Order Independence in Dynamic Games

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Abstract

In this paper we investigate the order independence of iterated reduction procedures in dynamic games. We distinguish between two types of order independence: with respect to strategies and with respect to outcomes. The first states that the specific order of elimination chosen should not affect the final set of strategy combinations, whereas the second states that it should not affect the final set of reachable outcomes in the game. We provide sufficient conditions for both types of order independence: monotonicity, and monotonicity on reachable histories, respectively.

We use these sufficient conditions to explore the order independence properties of various reduction procedures in dynamic games: the extensive-form rationalizability procedure (Pearce (1984), Battigalli (1997)), the backward dominance procedure (Perea (2014)) and Battigalli and Siniscalchi's (1999) procedure for jointly rational belief systems (Reny (1993)). We finally exploit these results to prove that every outcome that is reachable under the extensive-form rationalizability procedure is also reachable under the backward dominance procedure.

Keywords: Dynamic games, reduction operators, reduction procedures, elimination procedures, monotonicity, extensive-form rationalizability, backward dominance, jointly rational belief systems.

JEL Classification: C72, C73

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1 Introduction

Game theory is full of reduction procedures that recursively eliminate choices or strategies from a game. Examples in static games include the iterated elimination of strictly and weakly dominated choices, Pearce's (1984) definition of rationalizable choices in Definition 1, Bernheim's (1984) characterization of rationalizable choices in Proposition 3.2, and the Dekel-Fudenberg procedure (Dekel and Fudenberg (1990)) which proceeds by first eliminating all weakly dominated choices, and then continues with the iterated elimination of strictly dominated choices. In dynamic games we can think of the extensive-form rationalizability procedure (Pearce (1984) and Battigalli (1997)), the iterated conditional dominance procedure (Shimoji and Watson (1998)), the backward dominance procedure (Perea (2014)), the backwards rationalizability procedure (Penta (2016)) and Battigalli and Siniscalchi's (1999) procedure for jointly rational belief systems (Reny (1993)).

Some of these reduction procedures display the special property that the final outcome does not depend upon the order or speed by which we eliminate choices or strategies from the game. We refer to this property as *order independence.* This property is important both from a conceptual and computational point of view. On a conceptual level it reveals a form of invariance or robustness, which can be exploited for computational purposes by choosing an order of elimination which is convenient for the specific game at hand. Consider, for instance, the backward dominance procedure for dynamic games. A computationally efficient order of elimination for this procedure, especially for large games, turns out to be the *backwards order*, in which we start by eliminating strategies at the end of the game, after which we work our way backwards until reaching the beginning of the game. As the backward dominance procedure is order independent (see Perea (2014)), using this more convenient order of elimination does not affect the eventual output of the procedure.

In static games the notion of order independence is relatively well-understood, and has been investigated by Gilboa, Kalai and Zemel (1990), Apt (2004, 2011) and Luo, Qian and Qu (2016), among others. An important objective of each of these papers is to identify monotonicity conditions on the reduction operator that imply order independence. The weakest monotonicity condition among these is Luo, Qian and Qu's (2016) *1-monotonicity**, which states that if the set of choice combinations E is possible in some elimination order, and D is obtained from E by the elimination of some choices that can be eliminated from E, then the reduction of D must be contained in the reduction of E. Luo, Qian and Qu (2016) show that every reduction operator that satisfies 1-monotonicity* is guaranteed to be order independent. Since it can be shown that the iterated elimination of strictly dominated choices and Pearce's and Bernheim's procedures for rationalizability satisfy 1-monotonicity*, it follows that each of these procedures is order independent.

To the best of our knowledge, order independence in *dynamic* games has not been investigated yet on a systematic basis, and the goal of this paper is to fill that gap. We proceed in four steps. We start by introducing the general notion of a *reduction operator* for dynamic games, and show

how the extensive-form rationalizability procedure, the backward dominance procedure, and the procedure for jointly rational belief systems can be characterized by the iterated application of a specific reduction operator.

Secondly, we present some sufficient conditions for order independence. To that purpose, we distinguish between two types of order independence: order independence with respect to *strategies* and order independence with respect to *outcomes*. The first states that the set of *strategy combinations* obtained at the end does not depend upon the order or speed by which we eliminate strategies from the game, whereas the latter requires the set of *induced outcomes* to be independent of the order or speed of elimination. Clearly, order independence with respect to strategies implies order independence with respect to outcomes, but not *vice versa.* For both types of order independence we provide a sufficient condition, to which we refer as *monotonicity* and *monotonicity on reachable histories,* respectively. Monotonicity reduces to 1-monotonicity* if the game at hand is a static game, whereas monotonicity on reachable histories is an analogue to the condition with the same name in Perea (2016), where the condition is applied to a different type of reduction operator. We will say more about these relations below. These sufficient conditions are of great practical value when exploring order independence, since verifying these conditions is much more elementary than checking for order independence directly.

Thirdly, we use these sufficient conditions to verify whether the various reduction procedures listed above are order independent with respect to strategies or outcomes. More precisely, we show that the reduction operators underlying the backward dominance procedure and the procedure for jointly rational belief systems are monotone, whereas the reduction operator for the extensive-form rationalizability procedure is monotone on reachable histories. As a consequence, we conclude that the first two procedures are order independent with respect to strategies and the latter is order independent with respect to outcomes. This thus confirms Perea's (2014) result that the backward dominance procedure is order independent with respect to strategies. The last result, that the extensive-form rationalizability procedure is order independent with respect to outcomes, can also be found in Perea (2016) and is closely related to Chen and Micali (2013) who prove that the iterated conditional dominance procedure is order independent with respect to outcomes. In fact, Shimoji and Watson (1998) have shown that their iterated conditional dominance procedure characterizes extensive-form rationalizability, and hence our result above may be seen as a confirmation of Chen and Micali's theorem. By means of an example we show that the extensive-form rationalizability procedure is not order independent with respect to strategies.

We finally use the order independence properties of the backward dominance procedure and the extensive-form rationalizability procedure to derive a general relationship between both procedures. More precisely, we prove that every outcome induced by extensive-form rationalizable strategies is also induced by strategies surviving the backward dominance procedure. Since Battigalli and Siniscalchi (2002) have shown that the epistemic conditions of common strong belief in rationality (within a complete type structure) characterize extensive-form rationalizability, and Perea (2014) has proven that the backward dominance strategies can be characterized by common belief in future rationality, it follows that every outcome that is possible under common strong belief in rationality is also possible under common belief in future rationality. This is precisely the content of Theorem 9.4.2 in Perea (2012). From a reasoning perspective this result is rather intriguing, as extensive-form rationalizability is a typical *forward induction* concept whereas backward dominance displays a natural form of *backward induction* reasoning. As such, we show that in terms of outcomes, forward induction reasoning is more restrictive than backward induction reasoning, provided we identify forward and backward induction reasoning with the two concepts above. This result does not hold in terms of strategies, though.

In dynamic games with perfect information and without relevant ties, it is well-known that the backward dominance procedure leads to the unique backward induction strategies (see Perea (2014)), and hence to the unique backward induction outcome. In view of the result above, we may thus conclude that for such games the extensive-form rationalizability procedure must also uniquely lead to the backward induction outcome – a remarkable result first proven by Battigalli (1997).

To conclude, let us briefly compare our approach in this paper to that of static games. A reduction operator for static games is a relatively simple object, as it assigns to every product of choice sets – one choice set for every player – a product of reduced choice sets, obtained by eliminating some (or no) choices for each of the players. In dynamic games the situation becomes more complex, as for most reduction procedures the set of strategies that can be eliminated at some history crucially depends upon the available sets of strategies at *other* histories. To account for this interdependence, we must explicitly list the set of available strategies at *every* history in the game. The domain on which a reduction operator works therefore contains products of strategy sets that assign to every history in the game a set of strategies for every player that is active there. To make our analysis easier, we additionally specify at the beginning of the game a set of strategies for *all* players, irrespective of whether they are active or not at the beginning. These strategies intuitively represent the strategies that can eventually be chosen by the players in the dynamic game. A dynamic game reduction operator can then be defined as a mapping that assigns to every such product of strategy sets a product of reduced strategy sets, obtained by eliminating some (or no) strategies for each of the players at each of the respective histories.

The condition of *monotonicity* we introduce, and for which we show that it implies order independence with respect to strategies, is a rather direct extension of Luo, Qian and Qu's (2016) notion of *1-monotonicity** to the case of dynamic games. It states, for a given reduction operator r, that whenever we take a product of strategy sets E that is possible in an elimination order of r, and D is obtained from E by eliminating some, but not necessarily all, strategies that can be eliminated from E according to r, then the reduction of D must be contained in the reduction of E. In Theorem 4.1 we show that this condition guarantees that the reduction operator r is order independent with respect to strategies. That is, the order and speed of elimination we choose is inessential for the final sets of strategies obtained.

In turn, our condition of *monotonicity on reachable histories* is a rather direct adaptation of Perea's (2016) condition of monotonicity on reachable histories to our specific setting. Roughly speaking, it says that the monotonicity condition spelled out above should hold once we restrict to histories in the game that are reachable under the strategies in D. More precisely, consider a reduction operator r and a product of strategy sets E. A partial reduction of E is a product of strategy sets D obtained from E by eliminating some, but not necessarily all, strategies that can be eliminated from E according to r. Monotonicity on reachable histories then states the following: If we take a product of strategy sets E that is possible in an elimination order of r, and a product of strategy sets D that is behaviorally equivalent, on the histories reachable under D, to a partial reduction of E, then the reduction of D must be contained in the reduction of E if we restrict to the histories that are reachable under D. We show in Theorem 5.1 that every reduction operator that is monotone on reachable histories will be order independent with respect to outcomes. That is, the set of reachable outcomes in the dynamic game does not depend on the specific order and speed of elimination chosen.

The outline of this paper is as follows. In Section 2 we lay out the model of a dynamic game, the definition of conditional beliefs, and the notion of rational choice. In Section 3 we present the general definition of a reduction operator for dynamic games, and show how the extensive-form rationalizability procedure, the backward dominance procedure and the procedure for jointly rational belief systems can be characterized by the iterated application of a specific reduction operator. In Section 4 we define order independence with respect to strategies, introduce the notion of monotonicity, and show that it implies order independence with respect to strategies. We also prove that the reduction operators underlying the backward dominance procedure and the procedure for jointly rational belief systems are monotone, thereby showing that these two procedures are order independent with respect to strategies. Similarly, we define in Section 5 order independence with respect to outcomes, present the condition of monotonicity on reachable histories, and show that it implies order independence with respect to outcomes. We prove, moreover, that the reduction operator characterizing extensive-form rationalizability is monotone on reachable histories, implying that the extensive-form rationalizability procedure is order independent with respect to outcomes. In Section 6 we use some of the results above to show that every outcome that is reachable under extensive-form rationalizability is also reachable under the backward dominance procedure. In Section 7 we conclude the paper with some final remarks. All proofs are collected in Section 8.

2 Dynamic Games

In this section we present some basic definitions that we will need for the rest of the paper, including the formal model of a dynamic game, the notion of strategies and conditional belef vectors, and the definition of rational choice in a dynamic game.

2.1 Definition

In this paper we will focus on finite dynamic games with observable past choices. Such games allow for simultaneous moves, but at every stage of the game every active player knows exactly which choices have been made by the opponents in the past. Formally, a *finite dynamic game with observable past choices* is a tuple

$$G = (I, H, Z, (H_i)_{i\in I}, (C_i(h))_{i\in I, h\in H_i}, (u_i)_{i\in I})$$

where

(a) $I = \{1, 2, ..., n\}$ is the finite set of *players*;

(b) H is the finite set of *histories*, consisting of *non-terminal* and *terminal* histories. At every non-terminal history, one or more players must make a choice, whereas at every terminal history the game ends. By $\emptyset$ we denote the history that marks the beginning of the game;

(c) $Z \subseteq H$ is the set of terminal histories;

(d) $H_i \subseteq H$ is the set of non-terminal histories where player i must make a choice. For a given non-terminal history h, we denote by $I(h) := \{i \in I \mid h \in H_i\}$ the set of *active* players at h. We allow $I(h)$ to contain more than one player, that is, we allow for *simultaneous moves.* At the same time, we require $I(h)$ to be non-empty for every non-terminal history h;

(e) $C_i(h)$ is the finite set of choices available to player i at a history $h \in H_i$; and

(f) $u_i : Z \to \mathbb{R}$ is player i's utility function, assigning to every terminal history $z \in Z$ some utility $u_i(z)$.

