delta(C_1^2(v_1', r_1')) = \Delta(\{r, y\}). \end{aligned}$$

Note that at your view v_1, your choice b is optimal for the belief $\beta_1 \in B_1^3(v_1, r_1)$ that assigns probability 1 to r, and your choice g is optimal for the belief $\beta_1' \in B_1^3(v_1, r_1)$ that assigns probability 0.4 to b and probability 0.6 to r. Hence, your choices b and g both survive this round at (v_1, r_1). Barbara's choice y, however, is not optimal at her view v_2 for any belief in $B_2^3(v_2, r_2)$. All other remaining choices survive at the respective pairs of views and view-types. Hence, the new sets of choices are

$$\begin{aligned} C_1^3(v_1, r_1) &= \{b, g\},\ C_1^3(v_1', r_1') = \{r, y\}, \\ C_2^3(v_2, r_2) &= \{r\},\ C_2^3(v_2', r_2') = \{b, r, y\}. \end{aligned}$$

$$R_1 = \{r_1, r_1'\}, \quad R_2 = \{r_2, r_2'\}$$

$$\begin{array}{rcl} p_1(r_1) & = & (0.8) \cdot (v_2, r_2) + (0.2) \cdot (v_2', r_2') \\ p_1(r_1') & = & (v_2', r_2') \end{array}$$

$$\begin{array}{rcl} p_2(r_2) & = & (v_1, r_1) \\ p_2(r_2') & = & (v_1', r_1') \end{array}$$

Table 4: Alternative belief hierarchy on views in "Going to a party, version 2"

Round 4. The new sets of beliefs are

$$\begin{array}{rcl} B_1^4(v_1, r_1) & = & (0.5) \cdot \Delta(C_2^3(v_2, r_2)) + (0.5) \cdot \Delta(C_2^3(v_2', r_2')) \\ & = & (0.5) \cdot \Delta(\{r\}) + (0.5) \cdot \Delta(\{b, r, y\}) \\ & = & \{\beta_1 \in \Delta(\{b, r, y\}) \mid \beta_1(r) \geq 0.5\}, \\ B_1^4(v_1', r_1') & = & \Delta(C_2^3(v_2', r_2')) = \Delta(\{b, r, y\}), \\ B_2^4(v_2, r_2) & = & \Delta(C_1^3(v_1, r_1)) = \Delta(\{b, g\}), \\ B_2^4(v_2', r_2') & = & \Delta(C_1^3(v_1', r_1')) = \Delta(\{r, y\}). \end{array}$$

Note that at your view v_1, your choice b is optimal for the belief $\beta_1 \in B_1^4(v_1, r_1)$ that assigns probability 1 to r, and your choice g is optimal for the belief $\beta_1' \in B_1^4(v_1, r_1)$ that assigns probability 0.4 to b and probability 0.6 to r. Hence, your choices b and g both survive this round at (v_1, r_1). All other remaining choices survive at the respective pairs of views and view-types. Hence, the new sets of choices are

$$\begin{array}{rcl} C_1^4(v_1, r_1) & = & \{b, g\}, \ C_1^4(v_1', r_1') = \{r, y\}, \\ C_2^4(v_2, r_2) & = & \{r\}, \ C_2^4(v_2', r_2') = \{b, r, y\}, \end{array}$$

and the procedure terminates.

We thus conclude that you can rationally choose *blue* and *green* under common belief in rationality with the view v_1 and the belief hierarchy on views induced by r_1.

Suppose now that we modify the belief hierarchy on views, by considering the alternative epistemic model for views $\hat{M}^{view}$ in Table 4. This corresponds to a case where you are fairly certain that you told Barbara about your new clothes. More precisely, the view-type r_1 assigns probability 0.8 (instead of 0.5) to the event that you informed Barbara about your new *blue* and *green* outfits, and assigns probability 0.2 to the event that you did not. Everything else remains the same.

If we apply the procedure to this new situation, then it may be verified that the sets of choices in rounds 1 and 2 are the same as above. Hence, we obtain that the new set of beliefs in round 3 at (v_1, r_1) is given by

$$\begin{aligned} B_1^3(v_1, r_1) &= (0.8) \cdot \Delta(C_2^2(v_2, r_2)) + (0.2) \cdot \Delta(C_2^2(v_2', r_2')) \\ &= (0.8) \cdot \Delta(\{r, y\}) + (0.2) \cdot \Delta(\{b, r, y\}) \\ &= \{\beta_1 \in \Delta(\{b, r, y\}) \mid \beta_1(b) \leq 0.2\}. \end{aligned}$$

Therefore, your choice g is no longer optimal at your view v_1 for any belief in $B_1^3(v_1, r_1)$, which means that the new set of choices at (v_1, r_1) becomes

$$C_1^3(v_1, r_1) = \{b\}.$$

Hence, in this new situation you can only rationally wear *blue* (and not *green*) under common belief in rationality with the view v_1 and the belief hierarchy on views induced by r_1.

6 Related Literature

Roughly speaking, the literature on unawareness can be divided into two categories. The first category explores the logical foundations of unawareness in a single agent and multi-agent setting, without an explicit reference to games, whereas the second category applies the logic of unawareness to games. For a survey of this literature we refer the reader to Schipper (2014),

An important question being addressed by the first category is how unawareness can be modeled in a meaningful way, both syntactically and semantically. See, for instance, Fagin and Halpern (1988), Dekel, Lipman and Rustichini (1998), Modica and Rustichini (1999), Halpern (2001), Heifetz, Meier and Schipper (2006, 2008), Halpern and Rêgo (2008) and Li (2009).

A general conclusion in this literature is that in a multi-agent setting, every agent must be endowed with his own, *subjective* state space that only contains those objects he is aware of, and which therefore may be substantially smaller than the *full* state space. This principle is also reflected in our definition of a game with unawareness, and how we set up an epistemic model to encode belief hierarchies about choices and views.

To model a game with unawareness, we assume for every player a finite collection of possible views on the game. The implicit understanding is that a player with a certain view only has mental access to those choices that are part of his view, and to those views in the model that are smaller than his own. In other words, the subjective state space for a player with view v_i only contains the choices inside v_i, and the views for the opponents and himself that are contained in v_i.

