

EPICENTER Spring Course on Epistemic Game Theory

Chapter 8: Belief in the Opponents' Future Rationality

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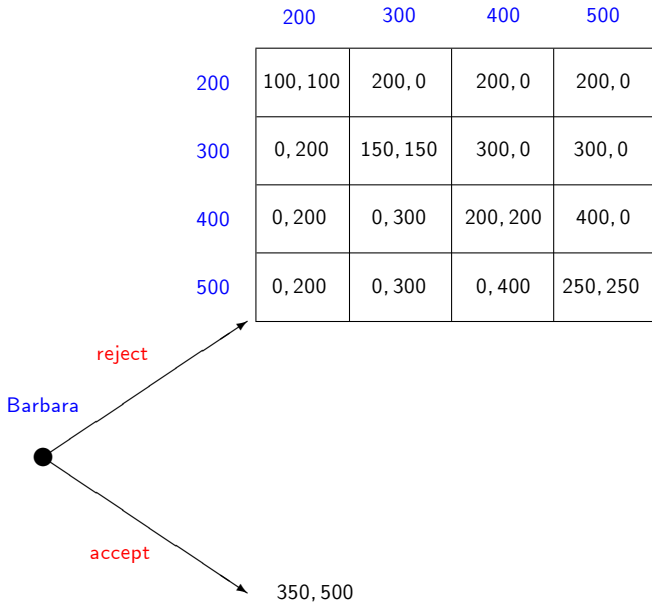
June 2017

- In a **dynamic game**, players may choose **one after the other**.
- Before you make a choice, you may (partially) **observe** what your opponents have chosen so far.
- It may happen that your **initial belief** about the opponents' choices will be **contradicted** later on.
- Then you must **revise** your belief about the opponents' choices. But **how?**
- There may be **several** plausible ways to revise your belief.

Example: Painting Chris' house

Story

- Chris is planning to **paint** his house tomorrow, and needs someone to **help** him.
- You and Barbara are both interested. This evening, both of you must come to Chris' house, and whisper a **price** in his ear. Price must be either **200, 300, 400** or **500 euros**.
- Person with **lowest price** will get the job. In case of a **tie**, Chris will toss a coin.
- Before you leave for Chris' house, Barbara gets a **phone call** from a colleague, who asks her to repair his car tomorrow at a price of **350 euros**.
- Barbara must decide whether or not to **accept** the colleague's offer.



	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250

Barbara



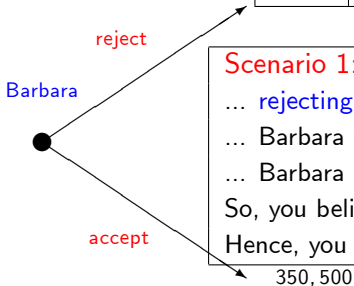
reject

Initially, you believe that Barbara **accepts** the offer.
 What if you observe that she has **rejected** the offer?
 Then, you must **revise** your belief.
 But **how**?

accept

350, 500

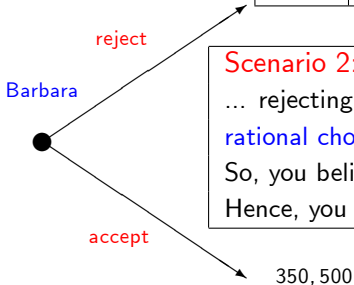
	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Scenario 1: You believe that ...
 ... rejecting offer was a **mistake** by Barbara,
 ... Barbara **will choose rationally** from now on
 ... Barbara believes that **you** choose **rationally**.
 So, you believe that Barbara chooses **200** or **300**.
 Hence, you will choose price **200**.

350, 500

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Scenario 2: You believe that ...
 ... rejecting colleague's offer was a
 rational choice for Barbara.
 So, you believe that Barbara chooses price 400.
 Hence, you will choose price 300.

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250

Barbara

reject



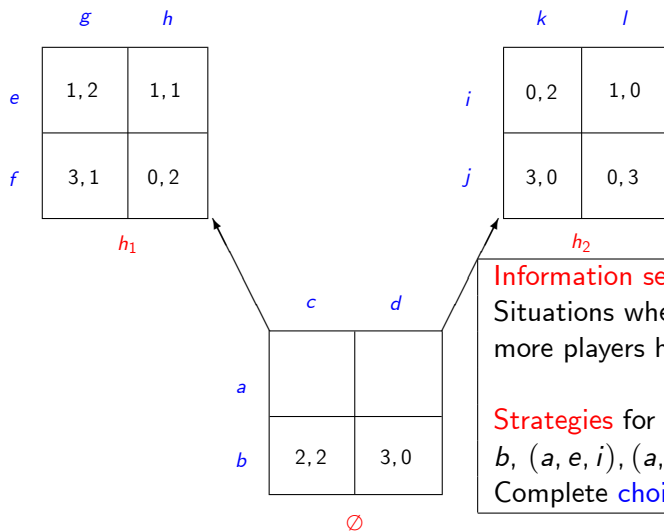
accept

So, your **choice** crucially depends on **how** you **revise your belief** about Barbara.

Both ways of revising your belief seem **plausible**.