For every non-terminal history h and choice combination $(c_i)_{i\in I(h)}$ in $\times_{i\in I(h)} C_i(h)$, we denote by $h' = (h, (c_i)_{i\in I(h)})$ the (terminal or non-terminal) history that immediately follows this choice combination at h. In this case, we say that h' immediately follows h. We say that a history h *follows* a non-terminal history h' if there is a sequence of histories $h^1, ..., h^K$ such that $h^1 = h'$, $h^K = h$, and h^{k+1} immediately follows h^k for all $k \in \{1, ..., K-1\}$. A history h is said to *weakly follow* h' if either h follows h' or $h = h'$. In the obvious way, we can then also define what it means for h to (*weakly*) *precede* another history h'.

We view a strategy for player i as a *plan of action* (Rubinstein (1991)), assigning choices only to those histories $h \in H_i$ that are not precluded by previous choices. Formally, consider a set of non-terminal histories $\hat{H}_i \subseteq H_i$, and a mapping $s_i : \hat{H}_i \to \cup_{h\in\hat{H}_i} C_i(h)$ assigning to every history $h \in \hat{H}_i$ some available choice $s_i(h) \in C_i(h)$. We say that a history $h \in H$ is *reachable* under s_i if at every history $h' \in \hat{H}_i$ preceding h, the choice $s_i(h')$ is the unique choice that leads to h. The mapping $s_i : \hat{H}_i \to \cup_{h\in\hat{H}_i} C_i(h)$ is called a *strategy* if $\hat{H}_i$ contains exactly those histories in H_i that are reachable under s_i.

By S_i we denote the set of strategies for player i. For every history $h \in H$ and player i, we denote by $S_i(h)$ the set of strategies for player i under which h is reachable. Similarly, for a given strategy s_i we denote by $H_i(s_i)$ the set of histories in H_i that are reachable under s_i.

2.2 Conditional Belief Vectors

At every history where player i is active, he is assumed to hold a conditional probabilistic belief about the strategy choices of his opponents. These conditional beliefs provide the basis upon which he will make his choices in the dynamic game. In particular, he may change – and often must change – his belief if the game moves from one history to another. Such a collection of conditional beliefs is called a conditional belief vector.

To formally define it, we need some additional pieces of notation. For a finite set X, we denote by $\Delta(X)$ the set of probability distributions on X. For a player i and history $h \in H_i$, let $S_{-i}(h) := \times_{j \neq i} S_j(h)$ be the set of opponents' strategy combinations under which h is reachable.

A *conditional belief vector* for player i is tuple $b_i = (b_i(h))_{h \in H_i}$ where $b_i(h) \in \Delta(S_{-i}(h))$ for every $h \in H_i$. Here, $b_i(h)$ represents the conditional probabilistic belief that i holds at h about the opponents' strategy choices. We say that the conditional belief vector b_i satisfies *Bayesian updating* if for every $h, h' \in H_i$ where h' follows h and $b_i(h)(S_{-i}(h')) > 0$, it holds that

$$b_i(h')(s_{-i}) = \frac{b_i(h)(s_{-i})}{b_i(h)(S_{-i}(h'))} \text{ for all } s_{-i} \in S_{-i}(h').$$

By B_i we denote the set of conditional belief vectors for player i that satisfy Bayesian updating.

For a given conditional belief vector b_i, a set $E \subseteq S_{-i}$ of opponents' strategy combinations, and a history $h \in H_i$, we say that $b_i(h)$ *believes* E if $b_i(h)(E) = 1$. Moreover, we say that $b_i(h)$ *strongly believes* E if $b_i(h)(E) = 1$ whenever $S_{-i}(h) \cap E \neq \emptyset$. That is, $b_i(h)$ assigns full probability to E whenever E is logically consistent with the event that h has been reached. We say that the conditional belief vector $b_i = (b_i(h))_{h \in H_i}$ strongly believes the event E if $b_i(h)$ strongly believes E for every $h \in H_i$.

2.3 Rationality

We finally define what it means for a strategy to be rational, at a given history, for a conditional belief vector. Before doing so, we must first formalize the expected utility generated by a strategy, at a given history, under a conditional belief vector. For a strategy combination $s = (s_i)_{i \in I}$ we denote by $z(s)$ the induced terminal history. For a history $h \in H_i$, a strategy $s_i \in S_i(h)$, and a conditional belief $b_i(h) \in \Delta(S_{-i}(h))$, we denote by

$$u_i(s_i, b_i(h)) := \sum_{s_{-i} \in S_{-i}(h)} b_i(h)(s_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

the induced *expected utility* at h. We say that strategy s_i is *rational at* h for the conditional belief $b_i(h)$ if $u_i(s_i, b_i(h)) \geq u_i(s_i', b_i(h))$ for all $s_i' \in S_i(h)$. That is, strategy s_i yields the highest possible expected utility at h under the belief $b_i(h)$.

For a given strategy s_i, conditional belief vector $b_i = (b_i(h))_{h \in H_i}$ and collection $G \subseteq H$ of histories, we say that strategy s_i is *rational at* G for b_i if s_i is rational at every $h \in G \cap H_i(s_i)$

for $b_i(h)$. Finally, we say that strategy s_i is *rational* for the conditional belief vector b_i if s_i is rational at H for b_i.

3 Reduction Operators

In this section we will define the general notion of a *reduction operator* for dynamic games. Subsequently, we show how the extensive-form rationalizability procedure (Pearce (1984), Battigalli (1997)), the backward dominance procedure (Perea (2014)), and the procedure for jointly rational belief systems (Battigalli and Siniscalchi (2002), Reny (1993)) can be characterized by the iterated application of an appropriately defined reduction operator.

3.1 Definition

Intuitively, a reduction operator is a mapping that eliminates strategies from a dynamic game. For static games a reduction operator is rather easy to define, as it assigns to every product of choice sets – one choice set for every player – a product of *reduced* choice sets, obtained by eliminating some (or no) choices for each of the players.

The straightforward extension to dynamic games would be to define a reduction operator as a mapping that assigns to every product of *strategy* sets – one strategy set for each player – a product of reduced strategy sets. Although this approach would still work for certain reduction procedures in dynamic games, like extensive-form rationalizability (see Perea (2016)) or jointly rational belief systems, is would no longer be sufficient for other procedures like the backward dominance procedure, which cannot easily be characterized by the iterated application of such a simple reduction operator. The reason is that in the backward dominance procedure, in order to evaluate whether a strategy can be eliminated at a certain history, one needs to rely on the sets of remaining strategies at histories that *follow*. This interdependence cannot easily be captured by a reduction operator of the simple kind described above, and thus calls for a more complex type of reduction operator.

A possible resolution to this problem would be to explicitly list, for *every* history in the game, a subset of strategies for every player that is active there. This subset of strategies could be interpreted as the collection of strategies that the player could reasonably choose if that particular history were to be reached. Hence, instead of listing one subset of strategies for every player, we list for every player a subset of strategies at *every history* where he is active. It turns out that this extended version of a product of strategy sets will be sufficient to easily characterize procedures like backward dominance as well. This is therefore the approach that we will adopt.

To make our analysis easier, we additionally list a subset of strategies for every player at the *beginning* of the game, independent of whether this player is active there or not. These strategies may be interpreted as the strategies that the player can reasonably choose before the game starts. This modelling choice is not crucial for our results, but makes the analysis more tractable.

A reduction operator can then be defined as a mapping that assigns to every such extended product of strategy sets, describing strategy sets at every history of the game, a product of *reduced* strategy sets. Formally, this can be defined as follows.

For every player i we define $H_i^* := H_i \cup \{\emptyset\}$, including the histories where player i is active and the beginning of the game. These are the histories at which we will describe a set of strategies for player i. A *product of strategy sets* is a Cartesian product

$$D = \times_{i \in I, h \in H_i^*} D_{ih} \text{ with } D_{ih} \subseteq S_i(h) \text{ for every } i \in I \text{ and } h \in H_i^*.$$

It thus prescribes, for every player i and history $h \in H_i^*$, a subset $D_{ih} \subseteq S_i(h)$ of player i strategies under which h is reachable. In particular, it prescribes for every player i a set of strategies $D_{i\emptyset} \subseteq S_i$ at the beginning of the game $\emptyset$, even if player i is not active at $\emptyset$. We allow the sets of strategies D_{ih} to be empty. By $D_\emptyset := \times_{i \in I} D_{i\emptyset}$ we denote the set of strategy combinations obtained at the beginning of the game.

For a given product of strategy sets $D = \times_{i \in I, h \in H_i^*} D_{ih}$, collection of histories $G \subseteq H$, and player i, we define the set of strategies

$$D_i(G) := \{s_i \in S_i \mid s_i \in D_{ih} \text{ for all } h \in H_i^*(s_i) \cap G\},$$

where $H_i^*(s_i) := H_i(s_i) \cup \{\emptyset\}$. That is, $D_i(G)$ contains those strategies for player i that are "contained in" D at all histories in G where these strategies are defined.

Definition 3.1 (Reduction operator) *A reduction operator is a mapping r that assigns to every product of strategy sets D a product of strategy sets $r(D) \subseteq D$ contained in it.*

For every $k \geq 1$, we denote by

$$r^k(D) := \underbrace{(r \circ ... \circ r)}_{k \text{ times}}(D)$$

the k-fold application of r to the product of strategy sets D, and we set $r^0(D) := D$.

Let $S^* := \times_{i \in I, h \in H_i^*} S_i(h)$ be the full product of strategy sets, and let $R \subseteq \times_{i \in I} S$ be a set of strategy combinations for the players. We say that the reduction operator r *yields the set R of strategy combinations* if

$$R = \cap_{k \geq 0} (r^k(S^*))_\emptyset.$$

That is, R is the set of strategy combinations obtained at the beginning of the game if we iteratively apply the reduction operator r to the full product of strategy sets.

3.2 Extensive-form Rationalizability

The *extensive-form rationalizability* procedure (Pearce (1984), Battigalli (1997)) iteratively eliminates strategies and conditional belief vectors from the game, as follows. Recall that B_i denotes the set of all conditional belief vectors for player i that satisfy Bayesian updating. We start by setting $S_i^{er,0} := S_i$ and $B_i^{er,0} := B_i$ for every player i. For every $k \geq 1$ and every player i we recursively define

$$\begin{aligned} S_i^{er,k} &: = \{s_i \in S_i^{er,k-1} \mid s_i \text{ rational for some } b_i \in B_i^{er,k-1}\}, \\ B_i^{er,k} &: = \{b_i \in B_i^{er,k-1} \mid b_i \text{ strongly believes } S_{-i}^{er,k}\}, \end{aligned}$$

where $S_{-i}^{er,k} := \times_{j \neq i} S_j^{er,k}$. By $S_i^{er} := \cap_{k \geq 0} S_i^{er,k}$ we denote the set of extensive-form rationalizable strategies.

Perea (2016) has shown that the extensive-form rationalizable strategies can be characterized by the iterated application of the *strong belief* reduction operator. This operator, however, is defined on a different class of products of strategy sets than the class we consider here. The products of strategy sets studied in Perea (2016) have the form $\hat{D} = \times_{i \in I} \hat{D}_i$, where $\hat{D}_i \subseteq S_i$ is a set of strategies for every player i. These products of strategy sets are thus sparser than the ones we use, since only one set of strategies is defined for every player. In contrast, we define a set of strategies for every player i and every *history* where player i is active. We refer to the products of strategy sets considered in Perea (2016) as *simple.*

For a given simple product of strategy sets $\hat{D}$, we denote by $H(\hat{D})$ the set of histories that are reachable by strategy combinations $(s_i)_{i \in I}$ in $\hat{D}$. The *strong belief* reduction operator sb, as defined in Perea (2016), assigns to every simple product of strategy sets $\hat{D} = \times_{i \in I} \hat{D}_i$ the reduced simple product of strategy sets $sb(\hat{D}) = \times_{i \in I} sb_i(\hat{D})$, where

$$sb_i(\hat{D}) := \{s_i \in \hat{D}_i \mid s_i \text{ rational at } H(\hat{D}) \text{ for some } b_i \in B_i \text{ that strongly believes } \hat{D}_{-i}\}$$

for every player i.

Theorem 4.1 in Perea (2016) shows that for every k, the strategies surviving round k of the extensive-form rationalizability procedure are exactly the strategies obtained by the k-fold successive application of the strong belief reduction operator. In the statement of this theorem below, we denote by $S^{er,k} := \times_{i \in I} S_i^{er,k}$ the set of strategy combinations surviving round k of the extensive-form rationalizability procedure, and we denote by $S := \times_{i \in I} S_i$ the full simple product of strategy sets.