Similarly, in the epistemic model we use to encode belief hierarchies on choices and views, the implicit understanding is that a player with view v_i only has mental access to choices in v_i, opponents' views that are contained in v_i, and types (hence, belief hierarchies) that only reason about views that are contained in v_i. The latter objects thus constitute the subjective state space for player i in the epistemic model if his view is v_i, and may thus be substantively smaller than the full epistemic model.

Papers in the second category deal specifically with static or dynamic games with unawareness, and can thus be seen as applications of the logic of unawareness. See, for instance, Feinberg (2004, 2012), Čopič and Galeotti (2006), Rêgo and Halpern (2012), Heifetz, Meier and Schipper (2013), Grant and Quiggin (2013), Halpern and Rêgo (2014) and Schipper (2017). Our paper clearly falls within this category as well.

As we already mentioned in Section 2, an important difference between our way of modelling games with unawareness and that of most other papers is that we do not exogenously specify a unique belief hierarchy on views for every player. In fact, of the abovementioned papers only Čopič and Galeotti (2006) do not fix the belief hierarchies on views in their model. Moreover, we allow for *probabilistic* belief hierarchies on views, whereas most papers above – exceptions being Feinberg (2004), Rêgo and Halpern (2012) and Halpern and Rêgo (2014) – restrict to deterministic belief hierarchies on views. We find such probabilistic beliefs on views important, as it allows for cases where a player is truly uncertain about the precise view held by an opponent.

In terms of the approach adopted, this paper is one of the few to provide an *epistemic* analysis of the players' reasoning in games with unawareness, through the epistemic conditions of common belief in rationality. Another example is Guarino (2017, Chapter 3), who offers an epistemic characterization of *extensive-form rationalizability* (Pearce (1984), Battigalli (1997), Heifetz, Meier and Schipper (2013)) for dynamic games with unawareness.

Like our paper, also Feinberg (2012) and Heifetz, Meier and Schipper (2013) investigate the implications of common (strong) belief in rationality by studying the concepts of *rationalizability* and *extensive-form rationalizability*, respectively. One difference with our approach is that the latter papers do not investigate these concepts on an epistemic basis.

7 Concluding Remarks

The goal of this paper has been to investigate the reasoning of players in static games with unawareness through the basic concept of *common belief in rationality.* Our approach has been primarily *epistemic*, as we started by formulating the epistemic conditions that constitute common belief in rationality, and subsequently designed a recursive elimination procedure that characterizes exactly those choices that can rationally be made, for every possible view, under this epistemic concept. We did so for two scenarios: one in which we do not fix the players' belief hierarchies on views, and one in which we do.

An interesting open question is how one can epistemically characterize various *equilibrium* concepts that have been proposed for games with unawareness, such as *action-awareness equilibrium* (Čopič and Galeotti (2006)), *extended Nash equilibrium* (Feinberg (2012)), *generalized Nash equilibrium* (Halpern and Rêgo (2014)), *generalized sequential equilibrium* (Rêgo and

Halpern (2012)), *sequential equilibrium* (Grant and Quiggin (2013)) and *self-confirming equilibrium* (Schipper (2017)).

Another problem that could be addressed in the future is how one could formulate the backward induction concept of *common belief in future rationality* (Perea (2014)) for dynamic games with unawareness. Moreover, it could be explored how this concept would relate to the forward induction concept of *extensive-form rationalizability* as defined by Heifetz, Meier and Schipper (2013) for dynamic games with unawareness. These, and other, open problems are left for future research.

8 Appendix

8.1 Proofs of Section 4

For the proofs of Section 4 we heavily rely on Lemma 3 in Pearce (1984). We will present this result below within the framework of views, because we can then readily apply it for our specific purposes. Consider a view $v_i = (D_i, D_{-i})$, a choice $c_i \in D_i$ and a probabilistic belief $\beta_i \in \Delta(D_{-i})$ about the opponents' choice combinations. Choice c_i is said to be *optimal* for β_i with the view v_i if

$$\sum_{c_{-i} \in D_{-i}} \beta_i(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in D_{-i}} \beta_i(c_{-i}) \cdot u_i(c_i', c_{-i}) \text{ for all } c_i' \in D_i.$$

Lemma 3 in Pearce (1984) states that a choice is optimal for at least one belief, if and only if, the choice is not strictly dominated.

Lemma 8.1 (Pearce (1984)) *Consider a view $v_i = (D_i, D_{-i})$ and an available choice $c_i \in D_i$. Then, c_i is optimal for some probabilistic belief on D_{-i} with the view v_i, if and only if, c_i is not strictly dominated for the view v_i.*

As we will see, this result is the cornerstone to the proofs of Section 4.

Proof of Theorem 4.1. Note that in the iterated strict dominance procedure for unawareness, $C_i^{k+1}(v_i) \subseteq C_i^k(v_i)$ for every player i, every view $v_i \in V_i$ and every round $k \geq 0$. Since there are only finitely many choices and views in the game, the procedure must terminate after finitely many rounds. That is, there is some $K \geq 0$ such that $C_i^k(v_i) = C_i^K(v_i)$ and $C_{-i}^k(v_i) = C_{-i}^K(v_i)$ for every player i, view $v_i \in V_i$ and every $k \geq K$. As such, it is sufficient to show that $C_i^k(v_i)$ is always non-empty for every player i, every view $v_i \in V_i$ and every $k \geq 0$. We prove so by induction on k.

For $k = 0$, this is clear since $C_i^0(v_i) = C_i(v_i)$, which is non-empty.

Take now some $k \geq 1$ and assume that $C_j^{k-1}(v_j)$ is non-empty for every player j and every view $v_j \in V_j$. Consider some player i and some view $v_i \in V_i$. We show that $C_i^k(v_i)$ is non-empty.

For every opponent $j \neq i$, take some view $v_j \in V_j$ that is contained in v_i. Note that such view v_j exists by Definition 2.1. For every opponent $j \neq i$, take a choice $c_j \in C_j^{k-1}(v_j)$, which is possible because $C_j^{k-1}(v_j)$ is non-empty by the induction assumption. Then, by construction, the choice combination $(c_j)_{j \neq i}$ is in $C_{-i}^k(v_i)$. Let the choice $c_i \in C_i(v_i)$ be optimal, among all choices in $C_i(v_i)$, for the belief β_i that assigns probability 1 to $(c_j)_{j \neq i}$. Hence, $\beta_i \in \Delta(C_{-i}^k(v_i))$. By Lemma 8.1 it then follows that c_i is not strictly dominated for the reduced view $(C_i(v_i), C_{-i}^k(v_i))$. In particular, c_i is not strictly dominated for the reduced view $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$, and hence $c_i \in C_i^k(v_i)$. We thus conclude that $C_i^k(v_i)$ is non-empty.