350, 500

Dynamic games



Information sets: \emptyset , h_1 and h_2 .

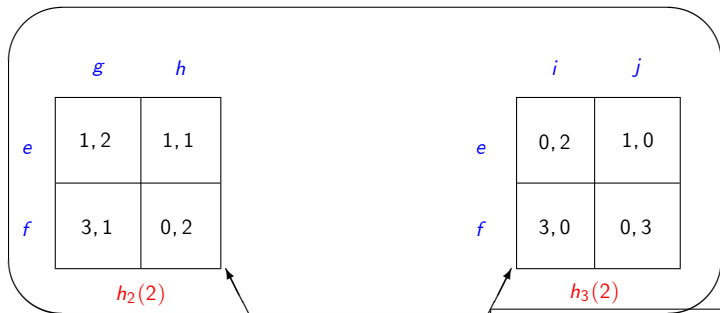
Situations where one or more players have to **choose**.

Strategies for player 1:

b , (a, e, i) , (a, e, j) , (a, f, i) , (a, f, j)

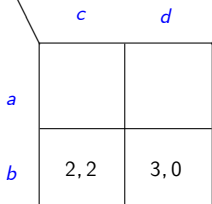
Complete **choice plan**.

$h_1(1)$



$h_2(2)$

$h_3(2)$



$\emptyset(1+2)$

Player 1 does **not** observe pl. 2's past choice.

Information sets for pl.1:
 \emptyset and h_1 .

Strategies for player 1:
 $b, (a, e), (a, f)$

- An **information set** for player i
- is a situation where player i must make a **choice**,
- describes the **information** that player i has about the opponents' **past choices**.
- H_i : collection of **information sets** for player i .
- $C_i(h)$: set of **available choices** at information set h .
- At an information set h , **more than one** player can make a choice.

Definition (Strategy)

A **strategy** for player i is a function s_i that assigns to each of his information sets $h \in H_i$ some **available choice** $s_i(h)$, **unless** h cannot be **reached** due to some choice $s_i(h')$ at an earlier information set $h' \in H_i$.

In the latter case, **no choice** needs to be specified at h .

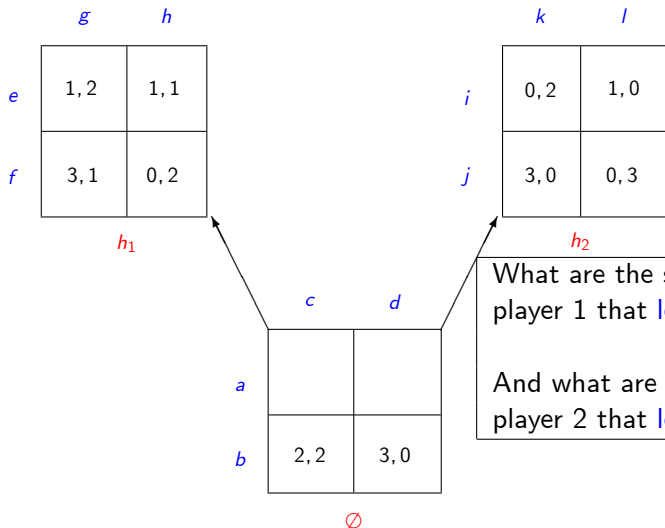
- This is **different** from the **classical** definition of a strategy!

Conditional beliefs

- In a **dynamic game**, a player holds at **each** of his information sets a **conditional belief** about the opponents' strategy choices.
- $S_{-i}(h)$: set of **opponents' strategy combinations** that lead to an information set $h \in H_i$.

Definition (Conditional belief)

A **conditional belief vector** b_i for player i about the opponents' strategies assigns to **every** information set $h \in H_i$ some **probability distribution** $b_i(h) \in \Delta(S_{-i}(h))$ on the **opponents' strategy combinations** that lead to h .



What are the strategies for player 1 that lead to h_1 ?

And what are the strategies for player 2 that lead to h_1 ?

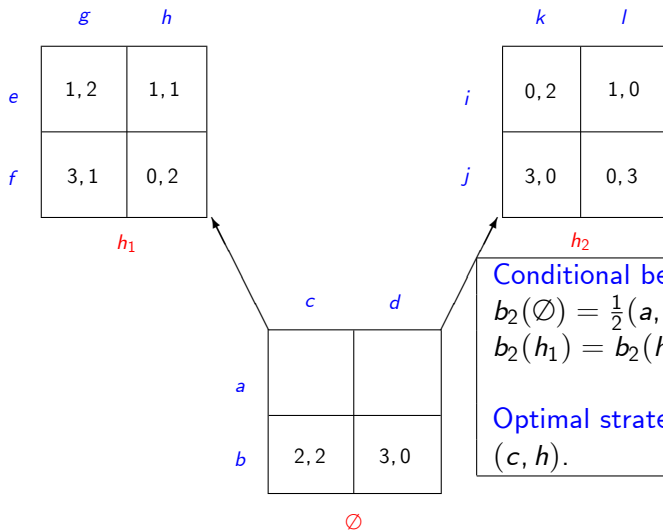
- A strategy s_i is **optimal** at an information set $h \in H_i$ that s_i leads to, for the belief $b_i(h)$, if

$$u_i(s_i, b_i(h)) \geq u_i(s'_i, b_i(h))$$

for all strategies s'_i that lead to h .