Theorem 3.1 (Theorem 4.1 in Perea (2016)) *For every $k \geq 0$ we have $S^{er,k} = (sb)^k(S)$.*

We will now "translate" the strong belief reduction operator into a reduction operator er that fits our framework, as follows. For a given product of strategy sets D, we denote by $H(D_\emptyset)$

the set of histories that are reachable by strategy combinations $(s_i)_{i \in I}$ in $D_\emptyset$. For every product of strategy sets $D = \times_{i \in I, h \in H_i^*} D_{ih}$ and every player i we set

$$er_{i\emptyset}(D) := \{s_i \in D_{i\emptyset} \mid s_i \text{ rational at } H(D_\emptyset) \text{ for some } b_i \in B_i \text{ that strongly believes } D_{-i\emptyset}\},$$

and we set

$$er_{ih}(D) := \emptyset \text{ for all } h \in H_i \backslash \{\emptyset\}.$$

We define

$$er(D) := \times_{i \in I, h \in H_i^*} er_{ih}(D).$$

Hence, by construction,

$$er_{i\emptyset}(D) = sb_i(D_\emptyset) \tag{3.1}$$

for every product of strategy sets D and every player i.

Together with Theorem 3.1 above, it immediately follows that for every k the set $S^{er,k}$ of strategy combinations surviving the first k rounds of extensive-form rationalizability is obtained by the k-fold application of the reduction operator er to the full product of strategy sets. Recall that $S^* := \times_{i \in I, h \in H_i^*} S_i(h)$ denotes the full product of strategy sets. By $(er^k(S^*))_\emptyset$ we represent the set of strategy combinations at $\emptyset$ in the product of strategy sets $er^k(S^*)$. The following result thus follows immediately from Theorem 3.1 and the definition of the reduction operator er.

Theorem 3.2 (Characterization of extensive-form rationalizability) *For every $k \geq 0$ we have that $S^{er,k} = (er^k(S^*))_\emptyset$. As a consequence, er yields the set $\times_{i \in I} S_i^{er}$ of extensive-form rationalizable strategy combinations.*

In a sense, the theorem above can be viewed as a reformulation of Theorem 4.1 in Perea (2016) within our specific framework of reduction operators.

3.3 Backward Dominance Procedure

Perea (2014) introduces the *backward dominance* procedure, and shows that it characterizes precisely those strategies that can rationally be chosen under *common belief in future rationality.* That is, these are precisely the strategies that result if players always believe that the opponents will choose rationally in the future, always believe that the other players always believe that their opponents will always choose rationally in the future, and so on.

The procedure is defined in terms of *decision problems.* Formally, a decision problem at a non-terminal history h is a tuple

$$E_h = \times_{i \in I} E_{ih}, \text{ where } E_{ih} \subseteq S_i(h) \text{ for all } i \in I.$$

That is, it prescribes for every player (active or non-active at h) a subset of strategies under which h is reachable. By $S_h := \times_{i \in I} S_i(h)$ we denote the full decision problem at h.

If player i is active at h, then a strategy $s_i \in E_{ih}$ is said to be *strictly dominated* in the decision problem E_h if there is some randomized strategy $\rho_i \in \Delta(E_{ih})$ such that

$$u_i(z(s_i, s_{-i})) < \sum_{s_i' \in E_{ih}} \rho_i(s_i') \cdot u_i(z(s_i', s_{-i})) \text{ for all } s_{-i} \text{ in } \times_{j \neq i} E_{jh}.$$

In the backward dominance procedure, we iteratively eliminate strategies from decision problems, as follows. For every non-terminal history h, let $E_h^{bd,0} := S_h$ be the full decision problem at h. Let $H^{fut}(h)$ be the set of histories that weakly follow h. For every round $k \geq 1$ we recursively define, for every non-terminal history h, the decision problem at h by

$$E_h^{bd,k} = \times_{i \in I} E_{ih}^{bd,k}$$

where for every player i,

$$E_{ih}^{bd,k} := \{s_i \in E_{ih}^{bd,k-1} \mid \text{there is no } h' \in H_i(s_i) \cap H^{fut}(h) \text{ such that } s_i \text{ is strictly dominated in } E_{h'}^{bd,k-1}\}.$$

By $S_i^{bd} := \cap_{k \geq 0} E_{i\emptyset}^{bd,k}$ we denote the set of strategies for player i that survive the backward dominance procedure at $\emptyset$. We call these strategies the *backward dominance* strategies.

We will now show that the backward dominance strategies are obtained by the iterated application of a certain reduction operator bd, defined as follows. Remember that for a given product of strategy sets $D = \times_{i \in I, h \in H_i^*} D_{ih}$, player i and collection of histories $G \subseteq H$, we defined $D_i(G) := \{s_i \in S_i \mid s_i \in D_{ih} \text{ for all } h \in H_i^*(s_i) \cap G\}$. For every product of strategy sets $D = \times_{i \in I, h \in H_i^*} D_{ih}$, every player i and every history $h \in H_i^*$, let

$$bd_{ih}(D) := \{s_i \in D_{ih} \mid s_i \text{ rational at } H^{fut}(h) \text{ for some conditional belief vector } b_i \text{ where } b_i(h') \text{ believes } D_{-i}(H^{fut}(h')) \text{ for all } h' \in H_i \cap H^{fut}(h)\},$$

where $D_{-i}(H^{fut}(h')) := \times_{j \neq i} D_j(H^{fut}(h'))$. We define

$$bd(D) := \times_{i \in I, h \in H_i^*} bd_{ih}(D).$$

We will show, for every k, that the strategies surviving round k of the backward dominance procedure are obtained by the k-fold application of the reduction operator bd to the full product of the strategy sets.

Theorem 3.3 (Characterization of backward dominance) *For every $k \geq 0$ and every player i we have that*

(a) $E_{ih}^{bd,k} = S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h))$ for every non-terminal history $h \in H$, and

(b) $E_{i\emptyset}^{bd,k} = (bd^k(S^*))_{i\emptyset}$.

As a consequence, bd yields the set $\times_{i \in I} S_i^{bd}$ of backward dominance strategy combinations.

Here, we have by definition that

$$(bd^k(S^*))_i(H^{fut}(h)) = \{s_i \in S_i \mid s_i \in (bd^k(S^*))_{ih'} \text{ for all } h' \in H_i^*(s_i) \cap H^{fut}(h)\}.$$

In Perea (2016) it has been shown that extensive-form rationalizability can be characterized by a simpler type of reduction operator, where we do not specify a set of strategies at every history, but only define one set of strategies for every player. In contrast, we see no easy way to characterize the backward dominance procedure by such a simple type of reduction operator.

3.4 Jointly Rational Belief Systems

Reny (1993) defines the notion of *jointly rational belief systems* for a collection $G \subseteq H$ of histories. Intuitively, it captures the idea that players believe at all histories in G that their opponents choose rationally, that players believe at all histories in G that the other players believe at all histories in G that their opponents choose rationally, and so on. Although Reny restricts to two-player games with perfect information, his concept can naturally be extended to all dynamic games with observable past choices. Moreover, Reny assumes that players hold conditional beliefs at *all* histories, also at those where they are not active. This assumption will automatically be satisfied within our model if we let players choose from a singleton choice set at those histories where in reality they are not active. We can thus safely drop this assumption by Reny without altering the concept in an essential way.

Consider an arbitrary collection of histories $G \subseteq H$, and consider for every player i a non-empty set of strategies $D_i \subseteq S_i$. The product $D = \times_{i \in I} D_i$ is called a *jointly rational belief system* for G is for every player i,

$$D_i = \{s_i \in S_i \mid s_i \text{ is rational for some conditional belief vector } b_i \text{ where } b_i(h) \text{ believes } D_{-i} \text{ for all } h \in H_i \cap G\}.$$

If we choose $G = \{\emptyset\}$ and assume that all players are active at the beginning of the game, then we obtain Ben-Porath's (1997) notion of *common certainty of rationality at the beginning of the game.* In general, however, a jointly rational belief system need not exist for every collection G of histories. For instance, there are only very few games where we can find a jointly rational belief system for the collection of *all* histories in the game. We refer the reader to Reny (1992a, 1993) for an extensive discussion of this issue.

Battigalli and Siniscalchi (1999) provide a reduction procedure that, for a given collection G of histories, yields the *largest* jointly rational belief system. In their procedure we set $S_i^{G,0} := S_i$ for every player i, and for every round $k \geq 1$ and player i we define

$$S_i^{G,k} = \{s_i \in S_i \mid s_i \text{ is rational for some conditional belief vector } b_i \text{ where } b_i(h) \text{ believes } S_{-i}^{G,k-1} \text{ for all } h \in H_i \cap G\}.$$

Then, the set $\cap_{k\geq 0}(\times_{i\in I}S_i^{G,k})$, provided it is non-empty, is the largest jointly rational belief system for G. If $\cap_{k\geq 0}(\times_{i\in I}S_i^{G,k})$ is empty, then there is no jointly rational belief system for G.

We will show that the largest jointly rational belief system for G can also be obtained by the recursive application of the reduction operator rG, to be defined below. For every product of strategy sets $D = \times_{i\in I, h\in H_i^*} D_{ih}$ and every player i, let

$$rG_{i\emptyset}(D) := \{s_i \in D_{i\emptyset} \mid s_i \text{ is rational for some conditional belief vector } b_i \text{ where } b_i(h) \text{ believes } D_{-i\emptyset} \text{ for all } h \in H_i \cap G\},$$

and set

$$rG_{ih}(D) := \emptyset \text{ for all } h \in H_i \backslash \{\emptyset\}.$$

We define

$$rG(D) := \times_{i\in I, h\in H_i^*} rG_{ih}(D).$$

The following theorem, which shows that the strategies that survive round k of Battigalli and Siniscalchi's procedure are precisely the strategies that result from the k-fold successive application of the reduction operator rG, follows directly from the definitions.

Theorem 3.4 (Characterization of jointly rational belief systems) *For every $k \geq 0$ and every player i we have that $S_i^{G,k} = (rG^k(S^*))_{i\emptyset}$. As a consequence, rG yields the largest jointly rational belief system for G.*

Similarly to the case of extensive-form rationalizability, also the reduction operator rG is of a simple type that does not use the sets of strategies prescribed at histories other than $\emptyset$. This operator could therefore be phrased equivalently within Perea's (2016) simpler framework, where a reduction operator works on *simple* products of strategy sets that only prescribe one set of strategies for every player.

4 Order Independence with Respect to Strategies

In this section we first define what it means for a reduction operator to be order independent with respect to strategies. Intuitively, it states that the final set of strategy combinations obtained at the beginning of the game, by iteratively applying the reduction operator, does not depend upon the order or speed by which we eliminate strategies. We next introduce the condition of monotonicity, and show that every monotone reduction operator is order independent with respect to strategies. We finally prove that the reduction operators yielding the backward dominance procedure and the procedure for jointly rational belief systems are monotone, thereby showing that these two reduction procedures are order independent with respect to strategies.

4.1 Definition

Informally, we say that a reduction operator r is order independent with respect to strategies if every possible elimination order for r yields the same set of strategies. By an elimination order for r we mean a sequence of successive *partial reductions,* in which at every round we eliminate some, but not necessarily all, strategies that can be eliminated according to r. More precisely, if $D = \times_{i\in I, h\in H_i^*} D_{ih}$ and $E = \times_{i\in I, h\in H_i^*} E_{ih}$ are two products of strategy sets, then D is a *partial reduction* of E if $r(E) \subseteq D \subseteq E$. Hence, D is obtained from E by eliminating some, but not necessarily all, strategies that can be eliminated according to r. We call $D = r(E)$ the *full reduction* of E.

Definition 4.1 (Elimination order for r) *An elimination order for a reduction operator r is a finite sequence of products of strategy sets $(D^0, D^1, ..., D^K)$ such that (a) $D^0 = S^*$, (b) D^{k+1} is a partial reduction of D^k for every $k \in \{0, ..., K-1\}$, and (c) $r(D^K) = D^K$.*

Part (c) makes sure that no further partial reductions are possible after round K. With this definition at hand we can formally define order independence with respect to strategies.

Definition 4.2 (Order independence with respect to strategies) *A reduction operator r is order independent with respect to strategies if for every two elimination orders $(D^0, ..., D^K)$ and $(E^0, ..., E^L)$ for r we have that $D^K_\emptyset = E^L_\emptyset$.*

A special elimination order for r is the "full speed" elimination order

$$(S^*, r(S^*), r^2(S^*), ..., r^K(S^*)),$$

where $r^{K+1}(S^*) = r^K(S^*)$, consisting of full reductions only. If the reduction operator r is order independent then every possible elimination order must eventually yield the same set of strategies as the full speed elimination order, but may possibly take more rounds to arrive there.

4.2 Sufficient Condition

For static games, Luo, Qian and Qu (2016) provide a sufficient condition for order independence with respect to strategies, which they call *1-monotonicity**. We will provide a "dynamic games version" of this condition, called *monotonicity,* and show that it implies independence with respect to strategies within our framework. In the formal definition below, we say that a product of strategy sets D is *possible in en elimination order for* r if there is an elimination order $(D^0, ..., D^K)$ for r such that $D = D^k$ for some $k \in \{0, ..., K\}$.