By induction, it follows that $C_i^k(v_i)$ is always non-empty for every player i, every view $v_i \in V_i$ and every round $k \geq 0$. As we have seen, this completes the proof. ■

Proof of Theorem 4.2. "Only if": For every player i and every view $v_i \in V_i$, let $C_i^{cbr}(v_i)$ be the set of choices in $C_i(v_i)$ that player i can rationally make under common belief in rationality with the view v_i. We show, by induction on k, that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$ for every $k \geq 0$, every player i and every view $v_i \in V_i$.

For $k = 0$ this is obviously true since $C_i^0(v_i) = C_i(v_i)$.

Now, consider some $k \geq 1$ and assume that $C_i^{cbr}(v_i) \subseteq C_i^{k-1}(v_i)$ for every player i and every view $v_i \in V_i$. Consider some player i, some view v_i, and some $c_i \in C_i^{cbr}(v_i)$. By the induction assumption we know that $c_i \in C_i^{k-1}(v_i)$. As $c_i \in C_i^{cbr}(v_i)$, there is some epistemic model $M = (T_j, b_j)_{j \in I}$ and some type $t_i \in T_i$ such that t_i is feasible for the view v_i, expresses common belief in rationality, and such that c_i is optimal for t_i within $C_i(v_i)$. Let $b_i^C(t_i)$ be the margainal of the belief $b_i(t_i)$ on C_{-i}. Then, in light of the above,

$$\sum_{c_{-i} \in C_{-i}(v_i)} b_i^C(t_i)(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}(v_i)} b_i^C(t_i)(c_{-i}) \cdot u_i(c_i', c_{-i}) \text{ for all } c_i' \in C_i(v_i). \quad (8.1)$$

Since the type t_i is feasible for the view v_i, and expresses common belief in rationality, we conclude that $b_i^C(t_i)((c_j)_{j \neq i}) > 0$ only if, for every $j \neq i$, choice c_j is in $C_j^{cbr}(v_j)$ for some view v_j that is contained in v_i. Since by the induction assumption we have that $C_j^{cbr}(v_j) \subseteq C_j^{k-1}(v_j)$, we conclude that $b_i^C(t_i)((c_j)_{j \neq i}) > 0$ only if, for every $j \neq i$, choice c_j is in $C_j^{k-1}(v_j)$ for some view v_j that is contained in v_i. Hence, by definition of the procedure, $b_i^C(t_i) \in \Delta(C_{-i}^k(v_i))$.

In view of (8.1) we thus conclude that $c_i \in C_i^{k-1}(v_i)$ is optimal for the belief $b_i^C(t_i) \in \Delta(C_{-i}^k(v_i))$ within the reduced view $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$. By Lemma 8.1 it then follows that c_i is not strictly dominated for the reduced view $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$, and hence $c_i \in C_i^k(v_i)$, by definition of the procedure. As this holds for every $c_i \in C_i^{cbr}(v_i)$, we conclude that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$, which was to show. By induction on k we conclude that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$ for every k, every player i and every view $v_i \in V_i$.

Now, consider a player i, a view $v_i \in V_i$, and a choice $c_i \in C_i(v_i)$ that can rationally be made under common belief in rationality with the view v_i. Then, $c_i \in C_i^{cbr}(v_i)$ and hence, by the analysis above, $c_i \in C_i^k(v_i)$ for every $k \geq 0$. Hence, (c_i, v_i) survives the procedure, which completes the proof of the "only if" direction.

"If": For every player i, and every view $v_i \in V_i$, let $C_i^\infty(v_i) := \cap_{k\geq 0} C_i^k(v_i)$ be the set of choices that survive the procedure for view v_i, and let $C_{-i}^\infty(v_i) := \cap_{k\geq 0} C_{-i}^k(v_i)$ be the set of opponents' choice combinations that survive the procedure at v_i. By Theorem 4.1 we know that all these sets $C_i^\infty(v_i)$ and $C_{-i}^\infty(v_i)$ are non-empty. We show that every choice in $C_i^\infty(v_i)$ can rationally be made under common belief in rationality with the view v_i.

By construction, every choice $c_i \in C_i^\infty(v_i)$ is not strictly dominated within the reduced view $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$. Hence, by Lemma 8.1, there is for every choice $c_i \in C_i^\infty(v_i)$ some belief $\beta_i^{c_i,v_i} \in \Delta(C_{-i}^\infty(v_i))$ such that c_i is optimal for $\beta_i^{c_i,v_i}$ within the reduced view $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$. We will show that, in fact, c_i is optimal for $\beta_i^{c_i,v_i}$ within the reduced view $(C_i(v_i), C_{-i}^\infty(v_i))$. Let $c_i^* \in C_i(v_i)$ be optimal for $\beta_i^{c_i,v_i}$ within the reduced view $(C_i(v_i), C_{-i}^\infty(v_i))$. Then, by Lemma 8.1, c_i^* is not strictly dominated for the reduced view $(C_i(v_i), C_{-i}^\infty(v_i))$, and hence c_i^* must be in $C_i^\infty(v_i)$. As c_i is optimal for $\beta_i^{c_i,v_i}$ within the reduced view $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$, it follows that

$$\sum_{c_{-i} \in C_{-i}^\infty(v_i)} \beta_i^{c_i,v_i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}^\infty(v_i)} \beta_i^{c_i,v_i}(c_{-i}) \cdot u_i(c_i^*, c_{-i}).$$

As c_i^* is optimal for $\beta_i^{c_i,v_i}$ within the reduced view $(C_i(v_i), C_{-i}^\infty(v_i))$, it follows that c_i is optimal for $\beta_i^{c_i,v_i}$ within the reduced view $(C_i(v_i), C_{-i}^\infty(v_i))$ as well.