Definition (Optimal strategy)

A strategy s_i is **optimal** for the conditional belief vector b_i , if at every information set $h \in H_i$ that s_i leads to, the strategy s_i is **optimal** for the belief $b_i(h)$.



Conditional belief for pl. 2:

$$b_2(\emptyset) = \frac{1}{2}(a, f, j) + \frac{1}{2}b$$

$$b_2(h_1) = b_2(h_2) = (a, f, j).$$

Optimal strategy for pl. 2:

(c, h) .

- We would like to model **hierarchies** of **conditional beliefs**.
- That is, we want to model
- the **conditional belief** that player i has, at every information set $h \in H_i$, about his opponents' **strategy choices**,
- the **conditional belief** that player i has, at every information set $h \in H_i$, about the **conditional belief** that opponent j has, at every information set $h' \in H_j$, about the **opponents' strategy choices**,
- and so on.

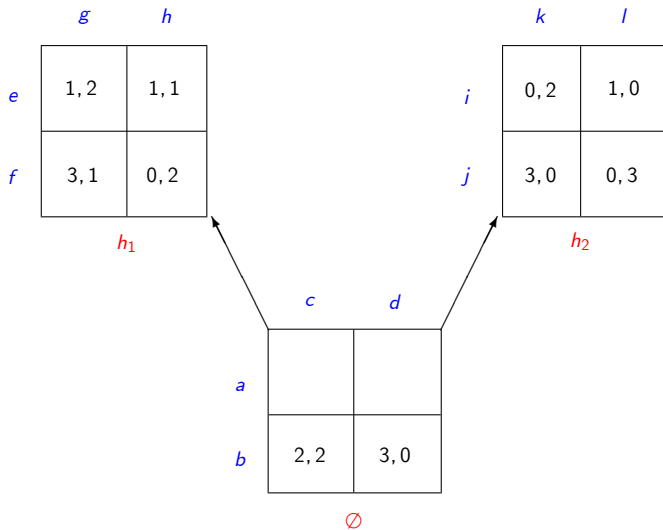
- Hence, in a **conditional belief hierarchy** you hold, at **each** of your **information sets**, a **conditional belief** about
 - the opponents' **strategy choices**, and
 - the opponents' **conditional belief hierarchies**.
- Like before, call a **(conditional) belief hierarchy** a **type**.
- Then, a **type** for you holds, at **each** of your **information sets**, a **conditional belief** about
 - the opponents' **strategy choices**, and
 - the opponents' **types**.
- This leads to an **epistemic model**.

Definition (Epistemic model)

An **epistemic model** for a dynamic game specifies for every player i a set T_i of possible **types**.

Moreover, every type t_i for player i specifies at every information set $h \in H_i$ a **probabilistic belief** $b_i(t_i, h)$ over the set $S_{-i}(h) \times T_{-i}$ of opponents' **strategy-type combinations**.

- Here, $b_i(t_i, h)$ represents the **conditional belief** that type t_i holds at information set $h \in H_i$ about the opponents' **strategy-type combinations**.
- From the epistemic model, we can **deduce** the **complete belief hierarchy** for every type.
- A type may **revise his belief** about the opponents' **strategies** during the game.
- A type may also **revise his beliefs** about the opponents' **beliefs** during the game.



Types

$$T_1 = \{t_1, \hat{t}_1\}, T_2 = \{t_2, \hat{t}_2\}$$

Beliefs for
player 1

$$b_1(t_1, \emptyset) = ((c, h), t_2)$$

$$b_1(t_1, h_1) = ((c, h), t_2)$$

$$b_1(t_1, h_2) = ((d, k), \hat{t}_2)$$

$$b_1(\hat{t}_1, \emptyset) = (0.3) \cdot ((c, g), t_2) + (0.7) \cdot ((d, l), \hat{t}_2)$$

$$b_1(\hat{t}_1, h_1) = ((c, g), t_2)$$

$$b_1(\hat{t}_1, h_2) = ((d, l), \hat{t}_2)$$

Beliefs for
player 2

$$b_2(t_2, \emptyset) = (b, t_1)$$

$$b_2(t_2, h_1) = ((a, f, i), t_1)$$

$$b_2(t_2, h_2) = ((a, f, i), t_1)$$

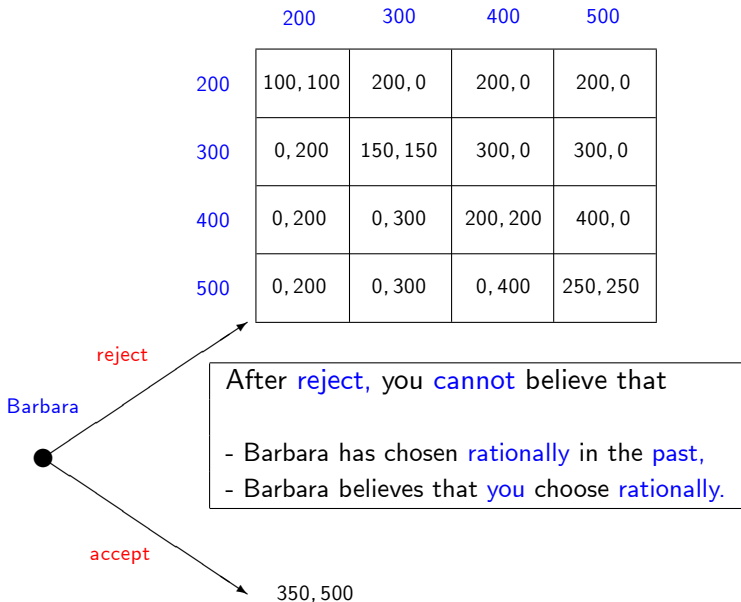
$$b_2(\hat{t}_2, \emptyset) = ((a, e, j), \hat{t}_1)$$