Definition 4.3 (Monotonicity) *A reduction operator r is monotone if for every two products of strategy sets D and E, where E is possible in an elimination order for r and D is a partial reduction of E, we have that $r(D) \subseteq r(E)$.*

If we apply this notion to a static game, containing only one non-terminal history $\emptyset$, then this reduces to 1-monotonicity* as defined in Luo, Qian and Qu (2016). In the following theorem we show that monotonicity is indeed a sufficient condition for independence with respect to strategies.

Theorem 4.1 (Sufficient condition for order independence w.r.t. strategies) *Every monotone reduction operator is order independent with respect to strategies.*

Since monotonicity is in general easy to verify, this result is very convenient for showing that certain reduction operators are order independent with respect to strategies.

4.3 Showing Order Independence with Respect to Strategies

In this section we will prove that the reduction operators bd and rG, characterizing the backward dominance procedure and jointly rational belief systems, respectively, are order independent with respect to strategies. To prove this, we rely on Theorem 4.1 and show that both reduction operators are monotone.

Theorem 4.2 (Monotone reduction operators) *The reduction operators bd and rG (for every $G \subseteq H$) are monotone.*

In the proof of this theorem we show, in fact, that both reduction operators satisfy an even stronger notion of monotonicity. We prove, for both operators r, that $r(D) \subseteq r(E)$ for all products of strategy sets D, E with $D \subseteq E$. Indeed, for proving $r(D) \subseteq r(E)$ we do not need the assumption that E is possible in an elimination order for r, nor that $r(E) \subseteq D$ (which is assumed if D is a partial reduction of E). For static games, this stronger notion of monotonicity is known as *hereditarity* (Gilboa, Kalai and Zemel (1990)) or *monotonicity* (Apt (2011)).

By combining Theorems 4.1 and 4.2 we immediately reach the following conclusion.

Corollary 4.1 (Order independence w.r.t. strategies) *The reduction operators bd and rG (for every $G \subseteq H$) are order independent with respect to strategies.*

In view of Theorems 3.3 and 3.4 it thus follows that for determining the strategies that survive the backward dominance procedure, or the strategies that are part of a jointly rational belief system for G, it is irrelevant which specific order of elimination we use. In particular, we may safely use the very convenient *backwards order of elimination* for the backward dominance procedure, where we start eliminating at the last non-terminal histories and then work our way backwards until we reach the beginning of the game.

We finally illustrate, by means of an example, that the reduction operator er, characterizing the extensive-form rationalizable strategies, is not order independent with respect to strategies, and hence cannot be monotone. Consider the perfect information game in Figure 1, which is

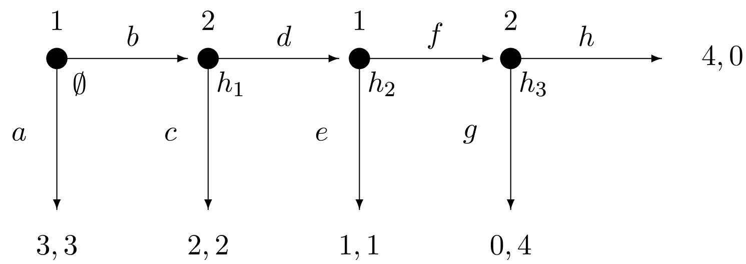

Figure 1: Reny's game

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based on Figure 3 in Reny (1992b). In this game, it may be verified that the "full speed" elimination order is given by

$$
\begin{aligned}
D^0 &= S^*,\\
D^1 &= er(D^0) = \{a, (b, f)\} \times \{c, (d, g)\} \times \emptyset \times \emptyset \times \emptyset,\\
D^2 &= er(D^1) = \{a\} \times \{(d, g)\} \times \emptyset \times \emptyset \times \emptyset.
\end{aligned}
$$

Here,

$$D^1 = \{a, (b, f)\} \times \{c, (d, g)\} \times \emptyset \times \emptyset \times \emptyset$$

means that

$$D^1_{1\emptyset} = \{a, (b, f)\}, \quad D^1_{2\emptyset} = \{c, (d, g)\}, \quad D^1_{2h_1} = \emptyset, \quad D^1_{1h_2} = \emptyset \text{ and } D^1_{2h_3} = \emptyset,$$

and similarly for D^2.

To see that D^1 and D^2 are as described above, note that for both players i the set $D^1_{i\emptyset}$ contains those strategies in S_i that are rational for some conditional belief vector in B_i. Strategy (b, e) cannot be rational for player 1 at $\emptyset$ since (b, e) yields him at most utility 2 there, whereas choosing a at $\emptyset$ gives him 3. Strategies a and (b, f), on the other hand, are both rational for some conditional belief vector in B_1. Therefore, $D^1_{1\emptyset} = \{a, (b, f)\}$. For player 2, strategy (d, h) is irrational at h_3, whereas strategies c and (d, g) are both rational for some conditional belief vector in B_2. Hence, $D^1_{2\emptyset} = \{c, (d, g)\}$.

By definition, $D^2_{i\emptyset}$ contains those strategies in $D^1_{i\emptyset}$ that are rational at $H(D^1_\emptyset)$ for some $b_i \in B_i$ that strongly believes $D^1_{-i\emptyset}$. Note that $D^1_\emptyset = \{a, (b, f)\} \times \{c, (d, g)\}$. Therefore, every conditional belief vector for player 1 that strongly believes $D^1_{2\emptyset}$ must at $\emptyset$ believe $\{c, (d, g)\}$. Consequently, only strategy a can be rational for such a conditional belief vector, and hence $D^2_{1\emptyset} = \{a\}$.

Similarly, every conditional belief vector for player 2 that strongly believes $D^1_{1\emptyset}$ must at h_1 and h_3 believe (b,f). As $H(D^1_\emptyset)$ contains all non-terminal histories, only strategy (d,g) can be rational at $H(D^1_\emptyset)$ for such a conditional belief vector. Therefore, $D^2_{2\emptyset} = \{(d,g)\}$.

It may be verified that $er(D^2) = D^2$, and hence the procedure stops at round 2. We thus conclude that (D^0, D^1, D^2) above is indeed the "full speed" elimination order for er. Moreover, $D^2_\emptyset = \{a\} \times \{(d,g)\}$, which are the extensive-form rationalizable strategies in this game.

An alternative elimination order for er in this game is the "backward induction sequence" $(E^0, ..., E^4)$ below, which mimicks the backward induction procedure:

$$\begin{aligned}
E^0 &= S^*, \\
E^1 &= \{a, (b,e), (b,f)\} \times \{c, (d,g)\} \times \emptyset \times \emptyset \times \emptyset, \\
E^2 &= \{a, (b,e)\} \times \{c, (d,g)\} \times \emptyset \times \emptyset \times \emptyset, \\
E^3 &= \{a, (b,e)\} \times \{c\} \times \emptyset \times \emptyset \times \emptyset, \\
E^4 &= \{a\} \times \{c\} \times \emptyset \times \emptyset \times \emptyset.
\end{aligned}$$

In a similar way as above, it may be verified that

$$\begin{aligned}
er(E^0) &= \{a, (b,f)\} \times \{c, (d,g)\} \times \emptyset \times \emptyset \times \emptyset, \\
er(E^1) &= \{a\} \times \{c, (d,g)\} \times \emptyset \times \emptyset \times \emptyset, \\
er(E^2) &= er(E^3) = er(E^4) = \{a\} \times \{c\} \times \emptyset \times \emptyset,
\end{aligned}$$

which implies that $(E^0, ..., E^4)$ is indeed an elimination order for er. Hence, $E^4_\emptyset = \{a\} \times \{c\}$, which are the backward induction strategies in this game. Since $D^2_\emptyset \neq E^4_\emptyset$, we conclude that the reduction operator er is not order independent with respect to strategies.

By Theorem 4.1 it must then be the case that er is not monotone. To see this, consider the products of strategy sets

$$D = \{a\} \times \{c, (d,g)\} \times \emptyset \times \emptyset \times \emptyset,$$

and

$$E = \{a, (b,e)\} \times \{c, (d,g)\} \times \emptyset \times \emptyset \times \emptyset.$$

Since $E = E^2$ in the backward induction elimination order $(E^0, ..., E^4)$ above, we know that E is possible in an elimination order for er. It may be verified that

$$\begin{aligned}
er(D) &= \{a\} \times \{c, (d,g)\} \times \emptyset \times \emptyset \times \emptyset, \\
er(E) &= \{a\} \times \{c\} \times \emptyset \times \emptyset \times \emptyset.
\end{aligned}$$

We thus conclude that $er(E) \subseteq D \subseteq E$, and hence D is a partial reduction of E. However, $er(D) \not\subseteq er(E)$ which implies that er is not monotone.

5 Order Independence with Respect to Outcomes

In the previous section we have seen that the *er* reduction operator, which yields the extensive-form rationalizable strategies, is not order independent with respect to strategies, and hence cannot be monotone. In this section we will show, however, that this operator is order independent with respect to *outcomes,* meaning that every possible elimination order yields the same set of induced *outcomes* (but not necessarily the same set of strategies). A similar result can be found in Chen and Micali (2013) who show that the iterated conditional dominance procedure (Shimoji and Watson (1998)), which also characterizes the extensive-form rationalizable strategies, is order independent with respect to outcomes. Our result can also be seen as an analogue to Perea (2016) where it is shown that the strong belief reduction operator, which characterizes the extensive-form rationalizable strategies and which has been discussed in Section 3, is order independent with respect to outcomes.

In this section we first provide a formal definition of order independence with respect to outcomes. We subsequently introduce the condition of monotonicity on reachable histories, and show that every reduction operator satisfying monotonicity on reachable histories is order independent with respect to outcomes. We finally show that the operator *er*, yielding extensive-form rationalizability, is monotone on reachable histories, relying heavily on results in Perea (2016). This implies that the extensive-form rationalizability procedure is order independent with respect to outcomes.

5.1 Definition

To formally define order independence with respect to outcomes we need the following notation. For a given set $\hat{D} \subseteq \times_{i \in I} S_i$ of strategy combinations, recall that we denote by $H(\hat{D})$ the set of histories reached by strategy combinations $(s_i)_{i \in I}$ in $\hat{D}$. Moreover, we denote by $Z(\hat{D}) := H(\hat{D}) \cap Z$ the set of terminal histories (or outcomes) that are reached by strategy combinations in $\hat{D}$.

Definition 5.1 (Order independence with respect to outcomes) *A reduction operator r is order independent with respect to outcomes if for every two elimination orders $(D^0, ..., D^K)$ and $(E^0, ..., E^L)$ for r we have that $Z(D^K_\emptyset) = Z(E^L_\emptyset)$.*

Hence, the outcomes induced by the set of strategy combinations at $\emptyset$ will always be the same, independent of the specific elimination order we choose. If we apply this definition to *simple* products of strategy sets, as discussed in Section 3, then we obtain the definition of order independence with respect to outcomes as given in Perea (2016).

5.2 Sufficient Condition

To formally define monotonicity on reachable histories, we need a new piece of terminology. Consider a strategy s_i for player i and a collection of histories $G \subseteq H$. By

$$s_i|_G := (s_i(h))_{h \in H_i(s_i) \cap G}$$

we denote the restriction of the strategy s_i to histories in G. Hence, $s_i|_G$ is a partial strategy that only prescribes a choice at histories that are reachable under s_i and that are part of G. Similary, for a set of strategies $D_i \subseteq S_i$ we denote by

$$D_i|_G := \{s_i|_G \mid s_i \in D_i\}$$

the restriction of the set D_i to G. Finally, for a product of strategy sets $D = \times_{i \in I, h \in H_i^*} D_{ih}$ we denote by

$$D|_G := \times_{i \in I, h \in H_i^*} D_{ih}|_G$$

the restriction of D to the histories in G.

Definition 5.2 (Monotonicity on reachable histories) *A reduction operator r is monotone on reachable histories if for every two products of strategy sets D and E where E is possible in an elimination order for r, and*

$$r(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)} \subseteq E|_{H(D_\emptyset)}$$

we have that

$$r(D)|_{H(D_\emptyset)} \subseteq r(E)|_{H(D_\emptyset)}.$$

In a sense, we require the monotonicity condition from the previous section only to hold at those histories that are reachable under $D_\emptyset$. Indeed, if we replace $H(D_\emptyset)$ by H, then we get exactly the definition of monotonicity discussed in the previous section. If we apply this definition to *simple* products of strategy sets, then we obtain precisely the notion of monotonicity on reachable histories as defined in Perea (2016).

In words, the condition states that if E is possible in an elimination order for r, and D is behaviorally equivalent, on the histories reachable under $D_\emptyset$, to a partial reduction of E, then the full reduction of D must be contained in the full reduction of E if we restrict to the histories reachable under $D_\emptyset$.

We will now show that monotonicity on reachable histories is a sufficient condition for order independence with respect to outcomes.