Moreover, since $\beta_i^{c_i,v_i} \in \Delta(C_{-i}^\infty(v_i))$ we know, by construction of the procedure, that $\beta_i^{c_i,v_i}$ only assigns positive probability to opponents' choices c_j where $c_j \in C_j^\infty(v_j[c_i, v_i, c_j])$ for some view $v_j[c_i, v_i, c_j] \in V_j$ contained in v_i. On the basis of these beliefs $\beta_i^{c_i,v_i}$ and views $v_j[c_i, v_i, c_j]$ we now construct the following epistemic model $M = (T_i, b_i)_{i\in I}$. Let the set of types for every player i be given by

$$T_i = \{t_i^{c_i,v_i} \mid v_i \in V_i \text{ and } c_i \in C_i^\infty(v_i)\}.$$

Moreover, for every player i and every type $t_i^{c_i,v_i} \in T_i$, let the belief $b_i(t_i^{c_i,v_i})$ on $C_{-i} \times V_{-i} \times T_{-i}$ be given by

$$b_i(t_i^{c_i,v_i})((c_j, v_j, t_j)_{j\neq i}) := \begin{cases} \beta_i^{c_i,v_i}((c_j)_{j\neq i}), & \text{if } v_j = v_j[c_i, v_i, c_j] \text{ and } t_j = t_j^{c_j,v_j} \\ & \text{for all } j \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, $t_i^{c_i,v_i}$ only assigns positive probability to combinations $(c_j, v_j, t_j^{c_j,v_j})$ for every opponent $j \neq i$, where $c_j \in C_j^\infty(v_j)$. Hence, in particular, $c_j \in C_j(v_j)$, which guarantees that condition (a) in Definition 3.1 is satisfied. Moreover, every type $t_i^{c_i,v_i}$ in M only assigns positive probability to opponents' views $v_j[c_i, v_i, c_j]$ which are contained in v_i. This implies that also condition (b) in Definition 3.1 is satisfied. Hence, we conclude that $M = (T_i, b_i)_{i\in I}$ so constructed is a well-defined epistemic model.

Note that every type $t_i^{c_i,v_i}$ has the belief $\beta_i^{c_i,v_i}$ about the opponents' choices. Since we have seen above that c_i is optimal for $\beta_i^{c_i,v_i}$ among all choices in $C_i(v_i)$, it follows that c_i is optimal for type $t_i^{c_i,v_i}$ among all choices in $C_i(v_i)$ as well. By construction, every type $t_i^{c_i,v_i}$ only assigns positive probability to combinations $(c_j, v_j, t_j^{c_j,v_j})$ for every opponent $j \neq i$, where $c_j \in C_j^\infty(v_j)$. Since we have seen that c_j is optimal for type $t_j^{c_j,v_j}$ among choices in $C_j(v_j)$, it follows that every type $t_i^{c_i,v_i}$ in the epistemic model believes in the opponents' rationality. As a consequence, every type in the epistemic model expresses *common* belief in rationality.

Take now some player i, and some choice-view pair (c_i, v_i) that survives the procedure. Then, $c_i \in C_i^\infty(v_i)$. Consider the type $t_i^{c_i,v_i}$ in the epistemic model constructed above. Note that type $t_i^{c_i,v_i}$ is feasible for the view v_i since, by construction, $t_i^{c_i,v_i}$ only assigns positive probability to opponents' views $v_j[c_i, v_i, c_j]$ that are contained in v_i. We have seen above that c_i is optimal for the type $t_i^{c_i,v_i}$ among choices in $C_i(v_i)$, and that type $t_i^{c_i,v_i}$ expresses common belief in rationality. It thus follows that c_i can rationally be chosen under common belief in rationality with the view v_i. This completes the proof. ■

8.2 Belief Hierarchies on Views Induced by Types

8.2.1 Epistemic Models for Views

Consider an epistemic model for views $M^{view} = (R_i, p_i)_{i \in I}$. We show how, for every player i and every view-type r_i, we can derive the induced belief hierarchy $h_i^{view}(r_i)$ on views. Formally, this belief hierarchy can be written as an infinite sequence of beliefs $h_i^{view}(r_i) = (h_i^1(r_i), h_i^2(r_i), ...)$, where $h_i^1(r_i)$ is the induced first-order belief, $h_i^2(r_i)$ is the induced second-order belief, and so on.

We will inductively define, for every n, the n-th order beliefs induced by types r_i in M^{view}, building upon the $(n-1)$-th order beliefs that have been defined in the preceding step. We start by defining the first-order beliefs.

For every player i, and every type $r_i \in R_i$, define the first-order belief $h_i^1(r_i) \in \Delta(V_{-i})$ by

$$h_i^1(r_i)(v_{-i}) := p_i(r_i)(\{v_{-i}\} \times R_{-i}) \text{ for all } v_{-i} \in V_{-i}.$$

Now, suppose that $n \geq 2$, and assume that the $(n-1)$-th order beliefs $h_i^{n-1}(r_i)$ have been defined for all players i, and every type $r_i \in R_i$. Let

$$h_i^{n-1}(R_i) := \{h_i^{n-1}(r_i) \mid r_i \in R_i\}$$

be the finite set of $(n-1)$-th order beliefs for player i induced by types in R_i. For every $h_i^{n-1} \in h_i^{n-1}(R_i)$, let

$$R_i[h_i^{n-1}] := \{r_i \in R_i \mid h_i^{n-1}(r_i) = h_i^{n-1}\}$$

be the set of types in R_i that have the $(n-1)$-th order belief h_i^{n-1}.

Let $h_{-i}^{n-1}(R_{-i}) := \times_{j \neq i} h_j^{n-1}(R_j)$, and for a given $h_{-i}^{n-1} = (h_j^{n-1})_{j \neq i}$ in $h_{-i}^{n-1}(R_{-i})$ let $R_{-i}[h_{-i}^{n-1}] := \times_{j \neq i} R_j[h_j^{n-1}]$.

For every type $r_i \in R_i$, let the n-th order belief $h_i^n(r_i) \in \Delta(V_{-i} \times h_{-i}^{n-1}(R_{-i}))$ be given by

$$h_i^n(r_i)(v_{-i}, h_{-i}^{n-1}) := p_i(r_i)(\{v_{-i}\} \times R_{-i}[h_{-i}^{n-1}])$$

for every $v_{-i} \in V_{-i}$ and every $h_{-i}^{n-1} \in h_{-i}^{n-1}(R_{-i})$.

Finally, for every type $r_i \in R_i$, we denote by

$$h_i^{view}(r_i) := (h_i^n(r_i))_{n \in \mathbb{N}}$$

the *belief hierarchy on views* induced by r_i.