$$b_2(\hat{t}_2, h_1) = ((a, e, j), \hat{t}_1)$$

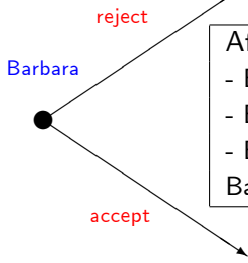
$$b_2(\hat{t}_2, h_2) = ((a, e, j), \hat{t}_1)$$

Common belief in future rationality

- We would like to extend the idea of **common belief in rationality** to **dynamic games**.
- **Problem:** At certain information sets, it may **not** be possible to believe that
 - opponent has chosen **rationally** in the **past**, or
 - opponent has chosen **rationally** in the **past**, and that the opponent believes that **you** choose **rationally**.
- Hence, **common belief in rationality at all information sets** is in general **not possible**.
- We must therefore look for a **weaker** definition of **common belief in rationality**.



	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



After **reject**, you **can** believe that

- Barbara will choose **rationally** in the **future**,
- Barbara believes that you will choose **rationally**,
- Barbara believes that you believe that Barbara will choose **rationally** in the **future**, etc.

Common belief in future rationality.

- You **believe in the opponents' future rationality** if you **always** believe that your opponents will make optimal choices at every **present** and **future** information set.

Definition (Belief in the opponents' rationality)

Type t_i believes at h that opponent j chooses **rationally at h'** if his conditional belief $b_i(t_i, h)$ only assigns **positive probability** to strategy-type pairs (s_j, t_j) for player j where strategy s_j is **optimal** for type t_j **at information set h'** .

Definition (Belief in the opponents' future rationality)

Type t_i believes at h in opponent j 's **future rationality** if t_i believes at h that j chooses rationally at **every** information set h' for player j that **weakly follows** h .

Type t_i **believes in the opponents' future rationality** if t_i believes, at **every** information set h for player i , in **every** opponent's future rationality.

- **Common belief in future rationality** means that you **always** believe that
- your opponents will choose **rationally now and in the future**,
- your opponents always believe that their opponents will choose **rationally now and in the future**,
- and so on.

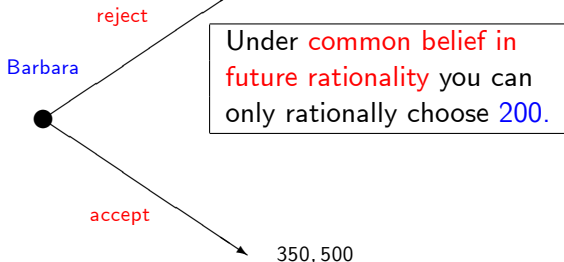
Definition (Common belief in future rationality)

- (1) Type t_i expresses **1-fold belief in future rationality** if t_i believes in the opponents' **future** rationality.
- (2) Type t_i expresses **2-fold belief in future rationality** if t_i assigns, at every information set $h \in H_i$, only **positive probability** to opponents' types that express **1-fold belief in future rationality**.

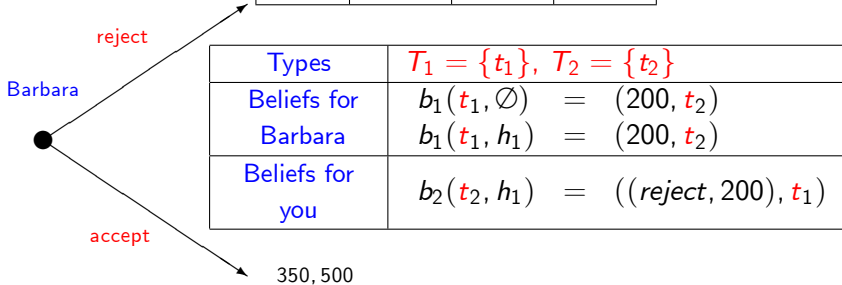
And so on.

Type t_i expresses **common belief in future rationality** if t_i expresses **k -fold belief** in future rationality for **every** k .

	200	300	400	500
200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



200	100, 100	200, 0	200, 0	200, 0
300	0, 200	150, 150	300, 0	300, 0
400	0, 200	0, 300	200, 200	400, 0
500	0, 200	0, 300	0, 400	250, 250



Both types express **common belief in future rationality**.

- We wish to find those **strategies** that you can rationally choose under **common belief in future rationality**.
- Can we construct an **algorithm** that helps us find these strategies?
- Yes! It will proceed by **iterately removing strategies** at the various **information sets** in the game.