Theorem 5.1 (Sufficient condition for order independence w.r.t. outcomes) *Every reduction operator that is monotone on reachable histories is order independent with respect to outcomes.*

This result mimicks Theorem 6.1 in Perea (2016), which shows the same implication for reduction operators on *simple* products of strategy sets.

5.3 Showing Order Independence with Respect to Outcomes

We now prove that the *er* reduction operator, which yields the extensive-form rationalizable strategies, is order independent with respect to outcomes. To that purpose we show that the operator is monotone on reachable histories, which, by Theorem 5.1, is sufficient to conclude that *er* is independent with respect to outcomes. Our proof relies heavily on Theorem 5.1 in Perea (2016), which shows that the reduction operator *sb* on simple products of strategy sets is monotone on reachable histories – a notion we will define below. Since the operator *er* is essentially a copy of *sb*, we can then easily conclude that *er* is monotone on reachable histories as well.

Before we can formally state Theorem 5.1 in Perea (2016), we need some extra definitions. Recall that a *simple product of strategy sets* is a Cartesian product $\hat{D} = \times_{i\in I}\hat{D}_i$, where $\hat{D}_i \subseteq S_i$ for every player i. A *simple reduction operator* $\hat{r}$ is a mapping that assigns to every simple product of strategy sets $\hat{D}$ a new simple product of strategy sets $\hat{r}(\hat{D}) \subseteq \hat{D}$. For two simple products of strategy sets $\hat{D}$ and $\hat{E}$ we say that $\hat{D}$ is a partial reduction of $\hat{E}$ if $\hat{r}(\hat{E}) \subseteq \hat{D} \subseteq \hat{E}$. An *elimination order* for a simple reduction operator $\hat{r}$ is a sequence $(\hat{D}^0, ..., \hat{D}^K)$ where (a) $\hat{D}^0 = \times_{i\in I}S_i$, (b) $\hat{D}^{k+1}$ is a partial reduction of $\hat{D}^k$ for every $k \in \{0, ..., K-1\}$, and (c) $\hat{r}(\hat{D}^K) = \hat{D}^K$. We say that a simple product of strategy sets $\hat{D}$ is *possible* in an elimination order for $\hat{r}$ if there is an elimination order $(\hat{D}^0, ..., \hat{D}^K)$ for $\hat{r}$ such that $\hat{D} = \hat{D}^k$ for some $k \in \{0, ..., K\}$.

Finally, we say that the simple reduction operator $\hat{r}$ is *monotone on reachable histories* is for every two simple products of strategy sets $\hat{D}$ and $\hat{E}$ where $\hat{E}$ is possible in an elimination order for $\hat{r}$ and

$$\hat{r}(\hat{E})|_{H(\hat{D})} \subseteq \hat{D}|_{H(\hat{D})} \subseteq \hat{E}|_{H(\hat{D})},$$

it holds that

$$\hat{r}(\hat{D})|_{H(\hat{D})} \subseteq \hat{r}(\hat{E})|_{H(\hat{D})}.$$

Theorem 5.2 (Theorem 5.1 in Perea (2016)) *The simple reduction operator sb is monotone on reachable histories.*

On the basis of this result it is now easy to prove that the reduction operator *er* is motonone on reachable histories as well.

Theorem 5.3 (Operator *er* is monotone on reachable histories) *The reduction operator er is monotone on reachable histories.*

On the basis of Theorem 5.1 we can immediately conclude that the reduction operator *er* is order independent with respect to outcomes.

Corollary 5.1 (Operator *er* is order independent w.r.t. outcomes) *The reduction operator er is order independent with respect to outcomes.*

This result is very similar to Corollary 6.1 in Perea (2016), which shows that the simple reduction operator *sb* is order independent with respect to outcomes as well. There is also a tight connection with Chen and Micali (2013) who show that the iterated conditional dominance procedure (Shimoji and Watson (1998)), which provides an alternative characterization of the extensive-form rationalizable strategies, is order independent with respect to outcomes as well.

6 Extensive-Form Rationalizability vs. Backward Dominance

In this section we will prove that every outcome that is reachable by extensive-form rationalizable strategies is also reachable by backward dominance strategies. That is, in terms of outcomes the concept of extensive-form rationalizability is at least as restrictive as the backward dominance procedure. We prove this result in three steps.

In Step 1 we present a special sequence of products of strategy sets $(D^0, ..., D^K)$, which we call the *backwards elimination order,* and show that it is an elimination order for the reduction operator *bd*. In Step 2 we extend the backwards elimination order by recursively applying the *er* operator to the last set D^K, and show that the *extended backwards elimination order* $(D^0, ..., D^{K+L})$ so obtained is an elimination order for the operator *er*. In Step 3 we use these findings to show that every extensive-form rationalizable outcome is also a backward dominance outcome, as follows. Since the operator *bd* yields the backward dominance strategies and is order independent with respect to strategies, it follows from Step 1 that $D^K_\emptyset$ contains all backward dominance strategies. In particular, $Z(D^K_\emptyset)$ is the set of backward dominance outcomes. Moreover, since the operator *er* yields the extensive-form rationalizable strategies and is order independent with respect to outcomes, we know from Step 2 that $Z(D^{K+L}_\emptyset)$ is the set of extensive-form rationalizable outcomes. As $Z(D^{K+L}_\emptyset) \subseteq Z(D^K_\emptyset)$, it follows that every extensive-form rationalizable outcome is a backward dominance outcome.

6.1 Backwards Elimination Order

We will define a particular sequence of products of strategy sets, which we call the *backwards elimination order,* and prove that it is an elimination order for the reduction operator *bd*. To define it, we need some new pieces of notation. Let L be the maximal number of consecutive non-terminal histories following $\emptyset$ in the game. For every $l \in \{0, ..., L\}$ we denote by H^l the set of non-terminal histories h such that every sequence of consecutive non-terminal histories following h contains at most l histories. Hence, H^0 contains the ultimate non-terminal histories, H^1 the ultimate and penultimate non-terminal histories, and so on. By construction, H^L contains all non-terminal histories in the game.

Take some $l \in \{0, ..., L\}$. By $bd[l]$ we denote the reduction operator which assigns to every product of strategy sets $D = \times_{i \in I, h \in H_i^*} D_{ih}$ the product of strategy sets $bd[l](D) = \times_{i \in I, h \in H_i^*} bd[l]_{ih}(D)$

where

$$bd[l]_{ih}(D) := \{s_i \in D_{ih} \mid s_i \text{ rational at } H^{fut}(h) \cap H^l \text{ for some conditional belief vector } b_i \\ \text{where } b_i(h') \text{ believes } D_{-i}(H^{fut}(h')) \text{ for all } h' \in H_i \cap H^{fut}(h) \cap H^l\},$$

for all players i and histories $h \in H_i^*$. Hence, in $bd[l]$ strategies are only restricted at histories in H^l. By construction, we have that $bd(D) \subseteq bd[l](D)$ for all products of strategy sets D.

Let $M := \sum_{i \in I, h \in H_i^*} |S_i(h)|$. Then, $(bd[l])^{M+1}(D) = (bd[l])^M(D)$ for every product of strategy sets D and every $l \in \{0, ..., L\}$. Set $K := (L+1) \cdot M$. For every $k \in \{1, ..., K\}$ let $l(k)$ be the unique number in $\{0, ..., L\}$ such that $k = l(k) \cdot M + m$ for some $m \in \{1, ..., M\}$.

Definition 6.1 (Backwards elimination order) *The backwards elimination order* $(D^0, ..., D^K)$ *is recursively given by* $D^0 := S^*$, *and*

$$D^k := bd[l(k)](D^{k-1})$$

for every $k \in \{1, ..., K\}$.

Hence, in the backwards elimination order we first restrict strategies at the ultimate non-terminal histories in the game, then we restrict strategies at the penultimate non-terminal histories in the game, until we reach the beginning of the game. We show that the backwards elimination order is an elimination order for the reduction operator *bd*.

Lemma 6.1 (Backwards elimination order is elimination order for *bd***)** *The backwards elimination order* $(D^0, ..., D^K)$ *defined above is an elimination order for the reduction operator bd.*

As a consequence, the backwards elimination order can always be used in a dynamic game to derive the backward dominance strategies. In a game with perfect information without relevant ties, the backwards elimination order reduces to the backward induction procedure. Therefore, the backward dominance strategies in such games coincide with the backward induction strategies. In general, the backwards elimination order turns out to be very convenient to derive the backward dominance strategies as it allows one to works backwards in the game, starting at the ultimate non-terminal histories in the game.

6.2 Extended Backwards Elimination Order

We now extend the backwards elimination order $(D^0, ..., D^K)$ defined above by recursively applying the *er* reduction operator to the final product of strategy sets D^K. As before, let $M := \sum_{i \in I, h \in H_i^*} |S_i(h)|$, implying that $er^{M+1}(D) = er^M(D)$ for every product of strategy sets D.

Definition 6.2 (Extended backwards elimination order) *The extended backwards elimination order is the sequence* $(D^0, ..., D^{K+M})$ *where* $(D^0, ..., D^K)$ *is the backwards elimination order, and* $D^{K+m} = er^m(D^K)$ *for every* $m \in \{1, ..., M\}$.

We show that this new sequence of products of strategy sets constitutes an elimination order for the reduction operator er.

Lemma 6.2 (Extended backwards elimination order is elimination order for er**)** *The extended backwards elimination order* $(D^0, ..., D^{K+M})$ *defined above is an elimination order for the reduction operator* er.

With this result we are now fully equipped to prove that every outcome reachable by extensive-form rationalizable strategies is also reachable by backward dominance strategies.

6.3 Extensive-form Rationalizable vs. Backward Dominance Outcomes

In previous sections we have shown that (a) the extensive-form rationalizable strategies are obtained by the iterative application of the *er* reduction operator (Theorem 3.2), (b) the backward dominance strategies are obtained by the iterative application of the *bd* reduction operator (Theorem 3.3), (c) the extended backwards elimination order is an elimination order for *er* (Lemma 6.2), (d) the backwards elimination order is an elimination order for *bd* (Lemma 6.1), (e) the *er* reduction operator is order independent with respect to outcomes (Corollary 5.1), and (f) the *bd* reduction operator is order independent with respect to strategies (Corollary 4.1). This will now enable us to show that every outcome that is reachable by extensive-form rationalizable strategies is also reachable by backward dominance strategies.

Let $M := \sum_{i \in I, h \in H_i^*} |S_i(h)|$, so that $er^{M+1}(D) = er^M(D)$ and $bd^{M+1}(D) = bd^M(D)$ for every product of strategy sets D. Let $S^{er} := \times_{i \in I} S_i^{er}$ be the set of extensive-form rationalizable strategy combinations. Then, by (a) we have that

$$S^{er} = (er^M(S^*))_\emptyset. \tag{6.1}$$

Let $(D^0, ..., D^{K+M})$ be the extended backwards elimination order. Since we know by (c) that this is an elimination order for er, it follows by (e) and (6.1) that

$$Z(D_\emptyset^{K+M}) = Z((er^M(S^*))_\emptyset) = Z(S^{er}). \tag{6.2}$$

On the other hand, let $S^{bd} := \times_{i \in I} S_i^{bd}$ be the backward dominance strategy combinations. Then, we know by (b) that

$$S^{bd} = (bd^M(S^*))_\emptyset. \tag{6.3}$$

Consider the backwards elimination order $(D^0, ..., D^K)$. Since we know by (d) that this is an elimination order for bd, it follows by (f) and (6.3) that

$$D_\emptyset^K = (bd^M(S^*))_\emptyset = S^{bd},$$

which implies that

$$Z(D_\emptyset^K) = Z(S^{bd}). \tag{6.4}$$

As $D_\emptyset^{K+M} \subseteq D_\emptyset^K$, we have that $Z(D_\emptyset^{K+M}) \subseteq Z(D_\emptyset^K)$, and hence it follows from (6.2) and (6.4) that

$$Z(S^{er}) = Z(D_\emptyset^{K+M}) \subseteq Z(D_\emptyset^K) = Z(S^{bd}).$$

That is, every outcome reachable by an extensive-form rationalizable strategy combination in S^{er} is also reachable by a backward dominance strategy combination in S^{bd}. We thus obtain the following result.

Theorem 6.1 (EFR vs. backward dominance outcomes) *Every outcome that is reachable by a combination of extensive-form rationalizable strategies is also reachable by a combination of backward dominance strategies.*

Hence, in terms of reachable outcomes the concept of extensive-form rationalizability is always at least as restrictive as backward dominance. We believe this result is interesting from a reasoning perspective, as extensive-form rationalizability is a natural instance of *forward induction* reasoning, whereas backward dominance is a typical *backward induction* concept. Therefore, the result states that in terms of reachable outcomes, forward induction reasoning is more restrictive than backward induction reasoning, provided we identify forward and backward induction reasoning with these two concepts.