8.2.2 Epistemic Models for Choices and Views

Consider an epistemic model for choices and views $M = (T_i, b_i)_{i \in I}$. We show how, for every player i and every type t_i, we can derive the induced belief hierarchy $h_i^{view}(t_i)$ on views. Formally, this belief hierarchy can be written as an infinite sequence of beliefs $h_i^{view}(t_i) = (h_i^1(t_i), h_i^2(t_i), ...)$, where $h_i^1(t_i)$ is the induced first-order belief on views, $h_i^2(t_i)$ is the induced second-order belief on views, and so on.

We will inductively define, for every n, the n-th order beliefs on views induced by types t_i in M, building upon the $(n-1)$-th order beliefs on views that have been defined in the preceding step. We start by defining the first-order beliefs.

For every player i, and every type $t_i \in T_i$, define the first-order belief on views $h_i^1(t_i) \in \Delta(V_{-i})$ by

$$h_i^1(t_i)(v_{-i}) := b_i(t_i)(C_{-i} \times \{v_{-i}\} \times T_{-i}) \text{ for all } v_{-i} \in V_{-i}.$$

Now, suppose that $n \geq 2$, and assume that the $(n-1)$-th order beliefs on views $h_i^{n-1}(t_i)$ have been defined for all players i, and every type $t_i \in T_i$. Let

$$h_i^{n-1}(T_i) := \{h_i^{n-1}(t_i) \mid t_i \in T_i\}$$

be the finite set of $(n-1)$-th order beliefs for player i induced by types in T_i. For every $h_i^{n-1} \in h_i^{n-1}(T_i)$, let

$$T_i[h_i^{n-1}] := \{t_i \in T_i \mid h_i^{n-1}(t_i) = h_i^{n-1}\}$$

be the set of types in T_i that have the $(n-1)$-th order belief h_i^{n-1}.

Let $h_{-i}^{n-1}(T_{-i}) := \times_{j \neq i} h_j^{n-1}(T_j)$, and for a given $h_{-i}^{n-1} = (h_j^{n-1})_{j \neq i}$ in $h_{-i}^{n-1}(T_{-i})$ let $T_{-i}[h_{-i}^{n-1}] := \times_{j \neq i} T_j[h_j^{n-1}]$.

For every type $t_i \in T_i$, let the n-th order belief on views $h_i^n(t_i) \in \Delta(V_{-i} \times h_{-i}^{n-1}(T_{-i}))$ be given by

$$h_i^n(t_i)(v_{-i}, h_{-i}^{n-1}) := b_i(t_i)(C_{-i} \times \{v_{-i}\} \times T_{-i}[h_{-i}^{n-1}])$$

for every $v_{-i} \in V_{-i}$ and every $h_{-i}^{n-1} \in h_{-i}^{n-1}(T_{-i})$.

Finally, for every type $t_i \in T_i$, we denote by

$$h_i^{view}(t_i) := (h_i^n(t_i))_{n \in \mathbb{N}}$$

the *belief hierarchy on views* induced by t_i.

8.3 Proofs of Section 5

Proof of Theorem 5.1. Note that $C_i^{k+1}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every player i, every view $v_i \in V_i$, every view-type $r_i \in R_i$ that is feasible for v_i, and every round $k \geq 0$. Since there are only finitely many choices, views and view-types in the game, the procedure must terminate after finitely many rounds. That is, there is some $K \geq 0$ such that $C_i^k(v_i, r_i) = C_i^K(v_i, r_i)$ for every player i, every view v_i, every view-type r_i that is feasible for v_i and every $k \geq K$. As such, it is sufficient to show that $C_i^k(v_i, r_i)$ is always non-empty for every player i, every view $v_i \in V_i$, every view-type $r_i \in R_i$ that is feasible for v_i, and every $k \geq 0$. We prove so by induction on k.

For $k = 0$, this is clear since $C_i^0(v_i, r_i) = C_i(v_i)$, which is non-empty.

Take now some $k \geq 1$ and assume that $C_j^{k-1}(v_j, r_j)$ is non-empty for every player j, every view $v_j \in V_j$ and every view-type r_j that is feasible for v_j. Consider some player i, some view $v_i \in V_i$ and some view-type $r_i \in R_i$ feasible for v_i. Then, $B_i^k(v_i, r_i)$ is non-empty, since the choice sets $C_j^{k-1}(v_j, r_j)$ are non-empty for every $j \neq i$, every $v_j \in V_j$ and every $r_j \in R_j$.

Now, take some $b_i \in B_i^k(v_i, r_i)$ and some choice $c_i \in C_i(v_i)$ that is optimal for b_i among choices in $C_i(v_i)$. Then, c_i will also be optimal for b_i among choices in $C_i^{k-1}(v_i, r_i)$, and hence $c_i \in C_i^k(v_i, r_i)$. We thus conclude that $C_i^k(v_i, r_i)$ is non-empty.

By induction, it follows that $C_i^k(v_i, r_i)$ is always non-empty for every player i, every view $v_i \in V_i$, every view-type $r_i \in R_i$ feasible for v_i, and every round $k \geq 0$. As we have seen, this completes the proof. ■

Proof of Theorem 5.2. "Only if": Assume, without loss of generality, that different types in M^{view} induce different belief hierarchies on views. For every player i, every view $v_i \in V_i$, and every view-type r_i that is feasible for v_i, let $C_i^{cbr}(v_i, r_i)$ be the set of choices in $C_i(v_i)$ that player i can rationally make under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i. We show, by induction on k, that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every $k \geq 0$, every player i, every view $v_i \in V_i$, and every view-type $r_i \in R_i$ that is feasible for v_i

For $k = 0$ this is obviously true since $C_i^0(v_i, r_i) = C_i(v_i)$.

Now, consider some $k \geq 1$ and assume that $C_i^{cbr}(v_i, r_i) \subseteq C_i^{k-1}(v_i, r_i)$ for every player i, every view $v_i \in V_i$ and every view-type $r_i \in R_i$ that is feasible for v_i. Consider some player i, some view v_i, some view-type r_i feasible for v_i, and some $c_i \in C_i^{cbr}(v_i, r_i)$. Then, there is some epistemic model $M = (T_j, b_j)_{j \in I}$ and some type $t_i \in T_i$ such that $h_i^{view}(t_i) = h_i^{view}(r_i)$, type t_i expresses common belief in rationality, and such that c_i is optimal for t_i within $C_i(v_i)$.