Step 1: 1-fold belief in future rationality.

- Which strategies can player i rationally choose if he expresses 1-fold belief in future rationality? That is, if he believes in the opponents' future rationality?
- Consider a type t_i that believes in the opponents' future rationality. Then, t_i believes at every information set $h \in H_i$ that opponent j chooses optimally at every information set $h' \in H_j$ that weakly follows h .
- A strategy s_j for player j is optimal at h' for some conditional belief at h' , if and only if, s_j is not strictly dominated within the full decision problem $\Gamma^0(h') = (S_j(h'), S_{-j}(h'))$ at h' .
- So, t_i assigns at h only positive probability to j 's strategies s_j that are not strictly dominated within any full decision problem $\Gamma^0(h')$ that weakly follows h , and at which j is active.

Step 1: 1-fold belief in future rationality.

- So, t_i assigns at h only positive probability to j 's strategies s_j that are not strictly dominated within any full decision problem $\Gamma^0(h')$ that weakly follows h , and at which j is active.
- At every information set $h \in H_i$, delete from the full decision problem $\Gamma^0(h)$ those strategies s_j that are strictly dominated within some full decision problem $\Gamma^0(h')$ that weakly follows h , and at which j is active. This gives the reduced decision problem $\Gamma^1(h)$.
- Hence, type t_i assigns at every information set $h \in H_i$ only positive probability to opponents' strategies in $\Gamma^1(h)$.
- So, every strategy that is optimal for t_i at h , must not be strictly dominated within the reduced decision problem $\Gamma^1(h)$.

Step 1: 1-fold belief in future rationality.

- So, every strategy that is optimal for t_i at h , must **not** be **strictly dominated** within the **reduced decision problem** $\Gamma^1(h)$.
- Let $\Gamma^2(\emptyset)$ be **reduced decision problem** at \emptyset which is obtained by **eliminating**, for every player i , those strategies that are **strictly dominated** within some **reduced decision problem** $\Gamma^1(h)$ at which i is active.
- **Conclusion:** Every strategy s_i that is **optimal** for some type t_i which expresses **1-fold belief in future rationality**, must be in $\Gamma^2(\emptyset)$.

Step 2: Up to 2-fold belief in future rationality.

- Which strategies can player i rationally choose if he expresses up to 2-fold belief in future rationality?
- Consider a type t_i that expresses up to 2-fold belief in future rationality. Then, t_i assigns at every $h \in H_i$ only positive probability to opponents' strategy-type pairs (s_j, t_j) where s_j is optimal for t_j at every $h' \in H_j$ that weakly follows h , and t_j expresses 1-fold belief in future rationality.
- We know from Step 1 that every such type t_j assigns at every $h' \in H_j$ only positive probability to opponents' strategies in $\Gamma^1(h')$.
- So, every such strategy s_j above must at every $h' \in H_j$ weakly following h not be strictly dominated within $\Gamma^1(h')$.

Step 2: Up to 2-fold belief in future rationality.

- So, every such strategy s_j above must at every $h' \in H_j$ weakly following h not be strictly dominated within $\Gamma^1(h')$.
- Let $\Gamma^2(h)$ be the reduced decision problem at h which is obtained from $\Gamma^1(h)$ by removing all strategies s_j which are strictly dominated within some $\Gamma^1(h')$ weakly following h , at which j is active.
- Then, type t_i will assign at h only positive probability to strategies s_j in $\Gamma^2(h)$.
- So, every strategy s_j which is optimal for t_i at h must not be strictly dominated within $\Gamma^2(h)$.

Step 2: Up to 2-fold belief in future rationality.

- So, every strategy s_i which is optimal for t_i at h must not be strictly dominated within $\Gamma^2(h)$.
- Let $\Gamma^3(\emptyset)$ be reduced decision problem at \emptyset which is obtained by eliminating, for every player i , those strategies that are strictly dominated within some reduced decision problem $\Gamma^2(h)$ at which i is active.
- **Conclusion:** Every strategy s_i that is optimal for some type t_i which expresses up to 2-fold belief in future rationality, must be in $\Gamma^3(\emptyset)$.

- Fix an **information set** h for player i .
- The **full decision problem** for player i at h is $\Gamma^0(h) = (S_i(h), S_{-i}(h))$, where $S_i(h)$ is the set of strategies for player i that lead to h , and $S_{-i}(h)$ is the set of opponents' strategy combinations that lead to h .
- A **reduced decision problem** for player i at h is $\Gamma(h) = (D_i(h), D_{-i}(h))$, where $D_i(h) \subseteq S_i(h)$ and $D_{-i}(h) \subseteq S_{-i}(h)$.

Algorithm (Backward dominance procedure)

Step 1. At every *full decision problem* $\Gamma^0(h)$, *eliminate* for every player i those strategies that are *strictly dominated* at some *full decision problem* $\Gamma^0(h')$ that *weakly follows* h and at which player i is *active*. This leads to *reduced decision problems* $\Gamma^1(h)$ at every information set h .