This result is not true in terms of strategies, however. To see this, consider the game in Figure 1. We have seen that the extensive-form rationalizable strategies are a for player 1 and (d, g) for player 2. The backward dominance strategies, in turn, coincide with the backward induction strategies in this game, which are a for player 1 and c for player 2. Hence, both concepts select a unique yet different strategy for player 2 in this game.

Battigalli and Siniscalchi (2002) have shown that extensive-form rationalizability is epistemically characterized by *common strong belief in rationality* within a complete type structure. Analogously, Perea (2014) has shown that the backward dominance strategies are epistemically characterized by *common belief in future rationality.* These insights, in combination with Theorem 6.1, imply that every outcome that is reachable under the epistemic conditions of common strong belief in rationality is also reachable under the epistemic conditions of common belief in future rationality. This is precisely the content of Theorem 9.4.2 in Perea (2012).

Now, consider a game with perfect information and without relevant ties (as defined in Battigalli (1997)). That is, at every non-terminal history there is only one active player, and for every player i, every non-terminal history $h \in H_i$, every two different choices $c_i, c_i' \in C_i(h)$, every terminal history z following c_i and every terminal history z' following c_i' we have that $u_i(z) \neq u_i(z')$. Theorem 6.1 in Perea (2014) shows that in every such game, the backward dominance strategies coincide with the unique backward induction strategies for every player. Consequently, the unique outcome reachable by a combination of backward dominance strategies is the backward induction outcome. By Theorem 6.1 in the present paper we may then conclude that the unique outcome reachable by extensive-form rationalizable strategies must be the backward induction outcome. This result, which is due to Battigalli (1997), is highly remarkable as extensive-form rationalizability is a forward induction concept for which the underlying reasoning is fundamentally different from backward induction reasoning.

Corollary 6.1 (Battigalli's theorem (1997)) *Consider a finite dynamic game with perfect information and without relevant ties. Then, the unique outcome reachable by a combination of extensive-form rationalizable strategies is the backward induction outcome.*

Alternative proofs for this result can be found in Battigalli (1997), Heifetz and Perea (2015) and Perea (2016). The result also follows from Chen and Micali (2013) who show that the iterated conditional dominance procedure (Shimoji and Watson (1998)) is order independent with respect to outcomes. As Shimoji and Watson (1998) have shown that their procedure characterizes the extensive-form rationalizable strategies, Battigalli's theorem follows from the latter two results in a similar fashion as outlined above.

7 Concluding Remarks

In this paper we have presented a new methodology to systematically explore the issue of order independence in dynamic games. To that purpose we have developed the general notion of a reduction operator for dynamic games, which enabled us to formally define order independence with respect to strategies and outcomes. We subsequently introduced the conditions of monotonicity and monotonicity on reachable histories, and showed that these imply order independence with respect to strategies and outcomes, respectively. We have used these tools to prove the order independence of various reduction procedures in dynamic games, and to explore the relationship, in terms of outcomes, between the extensive-form rationalizability procedure and the backward dominance procedure.

The results that have been proven in this paper were mostly known in one way or another. After all, the main objective of this paper was not to derive new results but rather to introduce a new general methodology for exploring the phenomenon of order independence in dynamic games. However, we are hopeful that the tools developed in this paper can be used in the future to prove new results for reduction procedures in dynamic games.

8 Proofs

Proof of Theorem 3.3. **(a)** We show the statement by induction on k. For $k = 0$, the statement is trivially true since $E_{ih}^{bd,0} = S_i(h) = S_i(h) \cap (bd^0(S^*))_i(H^{fut}(h))$ for every player i and every non-terminal history $h \in H$.

Take now some $k \geq 1$ and assume that $E_{ih}^{bd,k-1} = S_i(h) \cap (bd^{k-1}(S^*))_i(H^{fut}(h))$ for every player i and every non-terminal history $h \in H$. Take some player i and some non-terminal history $h \in H$. We show that (i) $E_{ih}^{bd,k} \subseteq S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h))$, and (ii) $S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h)) \subseteq E_{ih}^{bd,k}$.

(i) Take some $s_i \in E_{ih}^{bd,k}$. We show that $s_i \in S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h))$. Since, by definition, $s_i \in S_i(h)$, it only remains to show that $s_i \in (bd^k(S^*))_{ih'}$ for every $h' \in H_i^*(s_i) \cap H^{fut}(h)$.

Take some $h' \in H_i^*(s_i) \cap H^{fut}(h)$, and consider some $h'' \in H_i(s_i) \cap H^{fut}(h')$. Then, $h'' \in H_i(s_i) \cap H^{fut}(h)$. Since $s_i \in E_{ih}^{bd,k}$, we know from Lemma 8.14.6 in Perea (2012) that there is some conditional belief $b_i(h'') \in \Delta(E_{-ih''}^{bd,k-1})$ such that s_i is rational at h'' for $b_i(h'')$. By the induction assumption, $E_{-ih''}^{bd,k-1} = S_{-i}(h'') \cap (bd^{k-1}(S^*))_{-i}(H^{fut}(h''))$. Hence, s_i is rational at h'' for some $b_i(h'')$ that believes $(bd^{k-1}(S^*))_{-i}(H^{fut}(h''))$. Since this holds for every $h'' \in H_i(s_i) \cap H^{fut}(h')$, it follows that s_i is rational at $H^{fut}(h')$ for some conditional belief vector b_i where $b_i(h'')$ believes $(bd^{k-1}(S^*))_{-i}(H^{fut}(h''))$ for every $h'' \in H_i \cap H^{fut}(h')$.

Moreover, since $s_i \in E_{ih}^{bd,k}$ we know, by definition, that $s_i \in E_{ih}^{bd,k-1}$. Hence, by the induction assumption, $s_i \in (bd^{k-1}(S^*))_i(H^{fut}(h))$. Since $h' \in H_i^*(s_i) \cap H^{fut}(h)$, it follows that $s_i \in (bd^{k-1}(S^*))_{ih'}$.

We thus see that $s_i \in (bd^{k-1}(S^*))_{ih'}$ and s_i is rational at $H^{fut}(h')$ for some conditional belief vector b_i where $b_i(h'')$ believes $(bd^{k-1}(S^*))_{-i}(H^{fut}(h''))$ for every $h'' \in H_i \cap H^{fut}(h')$. Then, by definition, $s_i \in bd_{ih'}(bd^{k-1}(S^*))$. Since this holds for every $h' \in H_i^*(s_i) \cap H^{fut}(h)$, it follows that $s_i \in (bd^k(S^*))_i(H^{fut}(h))$. As we have already seen that $s_i \in S_i(h)$, it follows that $s_i \in S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h))$.

Since this holds for every $s_i \in E_{ih}^{bd,k}$, we conclude that $E_{ih}^{bd,k} \subseteq S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h))$.

(ii) Take some $s_i \in S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h))$. Then, in particular, $s_i \in S_i(h) \cap (bd^{k-1}(S^*))_i(H^{fut}(h))$, and hence by the induction assumption we have that $s_i \in E_{ih}^{bd,k-1}$. Now, take some $h' \in H_i(s_i) \cap H^{fut}(h)$. Since $s_i \in (bd^k(S^*))_i(H^{fut}(h))$, it follows that $s_i \in bd_{ih'}(bd^{k-1}(S^*))$. Hence, by definition, s_i is rational at $H^{fut}(h')$ for some b_i where $b_i(h'')$ believes $(bd^{k-1}(S^*))_{-i}(H^{fut}(h''))$ for all $h'' \in H_i \cap H^{fut}(h')$. Since $h' \in H_i(s_i) \cap H^{fut}(h')$, it follows that s_i is rational at h' for some $b_i(h') \in \Delta(S_{-i}(h) \cap (bd^{k-1}(S^*))_{-i}(H^{fut}(h')))$. By the induction assumption, $S_{-i}(h) \cap (bd^{k-1}(S^*))_{-i}(H^{fut}(h')) = E_{-ih'}^{bd,k-1}$. Hence, s_i is rational at h' for some $b_i(h') \in \Delta(E_{-ih'}^{bd,k-1})$. By Lemma 3 in Pearce (1984) it then follows that s_i is not strictly dominated in $E_{h'}^{bd,k-1}$.

Altogether, we see that $s_i \in E_{ih}^{bd,k-1}$ and that s_i is not strictly dominated in $E_{h'}^{bd,k-1}$ for every $h' \in H_i(s_i) \cap H^{fut}(h)$. Hence, by definition, $s_i \in E_{ih}^{bd,k}$. Since this holds for every $s_i \in S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h))$, it follows that $S_i(h) \cap (bd^k(S^*))_i(H^{fut}(h)) \subseteq E_{ih}^{bd,k}$. By induction on k, this completes the proof of (a).

(b) By (a) we know that $E_{i\emptyset}^{bd,k} = S_i(\emptyset) \cap (bd^k(S^*))_i(H^{fut}(\emptyset)) \subseteq (bd^k(S^*))_{i\emptyset}$. Hence, it only remains to show that $(bd^k(S^*))_{i\emptyset} \subseteq E_{i\emptyset}^{bd,k}$. We prove this by induction on k. For $k = 0$, the statement is true because $(bd^0(S^*))_{i\emptyset} = S_i = E_{i\emptyset}^{bd,0}$.

Take now some $k \geq 1$, and assume that $(bd^{k-1}(S^*))_{i\emptyset} \subseteq E_{i\emptyset}^{bd,k-1}$ for every player i. Consider some player i, and take some $s_i \in (bd^k(S^*))_{i\emptyset}$. Then, $s_i \in (bd^{k-1}(S^*))_{i\emptyset}$ and hence, by the induction assumption, $s_i \in E_{i\emptyset}^{bd,k-1}$.

As $s_i \in (bd^k(S^*))_{i\emptyset} = bd_{i\emptyset}(bd^{k-1}(S^*))$ we know, by definition, that s_i is rational at $H^{fut}(\emptyset)$ for some conditional belief vector b_i where $b_i(h)$ believes $(bd^{k-1}(S^*))_{-i}(H^{fut}(h))$ for every $h \in H_i \cap H^{fut}(\emptyset)$. Take some $h \in H_i(s_i) \cap H^{fut}(\emptyset)$. Then, s_i is rational at h for $b_i(h) \in \Delta(S_{-i}(h) \cap (bd^{k-1}(S^*))_{-i}(H^{fut}(h)))$. By (a) we know that $S_{-i}(h) \cap (bd^{k-1}(S^*))_{-i}(H^{fut}(h)) = E_{-ih}^{bd,k-1}$. Hence, s_i is rational at h for $b_i(h) \in \Delta(E_{-ih}^{bd,k-1})$. By Lemma 3 in Pearce (1984) it follows that s_i is not strictly dominated in $E_h^{bd,k-1}$.

We thus see that $s_i \in E_{i\emptyset}^{bd,k-1}$ and that s_i is not strictly dominated in $E_h^{bd,k-1}$ for every $h \in H_i(s_i) \cap H^{fut}(\emptyset)$. But then, by definition, $s_i \in E_{i\emptyset}^{bd,k}$. Since this holds for every $s_i \in (bd^k(S^*))_{i\emptyset}$, it follows that $(bd^k(S^*))_{i\emptyset} \subseteq E_{i\emptyset}^{bd,k}$. By induction on k, it follows that $(bd^k(S^*))_{i\emptyset} \subseteq E_{i\emptyset}^{bd,k}$ for every k. Since we have seen above that $E_{i\emptyset}^{bd,k} \subseteq (bd^k(S^*))_{i\emptyset}$ for every k, it follows that $E_{i\emptyset}^{bd,k} = (bd^k(S^*))_{i\emptyset}$ for every k. This completes the proof of (b).

From (b) it follows that

$$\cap_{k\geq 0}(bd^k(S^*))_\emptyset = \cap_{k\geq 0} \times_{i\in I} (bd^k(S^*))_{i\emptyset} = \cap_{k\geq 0} \times_{i\in I} E_{i\emptyset}^{bd,k} = \times_{i\in I} S_i^{bd},$$

and hence bd yields the set $\times_{i\in I} S_i^{bd}$ of backward dominance strategy combinations. ∎

Proof of Theorem 4.1. Consider some monotone reduction operator r. Take two products of strategy sets D and E. We prove the following property for r.

Claim 1. If E is possible in an elimination order for r, and D is a partial reduction of E, then $r(E)$ is a partial reduction of D.

Proof of Claim 1. Since r is monotone we know that $r(D) \subseteq r(E)$. Moreover, as D is a partial reduction of E it holds that $r(E) \subseteq D$. We thus see that $r(D) \subseteq r(E) \subseteq D$, and hence $r(E)$ is a partial reduction of D. The proof of the claim is thereby complete. $\diamondsuit$

Now, let $M := \sum_{i\in I, h\in H_i^*} |S_i(h)|$. Then, for every product of strategy sets D we necessarily have that $r^{M+1}(D) = r^M(D)$. Indeed, $r^{k+1}(D) \neq r^k(D)$ implies that there must be some player i, history $h \in H_i^*$ and strategy $s_i \in S_i(h)$ such that $s_i \in (r^k(D))_{ih} \backslash (r^{k+1}(D))_{ih}$.