Let $b_i^C(t_i)$ be the marginal of the belief $b_i(t_i)$ on C_{-i}. Later, we will show that $b_i^C(t_i) \in B_i^k(v_i, r_i)$. In order to do so, we need two preliminary observations.

First, since $h_i^{view}(t_i) = h_i^{view}(r_i)$, there is for every opponent j, and every view-type r_j that receives positive probability under $p_i(r_i)$, some set of types $T_j(r_j)$ such that

$$h_j^{view}(t_j) = h_j^{view}(r_j) \text{ for all } t_j \in T_j(r_j), \tag{8.2}$$

and

$$b_i(t_i)(\times_{j \neq i}(C_j \times \{v_j\} \times T_j(r_j))) = p_i(r_i)((v_j, r_j)_{j \neq i}) \tag{8.3}$$

for all $(v_j, r_j)_{j \neq i}$ in $V_{-i} \times R_{-i}$ with $p_i(r_i)((v_j, r_j)_{j \neq i}) > 0$. Here, we use the assumption above that different types in M^{view} induce different belief hierarchies on views.

Second, since t_i expresses common belief in rationality, we have that $b_i(t_i)((c_j, v_j, t_j)_{j \neq i}) > 0$ only if for every opponent $j \neq i$, type t_j expresses common belief in rationality, and c_j is optimal for t_j and v_j. Note that in this case, there must be some $r_j \in R_j$ with $t_j \in T_j(r_j)$, in view of (8.3). Hence, by (8.2), we know that $h_j^{view}(t_j) = h_j^{view}(r_j)$. Together with the facts that c_j is optimal for t_j and v_j, and t_j expresses common belief in rationality, it follows that $c_j \in C_j^{cbr}(v_j, r_j)$ in this case. By the induction assumption, $C_j^{cbr}(v_j, r_j) \subseteq C_j^{k-1}(v_j, r_j)$. We thus conclude that

$$b_i(t_i)((c_j, v_j, t_j)_{j \neq i}) > 0 \text{ only if } t_j \in T_j(r_j) \text{ and } c_j \in C_j^{k-1}(v_j, r_j) \tag{8.4}$$

for all opponents $j \neq i$.

We will now use (8.3) and (8.4) to prove that $b_i^C(t_i) \in B_i^k(v_i, r_i)$. That is, we must show that

$$b_i^C(t_i) = \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \beta_i^{(v_j, r_j)_{j \neq i}}, \tag{8.5}$$

where $\beta_i^{(v_j, r_j)_{j \neq i}} \in \Delta(\times_{j \neq i} C_j^{k-1}(v_j, r_j))$ for all $(v_j, r_j)_{j \neq i}$ with $p_i(r_i)((v_j, r_j)_{j \neq i}) > 0$.

Let

$$(V_{-i} \times R_{-i})^* := \{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i} \mid p_i(r_i)((v_j, r_j)_{j \neq i}) > 0\}.$$

For every $(v_j, r_j)_{j \neq i} \in (V_{-i} \times R_{-i})^*$, define $\beta_i^{(v_j, r_j)_{j \neq i}}$ by

$$\beta_i^{(v_j, r_j)_{j \neq i}}((c_j)_{j \neq i}) := \frac{b_i(t_i)(\times_{j \neq i}\{c_j\} \times \{v_j\} \times T_j(r_j))}{p_i(r_i)((v_j, r_j)_{j \neq i})}. \tag{8.6}$$

Then, it may be verified that $\beta_i^{(v_j, r_j)_{j \neq i}}$ is a probability distribution on C_{-i}, since

$$\beta_i^{(v_j, r_j)_{j \neq i}}(C_{-i}) = \frac{b_i(t_i)(\times_{j \neq i}\{C_j\} \times \{v_j\} \times T_j(r_j))}{p_i(r_i)((v_j, r_j)_{j \neq i})} = 1$$

because of (8.3).

We next show that $\beta_i^{(v_j,r_j)_{j\neq i}}$ only assigns positive probability to $(c_j)_{j\neq i} \in \times_{j\neq i} C_j^{k-1}(v_j, r_j)$. Indeed, suppose that $\beta_i^{(v_j,r_j)_{j\neq i}}((c_j)_{j\neq i}) > 0$. Then, by (8.6), $b_i(t_i)(\times_{j\neq i}\{c_j\} \times \{v_j\} \times T_j(r_j)) > 0$, and hence we conclude by (8.4) that $c_j \in C_j^{k-1}(v_j, r_j)$ for every $j \neq i$. Hence, $(c_j)_{j\neq i} \in \times_{j\neq i} C_j^{k-1}(v_j, r_j)$. We may thus conclude that

$$\beta_i^{(v_j,r_j)_{j\neq i}} \in \Delta(\times_{j\neq i} C_j^{k-1}(v_j, r_j)) \text{ for every } (v_j, r_j)_{j\neq i} \in (V_{-i} \times R_{-i})^*. \tag{8.7}$$

We finally show (8.5). By definition, for every $(c_j)_{j\neq i}$ in C_{-i}, we have that

$$\begin{aligned}
b_i^C(t_i)((c_j)_{j\neq i}) &= \sum_{(v_j,t_j)_{j\neq i} \in V_{-i}\times T_{-i}} b_i(t_i)((c_j, v_j, t_j)_{j\neq i}) \\
&= \sum_{(v_j,r_j)_{j\neq i} \in (V_{-i}\times R_{-i})^*} b_i(t_i)(\times_{j\neq i}\{c_j\} \times \{v_j\} \times T_j(r_j)) \\
&= \sum_{(v_j,r_j)_{j\neq i} \in (V_{-i}\times R_{-i})^*} p_i(r_i)((v_j, r_j)_{j\neq i}) \cdot \frac{b_i(t_i)(\times_{j\neq i}\{c_j\} \times \{v_j\} \times T_j(r_j))}{p_i(r_i)((v_j, r_j)_{j\neq i})} \\
&= \sum_{(v_j,r_j)_{j\neq i} \in (V_{-i}\times R_{-i})^*} p_i(r_i)((v_j, r_j)_{j\neq i}) \cdot \beta_i^{(v_j,r_j)_{j\neq i}}((c_j)_{j\neq i}),
\end{aligned}$$

which implies (8.5). Here, the second equality follows from (8.3), whereas the fourth equality follows from (8.6). But then, we conclude from (8.5) and (8.7) that $b_i^C(t_i) \in B_i^k(v_i, r_i)$.