Step 2. At every *reduced decision problem* $\Gamma^1(h)$, *eliminate* for every player i those strategies that are *strictly dominated* at some *reduced decision problem* $\Gamma^1(h')$ that *weakly follows* h and at which player i is *active*. This leads to new *reduced decision problems* $\Gamma^2(h)$ at every information set.

And so on. Continue until no more strategies can be eliminated in this way.

- The algorithm always stops within *finitely many steps*.
- At every information set, it yields a *nonempty set of strategies* for every player.
- The *order* in which we eliminate strategies – including the order in which we walk through the information sets – is *not important* for the final result!

Theorem (Algorithm “works”)

(1) For every $k \geq 1$, the *strategies* that can rationally be chosen by a type that expresses *up to k -fold belief in future rationality* are exactly the strategies that survive the *first $k + 1$ steps* of the *backward dominance procedure* at \emptyset .

(2) The *strategies* that can rationally be chosen by a type that expresses *common belief in future rationality* are exactly the strategies that survive the *full backward dominance procedure* at \emptyset .

- A strategy survives the *first $k + 1$ steps* of the *backward dominance procedure* at \emptyset if it is in the reduced decision problem $\Gamma^{k+1}(\emptyset)$.
- A strategy survives the *full backward dominance procedure* at \emptyset if it is in the reduced decision problem $\Gamma^k(\emptyset)$ for every k .

$\Gamma^0(h_1)$ 200 300 400 500

$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250

B

reject

accept

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
<i>accept</i>	350, 500	350, 500	350, 500	350, 500

350, 500

Step 1

$\Gamma^0(h_1)$ 200 300 400 500

$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250

B

reject

accept

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

350, 500

Step 1

$\Gamma^0(h_1)$ 200 300 400 500

$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250

B

reject

accept

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
<i>accept</i>	350, 500	350, 500	350, 500	350, 500

350, 500

Step 1

$\Gamma^0(h_1)$ 200 300 400 500

$(r, 200)$

100, 100

200, 0

200, 0

200, 0

$(r, 300)$

0, 200

150, 150

300, 0

300, 0

$(r, 400)$

0, 200

0, 300

200, 200

400, 0

$(r, 500)$

0, 200

0, 300

0, 400

250, 250

B

reject

accept

$\Gamma^0(\emptyset)$

200

300

400

500

$(r, 400)$

0, 200

0, 300

200, 200

400, 0

accept

350, 500

350, 500

350, 500

350, 500

350, 500

Step 1

$\Gamma^0(h_1)$ 200 300 400 500

$(r, 200)$

100, 100

200, 0

200, 0

200, 0

$(r, 300)$

0, 200

150, 150

300, 0

300, 0

$(r, 400)$

0, 200

0, 300

200, 200

400, 0

reject

B

●

accept

$\Gamma^0(\emptyset)$

200

300

400

500

$(r, 400)$

0, 200

0, 300

200, 200

400, 0

accept

350, 500

350, 500

350, 500

350, 500

350, 500

Step 1

$\Gamma^0(h_1)$ 200 300 400

$(r, 200)$	100, 100	200, 0	200, 0	
$(r, 300)$	0, 200	150, 150	300, 0	
$(r, 400)$	0, 200	0, 300	200, 200	

B

reject

accept

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
<i>accept</i>	350, 500	350, 500	350, 500	350, 500

350, 500

Step 1

$\Gamma^1(h_1)$ 200 300 400

$(r, 200)$

100, 100	200, 0	200, 0	
0, 200	150, 150	300, 0	
0, 200	0, 300	200, 200	

$(r, 300)$

$(r, 400)$

reject

B

$\Gamma^1(\emptyset)$	200	300	400
$(r, 400)$	0, 200	0, 300	200, 200
<i>accept</i>	350, 500	350, 500	350, 500