Take some arbitrary elimination order $(D^0, ..., D^K)$ for r. We show that $D_\emptyset^K = (r^M(S^*))_\emptyset$, which would imply that r is order independent with respect to strategies. To prove this statement we first show the following claim.

Claim 2. Take some $k \in \{0, ..., K-1\}$ and $m \geq 0$. Then, $r^m(D^{k+1})$ is a partial reduction of $r^m(D^k)$ and $r^{m+1}(D^k)$ is a partial reduction of $r^m(D^{k+1})$.

Proof of Claim 2. We prove the statement by induction on m. We start with $m = 0$. By definition, $r^0(D^{k+1}) = D^{k+1}$ is a partial reduction of $r^0(D^k) = D^k$. Since D^k is possible in an elimination order for r and D^{k+1} is a partial reduction of D^k, it follows by Claim 1 that $r(D^k)$ is a partial reduction of D^{k+1}.

Consider now some $m \geq 1$, and suppose that the statement in Claim 2 is true for $m - 1$. Note that $r^{m-1}(D^{k+1})$ is possible in an elimination order for r, since D^{k+1} is part of the elimination order $(D^0, ..., D^K)$. Since, by the induction assumption, $r^m(D^k)$ is a partial reduction of $r^{m-1}(D^{k+1})$ it follows by Claim 1 that $r^m(D^{k+1})$ is a partial reduction of $r^m(D^k)$.

Note, by a similar argument as above, that also $r^m(D^k)$ is possible in an elimination order for r. Since we have seen that $r^m(D^{k+1})$ is a partial reduction of $r^m(D^k)$, it follows by Claim 1 that $r^{m+1}(D^k)$ is a partial reduction of $r^m(D^{k+1})$.

By induction on m, the proof of the claim is complete. $\diamondsuit$

By the claim we know, for every $k \in \{0, ..., K-1\}$, that $r^M(D^{k+1})$ is a partial reduction of $r^M(D^k)$, and $r^{M+1}(D^k)$ is a partial reduction of $r^M(D^{k+1})$. This implies that $r^M(D^{k+1}) \subseteq r^M(D^k)$ and $r^{M+1}(D^k) \subseteq r^M(D^{k+1})$. Since $r^{M+1}(D^k) = r^M(D^k)$, it follows that $r^M(D^k) = r^M(D^{k+1})$.

Since this holds for every $k \in \{0, ..., K-1\}$, we conclude that $r^M(D^0) = r^M(D^K)$. As $D^0 = S^*$ and $r(D^K) = D^K$, it follows that $r^M(D^0) = r^M(S^*)$ and $r^M(D^K) = D^K$. Therefore, $D^K = r^M(S^*)$ and hence $D^K_\emptyset = (r^M(S^*))_\emptyset$. This completes the proof. ∎

Proof of Theorem 4.2. **(a)** Consider first the reduction operator bd. Take two products of strategy sets D and E such that $D \subseteq E$. We will show that $bd(D) \subseteq bd(E)$.

To that purpose, take some player i, some history $h \in H_i^*$ and some strategy $s_i \in bd_{ih}(D)$. Then, by definition, $s_i \in D_{ih}$ and s_i is rational at $H^{fut}(h)$ for some conditional belief vector b_i where $b_i(h') \in \Delta(D_{-i}(H^{fut}(h')))$ for all $h' \in H_i \cap H^{fut}(h)$. Since $D \subseteq E$ it follows that $D_{ih} \subseteq E_{ih}$ and $D_{-i}(H^{fut}(h')) \subseteq E_{-i}(H^{fut}(h'))$ for all $h' \in H_i \cap H^{fut}(h)$. Hence, $s_i \in E_{ih}$ and s_i is rational at $H^{fut}(h)$ for some conditional belief vector b_i where $b_i(h') \in \Delta(E_{-i}(H^{fut}(h')))$ for all $h' \in H_i \cap H^{fut}(h)$. By definition, this means that $s_i \in bd_{ih}(E)$. Since this holds for every player i, every history $h \in H_i^*$ and every strategy $s_i \in bd_{ih}(D)$, it follows that $bd(D) \subseteq bd(E)$.

Hence, $bd(D) \subseteq bd(E)$ for every two products of strategy sets D and E where $D \subseteq E$. In particular, this holds whenever E is possible in an elimination order for bd, and D is a partial reduction of E. Therefore, bd is monotone.

(b) Consider next the reduction operator rG for some fixed but arbitrary collection of histories $G \subseteq H$. Again, take two products of strategy sets D and E such that $D \subseteq E$. We will show that $rG(D) \subseteq rG(E)$.

Since $rG_{ih}(D) = \emptyset$ for every player i and every history $h \in H_i \backslash \{\emptyset\}$, it immediately follows that $rG_{ih}(D) \subseteq rG_{ih}(E)$ for all players i and all histories $h \in H_i \backslash \{\emptyset\}$.

It therefore remains to show that $rG_{i\emptyset}(D) \subseteq rG_{i\emptyset}(E)$ for all players i. Take some player i and some $s_i \in rG_{i\emptyset}(D)$. Then, $s_i \in D_{i\emptyset}$ and s_i is rational for some conditional belief vector b_i

where $b_i(h)$ believes $D_{-i\emptyset}$ whenever $h \in H_i \cap G$. Since $D \subseteq E$ we have that $D_{i\emptyset} \subseteq E_{i\emptyset}$ and $D_{-i\emptyset} \subseteq E_{-i\emptyset}$. Therefore, $s_i \in E_{i\emptyset}$ and s_i is rational for some conditional belief vector b_i where $b_i(h)$ believes $E_{-i\emptyset}$ whenever $h \in H_i \cap G$. We thus conclude that $s_i \in rG_{i\emptyset}(E)$.

Since this holds for every $s_i \in rG_{i\emptyset}(D)$, it follows that $rG_{i\emptyset}(D) \subseteq rG_{i\emptyset}(E)$. As we have already seen that $rG_{ih}(D) \subseteq rG_{ih}(E)$ for all players i and all histories $h \in H_i \backslash \{\emptyset\}$, it follows that $rG(D) \subseteq rG(E)$.

Hence, $rG(D) \subseteq rG(E)$ for all products of strategy sets D and E where $D \subseteq E$. In particular, this holds whenever E is possible in an elimination order for rG, and D is a partial reduction of E. Therefore, rG is monotone. This completes the proof. ■

Proof of Theorem 5.1. Consider some reduction operator r that is monotone on reachable histories. For every two products of strategy sets D and E we say that D is a *partial reduction on reachable histories of* E if

$$r(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)} \subseteq E|_{H(D_\emptyset)}.$$

We prove the following property of r.

Claim 1. If E is possible in an elimination order for r and D is a partial reduction on reachable histories of E, then $r(E)$ is a partial reduction on reachable histories of D.

Proof of Claim 1. Since D is a partial reduction on reachable histories of E, we have, by definition, that

$$r(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)} \subseteq E|_{H(D_\emptyset)}.$$

As r is monotone on reachable histories, it follows that

$$r(D)|_{H(D_\emptyset)} \subseteq r(E)|_{H(D_\emptyset)}.$$

Together with the assumption that $r(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)}$, we conclude that

$$r(D)|_{H(D_\emptyset)} \subseteq r(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)}. \tag{8.1}$$

Since $r(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)}$ it follows that $r(E)_\emptyset|_{H(D_\emptyset)} \subseteq D_\emptyset|_{H(D_\emptyset)}$. By Lemma 5.1 in Perea (2016) we may then conclude that $H(r(E)_\emptyset) \subseteq H(D_\emptyset)$. Together with (8.1) this yields

$$r(D)|_{H(r(E)_\emptyset)} \subseteq r(E)|_{H(r(E)_\emptyset)} \subseteq D|_{H(r(E)_\emptyset)},$$

and hence $r(E)$ is a partial reduction on reachable histories of D. This completes the proof of Claim 1. ◊

Let $M := \sum_{i \in I, h \in H_i^*} |S_i(h)|$. Then, as we have seen in the proof of Theorem 4.1, for every product of strategy sets D we necessarily have that $r^{M+1}(D) = r^M(D)$.

Take some arbitrary elimination order $(D^0, ..., D^K)$ for r. We show that $Z(D^K_\emptyset) = Z(r^M(S^*)_\emptyset)$, which would imply that r is order independent with respect to outcomes. To prove this statement we first show the following claim.

Claim 2. Take some $k \in \{0, ..., K-1\}$ and $m \geq 0$. Then, $r^m(D^{k+1})$ is a partial reduction on reachable histories of $r^m(D^k)$ and $r^{m+1}(D^k)$ is a partial reduction on reachable histories of $r^m(D^{k+1})$.

Proof of Claim 2. We prove the statement by induction on m. We start with $m = 0$. By definition, $r^0(D^{k+1}) = D^{k+1}$ is a partial reduction of $r^0(D^k) = D^k$, and hence, in particular, $r^0(D^{k+1})$ is a partial reduction on reachable histories of $r^0(D^k)$. Since D^k is possible in an elimination order for r and D^{k+1} is a partial reduction on reachable histories of D^k, it follows by Claim 1 that $r(D^k)$ is a partial reduction on reachable histories of D^{k+1}.

Consider now some $m \geq 1$, and suppose that the statement in Claim 2 is true for $m-1$. Note that $r^{m-1}(D^{k+1})$ is possible in an elimination order for r, since D^{k+1} is part of the elimination order $(D^0, ..., D^K)$. Since, by the induction assumption, $r^m(D^k)$ is a partial reduction on reachable histories of $r^{m-1}(D^{k+1})$ it follows by Claim 1 that $r^m(D^{k+1})$ is a partial reduction on reachable histories of $r^m(D^k)$.

Note, by a similar argument as above, that also $r^m(D^k)$ is possible in an elimination order for r. Since we have seen that $r^m(D^{k+1})$ is a partial reduction on reachable histories of $r^m(D^k)$, it follows by Claim 1 that $r^{m+1}(D^k)$ is a partial reduction on reachable histories of $r^m(D^{k+1})$.

By induction on m, the proof of the claim is complete. $\Diamond$

By the claim we know, for every $k \in \{0, ..., K-1\}$, that $r^M(D^{k+1})$ is a partial reduction on reachable histories of $r^M(D^k)$, and $r^{M+1}(D^k)$ is a partial reduction on reachable histories of $r^M(D^{k+1})$. This implies that

$$r^M(D^{k+1})|_{H(r^M(D^{k+1})_\emptyset)} \subseteq r^M(D^k)|_{H(r^M(D^{k+1})_\emptyset)}$$

and

$$r^{M+1}(D^k)|_{H(r^{M+1}(D^k)_\emptyset)} \subseteq r^M(D^{k+1})|_{H(r^{M+1}(D^k)_\emptyset)}.$$

Hence, in particular,

$$r^M(D^{k+1})_\emptyset|_{H(r^M(D^{k+1})_\emptyset)} \subseteq r^M(D^k)_\emptyset|_{H(r^M(D^{k+1})_\emptyset)}$$

and

$$r^{M+1}(D^k)_\emptyset|_{H(r^{M+1}(D^k)_\emptyset)} \subseteq r^M(D^{k+1})_\emptyset|_{H(r^{M+1}(D^k)_\emptyset)}.$$

By Lemma 5.1 in Perea (2016) it follows that $H(r^M(D^{k+1})_\emptyset) \subseteq H(r^M(D^k)_\emptyset)$ and $H(r^{M+1}(D^k)_\emptyset) \subseteq H(r^M(D^{k+1})_\emptyset)$. Since $r^{M+1}(D^k) = r^M(D^k)$, it follows that $H(r^M(D^{k+1})_\emptyset) = H(r^M(D^k)_\emptyset)$. In particular, this implies that $Z(r^M(D^{k+1})_\emptyset) = Z(r^M(D^k)_\emptyset)$.

Since this holds for every $k \in \{0, ..., K-1\}$, we conclude that $Z(r^M(D^0)_\emptyset) = Z(r^M(D^K)_\emptyset)$. As $D^0 = S^*$ and $r(D^K) = D^K$, it follows that $Z(r^M(D^0)_\emptyset) = Z(r^M(S^*)_\emptyset)$ and $Z(r^M(D^K)_\emptyset) = Z(D^K_\emptyset)$. Therefore, $Z(D^K_\emptyset) = Z(r^M(S^*)_\emptyset)$. This completes the proof. ■

Proof of Theorem 5.3. Take two products of strategy sets D and E where E is possible in an elimination order for er and

$$er(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)} \subseteq E|_{H(D_\emptyset)}.$$

We will show that

$$er(D)|_{H(D_\emptyset)} \subseteq er(E)|_{H(D_\emptyset)}.$$

By definition, $er_{ih}(D) = \emptyset$ for all players i and all histories $h \in H_i \backslash \{\emptyset\}$. Therefore, we trivially have that $er_{ih}(D)|_{H(D_\emptyset)} \subseteq er_{ih}(E)|_{H(D_\emptyset)}$ for all players i and all histories $h \in H_i \backslash \{\emptyset\}$. It thus remains to show that $er_\emptyset(D)|_{H(D_\emptyset)} \subseteq er_\emptyset(E)|_{H(D_\emptyset)}$.