Remember from above that c_i is optimal for t_i among choices in $C_i(v_i)$. Hence, c_i is optimal for the marginal belief $b_i^C(t_i) \in B_i^k(v_i, r_i)$ among choices in $C_i(v_i)$, which implies that $c_i \in C_i^k(v_i, r_i)$. As this holds for every $c_i \in C_i^{cbr}(v_i, r_i)$, we conclude that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$. By induction on k, we may then conclude that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every k.

Now, take some choice c_i that can rationally be chosen under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i. Then, by definition, $c_i \in C_i^{cbr}(v_i, r_i)$. By the conclusion above that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every k, it follows that $c_i \in C_i^k(v_i, r_i)$ for every k. Hence, (c_i, v_i, r_i) survives the procedure. This completes the "only if" direction.

"If": For every player i, every view $v_i \in V_i$, and every view-type $r_i \in R_i$ feasible for v_i, let $C_i^\infty(v_i, r_i) := \cap_{k\geq 0} C_i^k(v_i, r_i)$ be the set of choices that survive the procedure for view v_i and view-type r_i, and let $B_i^\infty(v_i, r_i) := \cap_{k\geq 0} B_i^k(v_i, r_i)$ be the set of beliefs that survive the procedure at v_i and r_i. By Theorem 5.1 we know that all these sets $C_i^\infty(v_i, r_i)$ and $B_i^\infty(v_i, r_i)$ are non-empty. We show that every choice in $C_i^\infty(v_i, r_i)$ can rationally be made under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i.

By construction, every choice $c_i \in C_i^\infty(v_i, r_i)$ is optimal for some belief $\beta_i^{c_i,v_i,r_i} \in B_i^\infty(v_i, r_i)$ among choices in $C_i^\infty(v_i, r_i)$. We will show that, in fact, c_i is optimal for $\beta_i^{c_i,v_i,r_i}$ among choices in

$C_i(v_i)$. Let $c_i^* \in C_i(v_i)$ be optimal for $\beta_i^{c_i,v_i,r_i}$ among choices in $C_i(v_i)$. Then, c_i^* is in $C_i^\infty(v_i, r_i)$. As c_i is optimal for $\beta_i^{c_i,v_i,r_i}$ among choices in $C_i^\infty(v_i, r_i)$, it follows that

$$\sum_{c_{-i} \in C_{-i}(v_i)} \beta_i^{c_i,v_i,r_i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}(v_i)} \beta_i^{c_i,v_i,r_i}(c_{-i}) \cdot u_i(c_i^*, c_{-i}).$$

As c_i^* is optimal for $\beta_i^{c_i,v_i,r_i}$ among choices in $C_i(v_i)$, it follows that c_i is optimal for $\beta_i^{c_i,v_i,r_i}$ among choices in $C_i(v_i)$ as well.

Moreover, since

$$\beta_i^{c_i,v_i,r_i} \in B_i^\infty(v_i, r_i) = \sum_{(v_j,r_j)_{j\neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j\neq i}) \cdot \Delta(\times_{j\neq i} C_j^\infty(v_j, r_j)),$$

there is for every $(v_j, r_j)_{j\neq i} \in V_{-i} \times R_{-i}$ that receives positive probability under $p_i(r_i)$, some belief $\gamma_i^{c_i,v_i,r_i}[(v_j, r_j)_{j\neq i}] \in \Delta(\times_{j\neq i} C_j^\infty(v_j, r_j))$ such that

$$\beta_i^{c_i,v_i,r_i} = \sum_{(v_j,r_j)_{j\neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j\neq i}) \cdot \gamma_i^{c_i,v_i,r_i}[(v_j, r_j)_{j\neq i}]. \tag{8.8}$$

On the basis of these beliefs $\beta_i^{c_i,v_i,r_i}$ we now construct the following epistemic model $M = (T_i, b_i)_{i\in I}$. Let the set of types for every player i be given by

$$T_i = \{t_i^{c_i,v_i,r_i} \mid v_i \in V_i,\ r_i \in R_i \text{ feasible for } v_i \text{ and } c_i \in C_i^\infty(v_i, r_i)\}.$$

Moreover, for every player i and every type $t_i^{c_i,v_i,r_i} \in T_i$, let the belief $b_i(t_i^{c_i,v_i,r_i})$ on $C_{-i} \times V_{-i} \times T_{-i}$ be given by

$$b_i(t_i^{c_i,v_i,r_i})((c_j, v_j, t_j)_{j\neq i}) := \begin{cases} \begin{array}{c} p_i(r_i)((v_j, r_j)_{j\neq i}) \cdot \\ \cdot \gamma_i^{c_i,v_i,r_i}[(v_j, r_j)_{j\neq i}]((c_j)_{j\neq i}) \end{array}, & \text{if } t_j = t_j^{c_j,v_j,r_j} \text{ for all } j \neq i, \\ 0, & \text{otherwise} \end{cases}. \tag{8.9}$$

Suppose that type $t_i^{c_i,v_i,r_i}$ assigns positive probability to some combination $(c_j, v_j, t_j^{c_j,v_j,r_j})_{j\neq i}$. Then, by (8.9), the belief $\gamma_i^{c_i,v_i,r_i}[(v_j, r_j)_{j\neq i}]$ assigns positive probability to the choice-combination $(c_j)_{j\neq i}$. Since $\gamma_i^{c_i,v_i,r_i}[(v_j, r_j)_{j\neq i}] \in \Delta(\times_{j\neq i} C_j^\infty(v_j, r_j))$, it follows that $c_j \in C_j^\infty(v_j, r_j)$, and hence $c_j \in C_j(v_j)$, for every $j \neq i$. Therefore, condition (a) in Definition 3.1 is satisfied. Moreover, by (8.9) it must be the case that $p_i(r_i)$ assigns positive probability to the combination $(v_j, r_j)_{j\neq i}$. As r_i is feasible for v_i, it follows that v_j is contained in v_i for every opponent $j \neq i$, which makes sure that condition (b) in Definition 3.1 is satisfied. Hence, we conclude that $M = (T_i, b_i)_{i\in I}$ so constructed is a well-defined epistemic model.