accept

350, 500

End of Step 1

$\Gamma^1(h_1)$ 200 300 400

$(r, 200)$

100, 100

200, 0

200, 0

$(r, 300)$

0, 200

150, 150

300, 0

$(r, 400)$

0, 200

0, 300

200, 200

reject

B



accept

$\Gamma^1(\emptyset)$

200

300

400

accept

350, 500

350, 500

350, 500

350, 500

Step 2

$\Gamma^1(h_1)$ 200 300 400

$(r, 200)$

100, 100	200, 0	200, 0	
0, 200	150, 150	300, 0	

$(r, 300)$

B

reject



accept

$\Gamma^1(\emptyset)$	200	300	400
<i>accept</i>	350, 500	350, 500	350, 500

350, 500

Step 2

$\Gamma^1(h_1)$ 200 300

$(r, 200)$

100, 100

200, 0

$(r, 300)$

0, 200

150, 150

reject

B



accept

$\Gamma^1(\emptyset)$

200

300

400

accept

350, 500

350, 500

350, 500

350, 500

Step 2

$\Gamma^2(h_1)$ 200 300

$(r, 200)$

100, 100

200, 0

$(r, 300)$

0, 200

150, 150

reject

B

accept

$\Gamma^2(\emptyset)$

200

300

accept

350, 500

350, 500

350, 500

End of Step 2

$\Gamma^2(h_1)$ 200 300

$(r, 200)$

100, 100	200, 0		

reject

B

accept

$\Gamma^2(\emptyset)$	200	300
<i>accept</i>	350, 500	350, 500

350, 500

Step 3

$\Gamma^2(h_1)$ 200

$(r, 200)$

100, 100			

reject

B

accept

$\Gamma^2(\emptyset)$	200	300
<i>accept</i>	350, 500	350, 500

350, 500

Step 3

$\Gamma^3(h_1)$ 200

$(r, 200)$

100, 100			

B

reject

accept

$\Gamma^3(\emptyset)$

200

accept

350, 500

350, 500

End of algorithm

Backwards order of elimination

- When we use the **backward dominance procedure**, the **order** in which we **eliminate strategies** is **not important** for the eventual result.
- In particular, it does not matter in which order we walk through the **information sets** when eliminating strategies.
- In many games, there is a very **convenient** order of elimination: **backwards order of elimination**.
- First, consider the **ultimate** information sets, and apply **iterated elimination** of strategies there.
- Then, consider **penultimate** information sets, and apply **iterated elimination** of strategies there.
- And so on, until we reach the **beginning** of the game.

- The backwards order of elimination works whenever the game is with observed past choices.

Definition (Game with observed past choices)

A dynamic game is with **observed past choices** if at every information set, the active players **know** precisely the choices made by the opponents in the **past**.

- However, the backwards order of elimination may **not** be **possible** if there are **unobserved past choices** in the game!

$\Gamma^0(h_1)$ 200 300 400 500

$(r, 200)$

100, 100	200, 0	200, 0	200, 0
----------	--------	--------	--------

$(r, 300)$

0, 200	150, 150	300, 0	300, 0
--------	----------	--------	--------

$(r, 400)$

0, 200	0, 300	200, 200	400, 0
--------	--------	----------	--------

$(r, 500)$

0, 200	0, 300	0, 400	250, 250
--------	--------	--------	----------

reject

B

●

accept

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
<i>accept</i>	350, 500	350, 500	350, 500	350, 500

350, 500

First, do **iterated elimination** of strictly dominated strategies at h_1 .

$\Gamma^0(h_1)$ 200

$(r, 200)$

100, 100			

B

reject

accept

$\Gamma^0(\emptyset)$	200	300	400	500
$(r, 200)$	100, 100	200, 0	200, 0	200, 0
$(r, 300)$	0, 200	150, 150	300, 0	300, 0
$(r, 400)$	0, 200	0, 300	200, 200	400, 0
$(r, 500)$	0, 200	0, 300	0, 400	250, 250
accept	350, 500	350, 500	350, 500	350, 500

350, 500

Then, **eliminate** these strategies **also** at \emptyset .

$\Gamma^0(h_1)$ 200

$(r, 200)$

100, 100			

B

reject

accept

$\Gamma^0(\emptyset)$	200
$(r, 200)$	100, 100
<i>accept</i>	350, 500

350, 500

Finally, do **elimination** of strictly dominated strategies at \emptyset .

$\Gamma^0(h_1)$ 200

$(r, 200)$

100, 100			

B

reject



accept

$\Gamma^0(\emptyset)$	200
<i>accept</i>	350, 500

350, 500

End of algorithm

- For dynamic games with **perfect information**, the **backward dominance procedure** reduces to a very **simple** procedure called **backward induction**.

Definition (Game with perfect information)

A dynamic game is with **perfect information** if at every information set there is only **one active player**, and this player always **knows** exactly what choices have been made by his opponents in the **past**.

Example: The shrinking treasure

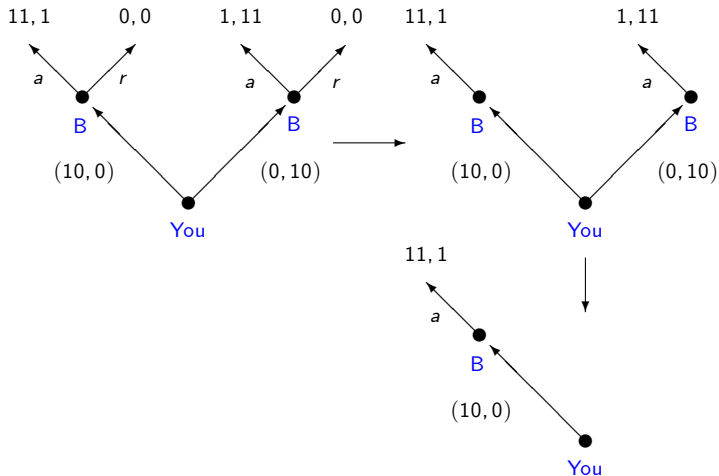
Story

- You, Barbara and Chris are on **holiday** in the Carribean.
- You and Barbara discover a **treasure** with **60 pieces** of gold. To divide the treasure, Chris proposes the following procedure.
- **Round 1:** Chris **takes away** 10 pieces. **You** must make a proposal to **Barbara** for how to divide the remaining **50 pieces** (with multiples of 10 pieces).
- **Barbara** can **accept** or **reject** your offer.
- If she **accepts**, you both get the amounts in the proposal, and a **bonus of 5 pieces** each from Chris for reaching an agreement so early.
- If Barbara **rejects**, we go to **Round 2**.