Since E is possible in an elimination order for er, there is an elimination order $(D^0, ..., D^K)$ for er such that $E = D^k$ for some $k \in \{0, ..., K\}$. We show that $(D^0_\emptyset, ..., D^K_\emptyset)$ is an elimination order for sb.

By definition, $D^0_\emptyset = \times_{i \in I} S_i$ since $D^0 = S^* = \times_{i \in I, h \in H^*_i} S_i(h)$. Take some $k \in \{0, ..., K-1\}$. Since D^{k+1} is a partial reduction of D^k with respect to er, we have that $er_\emptyset(D^k) \subseteq D^{k+1}_\emptyset \subseteq D^k_\emptyset$. Recall from (3.1) that $er_\emptyset(D^k) = sb(D^k_\emptyset)$, and hence $sb(D^k_\emptyset) \subseteq D^{k+1}_\emptyset \subseteq D^k_\emptyset$. This means that $D^{k+1}_\emptyset$ is a partial reduction of $D^k_\emptyset$ with respect to sb. Finally, by (3.1) we know that $sb(D^K_\emptyset) = er_\emptyset(D^K) = D^K_\emptyset$, since $er(D^K) = D^K$. We thus conclude that $(D^0_\emptyset, ..., D^K_\emptyset)$ is an elimination order for sb. Since $E = D^k$ we know, in particular, that $E_\emptyset = D^k_\emptyset$ and therefore $E_\emptyset$ is possible in an elimination order for sb.

Moreover, since $er(E)|_{H(D_\emptyset)} \subseteq D|_{H(D_\emptyset)} \subseteq E|_{H(D_\emptyset)}$ we know that $er_\emptyset(E)|_{H(D_\emptyset)} \subseteq D_\emptyset|_{H(D_\emptyset)} \subseteq E_\emptyset|_{H(D_\emptyset)}$. By (3.1) it holds that $er_\emptyset(E) = sb(E_\emptyset)$, and we therefore conclude that

$$sb(E_\emptyset)|_{H(D_\emptyset)} \subseteq D_\emptyset|_{H(D_\emptyset)} \subseteq E_\emptyset|_{H(D_\emptyset)}. \tag{8.2}$$

By Theorem 5.2 we know that sb is monotone on reachable histories. Recall that $E_\emptyset$ is possible in an elimination order for sb. It then follows from (8.2) that

$$sb(D_\emptyset)|_{H(D_\emptyset)} \subseteq sb(E_\emptyset)|_{H(D_\emptyset)}.$$

As $sb(D_\emptyset) = er_\emptyset(D)$ and $sb(E_\emptyset) = er_\emptyset(E)$ we conclude that

$$er_\emptyset(D)|_{H(D_\emptyset)} \subseteq er_\emptyset(E)|_{H(D_\emptyset)}.$$

Since we have seen above that $er_{ih}(D)|_{H(D_\emptyset)} \subseteq er_{ih}(E)|_{H(D_\emptyset)}$ for all players i and all histories $h \in H_i \backslash \{\emptyset\}$, it follows that

$$er(D)|_{H(D_\emptyset)} \subseteq er(E)|_{H(D_\emptyset)}.$$

We thus conclude that er is monotone on reachable histories. This completes the proof. ■

Proof of Lemma 6.1. By definition, $D^0 = S^*$. We next show that $bd(D^K) = D^K$. That is, we must show for every player i and every $h \in H_i^*$ that $bd_{ih}(D^K) = D_{ih}^K$. Take some player i and some history $h \in H_i^*$. Suppose that $h \in H^l$ for some $l \in \{0, ..., L\}$. Then, by construction, D_{ih}^k does not change anymore when $l(k) > l$, which occurs precisely when $k > l \cdot M + M$. Hence

$$D_{ih}^K = D_{ih}^{l \cdot M + M} = ((bd[l])^M (D^{l \cdot M}))_{ih}. \tag{8.3}$$

Moreover, since $h \in H^l$ we have that

$$\begin{aligned} bd_{ih}(D^K) &= bd[l]_{ih}(D^K) = bd[l]_{ih}(D^{l \cdot M + M}) = bd[l]_{ih}((bd[l])^M (D^{l \cdot M})) \\ &= ((bd[l])^M (D^{l \cdot M}))_{ih} = D_{ih}^K. \end{aligned}$$

Here, the first equality holds by definition of the operator $bd[l]$, the third equality by construction of the backwards elimination order, the fourth equality by the fact that $(bd[l])^{M+1}(D^{l \cdot M}) = (bd[l])^M (D^{l \cdot M})$, and the last equality by (8.3). Hence, $bd_{ih}(D^K) = D_{ih}^K$. Since this holds for every player i and every history $h \in H_i^*$, we have that $bd(D^K) = D^K$.

We finally show, for every $k \in \{0, ..., K-1\}$, that D^{k+1} is a partial reduction of D^k. That is, we must show that $bd(D^k) \subseteq D^{k+1} \subseteq D^k$. By definition we have that $D^{k+1} \subseteq D^k$, hence it only remains to show that $bd(D^k) \subseteq D^{k+1}$. By construction, $D^{k+1} = bd[l(k+1)](D^k)$. Since, by definition, $bd(D^k) \subseteq bd[l(k+1)](D^k)$ it follows that $bd(D^k) \subseteq D^{k+1}$. Hence, D^{k+1} is a partial reduction of D^k.

Altogether, we see that $(D^0, ..., D^K)$ is an elimination order for bd, which was to show. ■

Proof of Lemma 6.2. By definition we have that $D^0 = S^*$. Moreover,

$$er(D^{K+M}) = er(er^M(D^K)) = er^{M+1}(D^K) = er^M(D^K) = D^{K+M}.$$

It thus remains to show that D^{k+1} is a partial reduction of D^k (with respect to er) for every $k \in \{0, ..., K+M-1\}$. By construction, this holds for every $k \in \{K, ..., K+M-1\}$. It therefore suffices to show that D^{k+1} is a partial reduction of D^k for every $k \in \{0, ..., K-1\}$.

Take some $k \in \{0, ..., K-1\}$. We will show, for every player i and every history $h \in H_i^*$, that

$$er_{ih}(D^k) \subseteq D_{ih}^{k+1} \subseteq D_{ih}^k.$$

By construction we have that $D_{ih}^{k+1} \subseteq D_{ih}^k$. Moreover, by definition, it holds that $er_{ih}(D^k) = \emptyset$ whenever $h \neq \emptyset$. It thus remains to show that $er_{i\emptyset}(D^k) \subseteq D_{i\emptyset}^{k+1}$. Since, by definition, $D^{k+1} = bd[l(k+1)](D^k)$, we must show that

$$er_{i\emptyset}(D^k) \subseteq bd[l(k+1)]_{i\emptyset}(D^k). \tag{8.4}$$

Take some strategy $s_i \in er_{i\emptyset}(D^k)$. That is, $s_i \in D^k_{i\emptyset}$ and s_i is rational at $H(D^k_\emptyset)$ for some conditional belief vector $b_i \in B_i$ that strongly believes $D^k_{-i\emptyset}$. In order to show that $s_i \in bd[l(k+1)]_{i\emptyset}(D^k)$ we must prove that s_i is rational at $H^{l(k+1)}$ for some conditional belief vector b_i where $b_i(h)$ believes $D^k_{-i}(H^{fut}(h))$ for all $h \in H_i \cap H^{l(k+1)}$. That is, for every $h \in H_i(s_i) \cap H^{l(k+1)}$ we must find a conditional belief $b_i(h)$ that believes $D^k_{-i}(H^{fut}(h))$ and such that s_i is rational for $b_i(h)$ at h. We distinguish two cases: (i) $h \notin H^{l(k)-1}$, and (ii) $h \in H^{l(k)-1}$.

(i) Consider first some $h \in (H_i(s_i) \cap H^{l(k+1)}) \backslash H^{l(k)-1}$. Since h is not in $H^{l(k)-1}$, we know that all histories preceding h are not in $l(k)$. As, by construction of the backwards elimination order, $D^k_\emptyset$ only restricts the strategies at histories in $H^{l(k)}$, it follows that $h \in H(D^k_\emptyset)$. Recall from above that s_i is rational at $H(D^k_\emptyset)$ for some conditional belief vector $b_i \in B_i$ that strongly believes $D^k_{-i\emptyset}$. Since $h \in H(D^k_\emptyset)$ it follows that $S_{-i}(h) \cap D^k_{-i\emptyset} \neq \emptyset$, and therefore $b_i(h)$ believes $D^k_{-i\emptyset}$ if b_i strongly believes $D^k_{-i\emptyset}$. Hence, at $h \in H(D^k_\emptyset)$ strategy s_i is rational for a conditional belief $b_i(h)$ that believes $D^k_{-i\emptyset}$. By definition, $D^k_{-i\emptyset} \subseteq D^k_{-i}(H^{fut}(h))$, and hence s_i is rational at h for a conditional belief $b_i(h)$ that believes $D^k_{-i}(H^{fut}(h))$. This holds for every $h \in (H_i(s_i) \cap H^{l(k+1)}) \backslash H^{l(k)-1}$, and hence for every $h \in (H_i(s_i) \cap H^{l(k+1)}) \backslash H^{l(k)-1}$ there is a conditional belief $b_i(h)$ that believes $D^k_{-i}(H^{fut}(h))$ and such that s_i is rational for $b_i(h)$ at h.

(ii) Consider next some $h \in H_i(s_i) \cap H^{l(k)-1}$. Since $s_i \in D^k_{i\emptyset}$ and $D^k = bd[l(k)](D^{k-1})$ we have that $s_i \in bd[l(k)]_{i\emptyset}(D^{k-1})$. Hence, s_i is rational at $H^{l(k)}$ for some conditional belief vector b_i where $b_i(h')$ believes $D^{k-1}_{-i}(H^{fut}(h'))$ for all $h' \in H_i \cap H^{l(k)}$. Since $h \in H_i(s_i) \cap H^{l(k)-1}$ and $H^{l(k)-1} \subseteq H^{l(k)}$ we know that $h \in H_i(s_i) \cap H^{l(k)}$, and therefore s_i is rational at h for the belief $b_i(h)$ that believes $D^{k-1}_{-i}(H^{fut}(h))$.

Let $k = l(k) \cdot M + m$ for some $m \in \{1, ..., M\}$. Then, $k-1 \geq l(k) \cdot M = (l(k)-1) \cdot M + M$. As for every player j and every $h' \in H^*_j \cap H^{l(k)-1}$, the set $D^{k'}_{jh'}$ does not change anymore for $k' \geq (l(k)-1) \cdot M + M$, it follows that $D^{k-1}_{jh'} = D^k_{jh'}$ for every player j and every $h' \in H^*_j \cap H^{l(k)-1}$. Since $h \in H^{l(k)-1}$, it follows that every $h' \in H^{fut}(h)$ is in $H^{l(k)-1}$, and hence $D^{k-1}_{-i}(H^{fut}(h)) = D^k_{-i}(H^{fut}(h))$. Recall that s_i is rational at h for the belief $b_i(h)$ that believes $D^{k-1}_{-i}(H^{fut}(h))$. We thus conclude that s_i is rational at h for the belief $b_i(h)$ that believes $D^k_{-i}(H^{fut}(h))$, which was to show.

By combining (i) and (ii) we see that for every $h \in H_i(s_i) \cap H^{l(k+1)}$ there is a conditional belief $b_i(h)$ that believes $D^k_{-i}(H^{fut}(h))$ and such that s_i is rational for $b_i(h)$ at h. Hence, s_i is rational at $H^{l(k+1)}$ for some conditional belief vector b_i where $b_i(h)$ believes $D^k_{-i}(H^{fut}(h))$ for all $h \in H^{l(k+1)}$. Since we have seen above that $s_i \in D^k_{i\emptyset}$, we conclude that $s_i \in bd[l(k+1)]_{i\emptyset}(D^k)$. As this holds for every $s_i \in er_{i\emptyset}(D^k)$, it follows that $er_{i\emptyset}(D^k) \subseteq bd[l(k+1)]_{i\emptyset}(D^k)$, and hence (8.4) is true for every player i. As we have seen above, this implies that D^{k+1} is a partial reduction of D^k (with respect to er). Hence, we conclude that the extended backwards elimination order $(D^0, ..., D^{K+M})$ is an elimination order for er. This completes the proof. ■

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