We next show that every type $t_i^{c_i,v_i,r_i}$ holds the belief $\beta_i^{c_i,v_i,r_i}$ about the opponents' choices. Let $b_i^C(t_i^{c_i,v_i,r_i})$ be the marginal belief of type $t_i^{c_i,v_i,r_i}$ on C_{-i}. Then, for every $(c_j)_{j\neq i} \in C_{-i}$ we have that

$$
\begin{aligned}
b_i^C(t_i^{c_i,v_i,r_i})((c_j)_{j\neq i}) &= \sum_{(v_j,t_j)_{j\neq i}\in V_{-i}\times T_{-i}} b_i(t_i^{c_i,v_i,r_i})((c_j,v_j,t_j)_{j\neq i}) \\
&= \sum_{(v_j,r_j)_{j\neq i}\in V_{-i}\times R_{-i}} b_i(t_i^{c_i,v_i,r_i})((c_j,v_j,t_j^{c_j,v_j,r_j})_{j\neq i}) \\
&= \sum_{(v_j,r_j)_{j\neq i}\in V_{-i}\times R_{-i}} p_i(r_i)((v_j,r_j)_{j\neq i})\cdot \gamma_i^{c_i,v_i,r_i}[(v_j,r_j)_{j\neq i}]((c_j)_{j\neq i}) \\
&= \beta_i^{c_i,v_i,r_i}((c_j)_{j\neq i}),
\end{aligned}
$$

where the second and third equality follow from (8.9), and the last equality follows from (8.8). Hence, we conclude that $t_i^{c_i,v_i,r_i}$ holds the belief $\beta_i^{c_i,v_i,r_i}$ about the opponents' choices.

Note that, by construction, $c_i \in C_i^\infty(v_i,r_i)$ for every type $t_i^{c_i,v_i,r_i} \in T_i$. Since we have seen above that c_i is optimal for $\beta_i^{c_i,v_i,r_i}$ among choices in $C_i(v_i)$, and that $t_i^{c_i,v_i,r_i}$ holds the belief $\beta_i^{c_i,v_i,r_i}$ about the opponents' choices, it follows that c_i is optimal for $t_i^{c_i,v_i,r_i}$ among choices in $C_i(v_i)$. We use this to show that every type $t_i^{c_i,v_i,r_i}$ believes in the opponents' rationality. Suppose that $b_i(t_i^{c_i,v_i,r_i})((c_j,v_j,t_j^{c_j,v_j,r_j})_{j\neq i}) > 0$. Then, as we have just seen, c_j is optimal for type $t_j^{c_j,v_j,r_j}$ among choices in $C_j(v_j)$, and hence $t_i^{c_i,v_i,r_i}$ indeed believes in the opponents' rationality. As this holds for all types in the epistemic model M, we conclude that all types in M express *common* belief in rationality.

We finally show that every type $t_i^{c_i,v_i,r_i}$ has the belief hierarchy on views induced by the view-type r_i. For every $(v_j,r_j)_{j\neq i} \in V_{-i}\times R_{-i}$ we have that

$$
\begin{aligned}
\sum_{(c_j)_{j\neq i}\in C_{-i}} b_i(t_i^{c_i,v_i,r_i})((c_j,v_j,t_j^{c_j,v_j,r_j})_{j\neq i}) &= \sum_{(c_j)_{j\neq i}\in C_{-i}} p_i(r_i)((v_j,r_j)_{j\neq i})\cdot \\
&\quad \cdot\gamma_i^{c_i,v_i,r_i}[(v_j,r_j)_{j\neq i}]((c_j)_{j\neq i}) \\
&= p_i(r_i)((v_j,r_j)_{j\neq i})\cdot \\
&\quad \cdot \sum_{(c_j)_{j\neq i}\in C_{-i}} \gamma_i^{c_i,v_i,r_i}[(v_j,r_j)_{j\neq i}]((c_j)_{j\neq i}) \\
&= p_i(r_i)((v_j,r_j)_{j\neq i}), \qquad (8.10)
\end{aligned}
$$

where the first equality follows from (8.9), and the last equality follows from the fact that $\gamma_i^{c_i,v_i,r_i}[(v_j,r_j)_{j\neq i}]$ is a probability distribution on C_{-i}, and hence

$$\sum_{(c_j)_{j\neq i}\in C_{-i}} \gamma_i^{c_i,v_i,r_i}[(v_j,r_j)_{j\neq i}]((c_j)_{j\neq i}) = 1.$$

Equation (8.10) thus states that the probability that type $t_i^{c_i,v_i,r_i}$ assigns to the set of tuples $\{(v_j,t_j^{c_j,v_j,r_j})_{j\neq i} \mid (c_j)_{j\neq i} \in C_{-i}\}$ is the same as the probability that view-type r_i assigns to the tuple $(v_j, r_j)_{j\neq i}$. Since this holds for every type $t_i^{c_i,v_i,r_i}$ in the epistemic model M, we conclude that every type $t_i^{c_i,v_i,r_i}$ in M has the belief hierarchy on views induced by r_i. That is, $h_i^{view}(t_i^{c_i,v_i,r_i}) = h_i^{view}(r_i)$ for every type $t_i^{c_i,v_i,r_i}$ in M.

Take now some player i, and some triple (c_i, v_i, r_i) that survives the procedure. Then, $c_i \in C_i^\infty(v_i, r_i)$. Consider the type $t_i^{c_i,v_i,r_i}$ in the epistemic model constructed above. We first show that the type $t_i^{c_i,v_i,r_i}$ is feasible for the view v_i. Suppose that $b_i(t_i^{c_i,v_i,r_i})$ assigns positive probability to an opponent's view v_j. Then, by (8.9), $p_i(r_i)$ assigns positive probability to v_j. Since r_i is feasible for the view v_i, it must be the case that v_j is contained in v_i. Hence, type $t_i^{c_i,v_i,r_i}$ is feasible for the view v_i.

We have seen above that c_i is optimal for the type $t_i^{c_i,v_i,r_i}$ among choices in $C_i(v_i)$, that type $t_i^{c_i,v_i,r_i}$ expresses common belief in rationality, and that $h_i^{view}(t_i^{c_i,v_i,r_i}) = h_i^{view}(r_i)$. It thus follows that c_i can rationally be chosen under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i. This completes the proof. ■

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