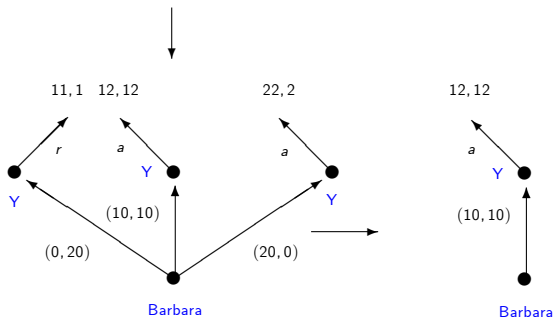
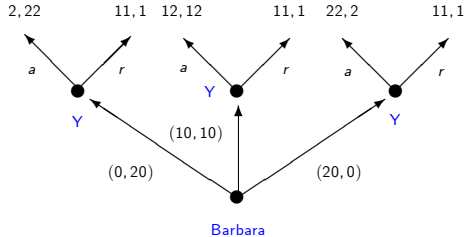
Story

- **Round 2:** Chris **takes away** another 10 pieces. **Barbara** must make a proposal to you for how to divide the remaining **40 pieces**.
- You can **accept** or **reject** the offer.
- If you **accept**, you both get the amounts in the proposal, and a **bonus of 4 pieces** each from Chris.
- If you **reject**, we go to **Round 3**.
- **Round 3:** Chris **takes away** another 10 pieces. **You** must make a proposal for how to divide the remaining **30 pieces**.
- And so on.

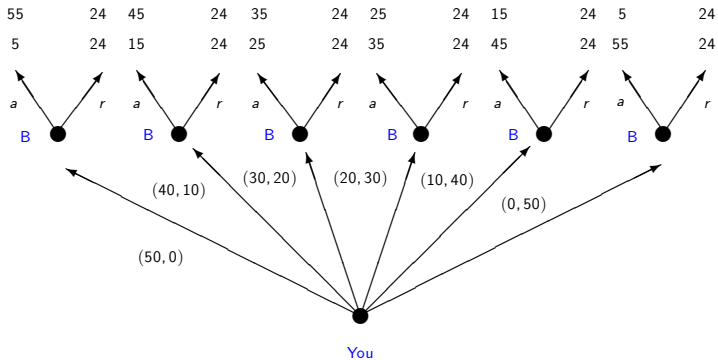
- Which strategies can you rationally choose under **common belief in future rationality**?
- This is a game with **perfect information**. In particular, it is a game with **observed past choices**.
- So, we can use the **backward dominance procedure** with the **backwards order of elimination**.



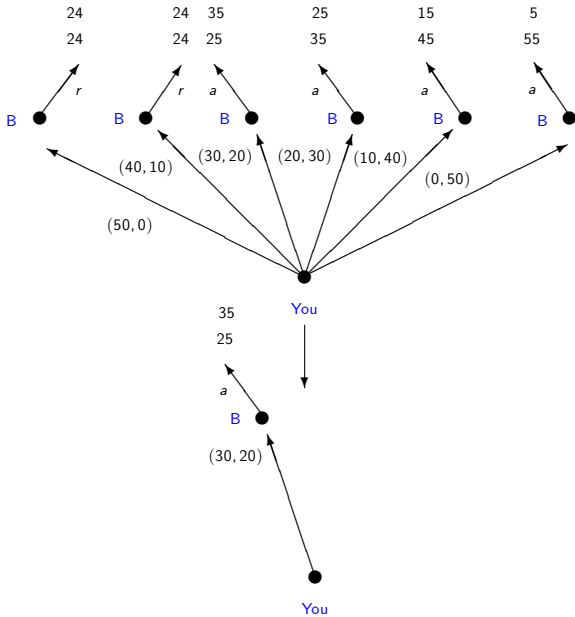
Round 5: 10 pieces left, bonus is 1, you make proposal



Round 4: 20 pieces left, bonus is 2, Barbara makes proposal



Round 1: 50 pieces left, bonus is 5, you make proposal



Round 1: 50 pieces left, bonus is 5, you make proposal

Definition (Backward induction procedure)

Consider a dynamic game with perfect information. At the beginning, we select at every **ultimate** information set all choices for the active player that are **optimal** at this information set. These are called the **backward induction choices** at the ultimate information sets.

We then select, at every **penultimate** information set, all choices for the active player that are **optimal** for some configuration of opponents' **backward induction choices** at the **ultimate** information sets. These are called the **backward induction choices** at the penultimate information sets.

And so on, until we reach the beginning of the game.

- A strategy is called a **backward induction strategy** if it consists of **backward induction choices**.
- For games with **perfect information**, the **backward dominance procedure** with the **backwards order of elimination** is equivalent to the **backward induction procedure**.

Theorem (Common belief in future rationality leads to backward induction)

Consider a dynamic game with *perfect information*.

Then, the strategies that can rationally be chosen under *common belief in future rationality* are exactly the *backward induction strategies*.

- If the game with perfect information is *generic* – that is, all utilities at the terminal histories are different – then there is a *unique* backward induction strategy for every player.
- In *non-generic* games with perfect information, there may be *more than one* backward induction strategy for a player.